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A van der Corput-type lemma for power bounded operators

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A VAN DER CORPUT-TYPE LEMMA FOR POWER BOUNDED OPERATORS

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Abstract. We prove a van der Corput-type lemma for power bounded Hilbert space operators. As a corollary we show that $N^{-1}\sum_{n=1}^{N} T^{p(n)}$ converges in the strong operator topology for all power bounded Hilbert space operators T and all polynomials p satisfying $p(\mathbb{N}_0) \subset \mathbb{N}_0$. This generalizes known results for Hilbert space contractions.

Similar results are true also for bounded strongly continuous semigroups of operators.

1. Introduction

By the mean ergodic theorem, the Cesàro means of the powers of a power bounded operator T on a reflexive Banach space converge in the strong operator topology to the projection onto $\ker(I - T)$ along $\overline{\operatorname{ran}(I - T)}$.

Frequently, the full sequence (T^n) of all powers of T can be replaced by a subsequence (T^{a_n}) where (a_n) is a given sequence of positive integers.

It is well known [BE], [BLRT] that $\frac{1}{N}\sum_{n=1}^{N} T^{a_n}$ converges in the strong operator topology for every unitary operator T (and more generally, using the dilation theory, for every Hilbert space contraction T) if and only if $\frac{1}{N}\sum_{n=1}^{N} \lambda^{a_n}$ converges for every complex number λ with $|\lambda| = 1$.

This condition, however, is in general difficult to verify. Nevertheless, it is known that $\frac{1}{N}\sum_{n=1}^{N} T^{p(n)}$ converges in the strong operator topology for all Hilbert space contractions and all polynomials p such that $p(\mathbb{N}_0) \subset \mathbb{N}_0$, where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. However, in general the limit operator is not a projection.

The main tools for the mean ergodic type results for subsequences are the spectral theory for unitary operators and the van der Corput lemma, see [EW], p. 184. Both of these tools are available only for unitary operators on Hilbert spaces. Using the dilation theory one can generalize these results to the setting of all contractions on Hilbert spaces. However, none of these tools is available for power bounded operators.

The aim of this paper is to prove a van der Corput-type lemma for power bounded operators on Hilbert spaces. As a corollary we obtain that $\frac{1}{N}\sum_{n=1}^{N} T^{p(n)}$ converges in the strong operator topology for all power bounded Hilbert space operators T and all polynomials p satisfying $p(\mathbb{N}_0) \subset \mathbb{N}_0$. Note that power bounded operators on Hilbert spaces are in general very different from contractions.

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Similar results are proved also for bounded strongly continuous semigroups of operators on Hilbert spaces.

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2. Discrete case

In this section we consider power bounded operators on Hilbert spaces and the convergence of Cesàro means with respect to subsequences of $\mathbb{N}$. The next result enables to reduce a given subsequence $(a_s)_{s=1}^\infty$ to sequences of differences $(a_{s+k} - a_s)_{s=1}^\infty$ for fixed $k \in \mathbb{N}$. If (a_s) is a polynomial subsequence, that is, there exists a polynomial p such that $a_s = p(s)$ for all $s \in \mathbb{N}$, then this result enables to reduce the degree of the polynomial.

Theorem 2.1. *Let T be a power bounded operator acting on a Hilbert space H and let $x \in H$. Let $(a_n)_{n=1}^\infty$ be a strictly increasing convex sequence of positive integers such that* $\sup\bigl\{ \frac{a_{2n}}{a_n} : n \in \mathbb{N} \bigr\} < \infty$. *Suppose that*

$$\lim_{N\to\infty} N^{-1} \sum_{j=1}^N T^{a_{j+k} - a_j} x = 0$$

for all $k \in \mathbb{N}$. Then

$$\lim_{N\to\infty} N^{-1} \sum_{j=1}^N T^{a_j} x = 0.$$

Proof. For all $j \in \mathbb{N}$ write $d_j = a_{j+1} - a_j$. Since the sequence (a_j) is convex, the sequence of differences (d_j) is increasing. Without loss of generality we may assume that $\|x\| = 1$. Let

$$M = \sup\{\|T^n\| : n \in \mathbb{N}_0\}.$$

Suppose on the contrary that there exists an $\eta > 0$ such that

$$\limsup_{N\to\infty} N^{-1} \Bigl\| \sum_{j=1}^N T^{a_j} x \Bigr\| > \eta.$$

Fix $k \in \mathbb{N}$ such that $k > \frac{20 M^4}{\eta^2}$. Let $c = \sup\{ \frac{a_{2n}}{a_n} : n \in \mathbb{N}\}$. Since the sequence (d_n) is increasing, we have

$$a_{2n} = a_n + d_n + d_{n+1} + \cdots + d_{2n-1} \geq a_n + n d_n$$

and so $\frac{a_n + n d_n}{a_n} \leq c$. Thus $\limsup_{n\to\infty} \frac{d_n}{a_n} \leq \limsup_{n\to\infty} \frac{c-1}{n} = 0$. Hence

$$\limsup_{n\to\infty} \frac{a_{2n+k} - a_{2n}}{a_n} \leq \limsup_{n\to\infty} \frac{k d_{2n+k-1}}{a_{2n+k-1}} \cdot \frac{a_{2n+k-1}}{a_{2n}} \cdot \frac{a_{2n}}{a_n} = 0$$

and $\lim_{n\to\infty} \frac{a_{2n+k} - a_{2n}}{a_n} = 0$.

Let $N_0 \in \mathbb{N}$ be such that $N_0 \geq \max\{ \frac{2kM}{\eta}, 4k\}$,

$$\frac{4M(a_{2N+k} - a_{2N})}{a_N} < k^{-1}$$

for all $N \geq N_0$ and

$$N^{-1}\Big\|\sum_{j=0}^{N} T^{a_{j+l}-a_j}x\Big\| < k^{-1} \tag{1}$$

for all $N \geq N_0$ and $l \in \{1, 2, \ldots, k-1\}$.

We need a lemma.

