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Abstract This paper deals with the relaxation of energies of media with structured deformations introduced by Del Piero & Owen [7–8]. Structured deformations provide a multiscale geometry that captures the contributions at the macrolevel of both smooth and non-smooth geometrical changes (disarrangements) at submacroscopic levels. The paper examines the special case of Choksi & Fonseca's energetics of structured deformations [4] in which the unrelaxed energy does not contain the bulk contribution. Thus the energy is purely interfacial, but of the general form. Some new properties of the relaxed energy densities are derived: (i) the bulk relaxed energy is the subadditive envelope of the unrelaxed interfacial energy and (ii) a broad sufficient condition is given for the relaxed interfacial energy to coincide with the original one. The relaxations of the specific interfacial energies of Owen & Paroni [14] and Barroso, Matias, Morandotti & Owen [3] are simple consequences of our general results.

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Keywords Structured deformations, relaxation, disarrangements, interfacial density, bulk density, functions of measures

Contents

1 Introduction

This paper deals with the relaxation of nonclassical continua modeled as media with structured deformations introduced by Del Piero & Owen [7–8]. ⋆⋆ In their original

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⋆⋆ The reader is referred to the proceedings [9] and to the recent survey [2] for additional references and for further developments.

setting, a structured deformation is a triplet $(\mathscr{K}, g, G)$ of objects of the following nature. The set $\mathscr{K} \subset \mathbb{R}^3$, the crack site, is a subset of vanishing Lebesgue measure of the reference region Ω, the map $g : \Omega \sim \mathscr{K} \to \mathbb{R}^3$, the deformation map, is piecewise continuously differentiable and injective, and G is a piecewise continuous map from $\Omega \sim \mathscr{K}$ to the set of invertible second order tensors describing deformation without disarrangements.

Within this context, *simple* deformations are triples $(\mathscr{K}, g, \nabla g)$ where g is a piecewise smooth injective map with jump discontinuities describing partial or full separation of pieces of the body. In view of this, in the general case of a structured deformation $(\mathscr{K}, g, G)$, the tensor

$$M = \nabla g - G,$$

the deformation due to disarrangements, measures the departure of $(\mathscr{K}, g, G)$ from the simple deformation $(\mathscr{K}, g, \nabla g)$.

Choksi & Fonseca [4] introduced into the theory of structured deformations energy considerations and the ideas of relaxation. For further studies in one and multidimensional settings see Del Piero [5–6]. It is well-known that the existing techniques of relaxation of the calculus of variations and continuum mechanics are unable to cope with the injectivity requirements. Accordingly, Choksi & Fonseca neglect the injectivity requirement; in addition, they assume weaker regularity. In their interpretation, structured deformations are pairs (g, G) where $g : \Omega \to \mathbb{R}^n$ is a special $\mathbb{R}^n$-valued map of bounded variation from the space $SBV(\Omega)$ and $G : \Omega \to \mathrm{Lin}$ is an integrable Lin-valued map from the space $L^1(\Omega)$.* Thus

$$SD(\Omega) := SBV(\Omega) \times L^1(\Omega)$$

is the set of all structured deformations. Structured deformations of the form $(g, \nabla g)$ with $g \in SBV(\Omega)$ are called *simple deformations* in this paper.

The relaxation starts from the energy

$$E(g) = \int_\Omega W(\nabla g)\, d\mathsf{V} + \int_{J_g} \psi([\![g]\!], \nu_g)\, d\mathsf{A}$$

of a simple deformation $(g, \nabla g)$. Here V and A are the Lebesgue measure and the $n-1$-dimensional Hausdorff measure in $\mathbb{R}^n$, ∇g is the absolutely continuous part of the derivative (= gradient) $\mathrm{D}g$ of g, while the singular part

$$\mathrm{D}^{\mathrm{s}} g := [\![g]\!] \otimes \nu_g \mathsf{A} \llcorner J_g$$

is a tensor-valued singular measure describing the discontinuities of g; that part is formed from the jump set $J_g \subset \Omega$ of g, the jump $[\![g]\!]$ of g on J_g, and the normal ν_g to J_g. The reader is referred to (4.2), below, for a detailed description of these objects. The material is characterized by the bulk energy density $W : \mathrm{Lin} \to \mathbb{R}$ and by the jump energy $\psi : \mathbb{D}_n \to \mathbb{R}$, where we denote

$$\mathbb{D}_n = \mathbb{R}^n \times \mathbb{S}^{n-1}.$$

* For brevity of notation, we omit the target spaces and write $SBV(\Omega) \equiv SBV(\Omega, \mathbb{R}^n)$ and $L^1(\Omega) \equiv L^1(\Omega, \mathrm{Lin})$. See Section 4 for more notation and detailed definitions.

The Approximation Theorem of Del Piero & Owen [7; Theorem 5.8] says that every structured deformation is a well-defined limit of simple deformations. In the framework of Choksi & Fonseca [4] (see also [16]) this means that corresponding to each structured deformation $(g, G) \in SD(\Omega)$ there exists a sequence $(g_k, \nabla g_k) \in SD(\Omega)$ (i.e., with g_k in $SBV(\Omega)$) such that

$$\left.\begin{aligned} &g_k \to g && \text{in } L^1(\Omega),\\ &\nabla g_k \overset{*}{\rightharpoonup} G && \text{in } \mathscr{M}(\Omega, \mathrm{Lin}),\\ &\sup\{|\nabla g_k|_{L^1(\Omega)} : k \in \mathbb{N}\} < \infty. \end{aligned}\right\} \tag{1.1}$$

The relaxed energy of a structured deformation $(g, G) \in SD(\Omega)$ is defined by

$$I(g, G) = \inf\{\liminf_{k \to \infty} E(g_k) : g_k \in SBV(\Omega) \text{ satisfies (1.1)}\}. \tag{1.2}$$

Thus the sequence approaching the above infimum realizes the most economical way to build up the deformation (g, G) using the approximations in SBV. The relaxation theorem of Choksi & Fonseca [4; Theorems 2.6 & 2.17] says that under some assumptions on W and ψ (a particular case of which is Assumption 1.1, below), the relaxed energy admits the integral representation

$$I(g, G) = \int_\Omega H(\nabla g, G)\, d\mathsf{V} + \int_{J_g} h([\![g]\!], \nu_g)\, d\mathsf{A} \tag{1.3}$$

where H and h are some functions determined explicitly in the cited theorems (particular cases are (2.1) and (2.2), below).

