

Higher gradient expansion for linear isotropic peridynamic materials[★]

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Abstract Peridynamics is a nonlocal continuum mechanics which replaces the differential operator embodied by the stress term $\operatorname{div} \mathbf{S}$ in Cauchy's equation of motion by a nonlocal force functional \mathfrak{L} to take into account long-range forces. The resulting equation of motion reads

$$\rho \ddot{\mathbf{u}} = \mathfrak{L} \mathbf{u} + \mathbf{b}, \quad (\mathbf{u} = \text{displacement}, \mathbf{b} = \text{body force}, \rho = \text{density}).$$

If the characteristic length δ of the interparticle interaction approaches 0, the operator \mathfrak{L} admits an expansion in δ which for a linear isotropic material reads

$$\mathfrak{L} \mathbf{u} = (\lambda + \mu) \nabla \operatorname{div} \mathbf{u} + \mu \Delta \mathbf{u} + \delta^2 \mathcal{A}_2 \cdot \nabla^4 \mathbf{u} + \delta^4 \mathcal{A}_3 \cdot \nabla^6 \mathbf{u} + \dots,$$

where λ and μ are the Lamé moduli of the classical elasticity, and the remaining higher order corrections contain products of the type $T_s \mathbf{u} := \mathcal{A}_s \cdot \nabla^{2s} \mathbf{u}$ of even order gradients $\nabla^{2s} \mathbf{u}$ (i.e., the collections of *all* partial derivatives of \mathbf{u} of order $2s$) and constant coefficients \mathcal{A}_s collectively forming a tensor of order $2s$. Symmetry arguments show that the terms $T_s \mathbf{u}$ have the form

$$\delta^{2s-2} (\lambda_s + \mu_s) \Delta^{s-1} \nabla \operatorname{div} \mathbf{u} + \delta^{2s-2} \mu_s \Delta^s \mathbf{u}$$

where λ_s and μ_s are scalar constants. This note determines explicitly λ_s and μ_s in terms of the properties of the material (i.e., of the operator \mathfrak{L}) in all dimensions n (typically, $n = 1, 2$ or 3).

AMS 2010 classification 35Q72, 74B05, 74B99, 74H10, 74H20, 74H25

Keywords Peridynamics, higher grade theories, nonlocal elastic material model, representation theorems for isotropic functions

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I Introduction

Peridynamics is a nonlocal continuum theory that does not use the spatial derivatives of the displacement field. This reformulation of elasticity theory to accommodate discontinuities and long-range forces was introduced by S. A. Silling in [16] and revised

[★] *Acknowledgement* The research was supported by Institutional research plan RVO 67985840.

and broadened by S. A. Silling, M. Epton, O. Weckner, J. Xu & E. Askari in [18]. It is a convenient tool to study defects such as cracks and interfaces. Peridynamics is similar in goals but different in form from the earlier versions of nonlocal mechanics summarized by I. A. Kunin in [11–12] and by A. C. Eringen in [9]. The reader is referred to the last three references for notes on the history and authorship.

To formulate the equation of motion, let $\Omega \subset \mathbb{R}^n$ (n arbitrary, but typically $n = 1, 2$ or 3), be the reference configuration of the body with material points labeled by $\mathbf{x} \in \Omega$. Let $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$ be the time-dependent displacement, and $\mathbf{b} = \mathbf{b}(\mathbf{x}, t)$ the body force. The motion is governed by the equation

$$\rho \ddot{\mathbf{u}} = \mathfrak{L} \mathbf{u} + \mathbf{b} \quad (1.1)$$

where ρ is the density and \mathfrak{L} an operator which describes the density of force ($\mathfrak{L} \mathbf{u}$)(\mathbf{x}) at the point \mathbf{x} exerted on \mathbf{x} by the rest of the body. The exact form of the operator \mathfrak{L} often differs in different authors; cf. e.g., [17, 21, 8, 19–20, 7, 1, 14, 5, 4]. It is hard to find compelling arguments for the preference of one form over another, and generality must be sought in this situation instead. We record the main forms of the operator \mathfrak{L} from [16] and [18], viz.,

$$\mathfrak{L} \mathbf{u}(\mathbf{x}) = \int_{\Omega} \mathbf{f}(\mathbf{u}(\mathbf{y}) - \mathbf{u}(\mathbf{x}), \mathbf{y} - \mathbf{x}) dV_{\mathbf{y}},$$

$$\mathfrak{L} \mathbf{u}(\mathbf{x}) = \int_{\Omega} (\mathbf{T}(\mathbf{u}(\mathbf{y}) - \mathbf{u}(\mathbf{x})) - \mathbf{T}(\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{y}))) dV_{\mathbf{y}},$$

respectively, where \mathbf{f} and \mathbf{T} are materially dependent functions of the indicated arguments. Throughout the paper, dV and dA denote the elements of the n -dimensional volume and $n - 1$ -dimensional area in \mathbb{R}^n .

The present note deals with the linearized isotropic case, with

$$\mathfrak{L} \mathbf{u}(\mathbf{x}) = \int_{\Omega} \mathbf{K}(\mathbf{y} - \mathbf{x})(\mathbf{u}(\mathbf{y}) - \mathbf{u}(\mathbf{x})) dV_{\mathbf{y}} \quad (1.2)$$

where the form of the kernel \mathbf{K} is dictated by the representation theorem of isotropic functions, i.e.,

$$\mathbf{K}(\mathbf{p}) = \psi(\mathbf{p})|\mathbf{p}|^2 \mathbf{1} + \omega(\mathbf{p})\mathbf{p} \otimes \mathbf{p} \quad (1.3)$$

$\mathbf{p} \in \mathbb{R}^n$, where ψ and ω are radial scalar functions determined by the properties of the material. A function η is said to be radial if the value $\eta(\mathbf{p})$ depends only on $|\mathbf{p}|$. The form (1.3) is used also by S. A. Silling in [16; §8] within the particular theory leading to Poisson's ratio $\nu = 1/4$ [#] and generally by T. Mengesha & Q. Du [13]. This is seemingly different from later proposal of S. A. Silling in [20; p. 104], also applied in [4], where \mathfrak{L} contains a double integration over the body while (1.2) does not. It will be shown in Section 2, below, that the discrepancy is only optical and (1.2)–(1.3) cover also that case.