Lemma 2.2. *There exists an $N \geq N_0$ such that*

$$N^{-1}\Big\|\sum_{j=N+1}^{2N} T^{a_j}x\Big\| > \eta.$$

Proof. Fix η_1 such that $\eta < \eta_1 < \limsup_{N'\to\infty} N'^{-1}\Big\|\sum_{j=1}^{N'} T^{a_j}x\Big\|$. Let $v \in \mathbb{N}$ be such that $\frac{M}{2^v} < \frac{\eta_1-\eta}{2}$. There exists an $N_2 \geq 4^v N_0$ such that

$$N_2^{-1}\Big\|\sum_{j=1}^{N_2} T^{a_j}x\Big\| > \eta_1.$$

Write $N_2 = 2^v \cdot N_1 + z$, where $0 \leq z < 2^v$. Then $N_1 \geq N_0$. Suppose on the contrary that $N^{-1}\Big\|\sum_{j=N+1}^{2N} T^{a_j}x\Big\| \leq \eta$ for all $N \geq N_0$. Then in particular,

$$\frac{1}{2^i N_1}\Big\|\sum_{j=2^iN_1+1}^{2^{i+1}N_1} T^{a_j}x\Big\| \leq \eta$$

for all $i \in \{0, 1, \ldots, v-1\}$. So

$$\begin{aligned} N_2^{-1}\Big\|\sum_{j=1}^{N_2} T^{a_j}x\Big\| &\leq N_2^{-1}\Big(\Big\|\sum_{j=1}^{N_1} T^{a_j}x\Big\| + \Big\|\sum_{j=N_1+1}^{2N_1} T^{a_j}x\Big\| + \cdots \\ &\qquad \cdots + \Big\|\sum_{j=2^{v-1}N_1+1}^{2^vN_1} T^{a_j}x\Big\| + \Big\|\sum_{j=2^vN_1+1}^{N_2} T^{a_j}x\Big\|\Big) \\ &\leq N_2^{-1}\Big(N_1M + \eta N_1 + 2\eta N_1 + \cdots + 2^{v-1}\eta N_1 + 2^v M\Big) \\ &\leq \frac{N_1M}{N_2} + \frac{2^v\eta N_1}{N_2} + \frac{2^vM}{N_2} \leq \eta + \frac{2M}{2^v} \leq \eta_1, \end{aligned}$$

which is a contradiction. □

Continuation of the proof of Theorem 2.1. Fix $N \geq N_0$ as in Lemma 2.2. Write for short $x_j = T^j x$ for all $j \in \mathbb{N}$. For all $r \in \{1, \ldots, a_N\}$ and $s \in \{N+1, \ldots, 2N\}$ write

$$u_{r,s} = x_r + x_{r+a_{s+1}-a_s} + \cdots + x_{r+a_{s+k-1}-a_s}.$$

Then

$$T^{a_s-r}u_{r,s} = x_{a_s} + x_{a_{s+1}} + \cdots + x_{a_{s+k-1}}.$$

Consider
$$A=\frac{1}{a_N N}\sum_{r=1}^{a_N}\sum_{s=N+1}^{2N}\|u_{r,s}\|^2.$$
We will estimate A from above and from below to obtain a contradiction.

First we consider a lower bound. Clearly
$$\begin{aligned}A&\geq\frac{1}{M^2a_NN}\sum_{r=1}^{a_N}\sum_{s=N+1}^{2N}\|x_{a_s}+x_{a_{s+1}}+\cdots+x_{a_{s+k-1}}\|^2\\&=\frac{1}{M^2N}\sum_{s=N+1}^{2N}\|x_{a_s}+x_{a_{s+1}}+\cdots+x_{a_{s+k-1}}\|^2.\end{aligned}$$
The Cauchy–Schwarz inequality and the triangular inequality then give
$$\begin{aligned}A&\geq\frac{1}{M^2}\Big(N^{-1}\sum_{s=N+1}^{2N}\|x_{a_s}+x_{a_{s+1}}+\cdots+x_{a_{s+k-1}}\|\Big)^2\\&\geq\frac{1}{M^2}\Big\|N^{-1}\sum_{s=N+1}^{2N}(x_{a_s}+x_{a_{s+1}}+\cdots+x_{a_{s+k-1}})\Big\|^2.\end{aligned}$$
Next
$$\begin{aligned}&\sum_{s=N+1}^{2N}\big(x_{a_s}+x_{a_{s+1}}+\cdots+x_{a_{s+k-1}}\big)\\&\qquad=\sum_{s=N+1}^{N+k-1}(s-N)x_{a_s}+\sum_{s=N+k}^{2N}kx_{a_s}+\sum_{s=2N+1}^{2N+k-1}(2N+k-s)x_{a_s}.\end{aligned}$$
Hence
$$A\geq\frac{1}{M^2N^2}\Big(k\Big\|\sum_{s=N+1}^{2N}x_{a_s}\Big\|-k^2M\Big)^2\geq\frac{1}{M^2}\Big(k\eta-\frac{k^2M}{N}\Big)^2\geq\Big(\frac{k\eta}{2M}\Big)^2$$
since $N\geq N_0\geq\frac{2kM}{\eta}$.

Next we estimate A from above. Using the inner product on H we write
$$A=\frac{1}{a_NN}\sum_{r=1}^{a_N}\sum_{s=N+1}^{2N}\sum_{j,j'=0}^{k-1}\langle x_{r+a_{s+j}-a_s},x_{r+a_{s+j'}-a_s}\rangle=B+\sum_{0\leq j<j'\leq k-1}C_{j,j'},$$
where
$$B=\frac{1}{a_NN}\sum_{r=1}^{a_N}\sum_{s=N+1}^{2N}\sum_{j=0}^{k-1}\|x_{r+a_{s+j}-a_s}\|^2\leq kM^2$$
and
$$C_{j,j'}=\frac{2}{a_NN}\operatorname{Re}\sum_{r=1}^{a_N}\sum_{s=N+1}^{2N}\langle x_{r+a_{s+j}-a_s},x_{r+a_{s+j'}-a_s}\rangle.$$
Fix $j,j'\in\{0,\ldots,k-1\}$ with $j<j'$. Let
$$\mathcal{B}=\big\{m\in\mathbb{N}:1+a_{N+1+j}-a_{N+1}\leq m\leq a_N+a_{2N+j}-a_{2N}\big\}.$$

For all $m \in \mathcal{B}$ let

$$\begin{aligned}\mathcal{A}_m &= \Big\{s \in \{N+1, \ldots, 2N\} : \text{there exists an } r \in \{1, \ldots, a_N\} \text{ such that } m = r + a_{s+j} - a_s\Big\} \\ &= \Big\{s \in \{N+1, \ldots, 2N\} : 1 \le m - a_{s+j} + a_s \le a_N\Big\} \\ &= \Big\{s \in \{N+1, \ldots, 2N\} : 1 + a_{s+j} - a_s \le m \le a_N + a_{s+j} - a_s\Big\}.\end{aligned}$$

Then

$$\begin{aligned}|C_{j,j'}| &\le \frac{2}{a_N N}\Big|\sum_{m\in\mathcal{B}}\Big\langle x_m, \sum_{s\in\mathcal{A}_m} x_{m+a_{s+j'}-a_{s+j}}\Big\rangle\Big| \\ &\le \frac{2M}{a_N N}\sum_{m\in\mathcal{B}}\Big\|\sum_{s\in\mathcal{A}_m} x_{m+a_{s+j'}-a_{s+j}}\Big\| \\ &\le \frac{2M^2}{a_N N}\sum_{m\in\mathcal{B}}\Big\|\sum_{s\in\mathcal{A}_m} x_{a_{s+j'}-a_{s+j}}\Big\|.\end{aligned}$$

Note that $\mathcal{A}_m$ is always an interval since the sequence $(a_{s+j} - a_s)_{s=1}^{\infty}$ is increasing.