This note deals with the relaxation of energy functions E for which the bulk contribution vanishes, i.e., with energy functions of the form

$$E(g) = \int_{J_g} \psi([\![g]\!], \nu_g)\, d\mathsf{A} \tag{1.4}$$

for each simple deformation $(g, \nabla g)$. Special cases of the energies of this type have been considered previously by Owen & Paroni [14] and Barroso, Matias, Morandotti & Owen [3], see Examples 1.4 and 1.5, below.

Throughout the paper, we assume that the energy E is given by (1.4), its relaxation I is defined by (1.2), and it will be proved I takes the form (1.3) where H and h will be determined by the function ψ alone. We make the following standing hypotheses about ψ.

Assumption 1.1

- The function $\psi : \mathbb{D}_n \to \mathbb{R}$ is continuous;
- we have $\psi(-a, -b) = \psi(a, b)$ and

$$c_1 |a| \le \psi(a, b) \le C_1 |a| \tag{1.5}$$

  for every $(a, b) \in \mathbb{D}_n$ and some $c_1 > 0$, $C_1 > 0$;
- the function $\psi(\cdot, \nu)$ is subadditive and positively homogeneous of degree 1 for each $\nu \in \mathbb{S}^{n-1}$.

The following two theorems are the main results of this note.

Theorem 1.2 *The function H is given by*

$$H(A,B) = \Phi(A-B), \quad A,\ B \in \mathrm{Lin},$$

where $\Phi : \mathrm{Lin} \to [0,\infty)$ is a subadditive and positively homogeneous function of degree 1; *in fact Φ is the biggest subadditive function on* Lin *satisfying*

$$\Theta(a \otimes b) \le \psi(a,b) \ \text{ for every } \ (a,b) \in \mathbb{D}_n \tag{1.6}$$

i.e.,

$$\begin{aligned}\Phi(M) = \sup\Big\{\Theta(M) : \Theta : \mathrm{Lin} \to [0,\infty) \ \text{ is subadditive and}\\ \Theta(a \otimes b) \le \psi(a,b) \ \text{ for every } \ (a,b) \in \mathbb{D}_n\Big\};\end{aligned} \tag{1.7}$$

equivalently,

$$\Phi(M) = \inf\Big\{\sum_{i=1}^{m} \psi(a_i,b_i) : m \in \mathbb{N}, (a_i,b_i) \in \mathbb{D}_n, \ \sum_{i=1}^{m} a_i \otimes b_i = M\Big\} \tag{1.8}$$

for every $M \in$ Lin.

To ease the statement of the next theorem, we extend the function $h : \mathbb{D}_n \to [0,\infty)$ by homogeneity with respect to the second variable, i.e., we define $\tilde h : \mathbb{R}^n \times \mathbb{R}^n \to [0,\infty)$ by $\tilde h(a,b) = |b|\, h\big(a, \mathrm{sgn}(b)\big)$ for every $a, b \in \mathbb{R}^n$, where

$$\mathrm{sgn}(b) = \begin{cases} b/|b| & \text{if } \ b \neq 0, \\ 0 & \text{else.}\end{cases}$$

Theorem 1.3

(i) *For every $a,\ b \in \mathbb{R}^n$, the functions $\tilde h(\cdot,b)$ and $\tilde h(a,\cdot)$ are subadditive and positively homogeneous of degree* 1.

(ii) *If there exists a subadditive and positively homogeneous function $\Lambda : \mathrm{Lin} \to [0,\infty)$ such that*

$$\psi(a,b) = \Lambda(a \otimes b) \tag{1.9}$$

for every $(a,b) \in \mathbb{D}_n$, then $h = \psi$.

Since the pointwise supremum of any family of subadditive functions is subadditive (e.g., [13; Theorem 7.2.2]), Equation (1.7) really defines a subadditive function. The proofs of Theorems 1.2 and 1.3 are given in Section 2, below. We now illustrate Theorems 1.2 and 1.3 on particular cases. These motivated the present study.

Example 1.4 ([14; Theorem 4, particular case $L = \mathrm{I}$]) *If*

$$\psi_{|\cdot|}(a,b) = |a \cdot b| \ \text{ and } \ \psi_{\pm}(a,b) = \{a \cdot b\}_{\pm}$$

for every $(a,b) \in \mathbb{D}_n$, where $\{\cdot\}_+$ and $\{\cdot\}_-$ denote the positive and negative parts of a real number, then

$$\Phi_{|\cdot|}(M) = |\mathrm{tr}\, M| \quad \text{ and } \quad \Phi_{\pm}(M) = \{\mathrm{tr}\, M\}_{\pm}, \tag{1.10}$$

$$h_{|\cdot|}(a,b) = |a \cdot b| \quad \text{ and } \quad h_{\pm}(a,b) = \{a \cdot b\}_{\pm}, \tag{1.11}$$

respectively, for every $M \in$ Lin *and $(a,b) \in \mathbb{D}_n$.*

As shown in [14], $\{\operatorname{tr} M\}_+$ is a volume density of disarrangements due to submacroscopic separations, $\{\operatorname{tr} M\}_-$ is a volume density of disarrangements due to submacroscopic switches and interpenetrations, and $|\operatorname{tr} M|$ is a volume density of all three of these non-tangential disarrangements: separations, switches, and interpenetrations. The evaluation in [14] of H (equivalently, of Φ) for (1.10) is rather complicated; a recent paper by Barroso, Matias, Morandotti & Owen [3] presents some simplification. Our version of the derivation is given in Section 3.

