§1. Kinematics of Fluids

Let $(T_1, T_2) \subset \mathbb{R}^1$ be a time interval

$\Omega_t \subset \mathbb{R}^3$ domain occupied by the fluid at time $t \in (T_1, T_2)$

Hypothesis: $\boxed{\text{exactly one fluid particle passes through each point } x \in \Omega_t \text{ at any time } t}$

Lagrangian Description of the Flow

consider the motion of each particular fluid particle

~~then~~

the trajectories of particles

(1.1) $$x = \varphi(X, t)$$

$$(x_i = \varphi_i(X, t),\ i = 1, 2, 3)$$

X. the reference determining the particle under consideration (usually is characterized in some coord. system by parameters X_1, X_2, X_3 which, together with time are called Lagrangean coordinates

(1.1) determines the position of the particle given by the reference X at time t.

Sometimes we use a more detailed description of the motion of fluid particles

$$(1.2) \qquad x = \varphi(X, t_0; t)$$

which determines, at time t, the position x of the particle passing through the point X at time t_0. Then

$$(1.3) \qquad X = \varphi(X, t_0; t_0)$$

provided the references are identical with the coordinates of particles at time t_0.

The Lagrangian description is used if we study the flow of a piece of fluid formed by the same particles at each time instant and filling a domain $\sigma(t) \subset \mathbb{R}^3$ at time t.

the velocity

$$(1.4)\quad \hat{v}(X,t) = \frac{\partial \varphi}{\partial t}(X,t)$$

the acceleration

$$(1.5)\quad \hat{a}(X,t) = \frac{\partial^2 \varphi}{\partial t^2}(X,t)$$

<u>Eulerian Description</u>

= based on the determination of the velocity $v(x,t)$ of the fluid particle passing through the point x at time t.

We can write

$$(1.6)\quad v(x,t) = \hat{v}(X,t) = \frac{\partial \varphi}{\partial t}(X,t),$$

where $x = \varphi(X,t)$.

Since it is not important that the fluid particle considered is given by the reference X, we will use the simplified notation $\tilde{\varphi}(t) = \varphi(X,t)$

Then $\overset{(1.7)}{\Rightarrow}$ $v(x,t) = \frac{d\tilde{\varphi}}{dt}(t)$, $x = \tilde{\varphi}(t) = (\tilde{\varphi}_1(t), \tilde{\varphi}_2(t), \tilde{\varphi}_3(t))$

time t, together with coordinates x_1, x_2, x_3 are <u>Eulerian coordinates</u>.

We suppose that the domain of definition of the mapping $v = v(x,t)$ is the set M:

$$(1.8) \begin{cases} v: M \longrightarrow \mathbb{R}^3 \\ M = \{(x,t);\ x \in \Omega_t,\ t \in (T_1, T_2)\} \subset \mathbb{R}^4 \end{cases}$$

Ω_t depends continuously on time t, the set M is open.

assume

$$(1.9) \quad v \in [C^1(M)]^3$$

<u>Acceleration</u>: $a(x,t) = \frac{d^2\tilde{\varphi}}{dt^2}(t) = \frac{d}{dt}[v(\tilde{\varphi}(t), t)] =$

$\boxed{x = \tilde{\varphi}(t)}$

$$= \frac{\partial v}{\partial t}(\tilde{\varphi}(t),t) + \sum_{i=1}^{3} \frac{\partial v}{\partial x_i}(\tilde{\varphi}(t),t)\frac{d\tilde{\varphi}_i}{dt}(t).$$

Hence

(1.9) $$a(x,t) = \frac{\partial v}{\partial t}(x,t) + \sum_{i=1}^{3} v_i(x,t)\frac{\partial v}{\partial x_i}(x,t),$$

(1.10) $$a = \frac{\partial v}{\partial t} + (v.\nabla)v$$

(1.11) $$\frac{d}{dt} = \underbrace{\frac{\partial}{\partial t}}_{\substack{\text{local}\\ \text{derivative}}} + \underbrace{v.\,\mathrm{grad}}_{\substack{\text{convective}\\ \text{derivative}}} \quad \ldots \text{ material derivative}$$

Remark:

In the Lagrangian conception this quantity is viewed as a function $F(X,t)$ determining the value of the quantity considered, found for the particle given by the reference X, at time t.

If we use the Eulerian description, the quantity is represented by a function F dependent on x and t. Then $F(x,t)$ denotes the value of the quantity at the point x and time t.

Describing the path of a fluid particle by the equation $x = \varphi(X,t) = \tilde{\varphi}(t)$, we can write

$$(1.12) \qquad \hat{F}(X,t) = \tilde{F}(t) = F(\varphi(X,t),t) = F(\tilde{\varphi}(t),t).$$

We express the rate of change of the quantity F bound to this particle as the time derivative of the function $\tilde{F}$.

$$(1.13) \qquad \frac{d\tilde{F}}{dt}(t) = \frac{\partial \hat{F}}{\partial t}(X,t)$$

$$= \frac{\partial F}{\partial t}(\tilde{\varphi}(t),t) + \sum \frac{\partial F}{\partial x_i}(\tilde{\varphi}(t),t)\frac{d\tilde{\varphi}_i}{dt}(t)$$

$$= \frac{\partial F}{\partial t}(x,t) + v(x,t)\cdot \operatorname{grad} F(x,t)$$

$\Rightarrow$ rate of change of the quantity F = the material derivative of F

$\Rightarrow$ the local derivative $\frac{\partial F}{\partial t} \Leftarrow$ from dependence of F on time t

$\Rightarrow$ $v \cdot \nabla F \ldots \Leftarrow$ transport of the quantity F by the moving fluid the material derivative is called the derivative along a trajectory of a fluid particle.

Relation between the Lagrangian and Eulerian Description of Flow

Lag. descr. $\Rightarrow$ Euler. descr.

express $v(x,t) = \hat{v}(X,t) = \frac{\partial \varphi}{\partial t}(X,t)$; $\varphi(\cdot,t)$ mapping.. denote $\phi(\cdot,t)$

(1.14) $x = \varphi(X,t) \Leftrightarrow X = \phi(x,t)$

(1.15) $v(x,t) = \hat{v}(\phi(x,t),t)$

In case the position of the particle at time t_0 is taken for its reference X and the Lagrangian description in the form

$x = \varphi(X, t_0; t)$, then mapping

$\phi(x,t)$ is the inverse mapping to $\varphi(\cdot, t_0; t)$.

the transition from the Eulerian description to Lagrangian is equivalent to the determination of the paths of fluid particles on the basis of a given velocity field $v(x,t)$. The trajectory of the fluid particle passing through a point $X \in \Omega_t$ at time $t_0 \in (T_1, T_2)$ is given as the solution of the initial value problem

$$\boxed{(1.16)\quad \frac{dx}{dt} = v(x,t), \qquad x(t_0) = X.}$$

1. Theorem:

Under assumption $v \in [C^1(M)]^3$ the following statements hold:

(1) For each $(X, t_0) \in M$ the problem (1.15) has exactly one maximal solution $\varphi(X, t_0; t)$ (defined for t from a certain interval $(\alpha_{X,t_0}, \beta_{X,t_0})$).

2. The mapping φ has continuous first order partial derivatives

with respect to x_1, x_2, x_3, t_0, t and

continuous derivative $\frac{\partial^2 \varphi}{\partial t \partial x_i}, \frac{\partial^2 \varphi}{\partial t_0 \partial x_i}, i = 1, 2, 3$ in

the domain of domain of definition

$\{(x, t_0, t); (x, t_0) \in M, t \in (\alpha_{x, t_0}, \beta_{x, t_0})\}$.