Define the sets

$$\begin{aligned}\mathcal{B}_0 &= \big\{m \in \mathcal{B} : N+1, 2N \in \mathcal{A}_m\big\} \\ &= \big\{m \in \mathcal{B} : \mathcal{A}_m = \{N+1, N+2, \ldots, 2N\}\big\}, \\ \mathcal{B}_1 &= \{m \in \mathcal{B} : N+1 \notin \mathcal{A}_m\}\end{aligned}$$

and

$$\mathcal{B}_2 = \{m \in \mathcal{B} : 2N \notin \mathcal{A}_m\}.$$

Note that $N+1 \in \mathcal{A}_m$ if and only if

$$1 + a_{N+1+j} - a_{N+1} \le m \le a_N + a_{N+1+j} - a_{N+1},$$

where the first inequality is satisfied automatically for all $m \in \mathcal{B}$. So

$$\mathcal{B}_1 = \big\{m \in \mathcal{B} : a_N + a_{N+1+j} - a_{N+1} < m \le a_N + a_{2N+j} - a_{2N}\big\}$$

and

$$\operatorname{card} \mathcal{B}_1 \le a_{2N+j} - a_{2N} \le a_{2N+k} - a_{2N}.$$

Similarly, $2N \in \mathcal{A}_m$ if and only if

$$1 + a_{2N+j} - a_{2N} \le m \le a_N + a_{2N+j} - a_{2N},$$

where the second inequality is satisfied automatically. So

$$\mathcal{B}_2 = \big\{m \in \mathcal{B} : 1 + a_{N+1+j} - a_{N+1} \le m < 1 + a_{2N+j} - a_{2N}\big\}$$

and $\operatorname{card} \mathcal{B}_2 \le a_{2N+j} - a_{2N} \le a_{2N+k} - a_{2N}$. Furthermore,

$$\begin{aligned}\mathcal{B}_0 &= \big\{m \in \mathcal{B} : 1 + a_{2N+j} - a_{2N} \le m \le a_N + a_{N+1+j} - a_{N+1}\big\} \\ &= \big\{m \in \mathcal{B} : 1 + d_{2N} + \cdots + d_{2N+j-1} \le m \le a_N + d_{N+1} + \cdots + d_{N+j}\big\}\end{aligned}$$

and card $\mathcal{B}_0 \le a_N$. Hence

$$|C_{j,j'}| \le \frac{2M^2}{a_N N}\Big(\sum_{j\in\mathcal{B}_0}\Big\|\sum_{s=N+1}^{2N} x_{a_{s+j'}-a_{s+j}}\Big\| + \sum_{m\in\mathcal{B}_1\cup\mathcal{B}_2}\Big\|\sum_{s\in\mathcal{A}_m} x_{a_{s+j'}-a_{s+j}}\Big\|\Big)$$

$$\le \frac{2M^2}{N}\Big\|\sum_{s=N+1}^{2N} x_{a_{s+j'}-a_{s+j}}\Big\| + \frac{4M^2}{a_N N}(a_{2N+k}-a_{2N})NM.$$

For the first term one estimates

$$\sum_{s=N+1}^{2N} x_{a_{s+j'}-a_{s+j}} = \sum_{s=1}^{2N+j} x_{a_{s+j'-j}-a_s} - \sum_{s=1}^{N+j} x_{a_{s+j'-j}-a_s},$$

and so by (1) one has

$$\Big\|\sum_{s=N+1}^{2N} x_{a_{s+j'}-a_{s+j}}\Big\| \le k^{-1}(2N+j) + k^{-1}(N+j) \le 3Nk^{-1}+2.$$

Hence

$$|C_{j,j'}| \le 6M^2k^{-1} + \frac{4M^2}{N} + \frac{4M^3}{a_N}(a_{2N+k}-a_{2N}) \le 8M^2k^{-1}$$

and we deduce the upper bound

$$A \le B + \sum_{0\le j<j'\le k-1} |C_{j,j'}| \le kM^2 + \binom{k}{2}\cdot 8M^2k^{-1} \le kM^2 + 4kM^2 = 5kM^2.$$

Since

$$5kM^2 < \Big(\frac{k\eta}{2M}\Big)^2,$$

we have a contradiction. □

Clearly it is sufficient to assume that the sequence (a_s) is increasing and convex only for all s sufficiently large.

Corollary 2.3. *Let T be a power bounded operator acting on a Hilbert space H and let $x \in H$. Let $(a_s)_{s=1}^\infty$ be a sequence in $\mathbb{N}_0$ and $N_0 \in \mathbb{N}$. Suppose that $a_{s+1} > a_s$ and $2a_{s+1} \le a_{s+2} + a_s$ for all $s \ge N_0$. Moreover, suppose* $\sup_{n>N_0} \frac{a_{2n}}{a_n} < \infty$ *and*

$$\lim_{N\to\infty} N^{-1}\sum_{j=N_0}^{N} T^{a_{j+k}-a_j}x = 0$$

for all $k \in \mathbb{N}$. Then

$$\lim_{N\to\infty} N^{-1}\sum_{j=1}^{N} T^{a_j}x = 0.$$

Denote by $\sigma_p(T)$ the point spectrum of an operator T.

Theorem 2.4. *Let T be a power bounded operator acting on a Hilbert space H such that $\sigma_p(T) \cap \{e^{2\pi it} : t \text{ rational}\} = \emptyset$. Let p be a non-constant polynomial which maps $\mathbb{N}_0$ to $\mathbb{N}_0$. Then*

$$\lim_{N\to\infty} N^{-1} \sum_{j=1}^{N} T^{p(j)} = 0$$

in the strong operator topology.

Proof. We prove the statement by induction on the degree of p.

If $\deg p = 1$, write $p(z) = \alpha_1 z + \alpha_0$ with integer coefficients α_1, α_0. Clearly $\alpha_1 \geq 1$ and $\alpha_0 \geq 0$. Then the statement follows from the mean ergodic theorem for the operator T^{α_1}.

Let $d \geq 1$ and suppose that the theorem is true for all polynomials of degree $\leq d$. Let p be a polynomial of degree $d+1$ satisfying $p(\mathbb{N}_0) \subset \mathbb{N}_0$. Set $a_n = p(n)$. Then Corollary 2.3 implies that

$$\lim_{N\to\infty} N^{-1} \sum_{j=1}^{N} T^{p(j)} = 0$$

in the strong operator topology. Now the theorem follows by induction. □

If we omit the condition on the spectrum of T, the Cesàro limit still exists.