Example 1.5 ([3; Equation (5.3)]) *If*

$$\psi(a, b) = |a \cdot p|$$

for $(a, b) \in \mathbb{D}_n$, *where* $p \in \mathbb{R}^n$ *is a fixed vector, then*

$$\Phi(M) = |M^{\mathrm{T}} p| \quad \text{and} \quad h(a, b) = |a \cdot p| \tag{1.12}$$

for any $M \in \operatorname{Lin}$ *and* $(a, b) \in \mathbb{D}_n$.

2 Proofs of Theorems 1.2 and 1.3

The following statement is a particular case $W = 0$ of the relaxation theorem of Choksi & Fonseca [4; Theorem 2.17].

Theorem 2.1 *The effective energies H and h are given by*

$$H(B, C) = \inf\{\Delta(u) : u \in \mathscr{A}(B, C)\}, \tag{2.1}$$

$$h(a, b) = \inf\{\Delta(u) : u \in \mathscr{B}(a, b, Q_b)\} \tag{2.2}$$

for any $B,\ C \in \operatorname{Lin}$ *and any* $(a, b) \in \mathbb{D}_n$, *where the objects occurring in these formulas are defined as follows*:

- *for any* $u \in SBV(\Omega)$ *and any bounded open set* $\Omega \subset \mathbb{R}^n$,

$$\Delta(u) := \int_{J_u \cap \Omega} \psi([\![u]\!], \nu_u)\, d\mathsf{A};$$

- *if* $Q = (-1/2, 1/2)^n$, *then*

$$\mathscr{A}(B, C) := \Big\{u \in SBV(Q) : u(x) = Bx \ \text{ if } \ x \in \partial Q,\ \int_Q \nabla u \, d\mathsf{V} = C\Big\}$$

- *if for every* $(a, b) \in \mathbb{D}_n$ *the map* $u_{a,b} : Q_b \to \mathbb{R}^n$ *is defined by*

$$u_{a,b}(x) = \frac{1}{2} a\big(\operatorname{sgn}(x \cdot b) + 1\big), \quad x \in Q_b,$$

  where Q_b *is any cube of unit edge, of center at* $0 \in \mathbb{R}^n$, *and of two faces normal to* b, *then*

$$\mathscr{B}(a, b, Q_b) := \big\{u \in SBV(Q_b), u = u_{a,b} \ \text{ on } \ \partial Q_b,\ \nabla u = 0 \ \text{ on } \ Q_b\big\}.$$

Theorem 2.1 will be employed in the proofs of Theorems 1.2 and 1.3, together with the constructions in the lemmas to be presented now.

Lemma 2.2 *We have*
$$H(B, C) = \Psi(B - C)$$
for any $B, C \in \mathrm{Lin}$, *where* $\Psi : \mathrm{Lin} \to [0, \infty)$ *is given by*
$$\Psi(M) = \inf\{\Delta(u) : u \in \mathscr{A}(M, 0)\} \tag{2.3}$$
for any $M \in \mathrm{Lin}$.

Proof It suffices to note that if $u \in \mathscr{A}(B, C)$ then v, given by $v(x) = u(x) - Cx$, $x \in Q$, satisfies $v \in \mathscr{A}(B - C, 0)$ and $\Delta(u) = \Delta(v)$. □

Remark 2.3 If the interfacial energy density ψ is of the special form (1.9) where $\Lambda : \mathrm{Lin} \to [0, \infty)$ is a subadditive and positively homogeneous function then $\Delta(u)$ is given by
$$\Delta(u) = \Lambda(\mathrm{D}^{\mathrm{s}} u)$$
where $\mathrm{D}^{\mathrm{s}} u := [\![u]\!] \otimes \nu_u \mathsf{A} \llcorner J_u$ is the singular part of the derivative $\mathrm{D}u$ of u and
$$\Lambda(\mathrm{D}^{\mathrm{s}} u) := \int_{J_u} \Lambda([\![u]\!] \otimes \nu_u)\, d\mathsf{A}$$
is an instance of Reshetnyak's [15] functional $\mu \mapsto \Lambda(\mu)$ of a measure $\mu \in \mathscr{M}(Q, \mathrm{Lin})$; see, e.g., [1; Equation (2.29)]. The subadditivity and positive homogeneity of degree 1 of Φ (asserted in Theorem 1.2) is then an instance of the general result [1; Proposition 2.37] asserting the same properties of the functional $\mu \mapsto \Lambda(\mu)$. Indeed, if $M_i \in \mathrm{Lin}$ and $u_i \in \mathscr{A}(M_i, 0)$, $i = 1, 2$, then $u_1 + u_2 \in \mathscr{A}(M_1 + M_2, 0)$ and therefore
$$\Phi(M_1 + M_2) \le \Lambda\big(\mathrm{D}^{\mathrm{s}}(u_1 + u_2)\big) = \Lambda(\mathrm{D}^{\mathrm{s}} u_1 + \mathrm{D}^{\mathrm{s}} u_2) \le \Lambda(\mathrm{D}^{\mathrm{s}} u_1) + \Lambda(\mathrm{D}^{\mathrm{s}} u_2);$$
taking the infimum over all $u_1 \in \mathscr{A}(M_1, 0)$, $u_2 \in \mathscr{A}(M_2, 0)$ gives
$$\Phi(M_1 + M_2) \le \Phi(M_1) + \Phi(M_2).$$
The positive homogeneity of degree 1 follows similarly. We note that the interfacial energies in Examples 1.4 and 1.5 have the form (1.9), but this is not the case generally. To prove the subadditivity and positive hemegeneity of degree 1 in the general case, we shall proceed in a different way, proving the formulas (1.7) and (1.8) also.

Lemma 2.4 *For any* $(a, b) \in \mathbb{D}_n$ *there exists a sequence* $u_k \in \mathscr{A}(a \otimes b, 0)$, $k = 1, \ldots,$ *such that*
$$\Delta(u_k) \to \psi(a, b) \tag{2.4}$$
as $k \to \infty$.

