Kurzweil-Stieltjes integral (Introduction to the modern theory of Stieltjes integration)

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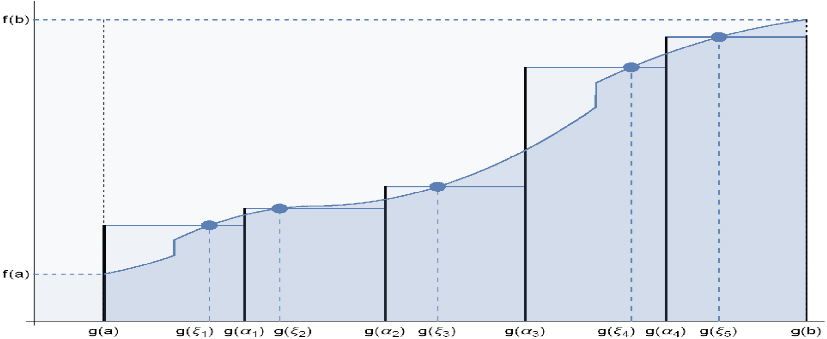
AREAS OF PLANAR REGIONS

Let $f : [a,b] \to \mathbb{R}$ be continuous and nonnegative,
$g : [a,b] \to \mathbb{R}$ be continuous and nondecreasing.

Consider the content **P** of the region $\{(x,y) \in \mathbb{R}^2 : x = g(t),\ 0 \le y \le f(t),\quad t \in [a,b]\}$.



$$S(\boldsymbol{\alpha}, \boldsymbol{\xi}) = \sum_{j=1}^{m} f(\xi_j)\,[g(\alpha_j) - g(\alpha_{j-1})]$$

$$a = \alpha_0 < \alpha_1 < \cdots < \alpha_m = b, \quad \boldsymbol{\xi} = \{\xi_1, \xi_2, \ldots, \xi_5\},\ \xi_j \in [\alpha_{j-1}, \alpha_j].$$

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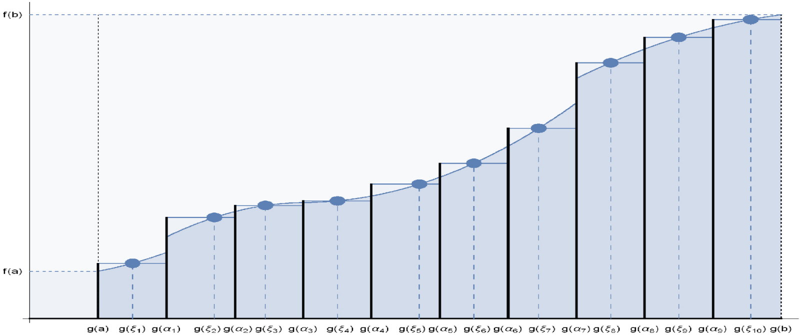
Consider the area **P** of the region $\{(x, y) \in \mathbb{R}^2 : x = g(t),\ 0 \leq y \leq f(t), \quad t \in [a, b]\}$.



$$\mathrm{S}(\boldsymbol{\alpha}, \boldsymbol{\xi}) = \sum_{j=1}^{m} f(\xi_j)\,[g(\alpha_j) - g(\alpha_{j-1})] \longrightarrow \mathbf{P} := \int_a^b f\, dg,$$

$$a = \alpha_0 < \alpha_1 < \cdots < \alpha_m = b, \quad \boldsymbol{\xi} = \{\xi_1, \xi_2, \ldots, \xi_5\},\ \xi_j \in [\alpha_{j-1}, \alpha_j].$$

Thomas Joannes Stieltjes
(*1856 - +1894)

- Moments (static, moment of inertia, etc).
- Line integrals of the 1st and 2nd kinds.
- Functional analysis:

Riesz

Φ is a continuous linear functional on $C([a, b])$ if and only if:

there is a function p of bounded variation on $[a, b]$ such that

$$\Phi(x) = \int_a^b x \; dp \quad \text{for any } x \in C([a, b]).$$

- $-\infty < a < b < \infty$,
- function $f\colon [a,b] \to \mathbb{R}$ is *regulated* on $[a,b]$, if
  $f(s+) := \lim_{\tau \to s+} f(\tau) \in \mathbb{R}$ for $s \in [a,b)$, $f(t-) := \lim_{\tau \to t-} f(\tau) \in \mathbb{R}$ for $t \in (a,b]$.
- $\Delta^+ f(s) = f(s+) - f(s)$, $\Delta^- f(t) = f(t) - f(t-)$, $\Delta f(t) = f(t+) - f(t-)$.
- $G([a,b])$ (or G) is the space of regulated functions on $[a,b]$.
  (G is Banach space with respect to the norm $\|f\|_\infty = \sup_{t\in[a,b]} \|f(t)\|$).
- $BV = BV([a,b]) = \left\{ f\colon [a,b] \to \mathbb{R} : \operatorname{var}_a^b f < \infty \right\}$ is the space of functions with bounded variation.
- function $f\colon [a,b] \to R$ is *finite step function*, if there is a division $a = \alpha_0 < \alpha_1 < \alpha_2 < \ldots < \alpha_m = b$ of $[a,b]$ such that f is constant on every (α_{j-1}, α_j),
  $S([a,b])$ (or S) is the set of finite step functions on $[a,b]$.
- Regulated functions are uniform limits of finite step functions, they have at most countably many points of discontinuity.
  Every function f of bounded variation is a difference $f = g - h$ of nondecreasing functions g and h.
- $S([a,b]) \subsetneqq BV([a,b]) \subsetneqq G([a,b])$.

- **tagged partition** of $[a, b]$: $P = (\boldsymbol{\alpha}, \boldsymbol{\xi})$,

$$\boldsymbol{\alpha} = \{a = \alpha_0 < \alpha_1 < \cdots < \alpha_m = b\},\ \boldsymbol{\xi} = \{\xi_1, \xi_2, \ldots, \xi_m\},\ \alpha_{j-1} \leq \xi_j \leq \alpha_j;$$

- **integral sum**: for $f, g : [a, b] \to \mathbb{R}$ and a tagged partition $P = (\boldsymbol{\alpha}, \boldsymbol{\xi})$ we put

$$\mathrm{S}(P) = \sum_{j=1}^{m} f(\xi_j)\,[g(\alpha_j) - g(\alpha_{j-1})].$$

- $\nu(P) = \nu(\boldsymbol{\alpha})\ (= m)$ is usually the number of the subintervals determined by P (or $\boldsymbol{\alpha}$) and $|\boldsymbol{\alpha}| = \max_j(\alpha_j - \alpha_{j-1})$.

Definition (Riemann-Stieltjes (RS) integral)

$$I = (\mathrm{RS}) \int_a^b f\,dg \iff \begin{cases} \text{for every } \varepsilon > 0 \text{ there is a } \delta > 0 \text{ such that} \\ \left| \mathrm{S}(P) - I \right| < \varepsilon \\ \text{for every } P = (\boldsymbol{\alpha}, \boldsymbol{\xi}) \text{ such that } |\boldsymbol{\alpha}| < \delta. \end{cases}$$

$$\int_c^c f\,dg = 0, \quad \int_b^a f\,dg = -\int_a^b f\,dg.$$

- If $g \in BV([a,b])$ and $\{f_n\} \subset C[a,b]$ is such that $f_n \rightrightarrows f$ on $[a,b]$, then
$$\lim_{n\to\infty} \int_a^b f_n \, dg = \int_a^b f \, dg \in \mathbb{R}.$$
- If $f \in C[a,b]$ and $\{g_n\} \subset BV([a,b])$ is such that $g_n \to g$ in $BV([a,b])$, then
$$\lim_{n\to\infty} \int_a^b f \, dg_n = \int_a^b f \, dg \in \mathbb{R}.$$
- (RS) $\int_a^b f \, dg \in \mathbb{R}$ for each $g \in BV([a,b])$ if and only if $f \in C[a,b]$.
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Jaroslav Kurzweil
(*1926)

