Published in: *Mathematical modelling of bodies with complicated bulk and boundary behavior,* pp. 217–237, M. Šilhavý (ed.),
University of Naples, Naples, 2008

Divergence measure vectorfields: their structure and the divergence theorem

M. Šilhavý
Institute of Mathematics of the AV ČR
Žitná 25
115 67 Prague 1
Czech Republic
e-mail: silhavy@math.cas.cz

Abstract A divergence measure vectorfield is an $\mathbf{R}^n$ valued measure on an open subset U of $\mathbf{R}^n$ whose weak divergence in U is a (signed) measure. The paper estabishes (i) the structure of the divergence measure vectorfields, (ii) the product rule for the product of the divergence measure by a function from $W^{1,\infty}(U)$, and (iii) the divergence theorem for the divergence measure vectorfields. In (i) it is shown that each divergence measure vectorfield can be decomposed into three measures with the following properties. The first measure is supported by a 1 dimensional countably rectifiable set and is absolutely continuous with respect to the 1 dimensional Hausdorff measure with a density that is in the approximate tangent space. The second measure is the 'Cantor part', which is singular with respect to the Lebesgue measure and vanishes on all sets of finite 1 dimensional Hausdorff measure. The third part is absolutely continuous with respect to the Lebesgue measure. In (ii) a formula for the product is given which is similar to that for smooth vectorfields with, however, the scalar product of the vectordfield and the gradient of the function replaced by a measure valued duality pairing between the divergence measure and the weak gradient of the function from $W^{1,\infty}(U)$. In (iii) it is shown that the surface integral of the normal component of the vectorfield occurring in the classical divergence theorem has to be replaced by a continuous linear functional on the space of Lipschitz functions on the boundary; the volume integral contains the duality pairing mentioned in (ii). This part broadly generalizes and simplifies the relevant results from [3–4]. Our proofs do not use the divergence theorem for smooth vectorfields and sets with a regular boundary; the boundary of U can be even fractal in the sense that the normal to ∂U cannot be defined.

Keywords Divergence measure vectorfields, normal trace, flat chains, currents

MSC 2000 classification 28A75, 28A80

Contents

1 Introduction

Let $U \subset \mathbf{R}^n$ be an open set; denote by $\operatorname{cl} U$ and ∂U the closure and boundary of U, respectively. An $\mathbf{R}^n$ valued measure $\mathbf{q}$ on U (i.e., and n tuple of signed measures on U) is said to be a ***divergence measure vectorfield*** on U if there exists a (signed) scalar valued measure $\operatorname{div}_U \mathbf{q}$ on U such that

$$\int_U \nabla \varphi \cdot d\mathbf{q} = -\int_U \varphi \, d \operatorname{div}_U \mathbf{q} \tag{1.1}$$

for every compactly supported class C^∞ function φ on U. The measure $\operatorname{div}_U \mathbf{q}$ is called the (weak) ***divergence*** of $\mathbf{q}$ in U. We denote by $\mathscr{D}\mathscr{M}(U)$ the set of all divergence measure vectorfields on U. We denote by $W^{1,\infty}(U)$ the Sobolev space of functions $f \in L^\infty(U, \mathbf{R}^n)$ whose weak gradient ∇f is in $L^\infty(U, \mathbf{R}^n)$ and note that each $f \in W^{1,\infty}(U)$ is represented by a continuous function; in the sequel we always use this continuous representative. Furthermore, if $K \subset \mathbf{R}^n$ let $\operatorname{Lip}_{\mathrm{B}}(K)$ be the set of all bounded Lipschitz functions on K and note that $\operatorname{Lip}_{\mathrm{B}}(U) \subset W^{1,\infty}(U)$ but the equality generally does not hold. If $0 \le k \le n$ and if μ is a scalar or vector valued measure we say that μ is $\mathscr{H}^k$ ***diffuse*** if $\mu(B) = 0$ for every Borel set B of finite k dimensional Hausdorff measure $\mathscr{H}^k$.[1]

The main results of this note establish the structure of divergence measure vectorfields, the product rule for divergence measure vectorfields and functions from $W^{1,\infty}(U)$ and a generalized divergence theorem, viz., if $\mathbf{q} \in \mathscr{D}\mathscr{M}(U)$

(i) then

$$\mathbf{q} = \boldsymbol{q}_s \mathscr{H}^1 \llcorner M + \mathbf{q}_c + \boldsymbol{q}_r \mathscr{L}^n \llcorner U \tag{1.2}$$

where

$$\left.\begin{array}{c} M \text{ is a countably } \mathscr{H}^1 \text{ rectifiable subset of } U, \\ \boldsymbol{q}_s \in L^1(\mathscr{H}^1 \llcorner M, \mathbf{R}^n) \text{ is tangential to } M, \\ \mathbf{q}_c \in \mathscr{M}(U, \mathbf{R}^n) \text{ is } \mathscr{H}^1 \text{ diffuse and } \mathscr{L}^n \text{ singular}, \\ \boldsymbol{q}_r \in L^1(U, \mathbf{R}^n); \end{array}\right\} \tag{1.3}$$

the three terms on the right hand side of (1.2) are the singular, 'Cantor,' and absolutely continuous parts of $\mathbf{q}$;

(ii) if $f \in W^{1,\infty}(U)$ then $f\mathbf{q} \in \mathscr{D}\mathscr{M}(U)$ and

$$\operatorname{div}_U(f\mathbf{q}) = f \operatorname{div}_U \mathbf{q} + \langle\!\langle \nabla f, \mathbf{q} \rangle\!\rangle_U \tag{1.4}$$

where $\langle\!\langle \nabla f, \mathbf{q} \rangle\!\rangle_U$ is a scalar valued measure on U that is absolutely continuous with respect to $\mathbf{q}$; the measure $\langle\!\langle \nabla f, \mathbf{q} \rangle\!\rangle_U$ is called the ***pairing*** of ∇f and $\mathbf{q}$;

(iii) there exists a continuous linear functional $\mathrm{N}_{\mathbf{q},U}(\cdot)$ on $\operatorname{Lip}_{\mathrm{B}}(\partial U)$ such that

$$\mathrm{N}_{\mathbf{q},U}(f|\partial U) = \int_U d\, \langle\!\langle \nabla f, \mathbf{q} \rangle\!\rangle_U + \int_U f d \operatorname{div}_U \mathbf{q} \tag{1.5}$$

for every $f \in \operatorname{Lip}_{\mathrm{B}}(\mathbf{R}^n)$, where $f|\partial U$ is the restriction of f to ∂U; the functional $\mathrm{N}_{\mathbf{q},U}(\cdot)$ is called the ***normal trace*** of $\mathbf{q}$ on ∂U. Here the continuity is understood in the sense of (3.7) (below).

Equation (1.2) is an analog of the decomposition of the gradient of a function of bounded variation into the jump, Cantor, and absolutely continuous parts; in

[1] We refer to Section 2 for further notation to be used in this introduction.

general all the three terms in (1.2) are different from $\mathbf{0}$. The decomposition (1.2) is proved by applying Federer's structure and support theorems. As shown below, this decomposition holds more generally for any measure that is also a flat 1 dimensional chain. In fact, the decomposition (1.2) is only the first member of a sequence of decompositions of measures representing flat k dimensional chains ($1 \le k \le n$). With k increasing, the decomposition (1.2) puts stronger restrictions on its members. The decomposition of the gradient of a function of bounded variation corresponds to $k = n - 1$. See Remark 5.7 (below). We note in passing that essentially any signed measure μ on $\mathbf{R}^n$ can be realized as a divergence of some $\mathbf{q} \in \mathscr{DM}(\mathbf{R}^n)$; in fact one can choose $\mathbf{q}$ in $\mathscr{DM}(\mathbf{R}^n) \cap L^p(\mathbf{R}^n, \mathbf{R}^n)$ for any p satisfying $1 \le p < n/(n-1)$ [26; Example 3.3(i)]. [2]

The product rule is a generalization of [4; Theorem 3.2] and the divergence theorem is a generalization of [3; Theorem 2.2], [4; Theorem 3.1] which also treat the divergence measure vectorfields. Under the condition that the boundary ∂U is Lipschitz deformable [3] and for a class of functions $f : \mathbf{R}^n \to \mathbf{R}$ with essentially Hölder continuous derivatives, the authors establish (1.5) with $\langle\langle \nabla f, \mathbf{q} \rangle\rangle_U$ replaced by [4] $\nabla f \cdot \mathbf{q}$ and with the normal trace defined on their class of functions. Being unaware of the pairing $\langle\langle \nabla f, \mathbf{q} \rangle\rangle_U$, the authors cannot treat the general Lipschitz f, for then $\nabla f \cdot \mathbf{q}$ is generally meaningless, which is one source of the restrictions; the other being the particular method of proof in [3–4], which is based on the classical divergence theorem for smooth vectorfields on regular domains. The present paper removes these restrictions by adopting a different method of proof, which does not use the classical divergence theorem. The boundary of our regions can be completely arbitrary, the existence of the normal to ∂U is irrelevant; thus ∂U can be fractal in the sense that the normal cannot be defined. For divergence measure vectorfields represented by functions from $L^p(U, \mathbf{R}^n), 1 \le p \le \infty$, the present approach removes the assumption of Lipschitz deformability of ∂U and shows that the results of [3–4] are equally valid for Lipschitz regions; however, these matters are not treated here.