Proof We denote by $\langle r \rangle$ the integral part of $r \in \mathbb{R}$. In the proof we shall repeatedly use the elementary fact that
$$\langle k t \rangle / k \to t \quad \text{as} \quad k \to \infty$$
uniformly in $t \in \mathbb{R}$. For any $k \in \mathbb{N}$ we put
$$Q_k = (1 - 1/k) Q, \quad c_k = \mathsf{V}(Q \sim Q_k)/\mathsf{V}(Q_k),$$
and define

$$u_k(x) = \begin{cases} a(b\cdot x) & \text{if } x \in Q \sim Q_k, \\ a\big(\langle k(b\cdot x)\rangle/k - c_k(b\cdot x)\big) & \text{if } x \in Q_k, \end{cases}$$

$x \in Q$. One has

$$\nabla u_k(x) = \begin{cases} a\otimes b & \text{if } x \in Q \sim Q_k, \\ -c_k a\otimes b & \text{if } x \in Q_k; \end{cases}$$

hence $\int_Q \nabla u_k \, d\mathsf{V} = 0$ and as $u_k(x) = (a\otimes b)x$ if $x \in \partial Q$, we have $u_k \in \mathscr{A}(a\otimes b, 0)$. Furthermore,

$$J_{u_k} = \partial Q_k \cup L_k \quad \text{where} \quad L_k = \{x \in Q_k : k(x\cdot b) \in \mathbb{Z}\}$$

and

$$[\![u_k]\!] \otimes \nu_{u_k} = \begin{cases} a\otimes \nu_{Q_k}\varphi_k(x) & \text{if } x \in \partial Q_k, \\ a\otimes b/k & \text{if } x \in L_k \end{cases}$$

where ν_{Q_k} is the outer normal to ∂Q_k and

$$\varphi_k(x) = (1 + c_k)(b\cdot x) - \langle k(b\cdot x)\rangle/k.$$

Hence

$$\Delta(u_k) = \int_{\partial Q_k} \psi\big(a\varphi_k(x), \nu_{Q_k}\big)\, d\mathsf{A} + \psi(a, b)\mathsf{A}(L_k)/k. \tag{2.5}$$

We now consider the limit $k \to \infty$. By $(1.5)_2$, $0 \le \psi\big(\varphi_k(x), \nu_{Q_k}\big) \le C_1|a||\varphi_k(x)|$ and since $(b\cdot x) - \langle k(b\cdot x)\rangle/k \to 0$ uniformly in x and $c_k \to 0$, we have $|\varphi_k(x)| \to 0$ uniformly and thus

$$\int_{\partial Q_k} \psi\big(a\varphi_k(x), \nu_{Q_k}\big)\, d\mathsf{A} \to 0. \tag{2.6}$$

Next observe that

$$\mathsf{A}(L_k)/k \to 1. \tag{2.7}$$

To see that, we introduce a sequence of scalar piecewise constant functions $\omega_k : Q_k \to \mathbb{R}$ by putting $\omega_k(x) = \langle k(b\cdot x)\rangle/k$, $x \in Q_k$. Using $J_{\omega_k} = L_k$ and $[\![\omega_k]\!] \otimes \nu_{\omega_k} = b/k$ on J_{ω_k}, we see that the (scalar version of the) Gauss-Green theorem (4.1) for ω_k reads

$$\int_{Q_k} d\mathrm{D}\omega_k \equiv \int_{J_{u_k}} [\![\omega_k]\!] \otimes \nu_{\omega_k}\, d\mathsf{A} \equiv b\,\mathsf{A}(L_k)/k = \int_{\partial Q_k} \omega_k \nu_{Q_k}\, d\mathsf{A}.$$

Transforming the last integral onto a common domain ∂Q and using $\omega_k(x) \to b\cdot x$ one obtains

$$b\,\mathsf{A}(L_k)/k \to \int_{\partial Q} (b\cdot x)\nu_Q \, d\mathsf{A}(x) = b,$$

and (2.5) follows. But (2.5), (2.6) and (2.7) give (2.4). □

Lemma 2.5 *If* $M \in \mathrm{Lin}$ *and* $(a_i, b_i) \in \mathbb{D}_n$, $i = 1, \ldots, m$, *satisfy*

$$M = \sum_{i=1}^{m} a_i \otimes b_i$$

and if for any pair of distinct indices $i, j \in \{1, \ldots, m\}$ *the pair* $\{b_i, b_j\}$ *is linearly independent (i.e.,* $b_i \neq b_j$ *and* $b_i \neq -b_j$*), then there exists a sequence* $u_k \in \mathscr{A}(M, 0)$, $k = 1, \ldots,$ *such that*

$$\Delta(u_k) \to \sum_{i=1}^{m} \psi(a_i, b_i) \tag{2.8}$$

as $k \to \infty$.

Proof By Lemma 2.4, for each $i \in \{1, \ldots, m\}$ there exists a sequence $u_k^i \in \mathscr{A}(a_i \otimes b_i, 0)$, $k = 1, \ldots,$ such that

$$\Delta(u_k^i) \to \psi(a_i, b_i) \tag{2.9}$$

as $k \to \infty$. Define $u_k := \sum_{i=1}^m u_k^i$ for every k. By the linear independence of pairs of different normals $\{b_i, b_j\}$, the jump sets of the maps u_k^i and u_k^j with $i \neq j$ intersect at the set of A-measure 0 for every k. Consequently, $\Delta(u_k) = \sum_{i=1}^m \Delta(u_k^i)$ and by (2.9) the sequence u_k has the required properties. □

***Proof of Theorem* 1.2** Let Ψ and $\Phi : \mathrm{Lin} \to [0, \infty)$ be given by (2.3) and (1.7), respectively. We first note that the two definitions of Φ in (1.7) and (1.8) are easily seen to be equivalent (omitted).

The main part of the proof of Theorem 1.2 is to establish $\Psi(M) = \Phi(M)$ for any $M \in \mathrm{Lin}$.