Notation

- **gauge**: $\delta : [a,b] \to (0,\infty)$;
- **tagged partition of interval**: $P = (\boldsymbol{\alpha}, \boldsymbol{\xi})$,

$$\boldsymbol{\alpha} = \{a = \alpha_0 < \alpha_1 < \cdots < \alpha_{\nu(P)} = b\}, \ \boldsymbol{\xi} = \{\xi_1, \xi_2, \ldots, \xi_{\nu(P)}\}, \ \alpha_{j-1} \le \xi_j \le \alpha_j;$$

- **integral sum**: for $f : [a,b] \to \mathbb{R}$, $g : [a,b] \to \mathbb{R}$ and $P = (\boldsymbol{\alpha}, \boldsymbol{\xi})$ we set

$$\mathrm{S}(P) = \sum_{j=1}^{\nu(P)} f(\xi_j)\,[g(\alpha_j) - g(\alpha_{j-1})].$$

- **δ-fine partition**: $P = (\boldsymbol{\alpha}, \boldsymbol{\xi})$ is δ-fine if $[\alpha_{j-1}, \alpha_j] \subset (\xi_j - \delta(\xi_j), \xi_j + \delta(\xi_j))$ for all j.

Definition

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- the integrals $\int_a^b f_n \, dg$ exist for all $n \in \mathbb{N}$,
- at least one of the following conditions is satisfied:
  - $g \in BV([a,b])$ and $f_n \rightrightarrows f$,
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THEN: the integral $\int_a^b f \, dg$ exists as well, and

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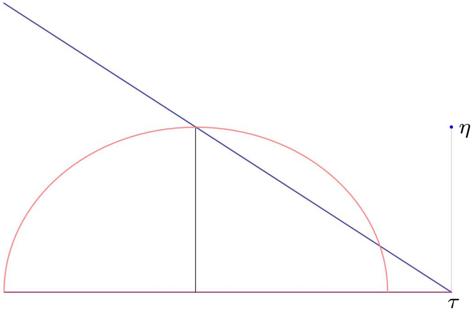
- $f(x) \equiv c,\ g : [a,b] \to \mathbb{R} \implies \int_a^b f\ dg = c\,[g(b) - g(a)].$
- $f : [a,b] \to \mathbb{R},\ g(x) \equiv c \implies \int_a^b f\ dg = 0.$
- $g : [a,b] \to \mathbb{R}$ regulated, $\tau \in [a,b]$ and $f = \chi_{[\tau,b]} \implies \int_\tau^b f\ dg = g(b) - g(\tau).$

  Let $\delta(x) = \begin{cases} \frac{1}{4}(\tau - x) & \text{for } x < \tau\,, \\ \eta & \text{for } x = \tau \end{cases}$

  and let $P = (\boldsymbol{\alpha}, \boldsymbol{\xi})$ be δ-fine. Then



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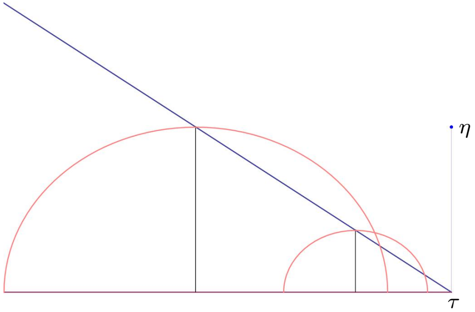

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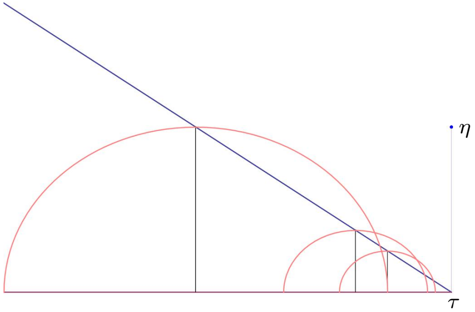