Let us now briefly discuss the pairing $\langle\langle \nabla f, \mathbf{q} \rangle\rangle_U$ and the normal trace $\mathrm{N}_{\mathbf{q},U}(\cdot)$. Of course, a comparison of (1.4) and (1.5) with the corresponding analogs for smooth vectorfields $\boldsymbol{q}$ shows that $\langle\langle \nabla f, \mathbf{q} \rangle\rangle_U$ is a generalization of the product $\nabla f \cdot \boldsymbol{q}$ and $\mathrm{N}_{\mathbf{q},U}(\cdot)$ is a generalization of the normal component of $\boldsymbol{q}$ on ∂U.

The pairing $\langle\langle \cdot, \cdot \rangle\rangle_U$ is introduced below in Theorem 3.2 independently of (1.4). It is shown that for any $\mathbf{q} \in \mathscr{DM}(U)$ and any bounded $\mathscr{L}^n$ measurable vectorfield z on U whose weak curl in U is a bounded measurable function one can associate a measure $\langle\langle z, \mathbf{q} \rangle\rangle_U$ that is a suitable generalization of the product $z \cdot \mathbf{q}$. For the existence of $\langle\langle z, \mathbf{q} \rangle\rangle_U$ is essential to know that *both* the divergence of $\mathbf{q}$ is a measure and the curl of z a bounded Lebesgue measurable function. In particular, if $f \in W^{1,\infty}(U)$ then $z := \nabla f$ is bounded and $\mathrm{curl}\, z = \mathbf{0}$. The pairing $\langle\langle z, \mathbf{q} \rangle\rangle_U$ reduces to the scalar product $z \cdot \mathbf{q}$ if z is continuous, but in general $z \cdot \mathbf{q}$ does not have any immediate meaning if z is defined only $\mathscr{L}^n$ almost everywhere on U and $\mathbf{q}$ is a general measure (see Example 3.9). The pairing $\langle\langle z, \mathbf{q} \rangle\rangle_U$ exists also under other (not strictly comparable) hypotheses

[2] For $p \ge n/(n-1)$ and $\mathbf{q} \in \mathscr{DM}(\mathbf{R}^n) \cap L^p(\mathbf{R}^n, \mathbf{R}^n)$ the measure $\mathrm{div}_{\mathbf{R}^n} \mathbf{q}$ is $\mathscr{H}^d$ diffuse where $d = n - p/(p-1)$ [26; Theorem 3.2].

[3] A condition stronger than the Lipschitz character of ∂U.

[4] We shall see that $\langle\langle \nabla f, \mathbf{q} \rangle\rangle_U = \nabla f \cdot \mathbf{q}$ if f is continuously differentiable.

on z and $\mathbf{q}$. Thus Anzellotti [2] establishes the existence of $\langle\langle z, \mathbf{q} \rangle\rangle_U$ if $z = \nabla f$ where f is a continuous function of bounded variation and $\mathbf{q} = \boldsymbol{q}\mathscr{L}^n \llcorner U$ where $\boldsymbol{q} \in L^\infty(U)$ and $\operatorname{div}_U \mathbf{q}$ is a measure, or if z is an $\mathbf{R}^n$ valued measure whose weak curl is a measure and $\mathbf{q} = \boldsymbol{q}\mathscr{L}^n \llcorner U$ where $\boldsymbol{q} \in L^\infty(U)$ and the weak divergence of $\boldsymbol{q}$ is in $L^n(U)$ (where n is the dimension of the space $\mathbf{R}^n$). We also refer to Témam [28; Chapter II, Section 7] and Kohn & Témam [14] for pairings between stresses $\boldsymbol{T}$ represented by an integrable function (the analog of $\boldsymbol{q}$) and the infinitesimal deformation tensor of a function of bounded deformation (the analog of z).

In general, the normal trace $\mathrm{N}_{\mathbf{q},U}(\cdot)$ is not represented by a measure (see Examples 3.8 and 3.9). A formula, generalizing that of [3–4], is given for $\mathrm{N}_{\mathbf{q},U}(g)$ in Theorem 3.6(i). Only under the additional conditions in Theorem 3.6(ii) the normal trace is represented by a measure. In various special cases an additional information is available on $\mathrm{N}_{\mathbf{q},U}(\cdot)$. Thus [27; Theorem 1.2, Chapter I] establishes (1.5) if U is a region with C^2 boundary, $\mathbf{q} = \boldsymbol{q}\mathscr{L}^n \llcorner U$ where $\boldsymbol{q} \in L^2(U, \mathbf{R}^n)$ and $\operatorname{div}_U \boldsymbol{q} \in L^2(U, \mathbf{R})$; moreover, $\langle\langle \nabla f, \mathbf{q} \rangle\rangle = \nabla f \cdot \boldsymbol{q}\mathscr{L}^n \llcorner U$, and $\mathrm{N}_{\boldsymbol{q},U}(\cdot)$ can be extended to a continuous linear functional on $W^{1/2,\,2}(\partial U, \mathbf{R})$. Further, Anzellotti [2; Section 1] shows that the trace of $\mathbf{q}$ on ∂U is given by the $\mathscr{H}^{n-1}$ integration of a bounded function if $\mathbf{q}$ is represented by a bounded function, and establishes (1.5) when f is a bounded continuous function of bounded variation. See also [5, 26], where the domain U can be a set of finite perimeter. If $\boldsymbol{q}$ is a (possibly unbounded) but integrable vectorfield with divergence a measure, the trace is a functional as above, but for "almost every domain" it is represented by a $\mathscr{H}^{n-1}$ integrable function on ∂U, see [24, 6, 16–17, 26].

The geometric integration theory of Whitney [29, 7] provides an abstract approach to the divergence theorem for generalized domains (flat n dimensional chains), and to *functions* $\boldsymbol{q} \in L^\infty(\mathbf{R}^n, \mathbf{R}^n)$ with the weak divergence in $L^\infty(\mathbf{R}^n, \mathbf{R})$ (flat $n-1$ dimensional cochains). See also [13]. By *further narrowing* the class of vectorfields $\boldsymbol{q}$ to functions with Lipschitz derivatives (possibly of higher order), the class of domains of integration is further extended to sharp chains and chainlets in Whitney [29] and Harrison [9–12], respectively. The flat and sharp chains and chainlets are generally not representable by sets, for example flat n dimensional chains are represented by Lebesgue integrable functions (the characteristic function of a Lebesgue measurable set inclusive) and a prototype of a chainlet is a dipole of two oppositely oriented regions at the same place. The present paper follows an opposite direction: the class of domains of integration is narrowed from flat n dimensional chains (or even sharp chains or chainlets) to classical sets U in $\mathbf{R}^n$ but at the same time the functions $\boldsymbol{q}$ of [29, 9–12] are generalized to the divergence measure vectorfields $\mathbf{q}$.

Fields of quantities represented by measures have mechanical motivations. The reader is referred to [3–5] for the motivation from the theory of hyperbolic systems of convervation laws. Another motivation comes from the notion of stress in continuum mechanics. The present approach readily generalizes to second order tensorfields $\mathbf{T}$ represented by measures, with $\mathbf{T}$ interpreted as the stress tensor. *First,* it was shown in [15] that the usage of the measure valued stress fields considerably simplifies the analysis of the statics of masonry materials. Measures permit solutions with stress concentrations on surfaces or lines in the masonry body and explicit solutions can be found for given boundary conditions. *Second,* normal traces of $\mathbf{T}$ (i.e., surface tractions) represented by measures with concentrations are found in some analytical solutions in linear elasticity with concentrated loads, as has been recently pointed out in [20–21]. *Finally,* the system of forces in bodies with material surfaces [8] can be considered as a single measure whose absolutely continuous part is the bulk stress and the singular part is the stress acting in the surface.

We close this introduction with a convention on the weak divergences. As introduced above, we denote the divergence of a $\mathbf{q} \in \mathscr{DM}(U)$ in U by $\operatorname{div}_U \mathbf{q}$; if $\mathbf{q} \in \mathscr{DM}(\mathbf{R}^n)$ we simplify the notation and write $\mathbb{div}\, \mathbf{q} := \operatorname{div}_{\mathbf{R}^n} \mathbf{q}$. By the symbol div without a subscript we denote the classical divergence operator on the smooth vectorfields.

2 Notation on sets and measures

Let Z be a finite-dimensional real inner product space. Below we shall need only the cases $Z = \mathbf{R}, Z = \mathbf{R}^n$ and $Z = \mathrm{Skw}$ where Skw is the set of all skew linear transformations from $\mathbf{R}^n$ to $\mathbf{R}^n$, which can be identified with the set of all skew n by n matrices, but it is preferable to assume that Z is arbitrary. A Z valued measure is any countably additive Z valued function μ defined on the system of all Borel subsets of $\mathbf{R}^n$. The case $Z = \mathbf{R}$ gives signed measures, see Rudin [22; Chapters 1 and 6], and the case $Z = \mathbf{R}^n$ gives vector valued measures [1; Chapter 1]; these can be identified with an n tuples of signed measures. If μ is a Z valued measure, we denote by $\|\mu\|$ the ***total variation*** of μ, i.e., the smallest nonnegative measure on $\mathbf{R}^n$ such that $|\mu(A)| \le \|\mu\|(A)$ for every Borel subset A of $\mathbf{R}^n$. We further denote by $\mathrm{M}(\mu)$ the ***mass*** of μ, defined by $\mathrm{M}(\mu) := \|\mu\|(\mathbf{R}^n)$. We say that μ is ***supported*** by a Borel set $U \subset \mathbf{R}^n$ if $\mu(B) = 0$ for every Borel set $B \subset \mathbf{R}^n$ such that $U \cap B = \emptyset$. Throughout the paper, *all measures are assumed to be defined on the whole of* $\mathbf{R}^n$, hence measures originally defined on a Borel set U are automatically extended by 0 outside U. We denote by $\mathscr{M}(U, Z)$ the set of all Z valued measures supported by U. If $\mu \in \mathscr{M}(U, Z)$ and $\alpha : U \to Z$ is a μ integrable function (i.e., α is $\|\mu\|$ measurable and $\int_U |\alpha|\, d\|\mu\| < \infty$) then $\int_U \alpha \cdot d\mu$ is a well defined number. In the same situation we define the product $\alpha \cdot \mu$ to be the signed measure given by

$$(\alpha \cdot \mu)(A) = \int_{A \cap U} \alpha \cdot d\mu$$

for every Borel subset A of $\mathbf{R}^n$.