To prove $\Psi(M) \le \Phi(M)$, we take any sequence $(a_i, b_i) \in \mathbb{D}_n$, $i = 1, \ldots, m$, $m \in \mathbb{N}$, such that $\sum_{i=1}^m a_i \otimes b_i = M$ and consider the infimum as in (1.8). It is easy to see that the same infimum is obtained if one considers only the sequences $(a_i, b_i) \in \mathbb{D}_n$ such that that for any part of distinct indices $i, j \in \{1, \ldots, m\}$ the pair $\{b_i, b_j\}$ is linearly independent. Otherwise one joins the members with the same or opposite value of b_i into a single term and to use the subadditivity of ψ with respect to the first variable to obtain possibly a smaller value of the sum in (1.8). Hence, for the given sequence $(a_i, b_i) \in \mathbb{D}_n$ satisfying the linear independency condition, we construct a sequence of maps $u_k \in \mathscr{A}(M, 0)$, $k = 1, \ldots,$ as in Lemma 2.5. Then

$$\Psi(M) \le \Delta(u_k)$$

by the definition of Ψ. Letting $k \to \infty$ and using (2.8), we obtain

$$\Psi(M) \le \sum_{i=1}^{m} \psi(a_i, b_i).$$

Taking the infimum over all sequences a_i, b_i, one obtains from the definition of Φ the inequality $\Psi(M) \le \Phi(M)$.

To prove $\Phi(M) \le \Psi(M)$, we let $u \in \mathscr{A}(M, 0)$ and observe preliminarily that

$$\int_{J_u} [\![u]\!] \otimes \nu_u \, d\mathsf{A} = \int_{\partial Q} u_{\partial\Omega} \otimes \nu_Q \, d\mathsf{A} = \int_{\partial Q} Mx \otimes \nu_Q \, d\mathsf{A}(x) = M \tag{2.10}$$

by the Gauss-Green theorem (4.1) with $\mathrm{D}u = [\![u]\!] \otimes \nu_u \mathsf{A} \llcorner J_u$, and $\Omega = Q$. The idea of the proof is to replace the integrals in

$$\Delta(u) = \int_{J_u} \psi([\![u]\!], v_u)\, d\mathsf{A} \quad \text{and} \quad M = \int_{J_u} [\![u]\!] \otimes v_u\, d\mathsf{A}$$

by finite (Lebesgue) sums to obtain the sums occurring in the definition (1.8) of Φ. The details can be as follows. Applying, e.g., [12; Corollary 1.77] to each component of the pair of maps $([\![u]\!], v_u) : J_u \to \mathbb{D}_m$, one obtains a sequence of simple maps $(s_k, b_k) : J_u \to \mathbb{R}^n \times \mathbb{R}^n$ such that, by applying the pointwise majorized convergence asserted in [12; Corollary 1.77], one has

$$\int_{J_u} \psi(s_k, b_k)\, d\mathsf{A} \to \int_{J_u} \psi([\![u]\!], v_u)\, d\mathsf{A} \quad \text{and} \quad \int_{J_u} s_k \otimes b_k\, d\mathsf{A} \to \int_{J_u} [\![u]\!] \otimes v_u\, d\mathsf{A}. \tag{2.11}$$

In view of the second relation, it is possible to modify the functions s_k, b_k slightly to obtain simple functions, denoted again by s_k, b_k, such that $(2.11)_1$ remains valid while $(2.11)_2$ is replaced by

$$\int_{J_u} s_k \otimes b_k\, d\mathsf{A} = \int_{J_u} [\![u]\!] \otimes v_u\, d\mathsf{A} = M \quad \text{and} \quad |b_k| = 1 \quad \text{for all } k.$$

The pair (s_k, b_k) has the form

$$(s_k, b_k) = \sum_{i=1}^{m_k} (s_k^i, b_k^i)\, 1_{S_k^i}$$

where the system $\{S_k^i : i = 1, \dots m_k\}$ is a partition of J_u and generally 1_S is the characteristic function of the set $S \subset \mathbb{R}^n$. Putting $a_k^i = \mathsf{A}(S_k^i) s_k^i$, for each k, we have a sequence (a_k^i, b_k^i), $i = 1, \dots, m_k$, such that

$$\sum_{i=1}^{m_k} a_k^i \otimes b_k^i = \int_{J_u} s_k \otimes b_k\, d\mathsf{A} = M$$

and

$$\sum_{i=1}^{m_k} \psi(a_k^i, b_k^i) = \int_{J_u} \psi(s_k, b_k)\, d\mathsf{A} \to \Delta(u) \quad \text{as} \quad k \to \infty.$$

As $u \in \mathscr{A}(M, 0)$ is arbitrary, we have $\Psi(M) \ge \Phi(M)$. □

***Proof of Theorem* 1.3*, Part* (i)** The definition (2.2) gives

$$\int_{J_u} \psi([\![u]\!], v_u)\, d\mathsf{A} \ge h(a, b) \tag{2.12}$$

for every $a \in \mathbb{R}^n$, $b \in \mathbb{S}^{n-1}$ and $u \in \mathscr{B}(a, b, Q_b)$. Througout the proof, let Q_b be a fixed cube and $u_{a,b}$ the map as in Theorem 2.1.

To prove that h is subadditive and positively homogeneous of degree 1 in the first variable, let $a_1, a_2 \in \mathbb{R}^n$, $b \in \mathbb{S}^{n-1}$ and put $a := a_1 + a_2$. Let H be the plane of normal b containing the origin and let $P : \mathbb{R}^n \to H$ be the orthogonal projection onto H. The intersection $Q_b \cap H$ is a square in H of unit edge and of center at 0. Let $S \subset H$ be a circle in H with origin 0 and of any radius $r > 0$ such that $S \subset Q_b$. The number r remains fixed throughout the proof. For any positive $\varepsilon < 1/2$ let C_ε be the (truncated) cylinder $C_\varepsilon = \{x \in \mathbb{R}^n : Px \in S,\ 0 < x \cdot b < \varepsilon\}$ and let $u_\varepsilon : Q_b \to \mathbb{R}^n$ be defined by

$$u_\varepsilon = \begin{cases} a_1 & \text{on} \quad C_\varepsilon, \\ u_{a,b} & \text{on} \quad Q_b \sim C_\varepsilon. \end{cases}$$

