Convergence of a numerical method for the compressible Navier-Stokes system

Bangwei She

based on the work with E. Feireisl and H. Mizerová

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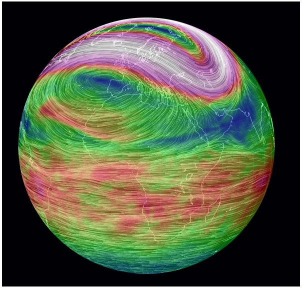
Fluid motion



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(1564-1642)

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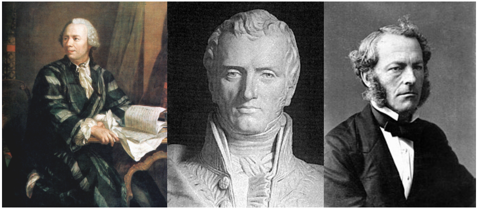

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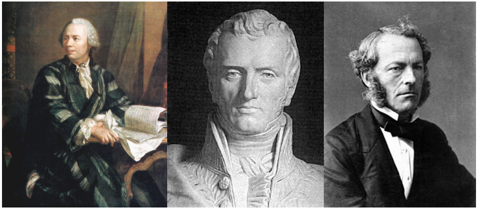

$$\partial_t \mathbf{U} + \mathrm{div}_x F(\mathbf{U}) = 0, \quad \mathbf{U}(0,x) = \mathbf{U}_0$$
$$\mathbf{U} = \mathbf{U}(t,x), \quad t \in (0,T), x \in R^d.$$

$$\begin{cases} \text{density} \ldots & \rho(t,\mathbf{x}) \\ \text{velocity} \ldots & \mathbf{u}(t,\mathbf{x}) \\ \text{energy} \ldots & e(t,\mathbf{x}) \end{cases}$$

Concepts of solution

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- Discontinuity $\longrightarrow$ *weak solutions*

$$\int_0^\infty \int_{R^d} \left(\mathbf{U} \frac{\partial \phi}{\partial t} + \sum_{k=1}^{d} \mathbf{F}_k\left(\mathbf{U}\right) \frac{\partial \phi}{\partial x_k} \right) + \int_{R^d} \mathbf{U}_0\, \phi\left(0,\cdot\right) = 0.$$

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- $m > 1$, or $d > 1$, infinitely many sols

De Lellis & Székelyhidi '12-'14 Chiodaroli, Feireisl '14-'15
Chiordaroli, De Lellis, Kreml '16

Convergence of numerical solutions

- **fundamental question in numerical analysis**
  - ? Does $\mathbf{U}_h \to \mathbf{U}$ as $h \to 0$?

- **open question for compressible flows**
  - Particularly for multi-d systems

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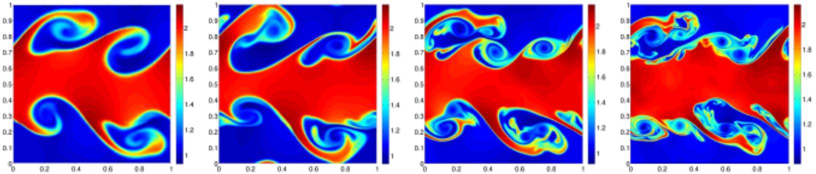
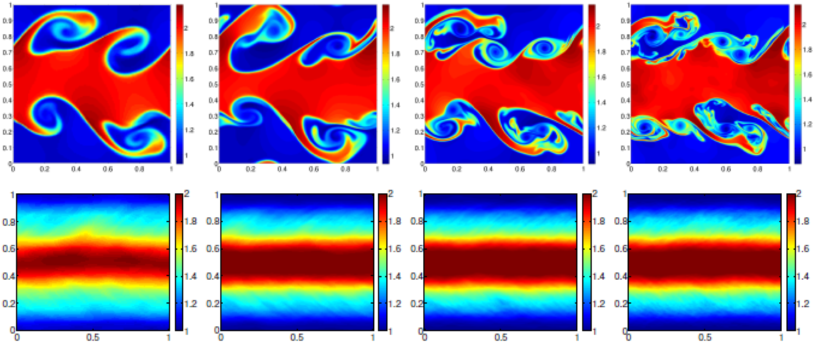
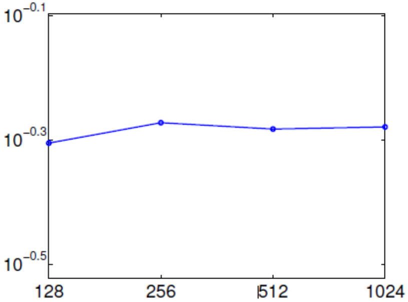
Certain numerical solutions of inviscid problems exhibit scheme independent oscillatory behaviour