We denote by $\mathscr{L}^n$ the Lebesgue measure in $\mathbf{R}^n$ ([7; Subsection 2.6.5] and if k is an integer, $0 \le k \le n$, we denote by $\mathscr{H}^k$ the k-dimensional Hausdorff measure in $\mathbf{R}^n$ [7; Subsections 2.10.2–2.10.6]. If A is a Borel set, and μ either a Z valued measure or a nonnegative Borel measure on $\mathbf{R}^n$ (such as $\mathscr{H}^k$ or $\mathscr{L}^n$), we denote by $\mu \llcorner A$ the ***restriction*** of μ to A, which is the measure defined by

$$(\mu \llcorner A)(B) = \mu(A \cap B),$$

for each Borel set $B \subset \mathbf{R}^n$. If ϕ is either a signed measure or a nonnegative Radon measure and α is a Z valued Borel map defined ϕ a.e. on A, with $|\alpha|\ \phi$ integrable, then $\alpha\phi \llcorner A$ denotes the Z valued measure on $\mathbf{R}^n$ defined by

$$(\alpha\phi \llcorner A)(B) = \int_{A \cap B} \alpha \, d\phi,$$

for each Borel set $B \subset \mathbf{R}^n$; thus

$$\int_A \beta \cdot d(\alpha\phi \llcorner A) = \int_A \beta \cdot \alpha \, d\phi$$

for any bounded Borel function $\beta : A \to Z$. Throughout the paper the integrals with unspecified domains of integration denote integrals over $\mathbf{R}^n$, $\int \ldots \equiv \int_{\mathbf{R}^n} \ldots$

If $U \subset \mathbf{R}^n$ is open we denote by $C_0(U, Z)$ the set of all Z valued continuous maps α on $\mathbf{R}^n$ such that the support $\operatorname{spt} \alpha$ of α is compact and contained in U. By $C_0^\infty(U, Z)$ we denote the set of all infinitely differentiable maps from $C_0(U, Z)$. If $\boldsymbol{x} \in \mathbf{R}^n$ and $\rho > 0$ then $\mathbf{B}(\boldsymbol{x}, \rho)$ denotes the open ball in $\mathbf{R}^n$ of center $\boldsymbol{x}$ and radius ρ.

If ϕ is a nonnegative Radon measure and $1 \le p \le \infty$, we denote by $L^p(\phi, Z)$ the usual Lebesgue spaces of (classes of equivalence) of p integrable Z valued functions on $\mathbf{R}^n$ relative to ϕ [7; Subsections 2.4.12–2.4.17]. We denote by $\| \cdot \|_{L^p(\phi)}$ the norm on $L^p(\phi, Z)$. If ϕ is supported by a Borel set U, we often consider the elements of $L^p(\phi, Z)$ as (classes of equivalence) of maps defined only on U. In the special case $\phi = \mathscr{L}^n \llcorner U$ we use the notation $L^p(U, Z) := L^p(\mathscr{L}^n \llcorner U, Z)$ and $\| \cdot \|_{L^p(U)} := \| \cdot \|_{L^p(\mathscr{L}^n \llcorner U)}$.

Let ω be a mollifier, let $\omega_\rho(\boldsymbol{x}) = \rho^{-n} \omega(\boldsymbol{x}/\rho)$ for any $\boldsymbol{x} \in \mathbf{R}^n$ and $\rho > 0$. If $\alpha : A \to Z$ is a locally $\mathscr{L}^n$ integrable map on a $\mathscr{L}^n$ measurable set $A \subset \mathbf{R}^n$ and $\rho > 0$, we define the ρ mollification α_ρ as a function on $\mathbf{R}^n$ given by

$$\alpha_\rho(\boldsymbol{x}) = \int_A \alpha(\boldsymbol{y}) \omega_\rho(\boldsymbol{x} - \boldsymbol{y}) \, d\mathscr{L}^n(\boldsymbol{y})$$

for every $\boldsymbol{x} \in \mathbf{R}^n$; thus α_ρ is the standard mollification of the extension by 0 of α from A to $\mathbf{R}^n$.

3 The main results

This section presents the summary of the main results without proofs, which are given in the subsequent sections, and some examples.

Throughout the rest of the paper, let $U \subset \mathbf{R}^n$ be an open set in $\mathbf{R}^n$.

If k is an integer with $0 \le k \le n$ and $M \subset \mathbf{R}^n$, we say that M is countably $\mathscr{H}^k$ rectifiable if M is $\mathscr{H}^k$ measurable and there exists a family of Lipschitz maps $\boldsymbol{\varphi}_i, i = 1, \ldots,$ with $\boldsymbol{\varphi}_i : \mathbf{R}^k \to \mathbf{R}^n$, such that

$$\mathscr{H}^k\Big(M \sim \bigcup_{i=1}^\infty \boldsymbol{\varphi}_i(\mathbf{R}^k)\Big) = 0.$$

We say that M is purely $\mathscr{H}^k$ unrectifiable if

$$\mathscr{H}^k(M \cap \boldsymbol{\varphi}(\mathbf{R}^k)) = 0$$

for every Lipschitz map $\boldsymbol{\varphi} : \mathbf{R}^k \to \mathbf{R}^n$; see [1; Definitions 2.57 and 2.64]; a countably $\mathscr{H}^k$ rectifiable set is a $\mathscr{H}^k$ measurable and $(\mathscr{H}^k, k)$ rectifiable set in the terminology of [7; Subsection 3.2.14] and a purely $\mathscr{H}^k$ unrectifiable set is a purely $(\mathscr{H}^k, k)$ unrectifiable set in the sense [7; Subsection 3.2.14].

If M is a countably $\mathscr{H}^k$ rectifiable set and $\boldsymbol{x} \in M$, we define the k dimensional approximate tangent cone $\operatorname{Tan}^k(M, \boldsymbol{x}) \subset \mathbf{R}^n$ to M at $\boldsymbol{x}$ by $\operatorname{Tan}^k(M, \boldsymbol{x}) := \operatorname{Tan}^k(\mathscr{H}^k \llcorner M, \boldsymbol{x}) \subset \mathbf{R}^n$ where the last object is defined in [7; Subsection 3.2.16] (for any measure ϕ in place of $\mathscr{H}^k \llcorner M$). We also refer to [1; Definitions 2.79 and 2.86].

It follows from [7; Theorem 3.2.19] that for $\mathscr{H}^k$ a.e. point of a countably $\mathscr{H}^k$ rectifiable set M the approximate k dimensional tangent cone is a k dimensional subspace of $\mathbf{R}^n$ which can be described as follows. If $\boldsymbol{\varphi} : \mathbf{R}^k \to \mathbf{R}^n$ is a Lipschitz map, $K \subset \mathbf{R}^k$ is $\mathscr{L}^k$ measurable such that $\boldsymbol{\varphi}|K$ is bilipschitz, and if $\nabla \boldsymbol{\varphi}(\boldsymbol{p})$ exists and is injective for $\mathscr{L}^k$ a.e. $\boldsymbol{p} \in K$ then

$$\operatorname{Tan}^k(M, \boldsymbol{x}) = \operatorname{range} \nabla \boldsymbol{\varphi}(\boldsymbol{\varphi}^{-1}(\boldsymbol{x}))$$

for $\mathscr{H}^k$ a.e. $\boldsymbol{x} \in \boldsymbol{\varphi}(K)$, see [7; Lemma 3.2.17]. This provides a complete description of $\operatorname{Tan}^k(M, \boldsymbol{x})$ for $\mathscr{H}^k$ a.e. point of M as the following result shows (see [7; Lemma 3.12.18]): If M is countably $\mathscr{H}^k$ rectifiable then there exists a countable family $\boldsymbol{\varphi}_i, i = 1, \dots,$ of Lipschitz maps from $\mathbf{R}^k$ to $\mathbf{R}^n$ and a family $K_i, i = 1, \dots,$ of compact subsets of $\mathbf{R}^k$ such that $\boldsymbol{\varphi}_i(K_i), i = 1, \dots,$ is a disjoint family of subsets of M,

$$\mathscr{H}^k\Big(M \sim \bigcup_{i=1}^{\infty} \boldsymbol{\varphi}_i(K_i)\Big) = 0,$$

and for each i, $\boldsymbol{\varphi}_i|K_i$ is bilipschitz, and $\nabla \boldsymbol{\varphi}_i(\boldsymbol{p})$ exists and is injective for each $\boldsymbol{p} \in K_i$.