Then

$$J_{u_\varepsilon} = \big((Q_b \cap H) \sim S\big) \cup S \cup (S + \varepsilon b) \cup S_\varepsilon$$

where $S_\varepsilon = \{x \in \mathbb{R}^n : Px \in \partial S,\ 0 < x \cdot b < \varepsilon\}$ is the mantle of the cylinder C_ε. Evaluating $[\![u_\varepsilon]\!]$ on each of the indicated parts of J_{u_ε} we obtain

$$\begin{aligned} \int_{J_{u_\varepsilon}} \psi([\![u_\varepsilon]\!], \nu_{u_\varepsilon})\, d\mathsf{A} = (1 - \mathsf{A}(S))\psi(a, b) + \mathsf{A}(S)\psi(a_1, b) \\ + \mathsf{A}(S)\psi(-a_2, -b) + \int_{S_\varepsilon} \psi(-a_2, \nu_{S_\varepsilon})\, d\mathsf{A}. \end{aligned}$$

Inequality (2.12) reads

$$\mathsf{A}(S)\big(\psi(a_1, b) + \psi(-a_2, -b)\big) + \int_{S_\varepsilon} \psi(-a_2, \nu_{S_\varepsilon})\, d\mathsf{A} \ge \mathsf{A}(S) h(a, b).$$

Letting $\varepsilon \to 0$ and using $\int_{S_\varepsilon} \psi(-a_2, \nu_{S_\varepsilon})\, d\mathsf{A} \to 0$ and $\psi(-a_2, -b) = \psi(a_2, b)$, we obtain

$$h(a_1 + a_2, b) \le h(a_1, b) + h(a_2, b),$$

i.e., the subadditivity in the first variable. The proof of the positive homogeneity of degree 1 in the first variable of ψ uses a similar but simpler construction. The details are omitted.

To prove that $\tilde{h}$ is subadditive and positively homogeneous of degree 1 in the second variable, let $a, b_1, b_2 \in \mathbb{R}^n$. The case of b_1 and b_2 linearly dependent being trivial, we assume that b_1 and b_2 are linearly independent and in addition that $b := b_1 + b_2$ is a unit vector. Let P be the infinite prism of triangular cross section with faces formed by planar strips of outer normals and widths, respectively, $-\operatorname{sgn}(b_1)$ and $|b_1|$, $-\operatorname{sgn}(b_2)$ and $|b_2|$, $-b$ and 1, such that the origin $0 \in \mathbb{R}^n$ is in the center of the face of normal $-b$. Let $R = \frac{1}{2} Q_b$ and for each $\varepsilon > 0$, let $P_\varepsilon = R \cap (\varepsilon P)$. If ε is sufficiently small, then $P_\varepsilon \subset \{x \in Q_b : 0 < x \cdot b < 1/2\}$. Let $u_\varepsilon : Q_b \to \mathbb{R}^n$ be defined by

$$u_\varepsilon = \begin{cases} -a & \text{on} \quad \{x \in Q_b : 0 < x \cdot b < 1/2\} \sim P_\varepsilon, \\ 0 & \text{on} \quad \{x \in Q_b : -1/2 < x \cdot b < 0\} \cup P_\varepsilon. \end{cases}$$

The inequality $\int_{J_{u_\varepsilon}} \psi\big([\![u]\!], \nu_{u_\varepsilon}\big)\, d\mathsf{A} \ge h(a, b)$ is easily found to take the form

$$\varepsilon\big(\tilde{h}(-a, -b_1) + \tilde{h}(-a, -b_2)\big) + V_\varepsilon \ge \varepsilon \tilde{h}(a, b) \tag{2.13}$$

where

$$V_\varepsilon = \int_{\varepsilon P \cap R} \psi(-a, \nu)\, d\mathsf{A}.$$

We divide Inequality (2.13) by ε and use $V_\varepsilon = \mathrm{O}(\varepsilon^2)$ to obtain

$$\tilde{h}(a, b_1 + b_2) \le \tilde{h}(a, b_1) + \tilde{h}(a, b_2).$$

The proof of the positive homogeneity of degree 1 is similar. □

***Proof of Theorem* 1.3*, Part* (ii)** Throughout the proof, let $(a, b) \in \mathbb{D}_n$ be arbitrary and let Q_b be a fixed cube and $u_{a,b}$ the map as in Theorem 2.1. If $u \in \mathscr{B}(a, b, Q_b)$, we apply the Gauss-Green theorem in the same way as in the proof of (2.10) to obtain

$$\int_{J_u} [\![u]\!] \otimes \nu_u \, d\mathsf{A} = a \otimes b. \tag{2.14}$$

Equations (1.9) and (2.14) and Jensen's inequality [12; Theorem 4.80] then give

$$\begin{aligned} \Delta(u) &:= \int_{J_u} \psi([\![u]\!], \nu_u) \, d\mathsf{A} \\ &= \int_{J_u} \Lambda([\![u]\!] \otimes \nu_u) \, d\mathsf{A} \\ &\geq \Lambda\Big(\int_{J_u} [\![u]\!] \otimes \nu_u \, d\mathsf{A} \Big) \\ &= \Lambda(a \otimes b) = \psi(a, b). \end{aligned}$$

