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for the Stokes system in planar domains**

Dagmar Medková

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ONE PROBLEM OF THE NAVIER TYPE FOR THE STOKES SYSTEM IN PLANAR DOMAINS

DAGMAR MEDKOVÁ

ABSTRACT. We study the problem $-\Delta \mathbf{u} + \nabla \rho = \mathbf{F}$, $\nabla \cdot \mathbf{u} = G$ in Ω , $\mathbf{u} \cdot \boldsymbol{\tau} = g$, $\rho = h$ on $\partial\Omega$, for a bounded simply connected Lipschitz domain in the plane. For $\mathbf{F} = 0$, $G = 0$, $g \in L^p(\partial\Omega)$, $h \in L^q(\partial\Omega)$ we study a solution in the sense of a nontangential limit. For $\mathbf{F} \in W^{s-1,q}(\Omega, \mathbb{R}^2)$, $G \in W^{s,q}(\Omega)$, $g \in W^{t-1/p,p}(\partial\Omega)$, $h \in W^{s-1/q,q}(\partial\Omega)$ with $t \leq s+1$ we prove the existence of a unique solution $(\mathbf{u}, \rho) \in W^{t,p}(\Omega, \mathbb{R}^2) \times W^{s,q}(\Omega)$. For $\mathbf{F} \in B_{s-1}^{q,r}(\Omega, \mathbb{R}^2)$, $G \in B_s^{q,r}(\Omega)$, $g \in B_{t-1/p}^{p,\beta}(\partial\Omega)$, $h \in B_{s-1/q}^{q,r}(\partial\Omega)$ with $t \leq s+1$ we prove the existence of a unique solution $(\mathbf{u}, \rho) \in B_t^{p,\beta}(\Omega, \mathbb{R}^2) \times B_s^{q,r}(\Omega)$. For $\mathbf{F} \in \mathcal{C}^{k-1,\gamma}(\overline{\Omega}, \mathbb{R}^2)$, $G \in \mathcal{C}^{k,\gamma}(\overline{\Omega})$, $h \in \mathcal{C}^{k,\gamma}(\partial\Omega)$, $g \in \mathcal{C}^{k+1,\gamma}(\partial\Omega)$ we prove the existence of a unique solution $(\mathbf{u}, \rho) \in \mathcal{C}^{k+1,\gamma}(\overline{\Omega}, \mathbb{R}^2) \times \mathcal{C}^{k,\gamma}(\overline{\Omega})$.

1. INTRODUCTION

Boundary value problems of Navier's type for the Stokes system are very interesting problems. This paper is devoted to one problem of this type. Let us suppose that $\Omega \subset \mathbb{R}^m$ is a bounded domain with connected Lipschitz boundary. We denote by $\mathbf{n} = \mathbf{n}^\Omega$ the outward unit normal vector of Ω . If \mathbf{v} is a vector, then $\mathbf{v}_\mathbf{n} = (\mathbf{v} \cdot \mathbf{n})\mathbf{n}$ is the normal part of \mathbf{v} , and $\mathbf{v}_\tau = \mathbf{v} - \mathbf{v}_\mathbf{n}$ is the tangential part of \mathbf{v} . There are two types of Navier's problem: I. It is given the normal part of the Dirichlet condition and the tangential part of the Neumann condition (or a corresponding Robin condition). II. It is given the tangential part of the Dirichlet condition and the normal part of the Neumann condition (or a corresponding Robin condition). Since the Stokes system has many Neumann conditions there are many Navier's problems.

The Navier problems corresponding to the Neumann condition $\partial \mathbf{u} / \partial \mathbf{n} - \rho \mathbf{n}$ (and the Robin condition $\partial \mathbf{u} / \partial \mathbf{n} - \rho \mathbf{n} + c\mathbf{u}$) are

$$-\Delta \mathbf{u} + \nabla \rho = \mathbf{f}, \quad \nabla \cdot \mathbf{u} = \chi \text{ in } \Omega, \quad \mathbf{u}_\tau = \mathbf{g}_\tau, \quad [\partial \mathbf{u} / \partial \mathbf{n}]_\mathbf{n} - \rho \mathbf{n} + c\mathbf{n} = \mathbf{h}_\mathbf{n} \text{ on } \partial\Omega$$

and

$$-\Delta \mathbf{u} + \nabla \rho = \mathbf{f}, \quad \nabla \cdot \mathbf{u} = \chi \text{ in } \Omega, \quad \mathbf{u}_\mathbf{n} = \mathbf{g}_\mathbf{n}, \quad [\partial \mathbf{u} / \partial \mathbf{n}]_\tau + c\mathbf{n}_\tau = \mathbf{h}_\tau \text{ on } \partial\Omega$$

(studied in [56]).