Theorem 2.5. *Let T be a power bounded operator acting on a Hilbert space H. Let p be a polynomial which maps $\mathbb{N}_0$ to $\mathbb{N}_0$. Then the limit*

$$\lim_{N\to\infty} N^{-1} \sum_{j=1}^{N} T^{p(j)}$$

exists in the strong operator topology.

Proof. By the Jacobs–Glicksberg–de Leeuw theorem, see [JL] or [Kre], p. 108–109, we decompose $H = H_1 \oplus H_2$ as a direct sum, where both H_1 and H_2 are subspaces invariant for T, $\sigma_p(T|H_2) \cap \{\alpha \in \mathbb{C} : |\alpha| = 1\} = \emptyset$ and $H_1 = \bigvee_{\alpha\in\mathbb{C},\, |\alpha|=1} \ker(T - \alpha I)$.

If $x \in H_2$ then $\lim_{N\to\infty} N^{-1} \sum_{j=1}^{N} T^{p(j)}x = 0$ by Theorem 2.4. Since the sequence $(N^{-1} \sum_{j=1}^{N} T^{p(j)})_{N=1}^{\infty}$ is uniformly bounded, it is sufficient to show that the limit

$$\lim_{N\to\infty} N^{-1} \sum_{j=1}^{N} T^{p(j)}x$$

exists for all $\alpha \in \mathbb{C}$ with $|\alpha| = 1$ and $x \in \ker(T - \alpha I)$.

Let $\alpha \in \mathbb{C}$ with $|\alpha| = 1$ and let $x \in \ker(T - \alpha)$. Write $\alpha = e^{2\pi it}$ with $t \in [0, 2\pi)$. If t is irrational, then $\lim_{N\to\infty} N^{-1} \sum_{j=1}^{N} T^{p(j)}x = 0$ by Theorem 2.4.

Now suppose that t is rational. Write $t = \frac{a}{b}$ with $a \in \mathbb{N}_0$ and $b \in \mathbb{N}$. Let $Tx = e^{2\pi ia/b}x$. Then the sequence $(T^{p(j)}x)_{j\in\mathbb{N}}$ is periodical with period b, so the limit

$$\lim_{N\to\infty} N^{-1} \sum_{j=1}^{N} T^{p(j)}x = b^{-1} \sum_{j=1}^{b} \alpha^{p(j)}x$$

exists. □

Remark 2.6. In general, the limit operator $\lim_{N\to\infty} \sum_{j=1}^N T^{p(j)}$ in the strong operator topology is not the projection onto $\ker(I-T)$. The simplest example is the operator T on the 1-dimensional space $\mathbb{C}$ defined by $Tz = iz$ for all $z \in \mathbb{C}$ and the quadratic polynomial given by $p(n) = n^2$. For n even we have $T^{n^2} z = z$, and for n odd, $T^{n^2} z = iz$. Hence $\lim_{N\to\infty} N^{-1} \sum_{j=1}^N T^{j^2} z = \frac{1+i}{2} z$ for all $z \in \mathbb{C}$ and $\lim_{N\to\infty} N^{-1} \sum_{j=1}^N T^{j^2}$ is not a projection.

For more details, see [KNS]. In fact, $\lim_{N\to\infty} \sum_{j=1}^N T^{p(j)}$ is a projection for all power bounded operators T if and only if p is linear.

3. Continuous case

In this section we discuss the mean ergodic theorem for bounded strongly continuous semigroups of operators on a Hilbert space. The situation is analogous to the discrete case. We repeat the argument since a unification (for example using a general scheme of [BLM]) would make the proofs less transparent.

Theorem 3.1. *Let $(T_t)_{t\geq 0}$ be a bounded strongly continuous semigroup of operators on a Hilbert space H and let $x \in H$. Let $f\colon [0,\infty) \to [0,\infty)$ be a differentiable function. Suppose there exists a $b \geq 0$ such that $f'|_{[b,\infty)}$ is strictly increasing and $f'(b) > 0$. Suppose that $\limsup_{t\to\infty} \frac{f(2t)}{f(t)} < \infty$. Moreover, suppose that*

$$\lim_{N\to\infty} \frac{1}{N} \int_b^N T_{f(t+\Delta)-f(t)} x \, \mathrm{d}t = 0$$

for all $\Delta \in (0,1]$. Then

$$\lim_{N\to\infty} \frac{1}{N} \int_0^N T_{f(t)} x \, \mathrm{d}t = 0.$$

Proof. Let $M = \sup\{\|T_t\| : t \in [0,\infty)\}$. Without loss of generality we may assume that $\|x\| = 1$, $f'(t) \geq 0$ for all $t \in [0,\infty)$ and f' is strictly increasing on $[0,\infty)$.

Suppose on the contrary that

$$\limsup_{N\to\infty} \Big\| \frac{1}{N} \int_0^N T_{f(t)} x \, \mathrm{d}t \Big\| > 0.$$

Then

$$\limsup_{N\to\infty} \frac{1}{N} \Big\| \int_N^{2N} T_{f(t)} x \, \mathrm{d}t \Big\| > 0. \tag{2}$$

The proof of (2) is analogous to that of Lemma 2.2; we omit the details.

For all $r, s \in [0,\infty)$ write

$$u_{r,s} = \int_0^1 T_{r+f(s+t)-f(s)} x \, \mathrm{d}t.$$

Then

$$T_{f(s)-r} u_{r,s} = \int_0^1 T_{f(s+t)} x \, \mathrm{d}t$$

if $r \le f(s)$. Let

$$A = \limsup_{N\to\infty} \frac{1}{Nf(N)} \int_0^{f(N)} \mathrm{d}r \int_N^{2N} \|u_{r,s}\|^2 \,\mathrm{d}s.$$

Then

$$\begin{aligned} A &\ge \limsup_{N\to\infty} \frac{1}{M^2 N f(N)} \int_0^{f(N)} \mathrm{d}r \int_N^{2N} ds \Big\| \int_0^1 T_{f(s+t)} x \,\mathrm{d}t \Big\|^2 \\ &= \limsup_{N\to\infty} \frac{1}{M^2 N} \int_N^{2N} \mathrm{d}s \Big\| \int_0^1 T_{f(s+t)} x \,\mathrm{d}t \Big\|^2. \end{aligned}$$

By the Cauchy–Schwarz inequality and the triangular inequality we have

$$\begin{aligned} A &\ge \limsup_{N\to\infty} \frac{1}{M^2} \Big(\int_N^{2N} \mathrm{d}s \, \frac{1}{N} \Big\| \int_0^1 T_{f(s+t)} x \,\mathrm{d}t \Big\| \Big)^2 \\ &\ge \limsup_{N\to\infty} \frac{1}{M^2 N^2} \Big\| \int_N^{2N} \mathrm{d}s \int_0^1 T_{f(s+t)} x \,\mathrm{d}t \Big\|^2. \end{aligned}$$