The present note deals with the limit of the theory if the range of the interparticle forces approaches zero, i.e., the limit of vanishing nonlocality. Our concern is

[#] Indeed, a closer examination of Silling's derivation of his [16; Eq. (61)] shows that he obtains (1.3) with ψ and ω satisfying the extra restriction

$$\omega(\mathbf{p}) = 2\psi(\mathbf{p}) + \mathbf{p} \cdot \mathbf{D}_{\mathbf{p}} \psi(\mathbf{p}),$$

which is the root of the unwelcome consequence $\nu = 1/4$.

(i) whether the theory approaches the classical isotropic linear elasticity

$$\rho \ddot{\mathbf{u}} = \mathfrak{N} \mathbf{u} + \mathbf{b}$$

with the Navier operator

$$\mathfrak{N} \mathbf{u} = (\lambda + \mu) \nabla \operatorname{div} \mathbf{u} + \mu \Delta \mathbf{u}, \quad \text{and} \quad (1.4)$$

(ii) how to determine the form of the higher-order corrections to \mathfrak{N} .

These questions were first addressed and solved by O. Weckner & R. Abeyaratne [21] in the unidimensional case and by E. Emmrich & O. Weckner [8] in dimensions $n = 1, 2$ and 3 within the original particular theory [16], which covers only materials with Poisson's ratio $\nu = \lambda / (2(\lambda + \mu)) = 1/4$. The considerations in [21] and [8] are based on the Taylor expansion of the difference $\mathbf{u}(\mathbf{y}) - \mathbf{u}(\mathbf{x})$ in (1.2), an approach used to solve Question (i) several times since [2–3, 7, 14, 4], the present note inclusive.

Question (ii) seems to be treated only in [21] and [8]. Both these reference show that the higher order corrections contain only gradients $\nabla^{2s} \mathbf{u}$ of even orders $2s$ ($s = 2, 3, \dots$) with the coefficients \mathcal{A}_s of the expansion

$$\mathfrak{L} \mathbf{u} = \mathfrak{N} \mathbf{u} + \delta^2 \mathcal{A}_2 \cdot \nabla^4 \mathbf{u} + \delta^4 \mathcal{A}_3 \cdot \nabla^6 \mathbf{u} + \dots,$$

where δ is the length scale of the theory. In the unidimensional case of [21] the coefficients \mathcal{A}_{2s} are just numbers and no problem arises. In the higher dimensional case in [8] the coefficients \mathcal{A}_s are tensors of order $2s$. These are determined for low values of s in [8] by introducing the polar coordinates in \mathbb{R}^2 or \mathbb{R}^3 , which results in complicated coordinate-dependent formulas which are hard to interpret.

In the present note we use the isotropy, apply elementary representation theorems for isotropic functions, and integrate over the unit spheres \mathbb{S}^{n-1} in \mathbb{R}^n to show that the expansion takes the form

$$\mathfrak{L} \mathbf{u} = \mathfrak{N} \mathbf{u} + \delta^2 \mathfrak{N}^{(2)} \mathbf{u} + \delta^4 \mathfrak{N}^{(3)} \mathbf{u} + \dots,$$

where $\mathfrak{N}^{(s)}$ are Navier operators of order $2s$ of the form

$$\mathfrak{N}^{(s)} \mathbf{u} = (\lambda_s + \mu_s) \Delta^{s-1} \nabla \operatorname{div} \mathbf{u} + \mu_s \Delta^s \mathbf{u} \quad (1.5)$$

with the Lamé moduli λ_s, μ_s determined explicitly by the peridynamic material, see (3.4) (below). Thus only very specific combinations of the partial derivatives of \mathbf{u} occur. The form (1.5) alone with unspecified coefficients λ_s, μ_s follows directly from the isotropy; however, to determine their explicit values requires an integration as outlined.

2 Linear isotropic peridynamic materials

In this section we discuss the form of the force operator \mathfrak{L} in (1.1) for a linear isotropic nonlocal material. We assume the form

$$\mathfrak{L} \mathbf{u}(\mathbf{x}) = \int_{\Omega} \mathbf{K}(\mathbf{y} - \mathbf{x})(\mathbf{u}(\mathbf{y}) - \mathbf{u}(\mathbf{x})) dV_{\mathbf{y}} \quad (2.1)$$

$\mathbf{x} \in \Omega$, where the kernel \mathbf{K} vanishes if the distance $|\mathbf{x} - \mathbf{y}|$ exceeds some characteristic length δ of the material. Then for points $\mathbf{x} \in \Omega$ whose distance from the boundary

of Ω exceeds δ , the integral in (2.1) over Ω can be replaced by the integral over \mathbb{R}^n . Later we shall deal with the limit $\delta \rightarrow 0$ and thus the distance from the boundary of every point $\mathbf{x} \in \Omega$ eventually exceeds δ . In other words, we identify $\Omega = \mathbb{R}^n$ and integrate over \mathbb{R}^n , i.e.,

$$\mathfrak{L} \mathbf{u}(\mathbf{x}) = \int \mathbf{K}(\mathbf{y} - \mathbf{x})(\mathbf{u}(\mathbf{y}) - \mathbf{u}(\mathbf{x})) dV_{\mathbf{y}}, \quad (2.2)$$

$\mathbf{x} \in \mathbb{R}^n$, where here and below, we abbreviate $\int := \int_{\mathbb{R}^n}$.