Thus the definition (2.2) gives $h = \psi$. □

3 Derivation of the examples

***Example* 1.4** Equation (1.10): We consider $\psi_{|\cdot|}(a, b) = |a \cdot b|$ first, and prove $(1.10)_1$. Clearly, the function $\Theta(M) = |\operatorname{tr} M|$ is a subadditive function satisfying (1.6) with $\psi = \psi_{|\cdot|}$ and hence (1.8) gives $\Phi_{|\cdot|}(M) \geq |\operatorname{tr} M|$ for any $M \in$ Lin. To prove the opposite inequality, we note that the definition (1.7) of $\Phi_{|\cdot|}$ gives

$$\psi_{|\cdot|}(a, b) = \Theta(a \otimes b) \leq \Phi_{|\cdot|}(a \otimes b) \leq \psi_{|\cdot|}(a, b)$$

for every $(a, b) \in \mathbb{D}_n$ and hence

$$\Phi_{|\cdot|}(a \otimes b) = |a \cdot b| \quad \text{and in particular} \quad \Phi_{|\cdot|}(a \otimes b) = 0 \quad \text{if} \quad a \cdot b = 0 \tag{3.1}$$

which determines $\Phi_{|\cdot|}$ on tensor products $a \otimes b$. To determine $\Phi_{|\cdot|}$ on a general $M \in$ Lin, we write $C = A + W$ where A and W are the symmetric and skew parts of C. Let $e_1, \ldots, e_n$ be an orthonormal basis of eigenvectors of A with the eigenvalues λ_i; then the equation $A = \sum_{i=1}^n \lambda_i e_i \otimes e_i$ can be rearranged as

$$A = (\operatorname{tr} M) e_1 \otimes e_1 + \sum_{i=2}^{n} \lambda_i \big(e_i \otimes e_1 - e_1 \otimes e_i - (e_1 + e_i) \otimes (e_1 - e_i) \big); \tag{3.2}$$

a combination with

$$W = \sum_{1 \leq i \neq j \leq n} W_{ij} e_i \otimes e_j, \quad W_{ij} = -W_{ji}, \tag{3.3}$$

yields

$$C = (\operatorname{tr} M) e_1 \otimes e_1 + \sum_{\alpha=1}^{(n-1)(n+3)} a_\alpha \otimes b_\alpha$$

where the collection of the dyads $a_\alpha \otimes b_\alpha$, $\alpha = 1, \ldots, (n-1)(n+3)$, is formed by the individual dyads in the sum in (3.2), which is altogether $3(n-1)$ dyads, and the dyads occurring in (3.3), i.e., $n(n-1)$ dyads. The only fact needed below is that these dyads satisfy $a_\alpha \cdot b_\alpha = 0$. The subadditivity of $\Phi_{|\cdot|}$ gives

$$\Phi_{|\cdot|}(C) \le \Phi_{|\cdot|}\big((\operatorname{tr} M)e_1 \otimes e_1\big) + \sum_{\alpha=1}^{(n-1)(n+3)} \Phi_{|\cdot|}(a_\alpha \otimes b_\alpha) \tag{3.4}$$

We now employ $(3.1)_1$ to find that $\Phi_{|\cdot|}\big((\operatorname{tr} M)e_1 \otimes e_1\big) = |\operatorname{tr} M|$, while all remaining terms on the right-hand side of (3.4) vanish by $(3.1)_2$. Hence $\Phi_{|\cdot|}(M) \le |\operatorname{tr} M|$. This completes the proof of $(1.10)_1$.

To prove the two equations in $(1.10)_2$ we employ $(1.10)_1$ as follows. One has $\psi_\pm(a,b) = \frac{1}{2}\big(|a \cdot b| \pm a \cdot b\big)$ and hence if $(a_i, b_i) \in \mathbb{D}_n$ and $M \in \mathrm{Lin}$ satisfy $\sum_{i=1}^m a_i \otimes b_i = M$ then

$$\sum_{i=1}^m \psi_\pm(a_i, b_i) = \frac{1}{2}\Big(\sum_{i=1}^m \psi_{|\cdot|}(a_i, b_i) \pm \operatorname{tr} M\Big).$$

Taking the infimum as in (1.8) and using the above evaluation of $\Phi_{|\cdot|}$ gives

$$\Phi_\pm(M) = \frac{1}{2}\big(\Phi_{|\cdot|}(M) \pm \operatorname{tr} M\big) = \frac{1}{2}\big(|\operatorname{tr} M| \pm \operatorname{tr} M\big) = \{\operatorname{tr} M\}_\pm$$

which is $(1.10)_2$.

Equation (1.11): We observe that $\psi_{|\cdot|}(a,b) = \Lambda_{|\cdot|}(a \otimes b)$ and $\psi_\pm(a,b) = \Lambda_\pm(a \otimes b)$ for every $(a,b) \in \mathbb{D}_n$ where $\Lambda_{|\cdot|}(M) = |\operatorname{tr} M|$ and $\Lambda_\pm(M) = \{\operatorname{tr} M\}_\pm$ for every $M \in \mathrm{Lin}$ and apply Theorem 1.3(ii). □

***Example* 1.5** Equation $(1.12)_1$: The function $\Theta(M) = |M^{\mathrm{T}} p|$ is a subadditive function satisfying (1.6) and we obtain in the same way as in the proof of Example 1.4 that $\Phi(M) \ge |M^{\mathrm{T}} p|$ for any $M \in \mathrm{Lin}$ and

$$\Phi(a \otimes b) = |a \cdot p|, \quad \text{and in particular} \quad \Phi(a \otimes b) = 0 \quad \text{if} \quad a \cdot p = 0. \tag{3.5}$$

To prove $\Phi(M) \le |M^{\mathrm{T}} p|$, we assume without loss in generality that $|p| = 1$ and let $\{p, e_2, \dots e_n\}$ be any orthonormal basis. In view of $\mathrm{I} = p \otimes p + \sum_{i=2}^n e_i \otimes e_i$ we have

$$M = \mathrm{I}M = p \otimes M^{\mathrm{T}} p + \sum_{i=2}^n e_i \otimes M^{\mathrm{T}} e_i;$$

normalizing the second members of the dyads, we obtain

$$M = |M^{\mathrm{T}} p| p \otimes \operatorname{sgn}(M^{\mathrm{T}} p) + \sum_{i=2}^n |M^{\mathrm{T}} e_i| e_i \otimes \operatorname{sgn}(M^{\mathrm{T}} e_i).$$

The subadditivity of Φ provides

$$\Phi(M) \le \Phi\big(|M^{\mathrm{T}} p| p \otimes \operatorname{sgn}(M^{\mathrm{T}} p)\big) + \sum_{i=2}^n |\Phi\big(M^{\mathrm{T}} e_i| e_i \otimes \operatorname{sgn}(M^{\mathrm{T}} e_i)\big) = |M^{\mathrm{T}} p|$$

by (3.5). Thus $\Phi(M) \le |M^{\mathrm{T}} p|$ and the proof of $(1.12)_1$ is complete.