Siddhartha Mishra





(a) nx = 128 (b) nx = 256 (c) nx = 512 (d) nx = 1024



$10^{-0.1}$
$10^{-0.3}$
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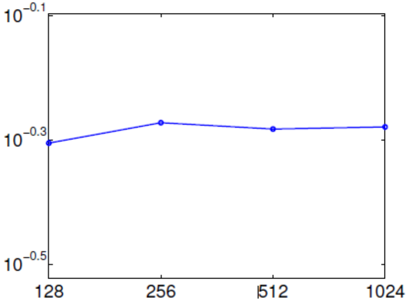

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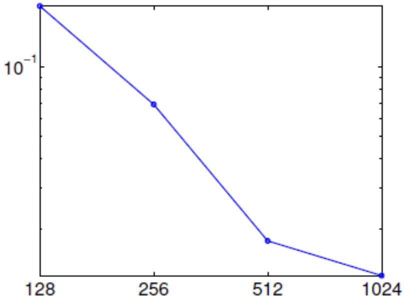

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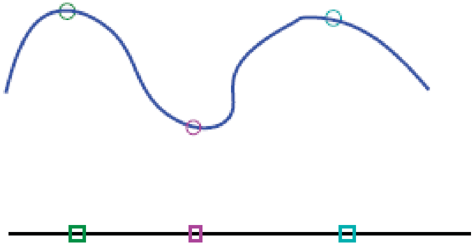

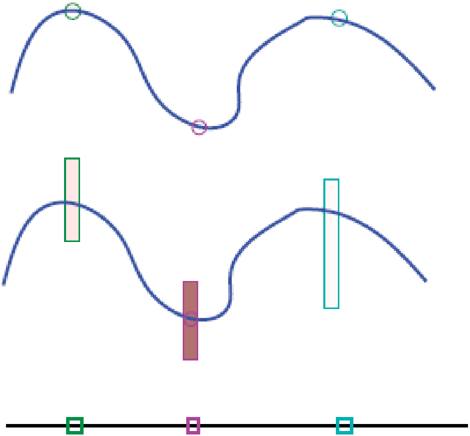

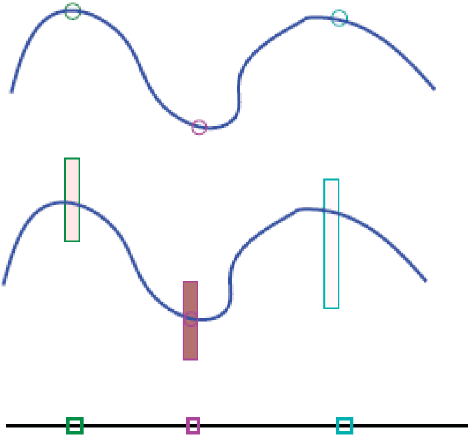

Measure-value solutions!

Feireisl, Lukacova, Mizerova, preprints.

Compressible isentropic Navier-Stokes

$$\rho_t + \mathrm{div}(\rho \mathbf{u}) = 0. \tag{1a}$$

$$(\rho \mathbf{u})_t + \mathrm{div}(\rho \mathbf{u} \otimes \mathbf{u}) = -\nabla p + \nabla \cdot \mathbb{S} \tag{1b}$$

$p = a\rho^\gamma$
$\mathbb{S} = 2\mu(\nabla \mathbf{u} + \nabla \mathbf{u}^T) + \eta \mathrm{div}\mathbf{u}\mathbf{I},$

$$\mathbf{u}|_{\partial\Omega} = 0 \quad \text{or} \quad \text{periodic}$$

$$\rho(\mathbf{x}, 0) = \rho_0 > 0, \quad \mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0.$$

Literature

Convergence to weak solutions

Karper [2013]: Convergence to a weak solution if $\boxed{\gamma > 3}$

Error estimates

Gallouet, Herbin, Maltese, Novotný [2015]

Convergence to smooth solutions + error estimates if $\boxed{\gamma > 3/2}$

Convergence to strong solutions

Feireisl, Lukáčová [2016]

Convergence via dissipative measure-valued solution for Physical relevant $\gamma \in (1, 2)$