If M is countably $\mathscr{H}^k$ rectifiable, we say that $\boldsymbol{q}_s \in L^1(\mathscr{H}^k \llcorner M, \mathbf{R}^n)$ is ***tangential*** to M if $\boldsymbol{q}_s(\boldsymbol{x}) \in \operatorname{Tan}^k(M, \boldsymbol{x})$ for $\mathscr{H}^k$ a.e. $\boldsymbol{x} \in M$.

3.1 Theorem. *Any* $\mathbf{q} \in \mathscr{DM}(U)$ *is of the form*

$$\mathbf{q} = \boldsymbol{q}_s \mathscr{H}^1 \llcorner M + \mathbf{q}_c + \boldsymbol{q}_r \mathscr{L}^n \llcorner U \tag{3.1}$$

where

$$\left.\begin{array}{c} M \text{ is a countably } \mathscr{H}^1 \text{ rectifiable subset of } U, \\ \boldsymbol{q}_s \in L^1(\mathscr{H}^1 \llcorner M, \mathbf{R}^n) \text{ is tangential to } M, \\ \mathbf{q}_c \in \mathscr{M}(U, \mathbf{R}^n) \text{ is } \mathscr{H}^1 \text{ diffuse and } \mathscr{L}^n \text{ singular,} \\ \boldsymbol{q}_r \in L^1(U, \mathbf{R}^n). \end{array}\right\} \tag{3.2}$$

The relationship of (3.1) to the decomposition of the derivative of a function of bounded variation into the jump, Cantor, and absolutely continuous parts is described in Remark 5.7 (below), but here we note the following special case. Let $n = 2$, let $u \in BV(U)$, and put $\mathbf{q} = \nabla u^\perp$ where $\boldsymbol{a}^\perp = (a_2, -a_1)$ for any $\boldsymbol{a} = (a_1, a_2) \in \mathbf{R}^2$. One finds that $\operatorname{div}_U \mathbf{q} = \mathbf{0}$ and thus $\mathbf{q} \in \mathscr{DM}(U)$. We have

$$\nabla u = \nabla^j u + \nabla^c u + \nabla^a u \tag{3.3}$$

where the three terms are the jump, Cantor, and $\mathscr{L}^2$ absolutely continuous parts of ∇u, with $\nabla^j u$ supported by a countable $\mathscr{H}^1$ rectifiable set M and of the form $\nabla^j u = [v]\boldsymbol{n}$ where $[v]$ is the jump of v on M and $\boldsymbol{n}$ the unit normal to M, i.e., the unit vector in the orthogonal complement of the approximate tangent space. The decomposition (3.3) then leads to (3.1)–(3.2) where $\boldsymbol{q}_s = [v]\boldsymbol{n}^\perp$ (which is tangential) and $\mathbf{q}_c = \nabla^c u^\perp$, $\boldsymbol{q}_r \mathscr{L}^n \llcorner U = \nabla^a u^\perp$. Since the three terms on the right hand side of (3.3) are known to be generally different from $\mathbf{0}$, so also are the three terms in (3.1).

If $\boldsymbol{\omega} \in C_0^\infty(U, \mathrm{Skw})$, we define the divergence $\operatorname{div} \boldsymbol{\omega}$ of $\boldsymbol{\omega}$ as an $\mathbf{R}^n$ valued function on $\mathbf{R}^n$ satisfying

$$\boldsymbol{a} \cdot \operatorname{div} \boldsymbol{\omega} = \operatorname{div}(\boldsymbol{\omega}^{\mathrm{T}} \boldsymbol{a})$$

for every $\boldsymbol{a} \in \mathbf{R}^n$ where the divergence on the right hand side is the classical divergence operator for vectorfields. We denote by $\mathscr{CF}(U)$ the set of all (Lebesgue classes of equivalence) of $\mathscr{L}^n$ measurable functions $\boldsymbol{z} \in L^\infty(U, \mathbf{R}^n)$ for which there exists a function $\operatorname{curl} \boldsymbol{z} \in L^\infty(U, \mathrm{Skw})$ such that

$$\int_U \boldsymbol{z} \cdot \operatorname{div} \boldsymbol{\omega} \, d\mathscr{L}^n = -\int_U \boldsymbol{\omega} \cdot \operatorname{curl} \boldsymbol{z} \, d\mathscr{L}^n$$

for each $\boldsymbol{\omega} \in C_0^\infty(U, \mathrm{Skw})$. If $\boldsymbol{z}$ is continuously differentiable in U then $\operatorname{curl} \boldsymbol{z} = \frac{1}{2}(\nabla \boldsymbol{z} - \nabla \boldsymbol{z}^{\mathrm{T}})$.

3.2 Theorem. *If $\boldsymbol{z} \in \mathscr{CF}(U)$ and $\mathbf{q} \in \mathscr{DM}(U)$, there exists a unique measure $\langle\!\langle \boldsymbol{z}, \mathbf{q} \rangle\!\rangle_U \in \mathscr{M}(U, \mathbf{R})$ such that*

$$\int_U \varphi \, d \, \langle\!\langle \boldsymbol{z}, \mathbf{q} \rangle\!\rangle_U = \lim_{\rho \to 0} \int_U \varphi \boldsymbol{z}_\rho \cdot d\mathbf{q} \tag{3.4}$$

for every $\varphi \in C_0^\infty(U, \mathbf{R})$. The measure $\langle\!\langle \boldsymbol{z}, \mathbf{q} \rangle\!\rangle_U$ is absolutely continuous with respect to $\|\mathbf{q}\|$.

Here $\boldsymbol{z}_\rho$ is the ρ mollification of $\boldsymbol{z}$. The principal value on the right hand side of (3.4) exists only because of the additional information that $\operatorname{div}_U \mathbf{q}$ is a measure and $\operatorname{curl} \boldsymbol{z}$ a bounded $\mathscr{L}^n$ measurable function: the proof of Proposition 4.3 (below) gives an explicit formula for the value of the limit, which involves both $\operatorname{div}_U \mathbf{q}$ and $\operatorname{curl} \boldsymbol{z}$. It is clear from (3.4) that $\langle\!\langle \boldsymbol{z}, \mathbf{q} \rangle\!\rangle_U = \boldsymbol{z} \cdot \mathbf{q}$ if either $\boldsymbol{z} \in \mathscr{CF}(U)$ is continuous or if $\mathbf{q} \in \mathscr{DM}(U)$ is of the form $\mathbf{q} = \boldsymbol{q} \mathscr{L}^n \llcorner U$ where $\boldsymbol{q} \in L^1(U, \mathbf{R}^n)$.

As mentioned in the introduction, we need the special case $\langle\!\langle \nabla f, \mathbf{q} \rangle\!\rangle_U$ where $f \in W^{1,\infty}(U)$. The pairing $\langle\!\langle \cdot, \cdot \rangle\!\rangle_U$ occurs in the following product rule for divergence measure vectorfields and Lipschitz functions.

3.3 Theorem. *If $\mathbf{q} \in \mathscr{DM}(U)$ and $f \in W^{1,\infty}(U)$ then $f\mathbf{q} \in \mathscr{DM}(U)$ and*

$$\operatorname{div}_U(f\mathbf{q}) = f \operatorname{div}_U \mathbf{q} + \langle\!\langle \nabla f, \mathbf{q} \rangle\!\rangle_U . \tag{3.5}$$

Also the following remark will be useful.

3.4 Remark. *If $\boldsymbol{z} \in \mathscr{CF}(U)$, $\mathbf{q} \in \mathscr{DM}(U)$, and $f \in W^{1,\infty}(U)$ then $f\boldsymbol{z} \in \mathscr{CF}(U)$ and*

$$\langle\!\langle f\boldsymbol{z}, \mathbf{q} \rangle\!\rangle_U = \langle\!\langle \boldsymbol{z}, f\mathbf{q} \rangle\!\rangle_U = f \, \langle\!\langle \boldsymbol{z}, \mathbf{q} \rangle\!\rangle_U .$$

If $K \subset \mathbf{R}^n$, we endow $\mathrm{Lip}_{\mathrm{B}}(K)$ with the norm

$$\|f\|_{\mathrm{Lip}_{\mathrm{B}}(K)} = \mathrm{Lip}(f) + \sup \{|f(\boldsymbol{x})| : \boldsymbol{x} \in K\}$$

for any $f \in \mathrm{Lip}_{\mathrm{B}}(K)$ and $\mathscr{DM}(U)$ with the norm

$$\|\mathbf{q}\|_{\mathscr{DM}(U)} = \mathrm{M}(\mathbf{q}) + \mathrm{M}(\operatorname{div}_U \mathbf{q})$$

for any $\mathbf{q} \in \mathscr{DM}(U)$.

The following form of the divergence theorem is the main result of the paper.

3.5 Theorem. *If* $\mathbf{q} \in \mathscr{DM}(U)$ *then there exists linear function* $\mathrm{N}_{\mathbf{q},U}(\cdot) : \mathrm{Lip}_{\mathrm{B}}(\partial U) \to \mathbf{R}$ *such that*

$$\mathrm{N}_{\mathbf{q},U}(f|\partial U) = \int_U d\,\langle\!\langle \nabla f, \mathbf{q} \rangle\!\rangle_U + \int_U f d\,\mathrm{div}_U\,\mathbf{q} \tag{3.6}$$

for every $f \in \mathrm{Lip}_{\mathrm{B}}(\mathbf{R}^n)$. *One has*

$$\mathrm{N}_{\mathbf{q},U}(g) \le \|\mathbf{q}\|_{\mathscr{DM}(U)} \|g\|_{\mathrm{Lip}_{\mathrm{B}}(\partial U)} \tag{3.7}$$

for all $g \in \mathrm{Lip}_{\mathrm{B}}(\partial U)$.