The Navier problems corresponding to the Neumann condition $T(\mathbf{u}, \rho)\mathbf{n} = [\nabla \mathbf{u} + (\nabla \mathbf{u})^T - \rho I]\mathbf{n}$ (and the Robin condition $T(\mathbf{u}, \rho)\mathbf{n} + c\mathbf{u}$) are

$$-\Delta \mathbf{u} + \nabla \rho = \mathbf{f}, \quad \nabla \cdot \mathbf{u} = \chi \text{ in } \Omega, \quad \mathbf{u}_\tau = \mathbf{g}_\tau, \quad [T(\mathbf{u}, \rho)\mathbf{n} + c\mathbf{u}]_\mathbf{n} = \mathbf{h}_\mathbf{n} \text{ on } \partial\Omega,$$

$$-\Delta \mathbf{u} + \nabla \rho = \mathbf{f}, \quad \nabla \cdot \mathbf{u} = \chi \text{ in } \Omega, \quad \mathbf{u}_\mathbf{n} = \mathbf{g}_\mathbf{n}, \quad [T(\mathbf{u}, \rho)\mathbf{n} + c\mathbf{u}]_\tau = \mathbf{h}_\tau \text{ on } \partial\Omega$$

(studied in [7], [14], [39], [42], [43], [57], [62]), [69]).

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In \mathbb{R}^3 we have $\Delta \mathbf{u} = \nabla(\nabla \cdot \mathbf{u}) - \nabla \times (\nabla \times \mathbf{u})$. This gives the Neumann condition for the Stokes system

$$\mathbf{n} \times (\nabla \times \mathbf{u}) + \rho \mathbf{n}.$$

Remark that $\mathbf{n} \times (\nabla \times \mathbf{u})$ is the tangential part of the Neumann condition and $\rho \mathbf{n}$ is the normal part of the Neumann condition. The corresponding Navier problems are

$$(1.1) \quad -\Delta \mathbf{u} + \nabla \rho = \mathbf{f}, \quad \nabla \cdot \mathbf{u} = \chi \text{ in } \Omega, \quad \mathbf{u}_\tau = \mathbf{g}_\tau, \quad \rho = h \text{ on } \partial\Omega,$$

and

$$-\Delta \mathbf{u} + \nabla \rho = \mathbf{f}, \quad \nabla \cdot \mathbf{u} = \chi \text{ in } \Omega, \quad \mathbf{u}_n = \mathbf{g}_n, \quad \mathbf{n} \times (\nabla \times \mathbf{u}) = \mathbf{n} \times \mathbf{h} \text{ on } \partial\Omega.$$

The corresponding Navier problems are also studied in planar domains. These problems were studied in [1], [2], [5], [6], [8], [9], [13], [15], [18], [40], [52] from theoretical and numerical point of view.

We gather what is known about the problem (1.1). J. M. Bernard studied in 2002 this problem in a bounded domain $\Omega \subset \mathbb{R}^3$ with boundary of class $C^{1,1}$ (see [13]). J. M. Bernard proved that for $\mathbf{f} \in L^2(\Omega, \mathbb{R}^3)$, $\chi = 0$ and $\mathbf{g}, \mathbf{h} \in H^{1/2}(\partial\Omega, \mathbb{R}^3)$ there exists a unique solution $(\mathbf{u}, \rho) \in H^1(\Omega, \mathbb{R}^3) \times H^1(\Omega)$. This result was generalized by Ch. Amrouche, P. Penel, N. Seloula in 2013 ([6]). For the same domains and $\mathbf{g}, \mathbf{h} \in W^{1-1/p,p}(\partial\Omega, \mathbb{R}^3)$, $\chi \in W^{1,p}(\Omega)$, $\mathbf{f} \in [H_\tau^{p'}(\text{curl}, \Omega)]'$ they proved that there exists a unique solution $(\mathbf{u}, p) \in W^{1,p}(\Omega, \mathbb{R}^3) \times W^{1,p}(\Omega)$. Moreover, they proved that for Ω with boundary of class $C^{2,1}$, $\mathbf{f} \in L^p(\Omega, \mathbb{R}^3)$ and $\mathbf{g} \in W^{2-1/p,p}(\partial\Omega, \mathbb{R}^3)$ the velocity $\mathbf{u} \in W^{2,p}(\Omega, \mathbb{R}^3)$. The problem (1.1) for planar domains has been studied in literature only from the numerical point of view. So, the goal of this paper is to study the Navier problem (1.1) on planar domains from theoretical point of view - i.e. to study the existence, uniqueness and regularity of solutions.