The author believes that the form of \mathbf{K} is dictated by the representation theorem of isotropic tensor-valued functions rather than by heuristic visualizations of inter-particle bonds. Thus, denoting by $O(n)$ the full orthogonal group on \mathbb{R}^n , we base our treatment on repeated uses of representation theorems for $O(n)$ -invariant scalar and tensor valued functions of a vector argument. Let f and \mathbf{G} be functions on \mathbb{R}^n and taking their values in \mathbb{R} and in the space of symmetric second order tensors on \mathbb{R}^n , respectively. We say that f and \mathbf{G} are $O(n)$ -invariant (equivalently, isotropic) if

$$f(\mathbf{Q}\mathbf{c}) = f(\mathbf{c}), \quad \mathbf{G}(\mathbf{Q}\mathbf{c}) = \mathbf{Q}\mathbf{G}(\mathbf{c})\mathbf{Q}^T$$

for all $\mathbf{c} \in \mathbb{R}^n$ and $\mathbf{Q} \in O(n)$. The well-known elementary representation theorems (a particular case of [6; §13.32; General Representation Theorem]) then say that f is radial, i.e., the value $f(\mathbf{c})$ depends only on $|\mathbf{c}|$ and \mathbf{G} has the form

$$\mathbf{G}(\mathbf{c}) = \eta(\mathbf{c})\mathbf{1} + \theta(\mathbf{c})\mathbf{c} \otimes \mathbf{c}, \quad (2.3)$$

$\mathbf{c} \in \mathbb{R}^n$, where η, θ are radial functions. Here $\mathbf{1}$ is the identity transformation on \mathbb{R}^n . Moreover, if f is a polynomial in the components of \mathbf{c} then f can be expressed as a polynomial in $|\mathbf{c}|^2$. The last is a particular case of [22; Statement T_n^m , §9], but also follows elementarily. Taking the trace of (2.3) and forming the scalar product of (2.3) with $\mathbf{c} \otimes \mathbf{c}$ we obtain

$$n\eta(\mathbf{c}) + |\mathbf{c}|^2\theta(\mathbf{c}) = \text{tr} \mathbf{G}(\mathbf{c}), \quad |\mathbf{c}|^2\eta(\mathbf{c}) + |\mathbf{c}|^4\theta(\mathbf{c}) = \mathbf{G}(\mathbf{c})\mathbf{c} \cdot \mathbf{c}$$

from which

$$\eta(\mathbf{c}) = \frac{|\mathbf{c}|^2 \text{tr} \mathbf{G}(\mathbf{c}) - \mathbf{G}(\mathbf{c})\mathbf{c} \cdot \mathbf{c}}{(n-1)|\mathbf{c}|^2}, \quad \theta(\mathbf{c}) = \frac{n\mathbf{G}(\mathbf{c})\mathbf{c} \cdot \mathbf{c} - |\mathbf{c}|^2 \text{tr} \mathbf{G}(\mathbf{c})}{(n-1)|\mathbf{c}|^4}. \quad (2.4)$$

Consequently, by the representation theorems,

$$\mathbf{K}(\mathbf{p}) = \psi(\mathbf{p})|\mathbf{p}|^2\mathbf{1} + \omega(\mathbf{p})\mathbf{p} \otimes \mathbf{p}, \quad (2.5)$$

$\mathbf{p} \in \mathbb{R}^n$, where ψ and ω are radial functions. We make the following permanent assumptions: if η stands for ψ or ω then

$$\eta(\mathbf{p}) = 0 \quad \text{if} \quad |\mathbf{p}| \geq 1 \quad \text{and} \quad \int |\eta(\mathbf{p})||\mathbf{p}|^3 dV_{\mathbf{p}} < \infty \quad (2.6)$$

Assumptions (2.6) guarantee that all the integrals that occur in our treatment will converge, see Remarks 3.2, below.

The above form (2.2) and (2.5) of \mathbf{K} is not the one assumed in [20; p. 104] and in [4; §2]. Rather, \mathfrak{L} is assumed to consist of two parts,

$$\mathfrak{L} = \mathfrak{L}_s + \mathfrak{L}_d$$

where the first operator takes the form

$$\mathfrak{L}_s \mathbf{u}(\mathbf{x}) = \int \mathbf{K}_d(\mathbf{y} - \mathbf{x})(\mathbf{u}(\mathbf{y}) - \mathbf{u}(\mathbf{x})) dV_{\mathbf{y}}, \quad (2.7)$$

with

$$\mathbf{K}_s(\mathbf{p}) = \zeta(\mathbf{p})\mathbf{p} \otimes \mathbf{p}, \quad (2.8)$$

$\mathbf{p} \in \mathbb{R}^n$, where ζ is a radial function. However, $\mathfrak{L}_d \mathbf{u}$ is given by more complicated expressions which involves double integrals over the body \mathbb{R}^n involving a materially dependent radial function γ . As pointed out in [4; Eq. (2.10)], for points at the distance at least 2δ from the boundary of the body, one has

$$\mathfrak{L}_d \mathbf{u}(\mathbf{x}) = \int \int \gamma(\mathbf{z} - \mathbf{x})\gamma(\mathbf{z} - \mathbf{y})(\mathbf{z} - \mathbf{x}) \otimes (\mathbf{z} - \mathbf{y})\mathbf{u}(\mathbf{y}) dV_z dV_y. \quad (2.9)$$

This can be rewritten as

$$\mathfrak{L}_d \mathbf{u}(\mathbf{x}) = \int \mathbf{C}(\mathbf{x}, \mathbf{y})\mathbf{u}(\mathbf{y}) dV_y \quad (2.10)$$

where

$$\mathbf{C}(\mathbf{x}, \mathbf{y}) = \int \gamma(\mathbf{z} - \mathbf{x})\gamma(\mathbf{z} - \mathbf{y})(\mathbf{z} - \mathbf{x}) \otimes (\mathbf{z} - \mathbf{y}) dV_z. \quad (2.11)$$