The right hand side of (3.6) depends only on the boundary values of f. The normal trace $\mathrm{N}_{\mathbf{q},U}(\cdot)$ is said to be (represented by) a ***measure*** if there exists a measure $\nu \in \mathscr{M}(\mathbf{R}^n, \mathbf{R})$, supported by ∂U, such that

$$\mathrm{N}_{\mathbf{q},U}(g) = \int g\, d\nu$$

for every $g \in \mathrm{Lip}_{\mathrm{B}}(\partial U)$. Examples 3.8 and 3.9 show that the normal trace is generally not a measure.

We now give a formula for the normal trace and a sufficient condition for the trace to be a measure.

3.6 Theorem. *Let* $\mathbf{q} \in \mathscr{DM}(U)$ *and let* $m : \mathbf{R}^n \to \mathbf{R}$ *be a nonnegative Lipschitz function with* $\mathrm{spt}\, m \subset \mathrm{cl}\, U$ *which is strictly positive on* U, *and for each* $\varepsilon > 0$ *let*

$$L_\varepsilon = \{\boldsymbol{x} \in U : 0 < m(\boldsymbol{x}) < \varepsilon\}.$$

Then

(i) *if* $f \in \mathrm{Lip}_{\mathrm{B}}(\mathbf{R}^n)$, *we have*

$$\mathrm{N}_{\mathbf{q},U}(f|\partial U) = -\lim_{\varepsilon \to 0} \varepsilon^{-1} \int_{L_\varepsilon} f\, d\,\langle\!\langle \nabla m, \mathbf{q} \rangle\!\rangle_U\,; \tag{3.8}$$

(ii) *if*

$$\liminf_{\varepsilon \to 0} \varepsilon^{-1} \|\, \langle\!\langle \mathbf{q}, \nabla m \rangle\!\rangle_U \,\|(L_\varepsilon) < \infty \tag{3.9}$$

then $\mathrm{N}_{\mathbf{q},U}(\cdot)$ *is a measure.*

A possible choice of m is

$$m(\boldsymbol{x}) = \begin{cases} \mathrm{dist}_{\partial U}(\boldsymbol{x}) & \text{if } \boldsymbol{x} \in U, \\ 0 & \text{if } \boldsymbol{x} \in \mathbf{R}^n \sim U. \end{cases}$$

A formula similar to (3.8) is found in [4; Equations (3.2)–(3.3)] in the context of regions with Lipschitz deformable boundary.

3.7 Corollary. *If* $\mathbf{q}_1, \mathbf{q}_2 \in \mathscr{DM}(U)$ *and* $\mathbf{q}_1 \llcorner U_\varepsilon = \mathbf{q}_2 \llcorner U_\varepsilon$ *for some* $\varepsilon > 0$ *where* $U_\varepsilon := \{\boldsymbol{x} \in U : \mathrm{dist}_{\partial U}(\boldsymbol{x}) < \varepsilon\}$ *then* $\mathrm{N}_{\mathbf{q}_1,U}(\cdot) = \mathrm{N}_{\mathbf{q}_2,U}(\cdot)$.

The following two examples are illustrate the nature of the normal trace in two particular cases.

3.8 Example. Let $1 \le \alpha < 3$ and let $\boldsymbol{q} : \mathbf{R}^2 \sim \{\mathbf{0}\} \to \mathbf{R}^2$ be defined by

$$\boldsymbol{q}(\boldsymbol{x}) = (x_2, -x_1)/|\boldsymbol{x}|^\alpha$$

for every $\boldsymbol{x} = (x_1, x_2) \in \mathbf{R}^2 \sim \{\mathbf{0}\}$ and let $U = \{\boldsymbol{x} = (x_1, x_2) \in \mathbf{R}^2 : |\boldsymbol{x}| < 1, x_2 < 0\}$. The classical divergence of $\boldsymbol{q}$ vanishes on $\mathbf{R}^2 \sim \{\mathbf{0}\}$. We have

$$\boldsymbol{q} \in L^p(U, \mathbf{R}^2) \quad \begin{cases} \text{for} \quad 1 \le p \le \infty & \text{if} \quad \alpha = 1, \\ \text{for} \quad 1 \le p < 2/(\alpha - 1) & \text{if} \quad 1 < \alpha < 3. \end{cases}$$

Thus $\mathsf{q} := \boldsymbol{q}\mathscr{L}^2 \llcorner U$ is a well defined $\mathbf{R}^2$ valued measure. It turns out that $\mathsf{q} \in \mathscr{DM}(U)$, $\operatorname{div}_U \mathsf{q} = 0$ and

$$\mathrm{N}_{\mathsf{q},U}(g) = \begin{cases} \int_{-1}^{1} g(t,0)\operatorname{sgn}(t)|t|^{1-\alpha}\,dt & \text{if} \quad 1 \le \alpha < 2, \\ \lim_{\varepsilon \to 0} \int_{\{\varepsilon < |t| \le 1\}} g(t,0)\operatorname{sgn}(t)|t|^{1-\alpha}\,dt & \text{if} \quad 2 \le \alpha < 3 \end{cases} \tag{3.10}$$

for any $g \in \mathrm{Lip}_{\mathrm{B}}(\partial U)$; thus if $1 \le \alpha < 2$ then $\mathrm{N}_{\mathsf{q},U}(\cdot)$ is a measure and if $2 \le \alpha < 3$ then $\mathrm{N}_{\mathsf{q},U}(\cdot)$ is not a measure. The principal value in $(3.10)_2$ exists for each Hölder continuous function $g : \partial U \to \mathbf{R}$ of exponent $\beta > \alpha - 2$, i.e., for each g satisfying $|g(\boldsymbol{x}) - g(\boldsymbol{y})| \le H|\boldsymbol{x} - \boldsymbol{y}|^\beta$ for some H and all $\boldsymbol{x}, \boldsymbol{y} \in \partial U$.

3.9 Example. Let U be a bounded open set and L a straight line in $\mathbf{R}^n$ of unit direction $\boldsymbol{e} \in \mathbf{R}^n$ such that $K := L \cap U \neq \emptyset$. Then $\mathscr{H}^1(K) < \infty$ and thus

$$\mathsf{q} := \boldsymbol{e}\mathscr{H}^1 \llcorner K$$

is a well defined $\mathbf{R}^n$ valued measure. One easily finds that $\mathsf{q} \in \mathscr{DM}(U)$ and $\operatorname{div}_U \mathsf{q} = 0$. We denote by $W^{1,\infty}(K)$ the Sobolev space on the 1 dimensional set $K \subset L$ and by h' the weak derivative of an element $h \in W^{1,\infty}(K)$. We note that if $f \in W^{1,\infty}(U)$ then the restriction $f|K$ of f to K belongs to $W^{1,\infty}(K)$ (recal our convention that the elements of $W^{1,\infty}(U)$ are represented by a continuous functions). It turns out that

$$\langle\!\langle \nabla f, \mathsf{q} \rangle\!\rangle_U = (f|K)'\mathscr{H}^1 \llcorner K \tag{3.11}$$

for any $f \in W^{1,\infty}(U)$. Furthermore, the relatively open subset K of L is a union of at most countably many segments $K_i \subset L$, $i \in I$, with endpoints $\boldsymbol{x}_i, \boldsymbol{y}_i \in L$ whose order we fix by $(\boldsymbol{y}_i - \boldsymbol{x}_i) \cdot \boldsymbol{e} > 0$. (The uncountable case arises if $L \cap \partial U$ has accumulation points.) It turns out that

$$\mathrm{N}_{\mathsf{q},U}(g) = \sum_{i \in I} \big(g(\boldsymbol{y}_i) - g(\boldsymbol{x}_i)\big) \tag{3.12}$$

for any $g \in \mathrm{Lip}_{\mathrm{B}}(\partial U)$. The sum in (3.12) is absolutely convergent since

$$\sum_{i \in I} |g(\boldsymbol{y}_i) - g(\boldsymbol{x}_i)| \le \sum_{i \in I} \mathrm{Lip}(g)|\boldsymbol{y}_i - \boldsymbol{x}_i| = \mathrm{Lip}(g)\mathscr{H}^1(K) < \infty;$$

if I is infinite then $\mathrm{N}_{\mathsf{q},U}(\cdot)$ is not represented by a measure.