We can rewrite the planar problem (1.1) as

$$\begin{aligned} -\Delta \mathbf{u} + \nabla \rho &= \mathbf{F}, \quad \nabla \cdot \mathbf{u} = G \quad \text{in } \Omega, \\ \mathbf{u} \cdot \boldsymbol{\tau} &= g, \quad \rho = h \quad \text{on } \partial\Omega. \end{aligned}$$

We study solutions of the problem in the scales of Sobolev spaces $W^{t,p}(\Omega, \mathbb{R}^2) \times W^{s,q}(\Omega)$ with $t > 1/p$, $s > 1/q$ and in the scale of Besov spaces $B_t^{p,\beta}(\Omega, \mathbb{R}^2) \times B_s^{q,r}(\Omega)$ with $t > 1/p$, $s > 1/q$. We also study classical solutions in spaces $C^{k,\alpha}(\Omega, \mathbb{R}^2) \times C^{l,\beta}(\Omega)$. To do so, we begin with study the problem for the homogeneous equations, i.e. for $\mathbf{F} = 0$, $G = 0$. We study the weakest possible solutions of the problem - L^p - L^q -solutions, i.e. solutions of the Stokes system such that the maximal function of the velocity \mathbf{u} is in $L^p(\partial\Omega)$, the maximal function of the pressure ρ is in $L^q(\partial\Omega)$, and the boundary conditions are fulfilled in the sense of nontangential limits. (Classical solutions of the problem are clearly L^p - L^q -solutions for arbitrary p and q . We shall see that solutions of the problem in $W^{t,p}(\Omega, \mathbb{R}^2) \times W^{s,q}(\Omega)$ or in $B_t^{p,\beta}(\Omega, \mathbb{R}^2) \times B_s^{q,r}(\Omega)$ with $t > 1/p$, $s > 1/q$ are L^p - L^q -solutions.) For L^p -solutions of the Dirichlet, Neumann or transmission problem for the Stokes system see for example [21], [35], [44], [58], [51], for the Brinkman system see for example [16], [37], [49], [60], for the Laplace equation see for example [17], [20], [30], [33], [34], [36], [45], [48], [59], [68].

The proofs in this paper are totally different than the proofs in [6] or in [51]. Using methods of complex analysis we reduce the original problem to two problems for Laplace equation: the Dirichlet problem and the Neumann problem. We gather

known results about these problems (and prove missing) and then we prove the unique solvability of the Navier problem for the Stokes system and regularity results.

2. FORMULATION OF THE PROBLEM

In the whole paper we assume that $\Omega \subset \mathbb{R}^2$ is a bounded domain with connected Lipschitz boundary. We denote by $\mathbf{n} = \mathbf{n}^\Omega$ the outward unit normal vector of Ω , and by $\tau = \tau^\Omega = (-n_2^\Omega, n_1^\Omega)$ the unit tangential vector of $\partial\Omega$.

First we remember the definition of an L^p -solution of the Dirichlet and the Neumann problem for the Laplace equation.

If $\mathbf{x} \in \partial\Omega$, $a > 0$ denote the nontangential approach regions of opening a at the point \mathbf{x} by

$$\Gamma_a(\mathbf{x}) = \{\mathbf{y} \in \Omega; |\mathbf{x} - \mathbf{y}| < (1 + a) \text{dist}(\mathbf{y}, \partial\Omega)\}.$$

If now \mathbf{v} is a vector function defined in Ω , we denote the nontangential maximal function of \mathbf{v} on $\partial\Omega$ by

$$M_a(\mathbf{v})(\mathbf{x}) = M_a^\Omega(\mathbf{v})(\mathbf{x}) = \sup\{|\mathbf{v}(\mathbf{y})|; \mathbf{y} \in \Gamma_a(\mathbf{x})\}.$$

It is well known that there exists $c > 0$ such that for $a, b > c$ and $1 \leq q < \infty$ there exist $C_1, C_2 > 0$ such that

$$\|M_a v\|_{L^q(\partial\Omega)} \leq C_1 \|M_b v\|_{L^q(\partial\Omega)} \leq C_2 \|M_a v\|_{L^q(\partial\Omega)}$$

for any measurable function v in Ω . (See, e.g. [33] and [61, p. 62].) We shall suppose that $a > c$ and write $\Gamma(\mathbf{x})$ instead of $\Gamma_a(\mathbf{x})$. Next, define the nontangential limit of \mathbf{v} at $\mathbf{x} \in \partial\Omega$

$$\mathbf{v}(\mathbf{x}) = \lim_{\Gamma(\mathbf{x}) \ni \mathbf{y} \rightarrow \mathbf{x}} \mathbf{v}(\mathbf{y})$$

whenever the limit exists.