Dissipative measure-valued solution

We say that a parameterized measure $\{\nu_{t,x}\}_{(t,x)\in(0,T)\times\Omega}$,

$$\nu \in L^\infty_{weak}\Big((0,T)\times\Omega; \mathcal{P}\left([0,\infty)\times R^N\right)\Big)$$

is a dissipative measure-valued solution of the Navier-Stokes system in $(0,T)\times\Omega$, if the following holds for a.a. $\tau\in(0,T)$, for any $\psi\in C^1\big((0,T)\times\Omega; R^d\big)$

$$\left[\int_\Omega \langle\nu_{t,x};\rho\rangle\psi(t,\cdot)dx\right]_{t=0}^{t=\tau} = \int_0^\tau\int_\Omega \Big(\langle\nu_{t,x};\rho\rangle\partial_t\psi + \langle\nu_{t,x};\rho\mathbf{u}\rangle\cdot\nabla_x\psi\Big)dxdt$$

$$\left[\int_\Omega \langle\nu_{t,x};\rho\mathbf{u}\rangle\psi(t,\cdot)dx\right]_{t=0}^{t=\tau} = \int_0^\tau\int_\Omega \Big(\langle\nu_{t,x};\rho\mathbf{u}\rangle\partial_t\psi - \mathcal{S}(\nabla\mathbf{u}):\nabla_x\psi dxdt$$

$$+\langle\nu_{t,x};\rho\mathbf{u}\otimes\mathbf{u}+p\mathbf{I}\rangle:\nabla_x\psi\Big)dxdt + \int_0^\tau\int_\Omega \mathcal{R}:\nabla_x\psi dxdt,$$

$$\left[\int_\Omega \langle\nu_{t,x};E\rangle\psi(t,\cdot)dx\right]_{t=0}^{t=\tau} + \int_0^\tau\int_\Omega \mathcal{S}(\nabla\mathbf{u}):\nabla_x\psi dxdt + \mathcal{D}(\tau)\le 0,$$

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where $\boxed{\int_0^\tau \|\mathcal{R}\|_{\mathcal{M}(\Omega)}dt \le \int_0^\tau \mathcal{D}(\tau)dt}$

Mesh

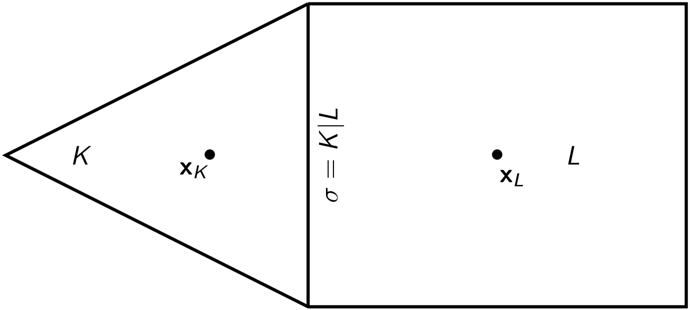

Figure : Dual grid

Mesh

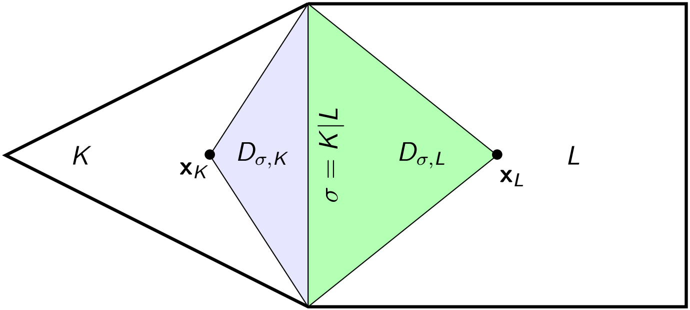

Figure : Dual grid

Discrete operators

Jump and Average

$$v^{\mathrm{out}}(x) = \lim_{\delta\to 0+} v(x+\delta \boldsymbol{n}),\ \ v^{\mathrm{in}}(x) = \lim_{\delta\to 0+} v(x-\delta \boldsymbol{n}),$$

$$\overline{v}(x) = \frac{v^{\mathrm{in}}(x)+v^{\mathrm{out}}(x)}{2},\ [\![v]\!] = v^{\mathrm{out}}(x) - v^{\mathrm{in}}(x)$$

Divergence

$$\mathrm{div}_h \mathbf{v}(\mathbf{x}) = \sum_{K\in\mathcal{T}_h} (\mathrm{div}_h \mathbf{v})_K 1_K, \quad (\mathrm{div}_h \mathbf{v})_K = \frac{1}{|K|}\sum_{\sigma\in\partial K} |\sigma| \overline{\mathbf{v}}_\sigma \cdot \boldsymbol{n}_\sigma,\ \forall\ \mathbf{v}\in Q_h.$$

