Adaptive mesh refinement and robust guaranteed error bounds

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Reaction-diffusion problem

$$-\Delta u + \kappa^2 u = f \quad \text{in } \Omega \subset \mathbb{R}^2$$
$$u = 0 \quad \text{on } \partial\Omega$$

- $\kappa \geq 0$ a constant, Ω polygon
- Robust and locally efficient upper bound on error

History

- R. Verfürth, 1998
- M. Ainsworth, I. Babuška, 1999
- S. Grosman, 2006
- I. Cheddadi, R. Fučík, M.I. Prieto, M. Vohralík, 2009
- M. Ainsworth, T. V., 2011
- T. Linß, 2014
- M. Ainsworth, T. V., 2014

Error estimator

Weak solution: $u \in V = H_0^1(\Omega)$:

$$(\nabla u, \nabla v) + \kappa^2 (u, v) = (f, v) \quad \forall v \in V$$

FEM solution: $u_h \in V_h \subset V$:

$$(\nabla u_h, \nabla v_h) + \kappa^2 (u_h, v_h) = (f, v_h) \quad \forall v_h \in V_h$$

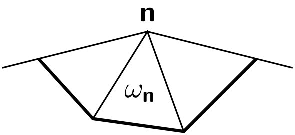
Error bound for $\kappa > 0$:

$$|\!|\!| u - u_h |\!|\!|^2 \le \sum_{K \in \mathcal{T}_h} [\eta_K(\boldsymbol{\tau}) + \mathrm{osc}_K(f)]^2 \quad \forall \boldsymbol{\tau} \in \mathbf{H}(\mathrm{div}, \Omega)$$

- $\eta_K^2(\boldsymbol{\tau}) = \|\boldsymbol{\tau} - \nabla u_h\|_K^2 + \kappa^{-2} \left\| \Pi_K f - \kappa^2 u_h + \operatorname{div} \boldsymbol{\tau} \right\|_K^2$
- $\mathrm{osc}_K(f) = \min\left\{ \frac{h_K}{\pi}, \frac{1}{\kappa} \right\} \|f - \Pi_K f\|_K$
- $V_h = \{ v_h \in V : v_h|_K \in P^1(K) \ \forall K \in \mathcal{T}_h \}$

Flux reconstruction for $\kappa > 0$

$$\mathbf{W}(\omega_{\mathbf{n}}) = \{\boldsymbol{\tau} \in \mathbf{H}(\mathrm{div}, \omega_{\mathbf{n}}) : \boldsymbol{\tau}|_K \in \mathbf{RT}_1(K),\\ \boldsymbol{\tau} \cdot \boldsymbol{\nu}_{\omega_{\mathbf{n}}} = 0 \text{ on edges } \gamma \subset \partial\omega_{\mathbf{n}},\ \mathbf{n} \notin \gamma\}$$



Flux reconstruction

$$\boldsymbol{\tau} \in \mathbf{H}(\mathrm{div}, \Omega), \quad \boldsymbol{\tau} = \sum_{\mathbf{n} \in \mathcal{N}_h} \boldsymbol{\tau}_{\mathbf{n}}, \quad \text{and } \boldsymbol{\tau}_{\mathbf{n}} \in \mathbf{W}(\omega_{\mathbf{n}}) \text{ minimizes}$$

$$\|\boldsymbol{\tau}_{\mathbf{n}} - \theta_{\mathbf{n}} \boldsymbol{\nabla} u_h\|^2_{\omega_{\mathbf{n}}} + \kappa^{-2} \left\|\theta_{\mathbf{n}}(\Pi f - \kappa^2 u_h) - \boldsymbol{\nabla}\theta_{\mathbf{n}} \cdot \boldsymbol{\nabla} u_h + \mathrm{div}\, \boldsymbol{\tau}_{\mathbf{n}}\right\|^2_{\omega_{\mathbf{n}}}$$

Equivalent to: find $\boldsymbol{\tau}_{\mathbf{n}} \in \mathbf{W}(\omega_{\mathbf{n}})$ such that

$$\kappa^{-2}(\mathrm{div}\, \boldsymbol{\tau}_{\mathbf{n}}, \mathrm{div}\, \mathbf{w}_h)_{\omega_{\mathbf{n}}} + (\boldsymbol{\tau}_{\mathbf{n}}, \mathbf{w}_h)_{\omega_{\mathbf{n}}}\\ = (\theta_{\mathbf{n}} \boldsymbol{\nabla} u_h, \mathbf{w}_h)_{\omega_{\mathbf{n}}} - \kappa^{-2}(\theta_{\mathbf{n}}(\Pi f - \kappa^2 u_h) - \boldsymbol{\nabla}\theta_{\mathbf{n}} \cdot \boldsymbol{\nabla} u_h, \mathrm{div}\, \mathbf{w}_h)_{\omega_{\mathbf{n}}}$$

