Function Spaces on Domains
(Selected topics)

Hans Triebel

University of Jena

Praha, Oct. 2011

$$\varphi_0 \in S(\mathbb{R}^n), \quad \varphi_0(x) = 1 \text{ if } |x| \leq 1, \quad \varphi_0(y) = 0 \text{ if } |y| \geq 3/2.$$

$$\varphi_k(x) = \varphi_0(2^{-k}x) - \varphi_0(2^{-k+1}x), \qquad k \in \mathbb{N}.$$

Dyadic resolution of unity: $\sum_{j=0}^{\infty} \varphi_j(x) = 1$.

Definition 1. $s \in \mathbb{R}$, $0 < p, q \leq \infty$. Then $B^s_{pq}(\mathbb{R}^n)$ collects all $f \in S'(\mathbb{R}^n)$ with

$$\|f \,|B^s_{pq}(\mathbb{R}^n)\| = \Big(\sum_{j=0}^{\infty} 2^{jsq} \big\| (\varphi_j \widehat{f})^\vee \,|L_p(\mathbb{R}^n) \big\|^q \Big)^{1/q} < \infty.$$

$s \in \mathbb{R}$, $0 < p < \infty$, $0 < q \leq \infty$. Then $F^s_{pq}(\mathbb{R}^n)$ collects all $f \in S'(\mathbb{R}^n)$ with

$$\|f \,|F^s_{pq}(\mathbb{R}^n)\| = \Big\| \Big(\sum_{j=0}^{\infty} 2^{jsq} \big| (\varphi_j \widehat{f})^\vee(\cdot) \big|^q \Big)^{1/q} |L_p(\mathbb{R}^n) \Big\| < \infty.$$

Remark 2. Recall: $S'(\mathbb{R}^n)$ tempered distributions in $\mathbb{R}^n$.

$$\Big(\int_\Omega |f(x)|^p \,\mathrm{d}x \Big)^{1/p} = \|f \,|L_p(\Omega)\|, \qquad \Omega \text{ domain in } \mathbb{R}^n.$$

$$\widehat{f}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\xi} f(x) \,\mathrm{d}x, \qquad Ff = \widehat{f}.$$

$F^{-1}f = f^\vee$: replace $-i$ by i, $(\varphi_j \widehat{f})^\vee(x)$ entire analytic function.

$B^s_{pq}(\mathbb{R}^n)$, $F^s_{pq}(\mathbb{R}^n)$ quasi-Banach spaces, independent of $\varphi = \{\varphi_j\}_{j=0}^\infty$.

Special cases:

$$(\Delta_h^1 f)(x) = f(x+h) - f(x), \quad \Delta_h^{k+1} f = \Delta_h^1 \Delta_h^k f, \qquad x \in \mathbb{R}^n, \quad h \in \mathbb{R}^n, \quad k \in \mathbb{N}.$$

Hölder-Zygmund spaces

$$\mathcal{C}^s(\mathbb{R}^n) = B^s_{\infty,\infty}(\mathbb{R}^n), \qquad s \in \mathbb{R}.$$

If $0 < s < m \in \mathbb{N}$ then (equivalent norms)

$$\|f \,|\mathcal{C}^s(\mathbb{R}^n)\|_m \sim \sup_{x \in \mathbb{R}^n} |f(x)| + \sup_{x \in \mathbb{R}^n; 0 < |h| \le 1} |h|^{-s} \big|\Delta_h^m f(x)\big|.$$

Besov spaces. $0 < p, q \le \infty$, $n\Big(\max(\frac{1}{p}, 1) - 1\Big) = \sigma_p < s < m \in \mathbb{N}$. Then

$$\|f \,|B^s_{pq}(\mathbb{R}^n)\| \sim \|f \,|L_p(\mathbb{R}^n)\| + \Big(\int_{|h| \le 1} |h|^{-sq} \big\|\Delta_h^m f \,|L_p(\mathbb{R}^n)\big\|^q \frac{\mathrm{d}h}{|h|^n}\Big)^{1/q}$$

$$\sim \|f \,|L_p(\mathbb{R}^n)\| + \Big(\int_0^1 t^{-sq} \sup_{|h| \le t} \big\|\Delta_h^m f \,|L_p(\mathbb{R}^n)\big\|^q \frac{\mathrm{d}t}{t}\Big)^{1/q}.$$

Classical Sobolev spaces. Littlewood-Paley assertion:

$$F^0_{p,2}(\mathbb{R}^n) = L_p(\mathbb{R}^n), \qquad 1 < p < \infty.$$

If $1 < p < \infty$ and $k \in \mathbb{N}_0$ then

$$F^k_{p,2}(\mathbb{R}^n) = W^k_p(\mathbb{R}^n),$$

equivalently normed by

$$\|f \,|W^k_p(\mathbb{R}^n)\| = \sum_{|\alpha| \le k} \|D^\alpha f \,|L_p(\mathbb{R}^n)\|.$$