Upwind flux

$$\mathrm{Up}[r,\mathbf{u}]_\sigma = r^{\mathrm{in}}(\mathbf{u}\cdot n)^+_\sigma + r^{\mathrm{out}}(\mathbf{u}\cdot n)^-_\sigma$$

$$r^+ = \max\{0,f\}, \quad r^- = \min\{0,f\}$$

$$Up[r,\mathbf{v}] = r^{\mathrm{up}}\mathbf{v}\cdot\boldsymbol{n} = r^{\mathrm{in}}[\overline{\mathbf{v}}\cdot\boldsymbol{n}]^+ + r^{\mathrm{out}}[\overline{\mathbf{v}}\cdot\boldsymbol{n}]^- = \overline{r}\ \overline{\mathbf{v}}\cdot\boldsymbol{n} - \frac{1}{2}|\overline{\mathbf{v}}\cdot\boldsymbol{n}|[\![r]\!],$$

Implicit FVM

$$\int_\Omega D_t \rho_h \phi_h - \sum_{\sigma \in \mathcal{E}_{int}} \int_\sigma F_h(\rho_h, \mathbf{u}_h) [\![\phi_h]\!] \mathrm{d}Sx = 0, \tag{2a}$$

$$\int_\Omega D_t(\rho_h \mathbf{u}_h) \cdot \phi_h - \sum_{\sigma \in \mathcal{E}_{int}} \int_\sigma \mathbf{F}_h(\rho_h \mathbf{u}_h, \mathbf{u}_h) \cdot [\![\phi_h]\!] \mathrm{d}Sx - \sum_{\sigma \in \mathcal{E}_{int}} \int_\sigma \overline{p_h} \boldsymbol{n} \cdot [\![\phi_h]\!] \mathrm{d}Sx$$
$$= -\mu \sum_{\sigma \in \mathcal{E}_{int}} \int_\sigma \frac{1}{d_\sigma} [\![\mathbf{u}_h]\!] \cdot [\![\phi_h]\!] \mathrm{d}Sx - (\mu + \lambda) \int_\Omega \mathrm{div}_h \mathbf{u}_h \, \mathrm{div}_h \phi_h \mathrm{d}x. \tag{2b}$$

Lemma 1 (Internal energy balance)

Let $(\rho_h, \mathbf{u}_h)$ *satisfy the discrete continuity equation* (2a), *then there exists* $\xi \in \mathrm{co}\{\rho_h^{k-1}, \rho_h^k\}$ *and* $\zeta \in \mathrm{co}\{\rho_K^k, \rho_L^k\}$ *for any* $\sigma = K|L \in \mathcal{E}_{int}$ *such that*

$$\int_\Omega D_t \mathcal{H}(\rho_h^k) - \sum_{\sigma \in \mathcal{E}_{int}} \int_\sigma \overline{\mathbf{u}_h^k} \cdot \boldsymbol{n} [\![p(\rho_h^k)]\!] \mathrm{d}Sx$$

$$= -\frac{\Delta t}{2} \int_\Omega \mathcal{H}''(\xi) |D_t \rho_h^k|^2 \mathrm{d}x - \frac{1}{2} \sum_{\sigma \in \mathcal{E}_{int}} \int_\sigma \mathcal{H}''(\zeta) [\![\rho_h^k]\!]^2 \left(h^\varepsilon + |\overline{\mathbf{u}_h} \cdot \boldsymbol{n}| \right) \mathrm{d}Sx,$$

where $\mathcal{H}(\rho) = \frac{p(\rho)}{\gamma - 1}$.

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- Positivity of numerical density
- Existence of numerical solution

Theorem 2 (Discrete energy inequality)

Let $(\rho_h, \mathbf{u}_h)$ *be a numerical solution obtained from the scheme* (2). *Then for any* $k = 1, \ldots, N_t$, *there exists* $\xi \in \mathrm{co}\{\rho_h^{k-1}, \rho_h^k\}$ *and* $\zeta \in \mathrm{co}\{\rho_K^k, \rho_L^k\}$ *for any* $\sigma = K|L \in \mathcal{E}_{int}$ *such that*