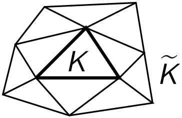
$$\forall \mathbf{w}_h \in \mathbf{W}(\omega_{\mathbf{n}})$$

Robust local efficiency

Theorem:

Consider regular family of triangulations ($h_K/\rho_K \le C \ \forall K \in \mathcal{T}_h, \forall \mathcal{T}_h$).
Then there exists $C > 0$ independent of h and κ such that

$$\eta_K^2(\boldsymbol{\tau}) \le C \left(|\!|\!| u - u_h |\!|\!|^2_{\widetilde{\widetilde{K}}} + \min\left(h_K^2, \kappa^{-2}\right) \left[\| f - \Pi f \|^2_{\widetilde{\widetilde{K}}} + \sum_{\mathbf{n} \in \mathcal{N}_K} \| \Pi f - f_{\mathbf{n}} \|^2_{\omega_{\mathbf{n}}} \right] \right)$$



Notation:

- $f_{\mathbf{n}}$ is the $L^2(\omega_{\mathbf{n}})$-projection of f onto $P_1(\omega_{\mathbf{n}})$
- $(\Pi f)|_K = \Pi_K f$ is the $L^2(K)$-projection of f onto $P_1(K)$

Case $\kappa = 0$

Braess–Schöberl reconstruction

$\boldsymbol{\tau} \in \mathbf{H}(\mathrm{div}, \Omega)$, $\quad \boldsymbol{\tau} = \sum_{\mathbf{n} \in \mathcal{N}_h} \boldsymbol{\tau}_{\mathbf{n}}$, $\quad$ and $\boldsymbol{\tau}_{\mathbf{n}} \in \mathbf{W}(\omega_{\mathbf{n}})$ minimizes

$$\|\boldsymbol{\tau}_{\mathbf{n}} - \theta_{\mathbf{n}} \boldsymbol{\nabla} u_h\|_{\omega_{\mathbf{n}}}^2$$

under the constraint

$$\Pi_{\omega_{\mathbf{n}}} \left(\theta_{\mathbf{n}}(\Pi f - \kappa^2 u_h)\right) - \boldsymbol{\nabla}\theta_{\mathbf{n}} \cdot \boldsymbol{\nabla} u_h + \mathrm{div}\, \boldsymbol{\tau}_{\mathbf{n}} = 0 \quad \text{in } \omega_{\mathbf{n}}$$

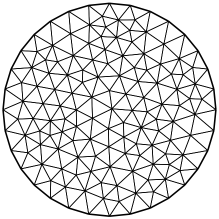
where $\Pi_{\omega_{\mathbf{n}}}$ is $L^2(\omega_{\mathbf{n}})$-projection onto $P_1^{\mathrm{disc}}(\omega_{\mathbf{n}})$

Error estimator

$$\eta_K(\boldsymbol{\tau}) = \|\boldsymbol{\tau} - \boldsymbol{\nabla} u_h\|_K$$

[Braess, Schöberl 2008]

Numerical example: circle



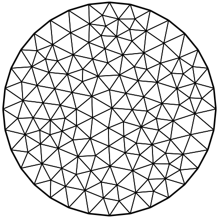
$$
\begin{aligned}
-\Delta u + \kappa^2 u &= 1 \quad \text{in } \Omega \\
u &= 0 \quad \text{on } \partial\Omega
\end{aligned}
$$

Exact solution

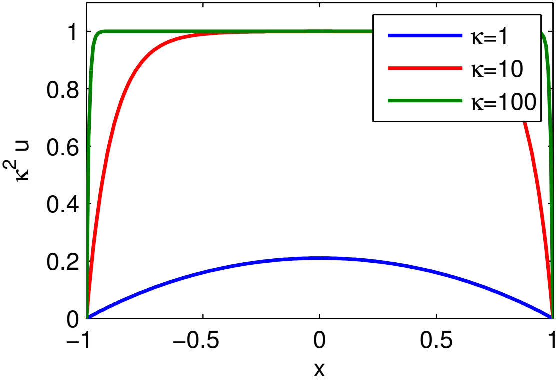
$$
u = \begin{cases} (1 - r^2)/4 & \text{for } \kappa = 0, \\ \kappa^{-2}\left[1 - I_0(\kappa r)/I_0(\kappa)\right] & \text{for } \kappa > 0, \end{cases} \qquad \text{where } r^2 = x^2 + y^2
$$

Numerical example: circle

$$-\Delta u + \kappa^2 u = 1 \quad \text{in } \Omega$$
$$u = 0 \quad \text{on } \partial\Omega$$