Observe that

$$\int \mathbf{C}(\mathbf{x}, \mathbf{y}) dV_y = \mathbf{0}. \quad (2.12)$$

To see it, note that $\int \gamma(\mathbf{z} - \mathbf{y})(\mathbf{z} - \mathbf{y}) dV_y = \int \gamma(\mathbf{v})\mathbf{v} dV_v = \mathbf{0}$ where the first equality results from the substitution $\mathbf{y} \mapsto \mathbf{v} = \mathbf{z} - \mathbf{y}$ and the second since the integrand is an even function of \mathbf{v} . Hence

$$\begin{aligned} \int \mathbf{C}(\mathbf{x}, \mathbf{y}) dV_y &= \int \gamma(\mathbf{z} - \mathbf{x})(\mathbf{z} - \mathbf{x}) \otimes \int \gamma(\mathbf{z} - \mathbf{y})(\mathbf{z} - \mathbf{y}) dV_y dV_z \\ &= \int \gamma(\mathbf{z} - \mathbf{x})(\mathbf{z} - \mathbf{x}) \otimes \int \gamma(\mathbf{v})\mathbf{v} dV_v dV_z = \mathbf{0}, \end{aligned}$$

which proves (2.12). Using (2.12), one rewrites (2.10) as

$$\mathfrak{L}_d \mathbf{u}(\mathbf{x}) = \int \mathbf{C}(\mathbf{x}, \mathbf{y})(\mathbf{u}(\mathbf{y}) - \mathbf{u}(\mathbf{x})) dV_y. \quad (2.13)$$

One readily finds from (2.11) that \mathbf{C} is translationally invariant, i.e., $\mathbf{C}(\mathbf{x} + \mathbf{a}, \mathbf{y} + \mathbf{a}) = \mathbf{C}(\mathbf{x}, \mathbf{y})$ for every $\mathbf{a}, \mathbf{x}, \mathbf{y} \in \mathbb{R}^n$; hence $\mathbf{C}(\mathbf{x}, \mathbf{y}) = \mathbf{K}_d(\mathbf{x} - \mathbf{y})$ with $\mathbf{K}_d = \mathbf{C}(\cdot, \mathbf{0})$, and (2.13) is converted into

$$\mathfrak{L}_d \mathbf{u}(\mathbf{x}) = \int \mathbf{K}_d(\mathbf{x} - \mathbf{y})(\mathbf{u}(\mathbf{y}) - \mathbf{u}(\mathbf{x})) dV_y. \quad (2.14)$$

By (2.11), \mathbf{K}_d is given by the symmetric expression

$$\mathbf{K}_d(\mathbf{p}) = \int \gamma(\mathbf{z} - \mathbf{p}/2)\gamma(\mathbf{z} + \mathbf{p}/2)(\mathbf{z} - \mathbf{p}/2) \otimes (\mathbf{z} + \mathbf{p}/2) dV_z$$

$\mathbf{p} \in \mathbb{R}^n$, as an easy translation of the integration variable \mathbf{z} in (2.11) shows. Further, if one expands the product $(\mathbf{z} - \mathbf{p}/2) \otimes (\mathbf{z} + \mathbf{p}/2)$ and observes that the mixed terms

$$[\int \gamma(\mathbf{z} - \mathbf{p}/2)\gamma(\mathbf{z} + \mathbf{p}/2)\mathbf{z} dV_z] \otimes \mathbf{p} \quad \text{and} \quad \mathbf{p} \otimes [\int \gamma(\mathbf{z} - \mathbf{p}/2)\gamma(\mathbf{z} + \mathbf{p}/2)\mathbf{z} dV_z]$$

vanish, one obtains

$$\mathbf{K}_d(\mathbf{p}) = \int \gamma(\mathbf{z} - \mathbf{p}/2)\gamma(\mathbf{z} + \mathbf{p}/2)(\mathbf{z} \otimes \mathbf{z} - \mathbf{p} \otimes \mathbf{p}/4) dV_z.$$

One observes from this that \mathbf{K}_d is an isotropic function. The representation theorem yields

$$\mathbf{K}_d(\mathbf{p}) = \rho(\mathbf{p})\mathbf{1} + \sigma(\mathbf{p})\mathbf{p} \otimes \mathbf{p} \quad (2.15)$$

$\mathbf{p} \in \mathbb{R}^n$, where ρ, σ are radial functions, given by

$$\rho(\mathbf{p}) = \frac{1}{(n-1)|\mathbf{p}|^2} \int \gamma(\mathbf{z} - \mathbf{p}/2)\gamma(\mathbf{z} + \mathbf{p}/2)(|\mathbf{z}|^2|\mathbf{p}|^2 - (\mathbf{z} \cdot \mathbf{p})^2) dV_z,$$

$$\sigma(\mathbf{p}) = \frac{1}{(1-n)|\mathbf{p}|^4} \int \gamma(\mathbf{z} - \mathbf{p}/2)\gamma(\mathbf{z} + \mathbf{p}/2)(|\mathbf{z}|^2|\mathbf{p}|^2 - n(\mathbf{z} \cdot \mathbf{p})^2 + (n-1)|\mathbf{p}|^4/4) dV_z,$$

see (2.4). To summarize, equations (2.9), (2.7), (2.14) and the representations (2.8), (2.15) show that the operator \mathfrak{L} from the references [20] and [4] is given by (2.2) with \mathbf{K} represented as in (2.5) with $\psi = \rho$ and $\omega = \zeta + \sigma$, as desired.

3 Asymptotic expansion for vanishing nonlocality

We wish to determine the asymptotic form of the theory under vanishing nonlocality. Thus we consider small values of the characteristic length scale δ of the material, i.e., we replace the original length scale δ by $\varepsilon\delta$ where the factor $\varepsilon > 0$ approaches 0. This amounts to passing from the space variable, say, \mathbf{p} , to \mathbf{p}/ε , as this makes the effective radius ε times smaller. Under this change, the integrals must be multiplied by an appropriate factor to compensate the change of volume.