$$
\begin{aligned}
& D_t \int_\Omega \left(\frac{1}{2} \rho_h^k |\mathbf{u}_h^k|^2 + \mathcal{H}(\rho_h^k) \right) \mathrm{d}x \\
& \qquad + \sum_{\sigma \in \mathcal{E}_{int}} \int_\sigma \frac{\mu}{d_\sigma} [\![\mathbf{u}_h^k]\!]^2 \mathrm{d}Sx + \int_\Omega \frac{\mu + \lambda}{d_\sigma} |\mathrm{div}_h \mathbf{u}_h^k|^2 \mathrm{d}x \\
& = - \frac{\Delta t}{2} \int_\Omega \mathcal{H}''(\xi) |D_t \rho_h^k|^2 - \sum_{\sigma \in \mathcal{E}_{int}} \int_\sigma \frac{h^\varepsilon}{2} \overline{\rho_h^k} [\![\mathbf{u}_h^k]\!]^2 \mathrm{d}Sx \\
& \quad - \frac{\Delta t}{2} \int_\Omega \rho_h^{k-1} |D_t \mathbf{u}_h^k|^2 \mathrm{d}x - \frac{1}{2} \sum_{\sigma \in \mathcal{E}_{int}} \int_\sigma (\rho_h^k)^{\mathrm{up}} |\overline{\mathbf{u}_h^k} \cdot \boldsymbol{n}| [\![\mathbf{u}_h^k]\!]^2 \\
& \quad - \frac{1}{2} \sum_{\sigma \in \mathcal{E}_{int}} \int_\sigma \mathcal{H}''(\zeta) [\![\rho_h^k]\!]^2 \left(h^\varepsilon + |\overline{\mathbf{u}_h^k} \cdot \boldsymbol{n}| \right) \mathrm{d}Sx \leq 0.
\end{aligned}
$$

Uniform bounds

Let $(\rho_h, \mathbf{u}_h)$ be a numerical solution obtained by the scheme (2) with $\gamma > 1,\ 0 < \varepsilon < 1$. Then we have

$$\|\rho_h \mathbf{u}_h^2\|_{L^\infty(L^1(\Omega))} \lesssim 1$$

$$\|\rho_h\|_{L^\infty(L^\gamma(\Omega))} \lesssim 1$$

$$\|p(\rho_h)\|_{L^\infty(L^1(\Omega))} \lesssim 1$$

$$\int_0^T \|\mathbf{u}_h\|_{1,\mathcal{T}}^2 \mathrm{d}t \lesssim 1$$

$$\|\mathrm{div}_h \mathbf{u}_h\|_{L^2 L^2}^2 \lesssim 1$$

$$\|\mathbf{u}_h\|_{L^2(L^6(\Omega))} \lesssim 1$$

Lemma 3 (Consistency)

Let $\rho_h^n, \mathbf{u}_h^n$ be the solution to the numerical scheme (2). Then

$$\int_\Omega \partial_h^t \rho_h^n \phi dx - \int_\Omega \rho_h^n \mathbf{u}_h^n \cdot \nabla_x \phi dx = \mathcal{O}(h^{\beta_1}), \beta_1 > 0.$$

$$\int_\Omega \partial_h^t (\rho_h \mathbf{u}_h)^n \cdot \mathbf{v} dx - \int_\Omega \rho_h^n \mathbf{u}_h^n \otimes \mathbf{u}_h^n : \nabla_x \mathbf{v} dx - \int_\Omega p(\rho_h^n) div_x \mathbf{v} dx$$
$$+\mu \int_\Omega (\nabla_h \mathbf{u}_h^n) : \nabla_x \mathbf{v} dx = \mathcal{O}(h^{\beta_2}), \beta_2 > 0.$$

Convergence

Theorem 4

Let $\gamma > 1, \Delta t \approx h,\ 0 < \varepsilon < 1$ *and the initial data satisfy*

$$\rho_0 \in L^\infty(R^d),\ \rho_0 \geq \underline{\rho} > 0 \text{ a.a. in } R^d,\ \mathbf{u}_0 \in L^2(R^d).$$

Then any Young measure $\nu_{t,x}$ *generated by the numerical sol of scheme* (2) *represents a dissipative measure-valued solution of NS (1).*

[1] Feireisl et.al. Dissipative measure-valued solutions to the compressible Navier–Stokes system. Calc. Vari. Partial Differ. Equ. 2016

Convergence

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Applying the weak-strong uniqueness [1] we conclude

Theorem 5

In addition to the hypotheses of Theorem 4, suppose the NS (1) endowed with the periodic boundary condition admits a regular solution. Then

$$\rho_h \to \rho \text{ (strongly) in } L^\gamma\left((0,T)\times K\right),$$

$$\mathbf{u}_h \to \mathbf{u} \text{ (strongly) in } L^2\left((0,T)\times K; R^d\right)$$

for any compact $K \subset \Omega$.

[1] Feireisl et.al. Dissipative measure-valued solutions to the compressible Navier–Stokes system. Calc. Vari. Partial Differ. Equ. 2016

Thank you for your attention!