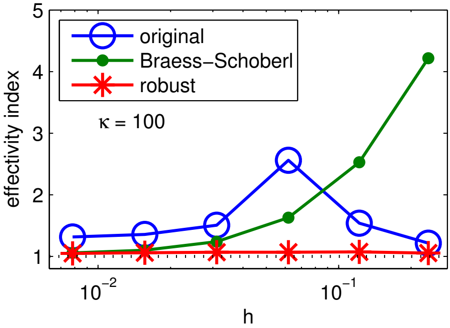


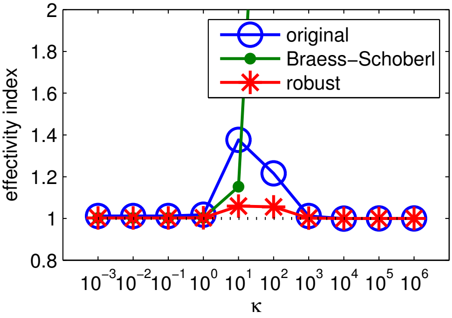
Exact solution



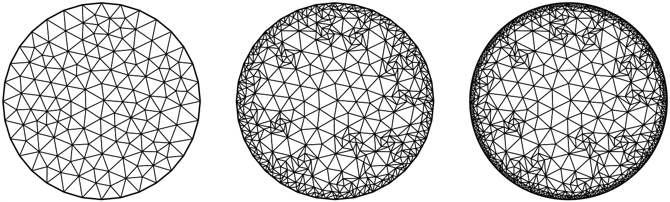
Numerical example: circle – effectivity

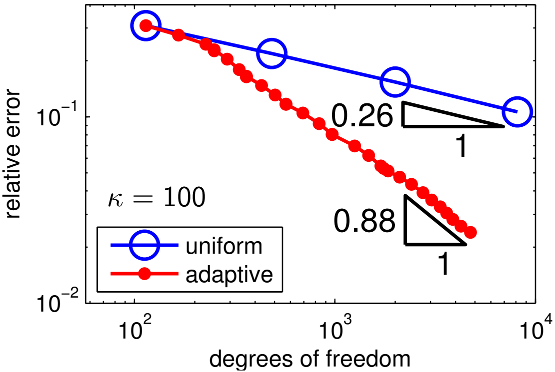
$$I_{\text{eff}} = \frac{\eta}{\|u - u_h\|}$$



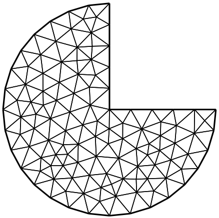


Numerical example: circle – adaptivity





Numerical example: L-shape



$$
\begin{aligned}
-\Delta u + \kappa^2 u &= f \quad \text{in } \Omega \\
u &= 0 \quad \text{on } \partial\Omega
\end{aligned}
$$

Exact solution

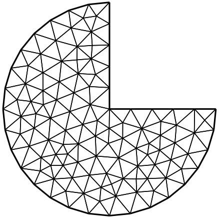
$$u = (r^{2/3} - r^2)\sin(2\theta/3 - \pi/3),$$

where

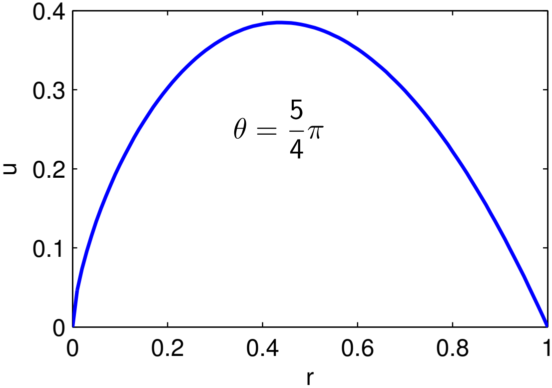
$$r \in [0,1], \quad \theta \in (0, 2\pi] \text{ are polar coordinates}$$

Numerical example: L-shape

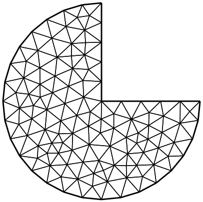
$$-\Delta u + \kappa^2 u = f \quad \text{in } \Omega$$
$$u = 0 \quad \text{on } \partial\Omega$$

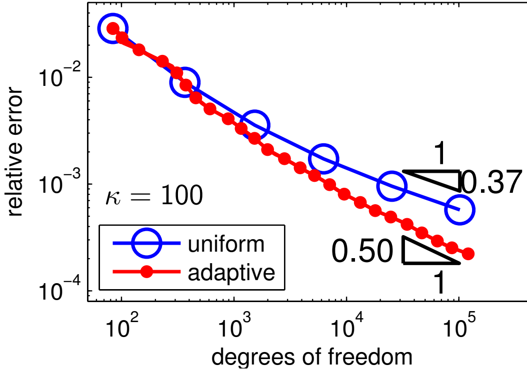


Exact solution



Numerical example: L-shape – adaptivity





Conclusions

- Simple flux reconstruction
- Local problems on patches
- Upper bound on error
- Proof of local efficiency
- Robustness

Thank you for your attention

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