Since

$$\int_N^{2N} \mathrm{d}s \int_0^1 T_{f(s+t)} x \,\mathrm{d}t = \int_N^{N+1} (s-N) T_{f(s)} x \,\mathrm{d}s + \int_{N+1}^{2N} T_{f(s)} x \,\mathrm{d}s + \int_{2N}^{2N+1} (2N+1-s) T_{f(s)} x \,\mathrm{d}s$$

one estimates

$$\Big\| \int_N^{2N} \mathrm{d}s \int_0^1 T_{f(s+t)} x \,\mathrm{d}t \Big\| \ge \Big\| \int_N^{2N} T_{f(s)} x \,\mathrm{d}s \Big\| - M.$$

Thus

$$\begin{aligned} A &\ge \limsup_{N\to\infty} \frac{1}{M^2} \Big(\frac{1}{N} \Big\| \int_N^{2N} T_{f(s)} x \,\mathrm{d}s \Big\| - \frac{M}{N} \Big)^2 \\ &= \frac{1}{M^2} \limsup_{N\to\infty} \Big(\frac{1}{N} \Big\| \int_N^{2N} T_{f(s)} x \,\mathrm{d}s \Big\| \Big)^2 > 0 \end{aligned}$$

by (2).

On the other hand,

$$\begin{aligned} A &= \limsup_{N\to\infty} \frac{1}{Nf(N)} \int_0^{f(N)} \mathrm{d}r \int_N^{2N} \mathrm{d}s \int_0^1 \mathrm{d}j \int_0^1 \big\langle T_{r+f(s+j)-f(s)} x, T_{r+f(s+j')-f(s)} x \big\rangle \,\mathrm{d}j' \\ &= \limsup_{N\to\infty} \frac{2 \,\mathrm{Re}}{Nf(N)} \int_0^1 \mathrm{d}j \int_j^1 \mathrm{d}j' \int_0^{f(N)} \mathrm{d}r \int_N^{2N} \big\langle T_{r+f(s+j)-f(s)} x, T_{r+f(s+j')-f(s)} x \big\rangle \,\mathrm{d}s. \end{aligned}$$

Setting $m = r + f(s+j) - f(s)$ we have

$$A \le \limsup_{N\to\infty} \frac{2}{Nf(N)} \Big| \int_0^1 \mathrm{d}j \int_j^1 \mathrm{d}j' \int_{\mathcal{B}_j} \mathrm{d}m \int_{\mathcal{A}_{j,m}} \big\langle T_m x, T_{m+f(s+j')-f(s+j)} x \big\rangle \,\mathrm{d}s \Big|,$$

where

$$\mathcal{B}_j = \big[f(N+j) - f(N), f(N) + f(2N+j) - f(2N) \big]$$

for all $j \in [0,1]$ and

$$\begin{aligned} \mathcal{A}_{j,m} &= \Big\{ s \in [N, 2N] : 0 \le m - f(s+j) + f(s) \le f(N) \Big\} \\ &= \Big\{ s \in [N, 2N] : f(s+j) - f(s) \le m \le f(N) + f(s+j) - f(s) \Big\} \end{aligned}$$

for all $m \in \mathcal{B}_j$. Let

$$\mathcal{B}'_j = \big\{m \in \mathcal{B}_j : \mathcal{A}_{j,m} = [N, 2N]\big\}.$$

Clearly $\mathcal{B}'_j = \{m \in \mathcal{B}_j : N, 2N \in \mathcal{A}_{j,m}\}$ since $\mathcal{A}_{j,m}$ is always an interval. By the definition of the set $\mathcal{A}_{j,m}$ we have

$$N \in \mathcal{A}_{j,m} \Longleftrightarrow f(N+j) - f(N) \le m \le f(N+j)$$

and

$$2N \in \mathcal{A}_{j,m} \Longleftrightarrow f(2N+j) - f(2N) \le m \le f(N) + f(2N+j) - f(2N).$$

So

$$\mathcal{B}'_j = \big[f(2N+j) - f(2N), f(N+j)\big]$$

and $\mu(\mathcal{B}_j \setminus \mathcal{B}'_j) \le 2\big(f(2N+j) - f(2N)\big) \le 2\big(f(2N+1) - f(2N)\big)$, where μ is the Lebesgue measure. Then

$$A \le \limsup_{N\to\infty} \frac{2}{N f(N)} \Big| \int_0^1 \mathrm{d}j \int_j^1 \mathrm{d}j' \int_{\mathcal{B}'_j} \mathrm{d}m \int_N^{2N} \big\langle T_m x, T_{m+f(s+j')-f(s+j)} x\big\rangle \, \mathrm{d}s \Big|$$

$$+ \limsup_{N\to\infty} \frac{2}{f(N)} 2M^2 \big(f(2N+1) - f(2N)\big).$$

We first consider the second term. Let $c = \limsup_{t\to\infty} \frac{f(2t)}{f(t)}$. For all $N \in \mathbb{R}$ large enough we have

$$c + 1 \ge \frac{f(2N)}{f(N)} = \frac{f(N) + \int_N^{2N} f'(t)\,\mathrm{d}t}{f(N)} \ge 1 + \frac{N f'(N)}{f(N)}$$

and hence $\limsup_{N\to\infty} \frac{f'(N)}{f(N)} \le \limsup_{N\to\infty} \frac{c}{N} = 0$. So

$$\limsup_{N\to\infty} \frac{f(2N+1) - f(2N)}{f(N)} \le \limsup_{N\to\infty} \frac{f'(2N+1)}{f(2N+1)} \cdot \frac{f(2N+1)}{f(2N)} \cdot \frac{f(2N)}{f(N)} = 0$$

and

$$\lim_{N\to\infty} \frac{f(2N+1) - f(2N)}{f(N)} = 0. \tag{3}$$

Thus

$$A \le \limsup_{N\to\infty} \frac{2}{N f(N)} \Big| \int_0^1 \mathrm{d}j \int_j^1 \mathrm{d}j' \int_{\mathcal{B}'_j} \mathrm{d}m \int_N^{2N} \big\langle T_m x, T_{m+f(s+j')-f(s+j)} x\big\rangle \, \mathrm{d}s \Big|.$$