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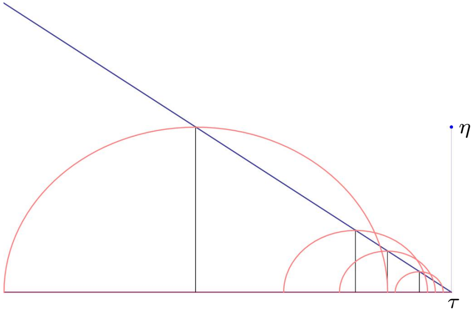

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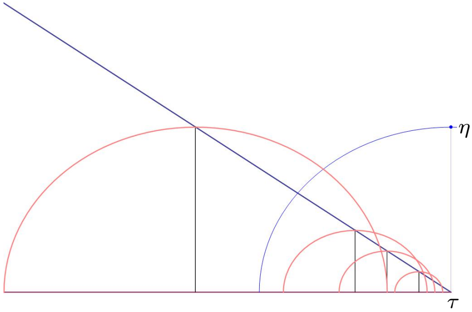

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  $\implies S(P) = [g(\tau) - g(\alpha_{\nu(P)-1})] \to [g(\tau) - g(\tau-)] \implies \int_a^\tau f\, dg = g(\tau) - g(\tau-)$

  $\implies \int_a^b f\, dg = g(b) - g(\tau) + g(\tau) - g(\tau-) = g(b) - g(\tau-).$

Integration of finite step functions

- $f(x) \equiv c,\ g : [a,b] \to \mathbb{R} \implies \int_a^b f\, dg = c\,[g(b) - g(a)].$
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Integration of finite step functions

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$$\int_a^b \chi_{[a,\tau]}\, dg = g(\tau+) - g(a), \quad \int_a^b \chi_{[a,\tau)}\, dg = g(\tau-) - g(a),$$

$$\int_a^b \chi_{[\tau]}\, dg = \begin{cases} g(b)-g(b-) & \text{for } \tau = b\,, \\ g(\tau+)-g(\tau-) & \text{for } \tau \in (a,b)\,, \\ g(b)-g(b-) & \text{for } \tau = b\,, \end{cases}$$

- $f:[a,b] \to \mathbb{R}$ $\tau \in [a,b] \implies$

$$\int_a^b f\, d\chi_{[a,\tau]} = \int_a^b f\, d\chi_{[a,\tau)} = -f(\tau), \quad \int_a^b f\, d\chi_{[\tau,b]} = \int_a^b f\, d\chi_{(\tau,b]} = f(\tau),$$

$$\int_a^b f\, d\chi_{[\tau]} = \begin{cases} -f(a) & \text{for } \tau = a\,, \\ 0 & \text{for } \tau \in (a,b)\,, \\ f(b) & \text{for } \tau = b\,. \end{cases}$$

- $f \in G([a,b]),\ g \in G([a,b]) \implies \int_a^b f\,dg \in \mathbb{R}$ and $\int_a^b g\,df \in \mathbb{R}$
  if at least one of f, g is a finite step function.
- If
  - $g \in BV([a,b])$,
  - $\int_a^b f_k\,dg$ exists for each k,
  - $f_k \rightrightarrows f$,

  then $\int_a^b f_k\,dg \to \int_a^b f\,dg \in \mathbb{R}$.
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- $f \in BV([a,b]),\ g \in G([a,b]) \implies \int_a^b f\,dg \in \mathbb{R}$.

Theorem

ASSUME: f and g are regulated on $[a,b]$ and at least one of them has a bounded variation.

THEN: both integrals $\int_a^b f\,dg$ and $\int_a^b g\,df$ exist.