We consider the material determined by the kernel \mathbf{K} of the form (2.5), whose length scale is equal to 1 by (2.6)₁. In view of this, we can consistently use the letter δ for the scaling factor previously denoted by ε , and define a family of rescaled operators \mathfrak{L}_δ , $\delta > 0$, by

$$\mathfrak{L}_\delta \mathbf{u}(\mathbf{x}) = \int \mathbf{K}_\delta(\mathbf{y} - \mathbf{x})(\mathbf{u}(\mathbf{y}) - \mathbf{u}(\mathbf{x})) dV_{\mathbf{y}} \quad (3.1)$$

where \mathbf{K}_δ are the rescaled kernels, defined for any $\mathbf{p} \in \mathbb{R}^n$ by

$$\mathbf{K}_\delta(\mathbf{p}) := \delta^{-n-2} \mathbf{K}(\mathbf{p}/\delta) = \delta^{-n-4} (\psi(\mathbf{p}/\delta) |\mathbf{p}|^2 \mathbf{1} + \omega(\mathbf{p}/\delta) \mathbf{p} \otimes \mathbf{p}).$$

To state the main result we use the following convention to simplify the formulas: since ψ and ω are radial, we have $\psi(\mathbf{p}) = \tilde{\psi}(r)$ and $\omega(\mathbf{p}) = \tilde{\omega}(r)$ where $r = |\mathbf{p}|$ and $\tilde{\psi}$ and $\tilde{\omega}$ are functions of a scalar argument. We write $\psi(r)$ and $\omega(r)$ for $\tilde{\psi}(r)$ and $\tilde{\omega}(r)$, e.g., $\int_0^\infty \psi(r) dr := \int_0^\infty \tilde{\psi}(r) dr$. No confusion can arise. Further, for any bounded function \mathbf{g} on \mathbb{R}^n with values in any normed space with the norm $|\cdot|$ we put

$$\|\mathbf{g}\|_\infty := \sup \{ |\mathbf{g}(\mathbf{p})| : \mathbf{p} \in \mathbb{R}^n \} < \infty.$$

Let, finally, κ_{n-1} be the area of the unit sphere $\mathbb{S}^{n-1} := \{\mathbf{p} \in \mathbb{R}^n : |\mathbf{p}| = 1\}$ in \mathbb{R}^n .

The main result of this note is as follows.

Theorem 3.1. *Let $k \geq 1$ be an integer and let $\mathbf{u} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ have bounded continuous derivatives of all orders $\leq 2k + 1$. Then $\mathfrak{L}_\delta \mathbf{u}$ is defined and bounded on \mathbb{R}^n for all $\delta > 0$ and*

$$\left. \begin{aligned} \mathfrak{L}_\delta \mathbf{u} &= \sum_{s=1}^k \delta^{2s-2} \mathfrak{N}^{(s)} \mathbf{u} + \delta^{2k} \mathfrak{G}_\delta^{(k)} \mathbf{u} \quad \text{on } \mathbb{R}^n \\ \text{where } \|\mathfrak{G}_\delta^{(k)} \mathbf{u}\|_\infty &\leq c \|\nabla^{2k+1} \mathbf{u}\|_\infty \quad \text{with } c \text{ independent of } \delta \text{ and } \mathbf{u}; \end{aligned} \right\} \quad (3.2)$$

here

$$\mathfrak{N}^{(s)} \mathbf{u} = (\lambda_s + \mu_s) \Delta^{s-1} \nabla \operatorname{div} \mathbf{u} + \mu_s \Delta^s \mathbf{u} \quad (3.3)$$

are the Navier operators of order $2s$ with the Lamé moduli of order s given by the equations

$$\lambda_s = \iota_s((2s-1)\omega_s - (n+2s)\psi_s), \quad \mu_s = \iota_s(\omega_s + (n+2s)\psi_s) \quad (3.4)$$

that involve a normalization constant ι_s and moments ψ_s and ω_s ,

$$\iota_s = \kappa_{n-1} \left/ 2^s s! \prod_{i=0}^{s-1} (2i+n) \right., \quad \eta_s := \int_0^\infty \eta(r) r^{n+2s+1} dr, \quad \text{with } \eta := \psi, \omega.$$

Remark 3.2. The first member of the sum in (3.2)₁ is the classical Navier operator $\mathfrak{N}^{(1)} \equiv \mathfrak{N}$ from (1.4), with the Lamé moduli

$$\lambda, \mu = \frac{\kappa_{n-1} \int_0^\infty (\omega(r) \mp (n+2)\psi(r)) r^{n+3} dr}{2n(2+n)}.$$

Remarks 3.3. (i) The hypothesis of Theorem 3.1 implies that \mathbf{u} is a Lipschitzian function, i.e.,

$$|\mathbf{u}(\mathbf{y}) - \mathbf{u}(\mathbf{x})| \leq \|\nabla \mathbf{u}\|_\infty |\mathbf{y} - \mathbf{x}|$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Thus the integrand in the definition of \mathfrak{L}_δ in (3.1) can be bounded by

$$\|\nabla \mathbf{u}\|_\infty |\mathbf{K}_\delta(\mathbf{x} - \mathbf{y})| |\mathbf{x} - \mathbf{y}| \leq \|\nabla \mathbf{u}\|_\infty \delta^{-n-4} |\mathbf{z}|^3 (n|\psi(\mathbf{z}/\delta)| + |\omega(\mathbf{z}/\delta)|).$$

where we abbreviate $\mathbf{z} = \mathbf{x} - \mathbf{y}$. Since

$$\begin{aligned} \int |\mathbf{K}_\delta(\mathbf{x} - \mathbf{y})| |\mathbf{x} - \mathbf{y}| dV_{\mathbf{y}} &\leq \delta^{-n-4} \int |\mathbf{z}|^3 (n|\psi(\mathbf{z}/\delta)| + |\omega(\mathbf{z}/\delta)|) dV_{\mathbf{z}} \\ &= \delta^{-1} \int |\mathbf{p}|^3 (n|\psi(\mathbf{p})| + |\omega(\mathbf{p})|) dV_{\mathbf{p}} \end{aligned}$$

and the last integral converges in view of the standing assumption (2.6), we see that the integral in the definition (3.1) of \mathfrak{L}_δ converges and defines a bounded function on \mathbb{R}^n with the bound $C\delta^{-1}\|\nabla \mathbf{u}\|_\infty$ where C is a constant independent of δ and \mathbf{u} .