Let $C = \big(\mathcal{B}'_j \setminus [0, f(N)]\big) \cup \big([0, f(N)] \setminus \mathcal{B}'_j\big)$ be the symmetrical difference of $\mathcal{B}'_j$ and $[0, f(N)]$. Then

$$\mu(C) \le f(2N+j) - f(2N) + f(N+j) - f(N) \le 2\big(f(2N+1) - f(2N)\big).$$

Hence using again (3) one deduces that

$$
\begin{aligned}
A \le \limsup_{N\to\infty} \frac{2}{Nf(N)}\Big|\int_0^1 \mathrm{d}j \int_j^1 \mathrm{d}j' \int_0^{f(N)} \mathrm{d}m \int_N^{2N} \big\langle T_m x, T_{m+f(s+j')-f(s+j)}x\big\rangle \,\mathrm{d}s\Big| \\
+ 2\big(f(2N+1)-f(2N)\big)N\frac{2M^2}{Nf(N)} \\
= \limsup_{N\to\infty} \frac{2}{Nf(N)}\Big|\int_0^{f(N)} \mathrm{d}m\Big\langle T_m x, \int_0^1 \mathrm{d}j \int_j^1 \mathrm{d}j' \int_N^{2N} T_{m+f(s+j')-f(s+j)}x\,\mathrm{d}s\Big\rangle\Big| \\
\le \limsup_{N\to\infty} \frac{2M^2}{N}\Big\|\int_0^1 \mathrm{d}j \int_j^1 \mathrm{d}j' \int_N^{2N} T_{f(s+j')-f(s+j)}x\,\mathrm{d}s\Big\| \\
\le \limsup_{N\to\infty} 2M^2 \int_0^1 \mathrm{d}j \int_j^1 \mathrm{d}j' \frac{1}{N}\Big\|\int_N^{2N} T_{f(s+j')-f(s+j)}x\,\mathrm{d}s\Big\|.
\end{aligned}
$$

Let $j, j' \in [0,1]$ with $j < j'$. Then

$$
\begin{aligned}
\Big\|\int_N^{2N} T_{f(s+j')-f(s+j)}x ds\Big\| &= \Big\|\int_{N+j}^{2N+j} T_{f(s+j'-j)-f(s)}x\,\mathrm{d}s\Big\| \\
&\le \Big\|\int_N^{2N} T_{f(s+j'-j)-f(s)}x\,\mathrm{d}s\Big\| \\
&\quad + \Big\|\int_N^{N+j} T_{f(s+j'-j)-f(s)}x\,\mathrm{d}s\Big\| + \Big\|\int_{2N}^{2N+j} T_{f(s+j'-j)-f(s)}x\,\mathrm{d}s\Big\| \\
&\le \Big\|\int_N^{2N} T_{f(s+j'-j)-f(s)}x\,\mathrm{d}s\Big\| + 2M.
\end{aligned}
$$

Setting $\Delta = j' - j$ we have

$$
\begin{aligned}
A &\le \limsup_{N\to\infty} 2M^2 \int_0^1 \mathrm{d}j \int_j^1 \mathrm{d}j' \frac{1}{N}\Big\|\int_N^{2N} T_{f(s+\Delta)-f(s)}x\,\mathrm{d}s\Big\| \\
&\le \limsup_{N\to\infty} 2M^2 \int_0^1 \mathrm{d}\Delta\,(1-\Delta)\frac{1}{N}\Big\|\int_N^{2N} T_{f(s+\Delta)-f(s)}x\,\mathrm{d}s\Big\| \\
&= \limsup_{N\to\infty} 2M^2 \int_0^1 \mathrm{d}\Delta\, g_N(\Delta),
\end{aligned}
$$

where $g_N(\Delta) = \frac{1-\Delta}{N}\Big\|\int_N^{2N} T_{f(s+\Delta)-f(s)}x\,\mathrm{d}s\Big\|$ for all $\Delta \in (0,1]$ and $N \in (0,\infty)$. For each $\Delta \in (0,1]$ we have

$$
\begin{aligned}
|g_N(\Delta)| &\le \frac{1}{N}\Big\|\int_0^{2N} T_{f(s+\Delta)-f(s)}x\,\mathrm{d}s - \int_0^N T_{f(s+\Delta)-f(s)}x\,\mathrm{d}s\Big\| \\
&\le \frac{2}{2N}\Big\|\int_0^{2N} T_{f(s+\Delta)-f(s)}x\,\mathrm{d}s\Big\| + \frac{1}{N}\Big\|\int_0^N T_{f(s+\Delta)-f(s)}x\,\mathrm{d}s\Big\|
\end{aligned}
$$

for all $N \in (0,\infty)$. So $\lim_{N\to\infty} g_N(\Delta) = 0$ by assumption. Clearly $|g_N(\Delta)| \leq M$ for all $N \in (0,\infty)$ and $\Delta \in (0,1]$. By the Lebesgue dominated convergence theorem, we deduce that $A = 0$, a contradiction. □

Lemma 3.2. *Let $\alpha \in (0,1]$, $c > 0$ and let $g \colon [0,\infty) \to \mathbb{R}$ be a differentiable function such that $\lim_{t\to\infty} \frac{g(t)}{t^\alpha} = 0$ and $\lim_{t\to\infty} \frac{g'(t)}{t^{\alpha-1}} = 0$. Define $f \colon [0,\infty) \to \mathbb{R}$ by*

$$f(t) = ct^\alpha + g(t).$$

Suppose that $f(t) \geq 0$ for all $t \in [0,\infty)$. Let $(T(t))_{t\geq 0}$ be a bounded strongly continuous semigroup of operators on a Hilbert space H. Then

$$\lim_{N\to\infty} \frac{1}{N} \int_0^N T_{f(t)}(I - T_\varepsilon) y \, \mathrm{d}t = 0$$

for all $\varepsilon > 0$ and $y \in H$.

Proof. Let $M = \sup\{\|T(t)\| : 0 \leq t < \infty\}$. Without loss of generality we may assume that $\|y\| = 1$.

Let $\delta > 0$. Choose $0 < \delta' < \min\{2^{-1/\alpha}\delta, \frac{c}{2}, \frac{c\alpha}{2^{1+1/\alpha}}\}$. There exists an $N_0 \in (0,\infty)$ such that $f(N_0) > \varepsilon + 1$, $|g(t)| < \delta' t^\alpha$ and $|g'(t)| < \delta' t^{\alpha-1}$ for all $t \geq N_0$. Then

$$|f'(t) - c\alpha t^{\alpha-1}| = |g'(t)| \leq \delta' t^{\alpha-1} < \delta t^{\alpha-1}$$

for all $t \geq N_0$. Set $\beta = \frac{\alpha-1}{\alpha}$. Clearly $\beta \leq 0$. For all $t \geq N_0$ there exists a ξ between $f(t)$ and ct^α such that

$$\big|f(t)^\beta - (ct^\alpha)^\beta\big| = |f(t) - ct^\alpha| \cdot |\beta \xi^{\beta-1}| = |g(t)| \cdot |\beta \xi^{\beta-1}|.$$