$\Omega \subset \mathbb{R}^n$ arbitrary domain (= open set), $\Omega \neq \mathbb{R}^n$, $g \in S'(\mathbb{R}^n)$, restriction to Ω:

$$g|\Omega \in D'(\Omega), \quad (g|\Omega)(\varphi) = g(\varphi), \quad \varphi \in D(\Omega).$$

Definition 3. (i) $A \in \{B, F\}$, $0 < p, q \leq \infty$, ($p < \infty$ for F-spaces), $s \in \mathbb{R}$. Then

$$A^s_{pq}(\Omega) = \big\{ f \in D'(\Omega) : \ f = g|\Omega \text{ for some } g \in A^s_{pq}(\mathbb{R}^n) \big\},$$

$$\| f \,|A^s_{pq}(\Omega) \| = \inf_{f = g|\Omega} \| g \,| A^s_{pq}(\mathbb{R}^n) \|.$$

(ii)

$$\widetilde{A}^s_{pq}(\overline{\Omega}) = \big\{ f \in A^s_{pq}(\mathbb{R}^n) : \ \mathrm{supp}\ f \subset \overline{\Omega} \big\},$$

$$\widetilde{A}^s_{pq}(\Omega) = \big\{ f \in D'(\Omega) : \ f = g|\Omega, \ g \in \widetilde{A}^s_{pq}(\overline{\Omega}) \big\},$$

$$\| f \,|\widetilde{A}^s_{pq}(\Omega) \| = \inf_{f = g|\Omega} \| g \,| \widetilde{A}^s_{pq}(\overline{\Omega}) \|.$$

(iii) $\mathring{A}^s_{pq}(\Omega)$ is the completion of $D(\Omega)$ in $A^s_{pq}(\Omega)$.

Remark 4. **Problems**: Extension,

$$\mathrm{ext} : \quad A^s_{pq}(\Omega) \hookrightarrow A^s_{pq}(\mathbb{R}^n), \qquad \mathrm{re} \circ \mathrm{ext} = \mathrm{id},$$

intrinsic descriptions, traces, decompositions, wavelet bases.

$\Omega \subset \mathbb{R}^n$ I-thick domain (interior thick): for any exterior cube $Q^e \subset \mathbb{R}^n \setminus \Omega$,

$$\ell(Q^e) \sim 2^{-j}, \quad \mathrm{dist}\,(Q^e, \Gamma) \sim 2^{-j} \quad \text{for all } j \in \mathbb{N},\ j \geq j_0 \in \mathbb{N},$$

where $\Gamma = \partial\Omega$, there is an interior cube $Q^i \subset \Omega$ with

$$\ell(Q^i) \sim 2^{-j}, \qquad \mathrm{dist}\,(Q^i, \Gamma) \sim \mathrm{dist}\,(Q^e, Q^i) \sim 2^{-j}.$$

Examples: bounded Lipschitz domains, C^∞ domains, but also Γ some fractals (snowflake etc.)

(Real) Lipschitz function h on $\mathbb{R}^{n-1}$, ($n \geq 2$):

$$|h(x') - h(y')| \leq c\,|x' - y'|, \qquad x', y' \in \mathbb{R}^{n-1}.$$

Special (graph) Lipschitz domain in $\mathbb{R}^n$,

$$x_n > h(x'), \qquad x = (x', x_n).$$

Bounded Lipschitz domain: $\Gamma = \partial\Omega$ covered by finitely many balls B_j, centred at Γ with

$$B_j \cap \Omega = B_j \cap \Omega_j, \qquad \Omega_j \text{ rotation of a special Lipschitz domain.}$$

Bounded C^∞-domain: same with $h \in C^\infty$.

A: quasi-Banach space. $P: A \hookrightarrow A$ projection if $P = P^2$. A subspace A_0 of A is called complemented if there is a projection P with $A_0 = PA$.

Theorem 5. Ω arbitrary domain in $\mathbb{R}^n$. $A^s_{pq}(\Omega)$ has the extension property if, and only if, $\widetilde{A}^s_{pq}(\Omega^c)$ is a complemented subspace of $A^s_{pq}(\mathbb{R}^n)$.