To justify (3.11), we let $\varphi \in C_0^\infty(U, \mathbf{R})$. If ρ is sufficiently small, we have $(\nabla f)_\rho = \nabla f_\rho$ on $\operatorname{spt}\varphi$ and the definition of q and an integration by parts shows that

$$\int_U \varphi(\nabla f)_\rho \cdot d\mathsf{q} = -\int_K (\varphi|K)' f_\rho|K\, d\mathscr{H}^1.$$

The limit $\rho \to 0$ using (3.4) for the left hand side and $f_\rho|K \to f|K$ uniformly for the right hand side gives

$$\int_U \varphi\, d\,\langle\!\langle \nabla f, \mathsf{q} \rangle\!\rangle_U = -\int_K (\varphi|K)' f|K\, d\mathscr{H}^1 = \int_K \varphi (f|K)'\, d\mathscr{H}^1.$$

To prove (3.12), we note that by $\operatorname{div}_U \mathsf{q} = 0$ and (3.11) the right hand side of (3.6) evaluates to

$$\int_U d\,\langle\!\langle \nabla f, \mathsf{q} \rangle\!\rangle_U = \int_K (f|K)'\, d\mathscr{H}^1 = \sum_{i \in I} \big(f(\boldsymbol{y}_i) - f(\boldsymbol{x}_i)\big).$$

3.10 Example. Let $U \subset \mathbf{R}^n$ be a bounded open set, let $\boldsymbol{\varphi} : \mathbf{R} \to \mathbf{R}^n$ be a bilipschitz map such that $\boldsymbol{\varphi}(t) \notin U$ for all sufficently small and all sufficiently large $t \in \mathbf{R}$ and whose derivative $\boldsymbol{\varphi}'$ satisfies $|\boldsymbol{\varphi}'(t)| = 1$ for $\mathscr{L}^1$ a.e. $t \in \mathbf{R}$. Let $M := \boldsymbol{\varphi}(\mathbf{R}) \cap U$, let $V := \boldsymbol{\varphi}^{-1}(M) \equiv \boldsymbol{\varphi}^{-1}(U)$, and let $(u_i, v_i), i \in I$, be a finite or countable system of all connected components of the open bounded set V, with $u_i < v_i$. Note that the index set I is infinite if $\boldsymbol{\varphi}(\mathbf{R}) \cap \partial U$ has accumulation points. Let $\mathbf{q} \in \mathscr{M}(U, \mathbf{R}^n)$. Then

(i) $\mathbf{q}$ is a divergence measure vectorfield supported by M if and only if there exists a $b \in BV(V)$ such that

$$\mathbf{q} = (b\boldsymbol{\varphi}') \circ \boldsymbol{\varphi}^{-1}\mathscr{H}^1 \llcorner M; \tag{3.13}$$

here $BV(V)$ is the space of functions of bounded variation on V;

moreover, if $\mathbf{q}$ is of the form (3.13) then

(ii) $\operatorname{div}_U \mathbf{q}$ is given by

$$\int_U m \, d \operatorname{div}_U \mathbf{q} = \int_V m \circ \boldsymbol{\varphi} \, db' \tag{3.14}$$

for each $m \in C_0^\infty(U, \mathbf{R})$ where $b' \in \mathscr{M}(V, \mathbf{R})$ is the measure representing the weak derivative of $b \in BV(V)$ (in other words, $\operatorname{div}_U \mathbf{q}$ is the pushforward $\boldsymbol{\varphi}_\# b'$ of the measure b' by $\boldsymbol{\varphi}$);

(iii) if $f \in W^{1,\infty}(U)$ then

$$\langle\!\langle \nabla f, \mathbf{q} \rangle\!\rangle = \big((f \circ \boldsymbol{\varphi})' b\big) \circ \boldsymbol{\varphi}^{-1}\mathscr{H}^1 \llcorner M; \tag{3.15}$$

we recall that we represent the elements of $W^{1,\infty}(U)$ by continuous functions; hence $f \circ \boldsymbol{\varphi} \in W^{1,\infty}(V)$; we use the symbol $(f \circ \boldsymbol{\varphi})'$ to denote the weak derivative of $f \circ \boldsymbol{\varphi}$;

(iv) if $g \in \operatorname{Lip}_B(\partial U)$ then

$$\mathrm{N}_{\mathbf{q},U}(g) = \sum_{i \in I} \big(g(\boldsymbol{\varphi}(v_i))b(v_i) - g(\boldsymbol{\varphi}(u_i))b(u_i)\big)$$

with an absolutely convergent right hand side, where

$$b(u_i) = \lim_{\substack{t \to u_i \\ t \in (u_i, v_i)}} b(t), \quad b(v_i) = \lim_{\substack{t \to v_i \\ t \in (u_i, v_i)}} b(t),$$

and we note that the limits exist and are finite since $b \in BV(V)$.

4 Newton homotopy and the pairing

To prove the existence of the duality pairing $\langle\!\langle z, \mathbf{q} \rangle\!\rangle_U$, we use the homotopy formula (4.3) (below), which allows us to reconstruct a $\mathbf{q} \in \mathscr{DM}(\mathbf{R}^n)$ from the locally integrable functions $\mathrm{H_N}\, \mathbf{q}$ and $\mathrm{H_N} \operatorname{div} \mathbf{q}$. Here $\mathrm{H_N}$ is the 'Newton homotopy' of scalar or vector valued measures defined by convolution integrals involving the $\mathbf{R}^n$ valued singular kernel $\boldsymbol{k} : \mathbf{R}^n \sim \{\mathbf{0}\} \to \mathbf{R}^n$ given by

$$\boldsymbol{k}(\boldsymbol{x}) = n^{-1}\kappa_n^{-1}\boldsymbol{x}/|\boldsymbol{x}|^n,$$

$\boldsymbol{x} \in \mathbf{R}^n \sim \{\mathbf{0}\}$ where κ_n is the volume of the unit ball in $\mathbf{R}^n$. The two Newton homotopies (4.1) and (4.2) and the homotopy formula (4.3) are special cases of Newton homotopies of measures with values in the space of r vectors [25]. While the definitions of $\mathrm{H_N}$ and (4.3) appear to be new, Murat [19; Proof of Lemme 2] and Anzellotti [2; Proof of Lemma 3.4] use Newton's kernel to solve the equation $\operatorname{curl} \boldsymbol{\mu} = \boldsymbol{f}$ in a similar context. See also [18; Theorems 2.7.3 and 3.7.2] for the scalar valued case, which, however, is algebraically different.

The convergence of the integrals defining $\mathrm{H_N}$ is governed by the convolutions of the total variations of the involved measures with the scalar valued (Riesz) kernel $k : \mathbf{R}^n \to (0, \infty]$ given by

$$k(\boldsymbol{x}) = n^{-1}\kappa_n^{-1}|\boldsymbol{x}|^{-n+1},$$

$\boldsymbol{x} \in \mathbf{R}^n$, where we put $|\boldsymbol{x}|^{-n+1} = \infty$ if $\boldsymbol{x} = \mathbf{0}$ and $n > 1$. For any $\boldsymbol{x} \in \mathbf{R}^n$ we denote by $k_{\boldsymbol{x}}, \boldsymbol{k}_{\boldsymbol{x}}$ the maps on $\mathbf{R}^n \sim \{\boldsymbol{x}\}$ given by $k_{\boldsymbol{x}} = k(\boldsymbol{x}-\boldsymbol{y}), \boldsymbol{k}_{\boldsymbol{x}}(\boldsymbol{y}) = \boldsymbol{k}(\boldsymbol{x}-\boldsymbol{y})$ for any $\boldsymbol{y} \in \mathbf{R}^n \sim \{\boldsymbol{x}\}$.

If μ is a nonnegative finite measure in $\mathbf{R}^n$, we define $\mathrm{G}_\mu : \mathbf{R}^n \to [0,\infty]$ by

$$\mathrm{G}_\mu(\boldsymbol{x}) = \int k_{\boldsymbol{x}}\, d\mu,$$

$\boldsymbol{x} \in \mathbf{R}^n$. An application of Fatou's lemma shows that G_μ is a lower semicontinuous function.

4.1 Remark. *If μ is a nonnegative finite measure then G_μ is a $\mathscr{L}^n$ locally integrable function on $\mathbf{R}^n$; in fact if $A \subset \mathbf{R}^n$ is $\mathscr{L}^n$ measurable then*

$$\int_A \mathrm{G}_\mu\, d\mathscr{L}^n \le \kappa_n^{-1/n}(\mathscr{L}^n(A))^{1/n}\mathrm{M}(\mu).$$

If $\phi, \mathbf{q}$ are measures with values in $\mathbf{R}$ and $\mathbf{R}^n$, respectively, we define the ***Newton homotopies*** $\mathrm{H_N}\,\phi$ of ϕ and $\mathrm{H_N}\,\mathbf{q}$ of $\mathbf{q}$ as functions with values in $\mathbf{R}^n$ and Skw by

$$\mathrm{H_N}\,\phi(\boldsymbol{x}) = \int \boldsymbol{k}_{\boldsymbol{x}}\, d\phi, \tag{4.1}$$

$$\mathrm{H_N}\,\mathbf{q}(\boldsymbol{x}) = -2\int \boldsymbol{k}_{\boldsymbol{x}} \wedge d\mathbf{q}, \tag{4.2}$$

for every $\boldsymbol{x} \in \mathbf{R}^n$ for which $\mathrm{G}_{\|\phi\|}(\boldsymbol{x}) < \infty$ or $\mathrm{G}_{\|\mathbf{q}\|}(\boldsymbol{x}) < \infty$. Here the wedge product of two vectors is defined by $\boldsymbol{a} \wedge \boldsymbol{b} = \frac{1}{2}(\boldsymbol{a} \otimes \boldsymbol{b} - \boldsymbol{b} \otimes \boldsymbol{a}) \in \mathrm{Skw}$ for any $\boldsymbol{a}, \boldsymbol{b} \in \mathbf{R}^n$. The integrands in (4.1) and (4.2) are bounded by $k_{\boldsymbol{x}}$; thus by Remark 4.1, the homotopies are $\mathscr{L}^n$ locally integrable functions.