(ii) The integrals

$$\eta_s = \int_0^\infty \eta(r) r^{n+2s+1} dr, \quad \text{with } \eta := \psi, \omega,$$

in the definition of ψ_s and ω_s converge in view of the standing assumption (2.6). Indeed, by (2.6)₁ we can replace the integration \int_0^∞ by \int_0^1 and since $s \geq 1$, we have $r^{2s+1} \leq r^2$ for $0 \leq r \leq 1$ and hence

$$\int_0^1 |\eta(r)| r^{n+2s+1} dr \leq \int_0^1 |\eta(r)| r^{n+2} dr = \kappa_{n-1}^{-1} \int |\eta(\mathbf{p})| |\mathbf{p}|^3 dV_{\mathbf{p}} < \infty$$

by (2.6)₂, as desired.

(iii) The above considerations and the proof of Theorem 3.1, below, show that the standing assumption (2.6) can be replaced by a weaker requirement that η decays rapidly at infinity in the sense that

$$\int |\mathbf{p}|^l |\eta(\mathbf{p})| dV_{\mathbf{p}} < \infty$$

for all integers $l \geq 2$. This covers the Gaussian kernel given by

$$\eta(\mathbf{p}) \sim e^{-|\mathbf{p}|^2}$$

up to a rescaling and even the family of kernels $\eta = \eta_q$ parametrized by a scalar parameter q where

$$\eta(\mathbf{p}) \sim e^{-|\mathbf{p}|^2} / |\mathbf{p}|^q, \quad -\infty < q < n+3,$$

which vanish at the origin if the parameter q is negative and which are singular at the origin if q is positive.

Proof of Theorem 3.1 To simplify the notation, let us establish the formula (3.2)₁ for $\mathfrak{L}_\delta \mathbf{u}(\mathbf{x})$ and related statements for $\mathbf{x} = \mathbf{0}$. Then

$$\begin{aligned} \mathfrak{L}_\delta \mathbf{u}(\mathbf{0}) &= \int \mathbf{K}_\delta(\mathbf{y})(\mathbf{u}(\mathbf{y}) - \mathbf{u}(\mathbf{0})) dV_{\mathbf{y}} \\ &= \delta^{-n-4} \int (\psi(\mathbf{y}/\delta)|\mathbf{y}|^2 \mathbf{1} + \omega(\mathbf{y}/\delta)\mathbf{y}^2)(\mathbf{u}(\mathbf{y}) - \mathbf{u}(\mathbf{0})) dV_{\mathbf{y}} \\ &= \delta^{-2} \int \mathbf{K}(\mathbf{p})(\mathbf{u}(\delta\mathbf{p}) - \mathbf{u}(\mathbf{0})) dV_{\mathbf{p}} \end{aligned} \quad (3.5)$$

where in the last line we have performed a substitution $\mathbf{y} \mapsto \mathbf{p} = \mathbf{y}/\delta$ and \mathbf{K} is given by (2.5). We apply Taylor's expansion to $\mathbf{u}(\delta\mathbf{p}) - \mathbf{u}(\mathbf{0})$ to obtain

$$\mathbf{u}(\delta\mathbf{p}) - \mathbf{u}(\mathbf{0}) = \sum_{i=1}^{2k} \frac{\delta^i}{i!} (\mathbf{p} \cdot \nabla)^i \mathbf{u}(\mathbf{0}) + \frac{\delta^{2k+2}}{(2k+1)!} \int_0^1 (\mathbf{p} \cdot \nabla)^{2k+1} \mathbf{u}(t\delta\mathbf{p}) dt.$$

Hence (3.5) provides

$$\mathfrak{L}_\delta \mathbf{u} = \mathfrak{T}_\delta^{(k)} \mathbf{u} + \mathfrak{R}_\delta^{(k)} \mathbf{u}$$

where

$$\begin{aligned} \mathfrak{T}_\delta^{(k)} \mathbf{u} &:= \sum_{i=1}^{2k} \frac{\delta^{i-2}}{i!} \int \mathbf{K}(\mathbf{p})(\mathbf{p} \cdot \nabla)^i \mathbf{u}(\mathbf{0}) dV_{\mathbf{p}}, \\ \mathfrak{R}_\delta^{(k)} \mathbf{u} &:= \frac{\delta^{2k}}{(2k+1)!} \int_0^1 \int \mathbf{K}(\mathbf{p})((\mathbf{p} \cdot \nabla)^{2k+1} \mathbf{u}(t\delta\mathbf{p})) dV_{\mathbf{p}} dt \end{aligned}$$

We analyze these two terms separately. We have

$$\mathfrak{R}_\delta^{(k)} \mathbf{u} = \delta^{2k} \mathfrak{S}_\delta^{(k)} \mathbf{u}$$

where

$$\mathfrak{S}_\delta^{(k)} \mathbf{u} = \frac{1}{(2k+1)!} \int_0^1 \int \mathbf{K}(\mathbf{p})((\mathbf{p} \cdot \nabla)^{2k+1} \mathbf{u}(t\delta\mathbf{p})) dV_{\mathbf{p}} dt. \quad (3.6)$$