- RS $\subset$ KS = PS.
- $(\mathrm{LS})\int_{[c,d]} f\,dg \in \mathbb{R} \implies$

$$\int_c^d f\,dg \in \mathbb{R} \quad \text{and} \quad (\mathrm{LS})\int_{[c,d]} f\,dg = f(c)\,\Delta^{-}g(c) + \int_c^d f\,dg + f(d)\,\Delta^{+}g(d).$$

- $\int_a^b f\,dg \in \mathbb{R},\ a \le c \le d \le b \implies$

$$\int_a^b f\,\chi_{[c,d]}\,dg = f(c)\,\Delta^{-}g(c) + \int_c^d f\,dg + f(d)\,\Delta^{+}g(d).$$

Theorem

ASSUME:

- $f,\, f_k \in G([a,b]), \quad g, g_k \in BV([a,b])$ for $k \in \mathbb{N}$,
- $f_k \rightrightarrows f, \quad g_k \rightrightarrows g$,
- $\alpha^* := \sup\{\mathrm{var}_a^b\, g_k\,; k \in \mathbb{N}\} < \infty$.

THEN: $\displaystyle\int_a^t f_k\, d\, g_k \rightrightarrows \int_a^t f\, d\, g$ on $[a,b]$.

Bounded convergence

ASSUME: $f \in G([a,b])$, $\{f_n\} \subset G([a,b])$ and

- $\|f_n\|_\infty \le M < \infty$ for $n \in \mathbb{N}$,
- $\lim_{n\to\infty} f_n(x) = f(x)$ for $x \in [a,b]$.

THEN:

$$\lim_{k\to\infty} \int_a^b f_n\, d\, g = \int_a^b f\, d\, g \quad \text{for every } g \in BV([a,b]).$$

Integration by parts

Let $f \in G[a,b]$, $g \in BV[a,b]$. Then both integrals

$$\int_a^b f\,dg \quad \text{and} \quad \int_a^b g\,df$$

exist and it holds

$$\int_a^b f\,dg + \int_a^b g\,df = f(b)\,g(b) - f(a)\,g(a) - \sum_{a \le t < b} \Delta^+ f(t)\,\Delta^+ g(t) + \sum_{a < t \le b} \Delta^- f(t)\,\Delta^- g(t)\,.$$

Substitution

Let $h \in BV[a,b]$, $f\colon [a,b] \to \mathbb{R}$ and $g\colon [a,b] \to \mathbb{R}$ are such that $\int_a^b f\,dg$ exists.

Then, if one from the integrals

$$\int_a^b h(t)\,d\Big[\int_a^t f\,dg\Big], \quad \int_a^b h\,f\,dg\,,$$

exists, the same is true also for the remaining one and

$$\int_a^b h(t)\,d\Big[\int_a^t f\,dg\Big] = \int_a^b h\,f\,dg\,.$$

Theorem (Hake)

- $\displaystyle\int_a^t f\,dg$ exists for every $t\in[a,b)$ and $\displaystyle\lim_{t\to b-}\Big(\int_a^t f\,dg+f(b)\,[g(b)-g(t)]\Big)=I\in\mathbb{R}$
$$\implies \int_a^b f\,dg=I.$$
- $\displaystyle\int_t^b f\,dg$ exists for every $t\in(a,b\,]$ and $\displaystyle\lim_{t\to a+}\Big(\int_t^b f\,dg+f(a)\,[g(t)-g(a)]\Big)=I\in\mathbb{R}$
$$\implies \int_a^b f\,dg=I.$$

Corollaries

- If $f\in G([a,b])$, $g\in G([a,b])$ and at least one of them has a bounded variation, then
$$h(t)=\int_a^t f\,dg \quad \text{is regulated on } [a,b]\,.$$
In particular, if $g\in BV([a,b])$, then also $h\in BV([a,b])$.
- $\Delta^+h(t)=f(t)\,\Delta^+g(t)$ for $t\in[a,b)$, $\quad\Delta^-h(s)=f(s)\,\Delta^-g(s)$ for $s\in(a,b]$.