Hence

$$\begin{aligned}
\big|\alpha c^{1/\alpha} f(t)^\beta - c\alpha t^{\alpha-1}\big| &= \alpha c^{1/\alpha} \big|f(t)^\beta - (ct^\alpha)^\beta\big| \\
&\leq \alpha c^{1/\alpha} |g(t)| \cdot |\beta| (c - \delta')^{\beta-1} (t^\alpha)^{\beta-1} \leq (1-\alpha) c^{1/\alpha} \delta' t^\alpha (c/2)^{\beta-1} t^{-1} \\
&\leq 2^{1/\alpha} \delta' t^{\alpha-1} < \delta t^{\alpha-1}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\big|\alpha c^{1/\alpha} f(t)^{\frac{\alpha-1}{\alpha}} - f'(t)\big| &\leq 2\delta t^{\alpha-1}, \\
\big|\alpha c^{1/\alpha} f(t)^{\frac{\alpha-1}{\alpha}}\big| &\geq c\alpha t^{\alpha-1} - 2^{1/\alpha}\delta' t^{\alpha-1} \geq \tfrac{1}{2} c\alpha t^{\alpha-1}
\end{aligned}$$

and

$$\Big|1 - \frac{f'(t)}{\alpha c^{1/\alpha} f(t)^{\frac{\alpha-1}{\alpha}}}\Big| = \Big|\frac{\alpha c^{1/\alpha} f(t)^{\frac{\alpha-1}{\alpha}} - f'(t)}{\alpha c^{1/\alpha} f(t)^{\frac{\alpha-1}{\alpha}}}\Big| \leq \frac{2\delta t^{\alpha-1}}{\frac{1}{2} c\alpha t^{\alpha-1}} < \frac{4\delta}{c\alpha}.$$

Define

$$A = \limsup_{N\to\infty} \frac{1}{N} \Big\| \int_0^N T_{f(t)}(I - T_\varepsilon) y \, \mathrm{d}t \Big\|.$$

Then

$$A = \limsup_{N\to\infty} \frac{1}{N} \Big\| \int_{N_0}^N T_{f(t)}(I - T_\varepsilon) y \, \mathrm{d}t \Big\| \leq B + C,$$

where

$$B = \limsup_{N\to\infty} \frac{1}{N}\Big\|\int_{N_0}^{N} \frac{f'(t)}{\alpha c^{1/\alpha} f(t)^{\frac{\alpha-1}{\alpha}}} T_{f(t)}(I-T_\varepsilon)y \,\mathrm{d}t\Big\|$$

and

$$C = \limsup_{N\to\infty} \frac{1}{N}\Big\|\int_{N_0}^{N} \Big(1 - \frac{f'(t)}{\alpha c^{1/\alpha} f(t)^{\frac{\alpha-1}{\alpha}}}\Big) T_{f(t)}(I-T_\varepsilon)y \,\mathrm{d}t\Big\|.$$

Obviously

$$C \le \frac{4\delta}{c\alpha}\cdot 2M = \frac{8M\delta}{c\alpha}.$$

To estimate B, substitute $s = f(t)$ and $\mathrm{d}s = f'(t)\,\mathrm{d}t$. Then

$$\begin{aligned}
B &= \limsup_{N\to\infty} \frac{1}{N}\Big\|\int_{f(N_0)}^{f(N)} \alpha^{-1}c^{-1/\alpha} s^{\frac{1-\alpha}{\alpha}} T_s(I-T_\varepsilon)y\,\mathrm{d}s\Big\| \\
&\le \limsup_{N\to\infty} \frac{1}{N}\Big\|\int_{f(N_0)}^{f(N_0)+\varepsilon} \alpha^{-1}c^{-1/\alpha} s^{\frac{1-\alpha}{\alpha}} T_s y\,\mathrm{d}s\Big\| \\
&\qquad + \limsup_{N\to\infty} \frac{1}{N}\Big\|\int_{f(N_0)+\varepsilon}^{f(N)} \alpha^{-1}c^{-1/\alpha} \Big(s^{\frac{1-\alpha}{\alpha}} - (s-\varepsilon)^{\frac{1-\alpha}{\alpha}}\Big) T_s y\,\mathrm{d}s\Big\| \\
&\qquad + \limsup_{N\to\infty} \frac{1}{N}\Big\|\int_{f(N)}^{f(N)+\varepsilon} \alpha^{-1}c^{-1/\alpha} (s-\varepsilon)^{\frac{1-\alpha}{\alpha}} T_s y\,\mathrm{d}s\Big\| \\
&= \limsup_{N\to\infty} \frac{1}{N}\Big\|\int_{f(N_0)+\varepsilon}^{f(N)} \alpha^{-1}c^{-1/\alpha} \Big(s^{\frac{1-\alpha}{\alpha}} - (s-\varepsilon)^{\frac{1-\alpha}{\alpha}}\Big) T_s y\,\mathrm{d}s\Big\|,
\end{aligned}$$

since

$$\frac{1}{N}\Big\|\int_{f(N_0)}^{f(N_0)+\varepsilon} \alpha^{-1}c^{-1/\alpha} s^{\frac{1-\alpha}{\alpha}} T_s y\,\mathrm{d}s\Big\| \le \frac{M\varepsilon}{N\alpha c^{1/\alpha}} (f(N_0)+\varepsilon)^{\frac{1-\alpha}{\alpha}} \to 0$$

and

$$\begin{aligned}
\frac{1}{N}\Big\|\int_{f(N)}^{f(N)+\varepsilon} \alpha^{-1}c^{-1/\alpha} (s-\varepsilon)^{\frac{1-\alpha}{\alpha}} T_s y\,\mathrm{d}s\Big\| &\le \frac{M\varepsilon}{N\alpha c^{1/\alpha}} f(N)^{\frac{1-\alpha}{\alpha}} \\
&\le \frac{M\varepsilon}{N\alpha c^{1/\alpha}}\cdot\big(\frac{c}{2}N^\alpha\big)^{\frac{1-\alpha}{\alpha}} \le \frac{2^{\frac{\alpha-1}{\alpha}}M\varepsilon}{\alpha c}\cdot N^{-\alpha} \to 0
\end{aligned}$$

as $N\to\infty$.