4.2 Proposition. *If $\mathbf{q} \in \mathscr{DM}(\mathbf{R}^n)$ then*

$$\mathbf{q} = \mathrm{H_N}\,\mathrm{div}\,\mathbf{q} + \mathrm{div}\,\mathrm{H_N}\,\mathbf{q} \tag{4.3}$$

in the sense of distributions, i.e., if $\boldsymbol{d} \in C_0^\infty(\mathbf{R}^n, \mathbf{R}^n)$ then

$$\int \boldsymbol{d} \cdot d\mathbf{q} = \int \boldsymbol{d} \cdot \mathrm{H_N}\,\mathrm{div}\,\mathbf{q}\, d\mathscr{L}^n - \int \mathrm{curl}\,\boldsymbol{d} \cdot \mathrm{H_N}\,\mathbf{q}\, d\mathscr{L}^n. \tag{4.4}$$

Prove first Proposition 4.3 and Remark 4.4 the case $U = \mathbf{R}^n$. Let $\mathbf{q} \in \mathscr{DM}(\mathbf{R}^n)$ be fixed; recalling that $\mathrm{H_N}\,\mathrm{div}\,\mathbf{q}$ and $\mathrm{H_N}\,\mathbf{q}$ are locally $\mathscr{L}^n$ integrable functions on $\mathbf{R}^n$, we let

$$F(\boldsymbol{w}) := \int \boldsymbol{w} \cdot \mathrm{H_N}\,\mathrm{div}\,\mathbf{q}\, d\mathscr{L}^n - \int \mathrm{curl}\,\boldsymbol{w} \cdot \mathrm{H_N}\,\mathbf{q}\, d\mathscr{L}^n$$

for any $\boldsymbol{w} \in \mathscr{CF}(\mathbf{R}^n)$ that vanishes outside a bounded set (depending on $\boldsymbol{w}$). Prove that for any $\boldsymbol{z} \in \mathscr{CF}(\mathbf{R}^n)$ and $\varphi \in C_0^\infty(\mathbf{R}^n, \mathbf{R})$ we have

$$\lim_{\rho \to 0} \int_U \varphi \boldsymbol{z}_\rho \cdot d\mathbf{q} = F(\varphi \boldsymbol{z}). \tag{4.5}$$

We now use the homotopy formula (4.3) to prove Theorem 3.2 in the following expanded form.

4.3 Proposition. *If $z \in \mathscr{CF}(U)$ and $\mathbf{q} \in \mathscr{DM}(U)$, there exists a unique measure $\langle\!\langle z, \mathbf{q} \rangle\!\rangle_U \in \mathscr{M}(U, \mathbf{R})$ such that* (3.4) *holds for every $\varphi \in C_0^\infty(U, \mathbf{R})$. The measure $\langle\!\langle z, \mathbf{q} \rangle\!\rangle_U$ is absolutely continuous with respect to $\|\mathbf{q}\|$; in fact there exists a unique function $\operatorname{den}_{\|\mathbf{q}\|, z} \in L^\infty(\|\mathbf{q}\|, \mathbf{R}^n)$ such that*

$$\langle\!\langle z, \mathbf{q} \rangle\!\rangle_U = \operatorname{den}_{\|\mathbf{q}\|, z} \|\mathbf{q}\|; \tag{4.6}$$

one has

$$\| \operatorname{den}_{\|\mathbf{q}\|, z} \|_{L^\infty(\|\mathbf{q}\| \llcorner V)} \le \|z\|_{L^\infty(V)} \tag{4.7}$$

for every open subset V of U. In particular,

$$\mathrm{M}(\langle\!\langle z, \mathbf{q} \rangle\!\rangle_U) \le \|z\|_{L^\infty(U)} \mathrm{M}(\mathbf{q}). \tag{4.8}$$

Recall that $\| \cdot \|_{L^\infty(V)}$ denotes the essential supremum norm on V with respect to the Lebesgue measure. We shall prove the proposition simultaneously with the following locality property of the pairing.

4.4 Remark. *If V is an open subset of U, if $z_1, z_2 \in \mathscr{CF}(U)$ coincide $\mathscr{L}^n$ a.e. on V and if $\mathbf{q}_1, \mathbf{q}_2 \in \mathscr{DM}(U)$ coincide on V in the sense that $\mathbf{q}_1 \llcorner V = \mathbf{q}_2 \llcorner V$ then $\langle\!\langle z_1, \mathbf{q}_1 \rangle\!\rangle_U$ and $\langle\!\langle z_2, \mathbf{q}_2 \rangle\!\rangle_U$ concide on V, i.e.,*

$$\langle\!\langle z_1, \mathbf{q}_1 \rangle\!\rangle_U \llcorner V = \langle\!\langle z_2, \mathbf{q}_2 \rangle\!\rangle_U \llcorner V. \tag{4.9}$$

5 The structure theorem

We employ the theory of flat chains [29; Part III], [7; Chapter Four] to prove Theorem 3.1.

We denote by $\mathscr{D}_1$ the set of all $\mathbf{R}^n$ valued distributions A, i.e., linear functions on $C_0^\infty(\mathbf{R}^n, \mathbf{R}^n)$ that are continuous in the Schwartz topology (see, e.g., [23; Chapter 6], [7; Subsection 4.1.1]) of $C_0^\infty(\mathbf{R}^n, \mathbf{R}^n)$, and denote by

$$\langle A, z \rangle$$

the value of A at $z \in C_0^\infty(\mathbf{R}^n, \mathbf{R}^n)$. Following Federer [7; Subsection 4.1.7], we call the elements of $\mathscr{D}_1$ the 1 ***dimensional currents.*** We can associate with any $\mathbf{q} \in \mathscr{M}(\mathbf{R}^n, \mathbf{R}^n)$ a current in $\mathscr{D}_1$ by

$$\langle A, z \rangle = \int_{\mathbf{R}^n} \langle z, d\mathbf{q} \rangle$$

for every $z \in C_0^\infty(\mathbf{R}^n, \mathbf{R}^n)$. Since $A = 0$ if and only if $\mathbf{q} = \mathbf{0}$, we identify $\mathbf{q}$ with A, define $\langle \mathbf{q}, z \rangle := \langle A, z \rangle$, interpret $\mathscr{M}(\mathbf{R}^n, \mathbf{R}^n)$ as a subset of $\mathscr{D}_1$ and write $\mathscr{M}(\mathbf{R}^n, \mathbf{R}^n) \subset \mathscr{D}_1$. Accordingly also $\mathscr{DM}(\mathbf{R}^n) \subset \mathscr{D}_1$.

If $z \in C_0^\infty(\mathbf{R}^n, \mathbf{R}^n)$, we define

$$|z|^\flat = \max\left\{ \|z\|_{L^\infty(\mathbf{R}^n)}, \|\operatorname{curl} z\|_{L^\infty(\mathbf{R}^n)} \right\},$$

and if $A \in \mathscr{D}_1$, we define

$$|A|^\flat = \sup\left\{ \langle A, z \rangle : z \in C_0^\infty(\mathbf{R}^n, \mathbf{R}^n), |z|^\flat \le 1 \right\};$$

we call $|A|^\flat$ the ***flat norm*** of A if $|A|^\flat < \infty$. We denote by $\mathscr{Q}_1$ the set of all $A \in \mathscr{D}_1$ with $|A|^\flat < \infty$, equipped with the flat norm, which makes it a Banach space. Noting that $\mathscr{DM}(\mathbf{R}^n) \subset \mathscr{Q}_1$, we put

$$\mathscr{C}_1^\flat := \text{ the closure of } \mathscr{DM}(\mathbf{R}^n) \text{ in } \mathscr{Q}_1^\flat \text{ with respect to } |\cdot|^\flat. \tag{5.1}$$

The elements of $\mathscr{C}_1^\flat$ are called ***flat* 1 *dimensional chains.*** We here note that on the 1 dimensional currents the boundary operator ∂A ([7; Subsection 4.1.7]) coincides, to within the − sign, with the divergence in the sense of distributions; thus the elements $\mathbf{q}$ of $\mathscr{DM}(\mathbf{R}^n)$ with compact support are Federer's ***normal currents,*** defined generally as currents such that both A and ∂A are represented by measures with compact support. Thus the definition (5.1) coincides with Federer's realization of flat chains as a subset of the space of currents (see [7; Subsections 4.1.12–4.1.19]); the isomorphy of this space with the original Whitney's [29] space of flat chains is explained in [7; pp. 377–378]. It can be shown that

$$\mathscr{C}_1^\flat \sim \mathscr{DM}(\mathbf{R}^n) \neq \emptyset,$$

and

$$\mathscr{Q}_1^\flat \sim \mathscr{C}_1^\flat \neq \emptyset.$$

The class of flat chains enjoys many strong geometric and analytic properties (such as Theorem 5.3, below) as amply demonstrated by [29; Part III], [7; Chapter Four]; many of them not true for the currents from the larger spaces $\mathscr{Q}_1^\flat$ and $\mathscr{D}_1$.

If m is any map defined on $\mathbf{R}^n$, we define its translation $\mathrm{T}_{\boldsymbol{a}} m$ by $\boldsymbol{a} \in \mathbf{R}^n$ by $\mathrm{T}_{\boldsymbol{a}} m(\boldsymbol{x}) = m(\boldsymbol{x} - \boldsymbol{a})$, $\boldsymbol{x} \in \mathbf{R}^n$. If $A \in \mathscr{D}_1$ and $\boldsymbol{a} \in \mathbf{R}^n$, we define the ***translation*** $\mathrm{T}_{\boldsymbol{a}} A$ of A by $\boldsymbol{a}$ by

$$\langle \mathrm{T}_{\boldsymbol{a}} A, \boldsymbol{z} \rangle = \langle A, \mathrm{T}_{-\boldsymbol{a}} \boldsymbol{z} \rangle$$

for every $\boldsymbol{z} \in C_0^\infty(\mathbf{R}^n, \mathbf{R}^n)$.