Remark 6. $\Omega^c = \mathbb{R}^n \setminus \Omega$ closed set.

$$\sigma_p = n\Big(\frac{1}{\min(p,1)} - 1\Big), \qquad \sigma_{pq} = n\Big(\frac{1}{\min(p,q,1)} - 1\Big).$$

Theorem 7. Ω I-thick in $\mathbb{R}^n$ ($n = 1$ bounded interval), $\overline{\Omega} \neq \mathbb{R}^n$, $|\Gamma| = 0$. For any $u > 0$ there is a common extension operator ext^u,

$$\mathrm{ext}^u : \quad B^s_{pq}(\Omega) \hookrightarrow B^s_{pq}(\mathbb{R}^n), \qquad \sigma_p < s < u,$$

$$\mathrm{ext}^u : \quad F^s_{pq}(\Omega) \hookrightarrow F^s_{pq}(\mathbb{R}^n), \qquad \sigma_{pq} < s < u.$$

Remark 8. Proof by wavelet expansions in $\widetilde{A}^s_{pq}(\Omega^c)$. Applies to fractal boundaries, snowflake. $s < 0$: E-thick (exterior-thick, same as I-thick, changing roles of Q^i and Q^e).

Theorem 9. (Rychkov, 1999) Ω bounded Lipschitz domain in $\mathbb{R}^n$. There is a universal extension operator ext for all spaces $A^s_{pq}(\Omega)$,

$$\mathrm{ext}: \quad A^s_{pq}(\Omega) \hookrightarrow A^s_{pq}(\mathbb{R}^n).$$

Remark 10. Proof by local means.

Ω domain in $\mathbb{R}^n$.

$$(\Delta^M_{h,\Omega} f)(x) = \begin{cases} (\Delta^M_h f)(x), & x + kh \in \Omega \text{ for all } k = 0, \ldots, M, \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 11. Ω bounded Lipschitz domain in $\mathbb{R}^n$. $0 < p, q \le \infty$, $\sigma_p < s < M \in \mathbb{N}$. Then $B^s_{pq}(\Omega)$ collects all $f \in L_{\overline{p}}(\Omega)$ with $\overline{p} = \max(1, p)$ with

$$\|f \,|L_{\overline{p}}(\Omega)\| + \Big(\int_0^1 t^{-sq} \sup_{|h| \le t} \big\| \Delta^M_{h,\Omega} f \,|L_p(\Omega) \big\|^q \frac{\mathrm{d}t}{t} \Big)^{1/q} < \infty$$

(equivalent quasi-norms).

Remark 12. Dispa 2003. But essentially covered by DeVore-Sharpley 1993 in the context of $L_p(\Omega)$, $0 < p < \infty$.

Ω E-thick domain in $\mathbb{R}^n$: Same as I-thick, but Q^i and Q^e changing roles.

Definition 13. Ω arbitrary domain in $\mathbb{R}^n$, $1 < p < \infty$, $k \in \mathbb{N}$.

(i) $W_p^k(\Omega)$ collects all $f \in L_p(\Omega)$ with

$$\|f \,|W_p^k(\Omega)\| = \sum_{|\alpha| \le k} \|D^\alpha f \,|L_p(\Omega)\| < \infty.$$

(ii) $\widetilde{W}_p^k(\Omega) = \widetilde{F}_{p,2}^k(\Omega)$.

Theorem 14. $1 < p < \infty$, $k \in \mathbb{N}$.

(i) Ω bounded Lipschitz domain. Then $W_p^k(\Omega) = F_{p,2}^k(\Omega)$.

(ii) Ω bounded E-thick domain, $|\Gamma| = 0$. Then $D(\Omega)$ dense in $\widetilde{W}_p^k(\Omega)$,

$$\mathrm{id} : \quad \widetilde{W}_p^k(\Omega) \hookrightarrow L_p(\Omega) \qquad \text{is compact,}$$

$$\|f \,|\widetilde{W}_p^k(\Omega)\| \sim \|f \,|W_p^k(\Omega)\| \sim \sum_{|\alpha| = k} \|D^\alpha f \,|L_p(\Omega)\|.$$

Remark 15. (i) Stein, 1970. (ii) proof by wavelets.

Now always Ω bounded C^∞ domain in $\mathbb{R}^n$, $n \geq 2$. $\Gamma = \partial\Omega$ compact $(n-1)$-dimensional C^∞ manifold. Standard method via local charts one can introduce any space $A^s_{pq}(\Gamma)$ by reduction to $A^s_{pq}(\mathbb{R}^{n-1})$.
Traces: Pointwise for smooth functions, inequalities, completion: $\varphi \in S(\mathbb{R}^n)$ restricted to Γ, $\varphi(\gamma)$, with $\gamma \in \Gamma$, denoted by $\mathrm{tr}_\Gamma \varphi$. For which $A^s_{pq}(\Omega)$ exists $A^\sigma_{uv}(\Gamma)$ with

$$\|\mathrm{tr}_\Gamma \varphi \,|A^\sigma_{uv}(\Gamma)\| \leq c \|\varphi \,|A^s_{pq}(\Omega)\| \qquad \text{for all} \quad \varphi \in S(\mathbb{R}^n)|\Omega?$$