If $\alpha = 1$ then $B = 0$. If $\alpha < 1$ and $s \in (\varepsilon,\infty)$, then there exists a $\xi \in (s-\varepsilon, s)$ such that

$$\big|s^{\frac{1-\alpha}{\alpha}} - (s-\varepsilon)^{\frac{1-\alpha}{\alpha}}\big| = \varepsilon\frac{1-\alpha}{\alpha}\xi^{\frac{1-2\alpha}{\alpha}}.$$

So

$$\begin{aligned}
B &\le \limsup_{N\to\infty} \frac{1}{N} f(N) M c^{-1/\alpha}\varepsilon\frac{1-\alpha}{\alpha^2}\max\{1, (f(N))^{\frac{1-2\alpha}{\alpha}}\} \\
&\le \limsup_{N\to\infty} \frac{M\varepsilon}{N\alpha^2 c^{1/\alpha}}\max\{f(N), f(N)^{\frac{1-\alpha}{\alpha}}\} \\
&\le \limsup_{N\to\infty} \frac{M\varepsilon}{N\alpha^2 c^{1/\alpha}}\max\{(2cN^\alpha, (2c)^{\frac{1-\alpha}{\alpha}}N^{1-\alpha}\} = 0.
\end{aligned}$$

Hence
$$A \le \frac{8M\delta}{c\alpha}.$$
Since $\delta > 0$ was arbitrary, we proved that
$$\lim_{N\to\infty} \frac{1}{N} \int_0^N T_{f(t)}(T_\varepsilon - I)y \, dt = 0.$$
□

Proposition 3.3. *Let $n \in \mathbb{N}_0$, $c > 0$ and $\alpha \in (n, n+1]$. Let $g\colon [0,\infty) \to \mathbb{R}$ be an $(n+1)-$times differentiable function such that $\lim_{t\to\infty} \frac{g^{(j)}(t)}{t^{\alpha-j}} = 0$ for all $j \in \{0,1,\ldots,n+1\}$. Define $f\colon [0,\infty) \to \mathbb{R}$ by*
$$f(t) = ct^\alpha + g(t).$$
Suppose that $f(t) \ge 0$ for all $t \in [0,\infty)$. Let $(T_t)_{t\ge 0}$ be a bounded strongly continuous semigroup of operators on a Hilbert space H. Let $\varepsilon > 0$ and $y \in H$. Then
$$\lim_{N\to\infty} \frac{1}{N} \int_0^N T_{f(t)}(I - T_\varepsilon)y \, dt = 0.$$

Proof. We prove the statement by induction on n. For $n = 0$ this was proved in Lemma 3.2.

Let $n \in \mathbb{N}$ and suppose that the statement is true for $n - 1$. We prove it for n.

Let f, g, T_t etc. satisfy all the required conditions. Then there exists an $N_0 \in \mathbb{N}$ such that $f'|_{[N_0,\infty)}$ is strictly increasing and positive. By Theorem 3.1, it is sufficient to show that
$$\lim_{N\to\infty} \frac{1}{N} \int_{N_0}^N T_{f(t+\Delta)-f(t)}(I - T_\varepsilon)y \, dt = 0$$
for all $\Delta \in (0,1]$.

Fix $\Delta \in (0,1]$. Define $F\colon [0,\infty) \to \mathbb{R}$ by
$$F(t) = f(t+\Delta) - f(t).$$
For all $t > 1$ the Taylor expansion gives
$$(t+\Delta)^\alpha - t^\alpha = t^\alpha \Big(1 + \frac{\Delta}{t}\Big)^\alpha - t^\alpha = t^\alpha \sum_{i=0}^\infty \binom{\alpha}{i} \frac{\Delta^i}{t^i} - t^\alpha = \Delta\alpha t^{\alpha-1} + h(t),$$
where $\binom{\alpha}{i} = \frac{\alpha(\alpha-1)\cdots(\alpha-i+1)}{i!}$ and
$$h(t) = t^{\alpha-2} \sum_{i=2}^\infty \binom{\alpha}{i} \frac{\Delta^i}{t^{2-i}}.$$
So we can write $F(t) = \Delta\alpha t^{\alpha-1} + G(t)$, where $G(t) = h(t) + \big(g(t+\Delta) - g(t)\big)$ for all $t \in (1,\infty)$.

Let $j \in \{0,\ldots,n\}$. Clearly
$$\lim_{t\to\infty} \frac{h^{(j)}}{t^{\alpha-1-j}} = 0.$$
Furthermore, for all $t > 1$ there exists a $\xi \in (t, t+\Delta)$ such that
$$g^{(j)}(t+\Delta) - g^{(j)}(t) = \Delta g^{(j+1)}(\xi).$$

Therefore

$$\lim_{t\to\infty} \frac{g^{(j)}(t+\Delta) - g^{(j)}(t)}{t^{\alpha-1-j}} = 0.$$

By the induction assumption we deduce that

$$\lim_{N\to\infty} \frac{1}{N} \int_{N_0}^{N} T_{f(t+\Delta)-f(t)}(I - T_\varepsilon)y \,\mathrm{d}t = \lim_{N\to\infty} \frac{1}{N} \int_{N_0}^{N} T_{F(t)}(I - T_\varepsilon)y \,\mathrm{d}t = 0.$$

Now the proposition follows by induction. □

Corollary 3.4. *Let $n \in \mathbb{N}$ and for all $i \in \{0, \ldots, n\}$ let $c_i \in \mathbb{R}$ and $\alpha_i \in [0, \infty)$. Suppose that $c_0 > 0$ and $\alpha_0 > \max\{\alpha_1, \ldots, \alpha_n\}$. Define $f\colon [0, \infty) \to \mathbb{R}$ by*

$$f(t) = \sum_{i=0}^{n} c_i t^{\alpha_i}.$$

Suppose that $f(t) \geq 0$ for all $t \in [0, \infty)$. Let $(T(t))_{t\geq 0}$ be a bounded strongly continuous semigroup of operators on a Hilbert space H. Then the limit

$$\lim_{N\to\infty} \frac{1}{N} \int_{0}^{N} T_{f(t)} \,\mathrm{d}t$$

exists in the strong operator topology and is equal to the projection P onto the kernel of the generator of $(T(t))$ with $\ker P = \overline{\bigcup_{\varepsilon>0}(T_\varepsilon - I)H}$.

Proof. Let A be the generator of the semigroup $(T(t))$. If $x \in \ker A$, then $T(t)x = x$ for all $t \geq 0$ and $\lim_{N\to\infty} \frac{1}{N}\int_0^N T_{f(t)}x \,\mathrm{d}t = x$.

Let $\varepsilon > 0$, $y \in H$ and set $x = (T_\varepsilon - I)y$. Then $\lim_{N\to\infty} \frac{1}{N}\int_0^N T_{f(t)}x \,\mathrm{d}t = 0$ by Proposition 3.3. Since the operators $\frac{1}{N}\int_0^N T_{f(t)} \,\mathrm{d}t$ are uniformly bounded, it is easy to see that $\lim_{N\to\infty} \frac{1}{N}\int_0^N T_{f(t)}x \,\mathrm{d}t = 0$ for all $x \in \overline{\bigcup_{\varepsilon>0}(T_\varepsilon - I)H}$.

Now the corollary follows from [EN], Lemma 4.4. □

Corollary 3.5. *Let $(T_t)_{t\geq 0}$ be a bounded strongly continuous semigroup of operators acting on a Hilbert space H, let A be its generator and P the projection onto $\ker A$ with $\ker P = \overline{\bigcup_{\varepsilon>0}(T_\varepsilon - I)H}$. Then*

$$\lim_{N\to\infty} \frac{1}{N} \int_{0}^{N} T_{t^\alpha} \,\mathrm{d}t = P$$

in the strong operator topology for all $\alpha > 0$.

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