Theorem (Hake)

- $\int_a^t f\,dg$ exists for every $t \in [a,b)$ and $\lim_{t\to b-}\Big(\int_a^t f\,dg + f(b)\,[g(b)-g(t)]\Big) = I \in \mathbb{R}$
$$\Longrightarrow \int_a^b f\,dg = I.$$
- $\int_t^b f\,dg$ exists for every $t \in (a,b\,]$ and $\lim_{t\to a+}\Big(\int_t^b f\,dg + f(a)\,[g(t)-g(a)]\Big) = I \in \mathbb{R}$
$$\Longrightarrow \int_a^b f\,dg = I.$$

Corollaries

- If $f \in G([a,b])$, $g \in G([a,b])$ and at least one of them has a bounded variation, then
$$h(t) = \int_a^t f\,dg \quad \text{is regulated on } [a,b]\,.$$
In particular, if $g \in BV([a,b])$, then also $h \in BV([a,b])$.
- $\Delta^+ h(t) = f(t)\,\Delta^+ g(t)$ for $t \in [a,b)$, $\Delta^- h(s) = f(s)\,\Delta^- g(s)$ for $s \in (a,b]$.

!!! For better understanding I refer to the **SAKS-HENSTOCK LEMMA** !!!

Riesz theorem

Φ is continuous linear functional on $C[a,b]$ $(\Phi \in (C[a,b])^*)$ $\Leftrightarrow$
there is $p \in BV([a,b])$ such that $p(a)=0$, p is right continuous on (a,b) $(p \in NBV([a,b]))$ and

$$\Phi(x) = \Phi_p(x) := \int_a^b x\, dp \quad \text{for every } x \in C[a,b].$$

Mapping $p \in NBV([a,b]) \to \Phi_p \in (C[a,b])^*$ is isometric isomorphism.

$G_L([a,b]) = \{x \in G([a,b]) : x(t-) = x(t) \text{ for } t \in (a,b]\}$

Theorem

Φ is continuous linear functional on $G_L([a,b])$ $(\Phi \in (G_L([a,b]))^*)$ $\Leftrightarrow$
there is $p \in BV([a,b])$ such that

$$\Phi(x) = \Phi_p(x) := p(b)\,x(b) - \int_a^b p\, dx \quad \text{for } x \in G_L[a,b].$$

Mapping $p \in BV([a,b]) \to \Phi_p \in (G_L([a,b]))^*$ is isomorphism.

$$\text{(L)} \qquad x(t) = \widetilde{x} + \int_{t_0}^{t} dA\,x + f(t) - f(t_0)\,, \quad t \in [a,b]\,.$$

Theorem

ASSUME:

- $A \in BV([a,b], \mathbb{R}^{n\times n})$ and $t_0 \in [a,b]$.
- $\det\,[I - \Delta^{-}A(t)] \neq 0$ for $t \in (t_0, b\,]$,

  $\det\,[I + \Delta^{+}A(s)] \neq 0$ for $s \in [a, t_0)$.

THEN: for each $f \in G([a,b], \mathbb{R}^n)$ and $\widetilde{x} \in \mathbb{R}^n$, (L) has 1! solution $x \in G([a,b], \mathbb{R}^n)$.

$$x_k(t) = \widetilde{x}_k + \int_a^t d[A_k]\,x + f_k(t) - f_k(a), \quad t \in [a,b].$$

$$x(t) = \widetilde{x} + \int_a^t d[A]\,x \; + f(t) - f(a), \qquad t \in [a,b].$$

$$A_k, A \in BV([a,b], \mathbb{R}^{n \times n}), \quad f_k, f \in G([a,b], \mathbb{R}^n), \quad \widetilde{x}_k, \widetilde{x} \in \mathbb{R}^n \quad \text{for } k \in \mathbb{N}\,.$$

Theorem

ASSUME:

- $\det[I - \Delta^- A(t)] \neq 0 \quad \text{for} \;\; t \in (a,b],$
- $A_k \rightrightarrows A \quad \text{on} \;\; [a,b], \quad \alpha^* := \sup\{\operatorname{var}_a^b A_k : k \in \mathbb{N}\} < \infty,$
- $\widetilde{x}_k \to \widetilde{x}, \quad f_k \rightrightarrows f \quad \text{on} \;\; [a,b].$

THEN: $x_k \rightrightarrows x \quad \text{on} \;\; [a,b].$

References

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