If, then tr_Γ defined by completion of $S(\mathbb{R}^n)|\Omega$ in $A^s_{pq}(\Omega)$. Recall $S(\mathbb{R}^n)|\Omega$ dense in $A^s_{pq}(\Omega)$ if $\max(p,q) < \infty$. Minor modification incorporated also $\max(p,q) = \infty$. If $s < 1/p$, then no traces:

Proposition 16. Ω bounded C^∞-domain. $0 < p, q < \infty$. If

$$\max\left(\tfrac{1}{p} - 1, n\left(\tfrac{1}{p} - 1\right)\right) < s < \tfrac{1}{p}$$

then

$$\mathring{A}^s_{pq}(\Omega) = A^s_{pq}(\Omega) = \widetilde{A}^s_{pq}(\Omega).$$

ν outer normal at Γ. Then $\mathrm{tr}_\Gamma \frac{\partial^j}{\partial \nu^j}$ makes sense near Γ for smooth functions.

$$\mathrm{tr}_\Gamma^r : \quad f \mapsto \left\{ \mathrm{tr}_\Gamma \frac{\partial^j f}{\partial \nu^j} : \ 0 \le j \le r \right\}, \qquad r \in \mathbb{N}_0.$$

Theorem 17. Let $1 \le p \le \infty$, $0 < q \le \infty$, $r \in \mathbb{N}_0$ and $r + \frac{1}{p} < s$. Then

$$\mathrm{tr}_\Gamma^r \, B^s_{pq}(\Omega) = \prod_{k=0}^{r} B^{s-\frac{1}{p}-k}_{pq}(\Gamma).$$

If, in addition, $p < \infty$ and $q \ge 1$, then

$$\mathrm{tr}_\Gamma^r \, F^s_{pq}(\Omega) = \prod_{k=0}^{r} B^{s-\frac{1}{p}-k}_{pp}(\Gamma).$$

Remark 18. Does there exist linear extension operators

$$\mathrm{ext}^r_\Gamma : \quad \prod_{k=0}^{r} B^{s-\frac{1}{p}-k}_{pq}(\Gamma) \hookrightarrow B^s_{pq}(\Omega)$$

such that

$$\mathrm{tr}^r_\Gamma \circ \mathrm{ext}^r_\Gamma = \mathrm{id} \qquad \text{identity in} \quad \prod_{k=0}^{r} B^{s-\frac{1}{p}-k}_{pq}(\Gamma)?$$

Similarly for F-spaces.

Theorem 19. For any $u \in \mathbb{N}$ there exists a common extension operator $\mathrm{ext}^{r,u}_\Gamma$ for all spaces in Theorem 17 with $r + \frac{1}{p} < s < u$.

Remark 20. Construction by wavelet frames for boundary spaces and wavelet-friendly extensions from Γ to Ω.

Again Ω bounded C^∞ domain in $\mathbb{R}^n$. Largest possible r for traces (and extensions): $r(s,p) = [s - \frac{1}{p}]$ if $s - \frac{1}{p} \notin \mathbb{N}_0$.

Theorem 21.

$$1 \le p < \infty, \quad -1 < s - \frac{1}{p} \notin \mathbb{N}_0, \quad \begin{cases} 0 < q < \infty, & B\text{-spaces}, \\ 1 \le q < \infty, & F\text{-spaces}. \end{cases}$$

Then

$$\widetilde{B}^s_{pq}(\Omega) = \mathring{B}^s_{pq}(\Omega) = \left\{ f \in B^s_{pq}(\Omega) : \ \mathrm{tr}_\Gamma^{r(s,p)} f = 0 \right\},$$

$$\widetilde{F}^s_{pq}(\Omega) = \mathring{F}^s_{pq}(\Omega) = \left\{ f \in F^s_{pq}(\Omega) : \ \mathrm{tr}_\Gamma^{r(s,p)} f = 0 \right\}.$$

Furthermore if in addition $s < u \in \mathbb{N}$ then

$$B^s_{pq}(\Omega) = \widetilde{B}^s_{pq}(\Omega) \times \mathrm{ext}_\Gamma^{r(s,p),u} \prod_{k=0}^{r(s,p)} B^{s-\frac{1}{p}-k}_{pq}(\Gamma),$$

$$F^s_{pq}(\Omega) = \widetilde{F}^s_{pq}(\Omega) \times \mathrm{ext}_\Gamma^{r(s,p),u} \prod_{k=0}^{r(s,p)} B^{s-\frac{1}{p}-k}_{pp}(\Gamma).$$

Remark 22. If $-1 < s - \frac{1}{p} < 0$ then $r(s,p) = -1$: interpretation according to Proposition 16.