Let us show that $\mathfrak{S}_\delta^{(k)} \mathbf{u}$ satisfies (3.2)₂ with

$$c = \frac{1}{(2k+1)!} \int_{|\mathbf{p}| \leq 1} |\mathbf{p}|^3 \chi(\mathbf{p}) dV_{\mathbf{p}} < \infty \quad (3.7)$$

where $\chi(\mathbf{p}) = n|\psi(\mathbf{p})| + |\omega(\mathbf{p})|$. To see it, we estimate the inner integrand in (3.6) by

$$\begin{aligned} |\mathbf{K}(\mathbf{p})| |\mathbf{p}|^{2k+1} |\nabla^{2k+1} \mathbf{u}(t\delta\mathbf{p})| &\leq |\mathbf{K}(\mathbf{p})| |\mathbf{p}|^{2k+1} \|\nabla^{2k+1} \mathbf{u}\|_\infty \\ &= \chi(\mathbf{p}) |\mathbf{p}|^{2k+3} \|\nabla^{2k+1} \mathbf{u}\|_\infty \end{aligned}$$

and hence

$$\left| \int_0^1 \int \mathbf{K}(\mathbf{p})((\mathbf{p} \cdot \nabla)^{2k+1} \mathbf{u}(t\delta\mathbf{p})) dV_{\mathbf{p}} dt \right| \leq \int_0^1 \int \chi(\mathbf{p}) |\mathbf{p}|^{2k+3} dV_{\mathbf{p}} dt \|\nabla^{2k+1} \mathbf{u}\|_\infty$$

which we combine with

$$\int_0^1 \int \chi(\mathbf{p}) |\mathbf{p}|^{2k+3} dV_{\mathbf{p}} dt = \int_{|\mathbf{p}| \leq 1} |\mathbf{p}|^{2k+3} \chi(\mathbf{p}) dV_{\mathbf{p}} \leq \int_{|\mathbf{p}| \leq 1} |\mathbf{p}|^3 \chi(\mathbf{p}) dV_{\mathbf{p}}$$

which is finite by (2.6). This proves that $\mathfrak{S}_\delta^{(k)} \mathbf{u}$ satisfies (3.2)₂ with c from (3.7).

Further to analyze $\mathfrak{T}_\delta^{(k)} \mathbf{u}$, we observe that the integrand in $\mathfrak{T}_\delta^{(k)} \mathbf{u}$ is an odd function of \mathbf{p} if i is odd, we have

$$\mathfrak{T}_\delta^{(k)} \mathbf{u} = \sum_{s=1}^k \frac{\delta^{2s-2}}{(2s)!} \int K(\mathbf{p})(\mathbf{p} \cdot \nabla)^{2s} \mathbf{u}(\mathbf{0}) dV_{\mathbf{p}}.$$

A substitution $\mathbf{p} \mapsto (r, \mathbf{q})$ where $r = |\mathbf{p}|$, $\mathbf{q} = \mathbf{p}/r$, provides

$$\mathfrak{T}_\delta^{(k)} \mathbf{u} = \sum_{s=1}^k \frac{\delta^{2s-2}}{(2s)!} (\psi_s \mathbf{I}_s + \omega_s \mathbf{J}_s) \quad (3.8)$$

where

$$\mathbf{I}_s = \int_{\mathbb{S}^{n-1}} (\mathbf{q} \cdot \nabla)^{2s} \mathbf{u}(\mathbf{0}) dA_{\mathbf{q}}, \quad \mathbf{J}_s = \int_{\mathbb{S}^{n-1}} \mathbf{q} \otimes \mathbf{q} (\mathbf{q} \cdot \nabla)^{2s} \mathbf{u}(\mathbf{0}) dA_{\mathbf{q}}.$$

To evaluate these integrals, we are going to use Formulas (4.1) and (4.2), below, with the vector \mathbf{c} identified formally with the “differential vector” ∇ . This step is justified by the Fourier transformation which converts the action of ∇ into the multiplication by the true vector $\mathbf{c} = i \boldsymbol{\xi}$ where $\boldsymbol{\xi}$ is the Fourier variable. Thus applying (4.1) and (4.2) with $\mathbf{c} = \nabla$ and $\mathbf{b} = \mathbf{u}$ and noting that $|\mathbf{c}|^2 = \nabla \cdot \nabla = \Delta$ and $\mathbf{c} \otimes \mathbf{c} = \nabla \otimes \nabla = \nabla \operatorname{div}$, we obtain from (4.1) and (4.2),

$$\int_{\mathbb{S}^{n-1}} (\mathbf{q} \cdot \nabla)^{2s} dA_{\mathbf{q}} = \varphi_s \Delta^s, \quad \int_{\mathbb{S}^{n-1}} \mathbf{q} \otimes \mathbf{q} (\mathbf{q} \cdot \nabla)^{2s} dA_{\mathbf{q}} = \alpha_s \Delta^s \mathbf{1} + \beta_s \Delta^{s-1} \nabla \operatorname{div}.$$

Hence $\mathbf{I}_s = \varphi_s \Delta^s \mathbf{u}$, $\mathbf{J}_s = \alpha_s \Delta^s \mathbf{u} + \beta_s \Delta^{s-1} \nabla \operatorname{div} \mathbf{u}$ and (3.8) reduces to

$$\mathfrak{T}_\delta^{(k)} \mathbf{u} = \sum_{s=1}^k \frac{\delta^{2s-2}}{(2s)!} (\psi_s \varphi_s \Delta^s + \omega_s (\alpha_s \Delta^s \mathbf{u} + \beta_s \Delta^{s-1} \nabla \operatorname{div} \mathbf{u})) \mathbf{u} \quad (3.9)$$

Invoking the definitions (4.3)–(4.4) of φ_s , α_s and β_s , we rewrite (3.9) as

$$\mathfrak{T}_\delta^{(k)} \mathbf{u} = \sum_{s=1}^k \delta^{2s-2} \iota_s \left((\psi_s (n+2s) + \omega_s) \Delta^s \mathbf{u} + 2s \Delta^{s-1} \nabla \operatorname{div} \mathbf{u} \right)$$

where $\iota_s = \varphi_s / ((n+2s)(2s)!)$ and (3.2), (3.3) and (3.4) follow. \square

4 Moments over the sphere

We now establish Formulas (4.1) and (4.2), below, that were used in Proof of Theorem 3.1 in Section 3.