Equation $(1.12)_2$: We apply Theorem 1.3(ii) in an obvious way. □

4 Notation; functions of bounded variation

We denote by $\mathbb{Z}$ the set of integers, by $\mathbb{N}$ the set of positive integers, by $\mathbb{S}^{n-1}$ the unit sphere in $\mathbb{R}^n$, by Lin the set of all linear transformations from $\mathbb{R}^n$ into itself, often identified with the set of $n \times n$ matrices with real elements. We use the symbols '$\cdot$' and '$|\cdot|$' to denote the scalar product and the euclidean norm on $\mathbb{R}^n$ and on Lin. The latter are defined by $B \cdot C := \operatorname{tr}(BC^{\mathrm{T}})$ and $|B| = \sqrt{B \cdot B}$ where $B^{\mathrm{T}} \in \mathrm{Lin}$ is the transpose of B and tr denotes the trace.

A real-valued function f defined on a vectorspace X is said to be subadditive if $f(x+y) \leq f(x) + f(y)$ for every $x, y \in \mathrm{X}$ and positively homogeneous of degree 1 if $f(tx) = tf(x)$ for every $t \geq 0$ and $x \in \mathrm{X}$.

If Ω is an open subset of $\mathbb{R}^n$, we denote by $L^1(\Omega)$ the space of Lin valued integrable maps on Ω. We denote by $\mathscr{M}(\Omega, \mathrm{Lin})$ the set of all (finite) Lin valued measures on Ω. If $\mu \in \mathscr{M}(\Omega, \mathrm{Lin})$, we denote by $\mu \llcorner B$ the restriction of μ to a Borel set $B \subset \Omega$. If $G, G_k \in L^1(\Omega)$, $k = 1, 2, \ldots$, we say that G_k converges to G in the sense of measures, and write

$$G_k \stackrel{*}{\rightharpoonup} G \quad \text{in } \mathscr{M}(\Omega, \mathrm{Lin})$$

if $\int_\Omega G_k \cdot H \, d\mathsf{V} \to \int_\Omega G \cdot H \, d\mathsf{V}$ for every continuous map $H : \mathbb{R}^n \to \mathrm{Lin}$ which vanishes outside Ω.

We state some basic definitions and properties of the spaces of maps of bounded variation and of special maps of bounded variation. For more details, see [1, 10, 17], and [11].

We define the set $BV(\Omega, \mathbb{R}^n)$ of maps of bounded variation as the set of all $u \in L^1(\Omega)$ such that there exists a measure $\mathrm{D}u \in \mathscr{M}(\Omega, \mathrm{Lin})$ satisfying

$$\int_\Omega u \cdot \operatorname{div} T \, d\mathsf{V} = -\int_\Omega T \cdot d\mathrm{D}u$$

for each class ∞ map $T : \mathbb{R}^n \to \mathbb{R}^{n \times n}$ which vanishes outside some compact subset of Ω. Here $\operatorname{div} T$ is an $\mathbb{R}^n$ valued map on Ω given by $(\operatorname{div} T)_i = \sum_{j=1}^n T_{ij,j}$ where the comma followed by an index j denotes the partial derivative with respect to j th variable. The measure $\mathrm{D}u$ is uniquely determined and called the weak (or generalized) derivative of u. We shall need the following form of the Gauss-Green theorem for BV: if Ω is a domain with lipschitzian boundary and $u \in BV(\Omega, \mathbb{R}^n)$ then there exist an A integrable map $u_{\partial\Omega} : \partial\Omega \to \mathbb{R}^n$ such that

$$\int_\Omega d\mathrm{D}u = \int_{\partial(\Omega)} u_{\partial\Omega} \otimes \nu_\Omega \, d\mathsf{A} \tag{4.1}$$

where ν_Ω is the outer normal to $\partial\Omega$. The map $u_{\partial\Omega}$ is determined to within a change on a set of A measure 0 and is called the trace of u.

We define the set $SBV(\Omega)$ of special maps of bounded variation as the set of all $u \in BV(\Omega, \mathbb{R}^n)$ for which $\mathrm{D}u$ has the form

$$\mathrm{D}u = \nabla u \, \mathsf{V} \llcorner \Omega + [\![u]\!] \otimes \nu_u \mathsf{A} \llcorner J_u \tag{4.2}$$

where ∇u, the absolutely continuous part of $\mathrm{D}u$, is a map in $L^1(\Omega)$ and the term

$$\mathrm{D}^{\mathrm{s}} u := [\![u]\!] \otimes \nu_u \mathsf{A} \llcorner J_u$$

on the right-hand side of (4.2) is called the jump (or singular) part of Du. The objects $J_u \subset \Omega$, $[\![u]\!] : J_u \to \mathbb{R}^n$ and $\nu_u : J_u \to \mathbb{S}^{n-1}$ are called the jump set of u, the jump of u and the normal to J_u, respectively. Here J_u is the set of all $x \in \Omega$ for which there exist $\nu_u(x) \in \mathbb{S}^{n-1}$ and $u^{\pm}(x) \in \mathbb{R}^n$ such that we have the approximate limits

$$u^{\pm}(x) = \underset{\substack{y \to x \\ y \in \mathbf{H}^{\pm}(x, \nu_u(x))}}{\operatorname{ap\,lim}} u(x),$$

where $\mathbf{H}^{\pm}(x, \nu_u(x)) = \{y \in \mathbb{R}^n : \pm(y - x) \cdot \nu_u(x) > 0\}$. For a given $x \in \Omega$, either the triplet $(\nu_u(x), u^+(x), u^-(x))$ does not exist or it is uniquely determined to within the change $(\nu_u(x), u^+(x), u^-(x)) \mapsto (-\nu_u(x), u^-(x), u^+(x))$. With one of these choices, one puts $[\![u]\!](x) = u^+(x) - u^-(x)$ and notes that $[\![u]\!](x) \otimes \nu_u(x)$ is unique.

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