5.1 Proposition. *If $A \in \mathscr{C}_1^\flat$ then $|\mathrm{T}_{\boldsymbol{a}} A - A|^\flat \to 0$ as $|\boldsymbol{a}| \to 0$.*

It is possible to show [25] that this property actually *characterizes* the elements of $\mathscr{C}_1^\flat$ among the elements of $\mathscr{Q}_1^\flat$.

5.2 Remark. If $\mathbf{q} \in \mathscr{M}(\mathbf{R}^n, \mathbf{R}^n)$ represents a flat chain and $f \in L^\infty(\|\mathbf{q}\|, \mathbf{R})$ then $f\mathbf{q}$ represents a flat chain also [7; Subsection 4.1.17] and clearly

$$|f\mathbf{q}|^\flat \le \mathbf{M}(f\mathbf{q}) \le \mathbf{M}(\mathbf{q}) \|f\|_{L^\infty(\|\mathbf{q}\|)};$$

in particular if E is a Borel subset of $\mathbf{R}^n$ then the measure $\mathbf{q} \llcorner E$ represents a flat chain; in particular, if $\mathbf{q}$ is a divergence measure vectorfield then $\mathbf{q} \llcorner E$ represents a flat chain but generally is not a divergence measure.

5.3 Theorem. ([7; Theorem 4.1.20 and Subsection 4.1.21]). *If $A \in \mathscr{C}_1^\flat$ then*

$$\mathscr{I}_1^1(\operatorname{spt} A) = 0 \quad \text{implies} \quad A = 0;$$

if additionally A is represented by a measure $\mathbf{q}$ and $E \subset \mathbf{R}^n$ then

$$\mathscr{I}_1^1(E) = 0 \quad \text{implies} \quad \|\mathbf{q}\|(E) = 0.$$

Here spt$A$ is the support of (the distribution) A and $\mathscr{I}_1^1$ is the 1 dimensional integral-geometric measure [7; Subsection 2.10.5 and Theorem 2.10.15]. We here note that $\mathscr{I}_1^1$ has a larger class of null sets than $\mathscr{H}^1$ ([7; Subsection 2.10.6]), viz., $\mathscr{I}_1^1$ vanishes on all purely unrectifiable sets, whereas $\mathscr{H}^1$ does not. This will be employed in the proof of the following proposition.

5.4 Proposition. *Let $\mathbf{q} \in \mathscr{M}(\mathbf{R}^n, \mathbf{R}^n)$ be supported by a $\mathscr{H}^1$ measurable set of countably finite $\mathscr{H}^1$ measure. Then $\mathbf{q}$ represents a flat 1 dimensional chain if and only if*

$$\mathbf{q} = \boldsymbol{q}_s \mathscr{H}^1 \llcorner M \tag{5.2}$$

where

$$\left. \begin{gathered} M \text{ is a countably } \mathscr{H}^1 \text{ rectifiable subset of } \mathbf{R}^n, \\ \boldsymbol{q}_s \in L^1(\mathscr{H}^1 \llcorner M, \mathbf{R}^n) \text{ is tangential to } M. \end{gathered} \right\} \tag{5.3}$$

5.5 Theorem. *If $A \in \mathscr{C}_1^\flat$ is represented by a measure $\mathbf{q} \in \mathscr{M}(\mathbf{R}^n, \mathbf{R}^n)$ then*

$$\mathbf{q} = \boldsymbol{q}_s \mathscr{H}^1 \llcorner M + \mathbf{q}_c + \boldsymbol{q}_r \mathscr{L}^n \tag{5.4}$$

where

$$\left. \begin{gathered} M \text{ is a countably } \mathscr{H}^1 \text{ rectifiable subset of } \mathbf{R}^n, \\ \boldsymbol{q}_s \in L^1(\mathscr{H}^1 \llcorner M, \mathbf{R}^n) \text{ is tangential to } M, \\ \mathbf{q}_c \in \mathscr{M}(\mathbf{R}^n, \mathbf{R}^n) \text{ is } \mathscr{H}^1 \text{ diffuse and } \mathscr{L}^n \text{ singular}, \\ \boldsymbol{q}_r \in L^1(\mathbf{R}^n, \mathbf{R}^n). \end{gathered} \right\} \tag{5.5}$$

5.6 Remark. By a method similar to that in the proof of Proposition 5.4 one can also prove the following: *If k is an integer satisfying $1 \le k \le n$ and $\mathbf{q} \in \mathscr{M}(\mathbf{R}^n, \mathbf{R}^n)$ is given by*

$$\mathbf{q} = \boldsymbol{q}_s \mathscr{H}^k \llcorner M \tag{5.6}$$

where

$$\left. \begin{gathered} M \text{ is a countably } \mathscr{H}^k \text{ rectifiable set}, \\ \boldsymbol{q}_s \in L^1(\mathscr{H}^k \llcorner M, \mathbf{R}^n); \end{gathered} \right\} \tag{5.7}$$

then $\mathbf{q}$ represents a 1 dimensional flat chain if and only if $\boldsymbol{q}_s$ is tangential to M.

In contrast to Proposition 5.4, here the countable rectifiability (5.7) of M and the absolute continuity (5.6) must be assumed (if $k > 1$). The above assertion generalizes the tangentiality assertions of [15; Proposition 1] from smooth manifolds to countably rectifiable sets. If $\mathbf{q}$ is a measure given by (5.6) and (5.7) with $1 < k < n$, then the decomposition (5.4)–(5.5) takes the following form: the first and the third terms vanish, while the second is equal to the present $\mathbf{q}$. This gives examples with nonvanishing Cantor part in (5.4). Of course, $\mathbf{q}$ generally is not a divergence measure vectorfield. A sufficient condition for $\mathbf{q} \in \mathscr{DM}(\mathbf{R}^n)$ is, e.g., that M is a class C^2 k dimensional manifold and $\boldsymbol{q}_s$ is tangential and relatively compactly supported in M. Thus we have examples of divergence measure vectorfields with nonvanishing Cantor parts.

5.7 Remark. Proposition 5.4 and Theorem 5.5 easily generalize to flat chains of any dimension k, $0 \le k \le n$, as follows: denoting by $\wedge_k$ the set of all k vectors on $\mathbf{R}^n$, and saying that $\boldsymbol{\alpha}_s \in L^1(\mathscr{H}^k \llcorner M, \wedge_k)$ is tangential to M if $\boldsymbol{\alpha}_s(\boldsymbol{x})$ can be written as a wedge product of some k tuple of vectors from $\operatorname{Tan}^k(M, \boldsymbol{x})$ for $\mathscr{H}^k$ a.e. $\boldsymbol{x} \in M$, we have the following statements:

(i) *if $\boldsymbol{\mu} \in \mathscr{M}(\mathbf{R}^n, \wedge_k)$ is supported by a set of countably finite $\mathscr{H}^k$ measure then $\boldsymbol{\mu}$ represents a flat k dimensional chain if and only if $\boldsymbol{\mu}$ is of the form*

$$\boldsymbol{\mu} = \boldsymbol{\alpha}_s \mathscr{H}^k \llcorner M$$

where

$$\left.\begin{array}{c} M \text{ is a countably } \mathscr{H}^k \text{ rectifiable set,} \\ \boldsymbol{\alpha}_s \in L^1(\mathscr{H}^k \llcorner M, \wedge_k) \text{ is tangential to } M; \end{array}\right\} \qquad (5.8)$$

(ii) *if A is a k dimensional flat chain represented by a measure $\boldsymbol{\mu} \in \mathscr{M}(\mathbf{R}^n, \wedge_k)$ then $\boldsymbol{\mu}$ is of the form*

$$\boldsymbol{\mu} = \boldsymbol{\alpha}_s \mathscr{H}^k \llcorner M + \boldsymbol{\mu}_c + \boldsymbol{\alpha}_r \mathscr{L}^n \qquad (5.9)$$

where

$$\left.\begin{array}{c} M \text{ is a countably } \mathscr{H}^k \text{ rectifiable set,} \\ \boldsymbol{\alpha}_s \in L^1(\mathscr{H}^k \llcorner M, \wedge_k) \text{ is tangential to } M, \\ \boldsymbol{\mu}_c \in \mathscr{M}(\mathbf{R}^n, \wedge_k) \text{ is } \mathscr{H}^k \text{ diffuse and } \mathscr{L}^n \text{ singular,} \\ \boldsymbol{\alpha}_r \in L^1(\mathbf{R}_n, \wedge_k). \end{array}\right\} \qquad (5.10)$$

We recall the class of rectifiable k dimensional currents, which, although defined differently ([7; Subsection 4.1.24]), is a subset of the class of the flat chains considered in (i). Namely, rectifiable currents are distinuished by the extra property that the mass $|\boldsymbol{\alpha}_s(\boldsymbol{x})|$ of $\boldsymbol{\alpha}_s(\boldsymbol{x})$ is an integer for $\mathscr{H}^k$ a.e. $\boldsymbol{x} \in M$; this follows from the equivalence of Conditions (1) and (4) in [7; Theorem 4.1.28]. We also note that one particular case of (ii) is well known. Namely, if $k = n - 1$ and $v \in BV(\mathbf{R}^n)$ then the measure ∇v can be identified, via the Hodge star operator, with an $n-1$ dimensional flat chain which is represented by a measure. Equation (5.9)–(5.10) then gives the classical decomposition of ∇v into the jump, Cantor, and absolutely continuous parts.

Acknowledgment The research has been supported by a grant of MIUR "Variational theory of microstructure, semiconvexity, and complex materials." The support is gratefully acknowledged.

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