We denote by \mathbf{B} and Γ the beta and gamma functions

$$\mathbf{B}(s, t) = \int_0^1 x^{s-1} (1-x)^{t-1} dx, \quad \Gamma(t) = \int_0^\infty x^{t-1} e^{-x} dx,$$

see, e.g., [15; Chapter 5] or [23; Chapter XII], whose elementary properties are used below without any further reference. The area of \mathbb{S}^{n-1} is given by

$$\kappa_{n-1} = 2\pi^{n/2} / \Gamma(n/2).$$

Proposition 4.1. *If s is a positive integer and $c \in \mathbb{R}^n$ then*

$$\int_{\mathbb{S}^{n-1}} (\mathbf{q} \cdot \mathbf{c})^{2s} dA_{\mathbf{q}} = \varphi_s |\mathbf{c}|^{2s}, \quad (4.1)$$

$$\int_{\mathbb{S}^{n-1}} \mathbf{q} \otimes \mathbf{q} (\mathbf{q} \cdot \mathbf{c})^{2s} dA_{\mathbf{q}} = \alpha_s |\mathbf{c}|^{2s} \mathbf{1} + \beta_s \mathbf{c} \otimes \mathbf{c} |\mathbf{c}|^{2s-2} \quad (4.2)$$

where

$$\varphi_s = (2s)! \pi^{n/2} / (2^{2s-1} s! \Gamma(s + n/2)), \quad (4.3)$$

$$\alpha_s = \varphi_s / (2s + n), \quad \beta_s = 2s \varphi_s / (2s + n). \quad (4.4)$$

Formula (4.1) is known, see, e.g., [10; p. 445], and (4.2) is an easy consequence. For convenience, a complete proof is given.

Proof Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be given by

$$f(\mathbf{c}) = \int_{\mathbb{S}^{n-1}} (\mathbf{q} \cdot \mathbf{c})^{2s} dA_{\mathbf{q}},$$

$\mathbf{c} \in \mathbb{R}^n$. The isotropy of the sphere implies the easily verifiable fact that f is invariant with respect $O(n)$; since f is also a polynomial in \mathbf{c} , the remarks on polynomial invariant functions at the beginning of Section 2 yield that f is expressible as a polynomial in $|\mathbf{c}|^2$. Since f is a polynomial of degree $2s$ in \mathbf{c} , one easily deduces that this polynomial must have the form of the right-hand side of (4.1) with some scalar constant φ_s . Let us show that φ_s is given by (4.3). Taking $\mathbf{c} = (1, 0, \dots, 0)$ in (4.1) we obtain

$$\varphi_s = \int_{\mathbb{S}^{n-1}} q_1^{2s} dA_{\mathbf{q}}$$

where q_1 is the first component of the vector \mathbf{q} . Performing the trivial integration perpendicular to the first axis in \mathbb{R}^n we obtain a one-dimensional integral

$$\varphi_s = 2\kappa_{n-2} \int_0^1 q_1^{2s} (1 - q_1^2)^{(n-3)/2} dq_1 \quad (4.5)$$

where

$$\kappa_{n-2} = \frac{(n-1)\pi^{(n-1)/2}}{\Gamma((n+1)/2)} = \frac{2\pi^{(n-1)/2}}{\Gamma((n-1)/2)} \quad (4.6)$$

is the area of the unit sphere in \mathbb{R}^{n-1} . The substitution $q_1 \mapsto x = q_1^2$ reduces the integral (4.5) into

$$\begin{aligned} \varphi_s / \kappa_{n-2} &= \int_0^1 x^{s-1/2} (1-x)^{(n-3)/2} dx \\ &= -\frac{(n-3)}{2s+1} \int_0^1 x^{s+1/2} (1-x)^{(n-5)/2} dx \\ &= \frac{(n-3)(n-5)}{(2s+1)(2s+3)} \int_0^1 x^{s+3/2} (1-x)^{(n-7)/2} dx \\ &= \frac{1}{(s+1/2)(s+3/2) \cdots (s+n/2)} \int_0^1 x^{s+n+1/2} (1-x)^{-2} dx \end{aligned}$$

$$\varphi_s = \kappa_{n-2} \int_0^1 x^{s-1/2} (1-x)^{(n-3)/2} dx = \frac{\kappa_{n-2} \Gamma(s+1/2) \Gamma((n-1)/2)}{\Gamma(s+n/2)} \quad (4.7)$$

where we have used

$$\mathbf{B}(s, t) = \frac{\Gamma(s) \Gamma(t)}{\Gamma(s+t)}.$$

Combining (4.7), (4.6) with

$$\Gamma(s+1/2) = (2s)! \pi^{1/2} / (2^{2s} s!)$$

we obtain (4.3).

To proceed to the proof of (4.2), let \mathbf{G} be a function on \mathbb{R}^n with values in the space of symmetric second order tensors defined by

$$\mathbf{G}(c) = \int_{\mathbb{S}^{n-1}} \mathbf{q} \otimes \mathbf{q} (\mathbf{q} \cdot c)^{2s} dA_{\mathbf{q}},$$

$c \in \mathbb{R}^n$. This an $O(n)$ invariant function and hence of the form (2.3) with η and θ given by the general formulas (2.4) which in the present case read

$$\eta(c) = (\varphi_s - \varphi_{s+1}) |c|^{2s} / (n-1), \quad \theta(c) = (n\varphi_{s+1} - \varphi_s) |c|^{2s-2} / (n-1).$$

where we have used (4.1) with s replaced by $s+1$. A combination with the identity $\varphi_{s+1} = (2s+1)\varphi_s / (2s+n)$, which follows from the elementary properties of the gamma function, leads to

$$\eta(c) = \alpha_s |c|^{2s}, \quad \theta(c) = \beta_s |c|^{2s-2}$$

and hence to (4.2) via the formula (2.3). \square

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