Let $h \in L^p(\partial\Omega)$, $1 < p < \infty$. We say that ρ is an L^p -solution of the Dirichlet problem for the Laplace equation

$$(2.1) \quad \Delta\rho = 0 \quad \text{in } \Omega, \quad \rho = h \quad \text{on } \partial\Omega$$

if $\rho \in \mathcal{C}^2(\Omega)$, $\Delta\rho = 0$ in Ω , $M_a^\Omega \rho \in L^p(\partial\Omega)$ and $h(\mathbf{x})$ is the nontangential limit of ρ at almost all $x \in \partial\Omega$.

Let $f \in L^p(\partial\Omega)$, $1 < p < \infty$. We say that φ is an L^p -solution of the Neumann problem for the Laplace equation

$$(2.2) \quad \Delta\varphi = 0 \quad \text{in } \Omega, \quad \frac{\partial\varphi}{\partial n} = f \quad \text{on } \partial\Omega$$

if $\varphi \in \mathcal{C}^2(\Omega)$, $\Delta\varphi = 0$ in Ω , $M_a^\Omega(\nabla\varphi) \in L^p(\partial\Omega)$, at almost all $\mathbf{x} \in \partial\Omega$ there exists a nontangential limit of $\nabla\varphi$ and $n(\mathbf{x}) \cdot \nabla\varphi(\mathbf{x}) = f(\mathbf{x})$.

We now define L^p - L^q -solution of our problem. Let $1 < p, q < \infty$, $g \in L^p(\partial\Omega)$, $h \in L^q(\partial\Omega)$. We say that (\mathbf{u}, ρ) is an L^p - L^q -solution of the problem

$$(2.3a) \quad -\Delta\mathbf{u} + \nabla\rho = 0, \quad \nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega,$$

$$(2.3b) \quad \mathbf{u} \cdot \tau = g, \quad \rho = h \quad \text{on } \partial\Omega,$$

if $\mathbf{u} = (u_1, u_2) \in \mathcal{C}^2(\Omega, \mathbb{R}^2)$, $\rho \in \mathcal{C}^1(\Omega)$ solve (2.3a), $M_a(\mathbf{u}) \in L^p(\partial\Omega)$, $M_a(\rho) \in L^q(\partial\Omega)$, there exist nontangential limits of \mathbf{u} and ρ at almost all points of $\partial\Omega$, and these limits satisfy the boundary conditions (2.3b).

3. BOUNDARY VALUE PROBLEMS FOR THE LAPLACE EQUATION

We reduce the problem (2.3) to the Dirichlet problem and the Neumann problem for the Laplace equation. So, we gather some results about these problems.

We need several function spaces. Let $1 < p, q < \infty$. If k is a nonnegative integer then $W^{k,p}(\Omega) = \{f \in L^p(\Omega); \partial^\alpha f \in L^p(\Omega) \forall |\alpha| \leq k\}$ is the classical Sobolev space. If $0 < \lambda < 1$ and $s = k + \lambda$, then $W^{s,p}(\Omega) = \{u \in W^{k,p}(\Omega); \|u\|_{W^{s,p}(\Omega)} < \infty\}$ where

$$\|u\|_{W^{s,p}(\Omega)} = \left[\|u\|_{W^{k,p}(\Omega)}^p + \sum_{|\alpha|=k} \int_{\Omega} \int_{\Omega} \frac{|\partial^\alpha u(x) - \partial^\alpha u(y)|^p}{|x-y|^{m+p\lambda}} dx dy \right]^{1/p}.$$

If $s \in \mathbb{R}$ then $B_s^{p,q}(\mathbb{R}^2)$ is a Besov space. (For the definition see for example [67].) Denote by $B_s^{p,q}(\Omega)$ the set of all distribution f on Ω for which there exists $F \in B_s^{p,q}(\mathbb{R}^2)$ such that $f = F$ on Ω , and define the norm

$$\|f\|_{B_s^{p,q}(\Omega)} = \inf\{\|F\|_{B_s^{p,q}(\mathbb{R}^2)}; f = F \text{ on } \Omega\}.$$

If k is a nonnegative integer, $0 < \lambda < 1$ and $s = k + \lambda$, then $W^{s,p}(\Omega) = B_s^{p,p}(\Omega)$ (see [63, Lemma 36.1] and [46, Proposition 7.6]). If $\epsilon > 0$ and $1 < r < \infty$, then $W^{s+\epsilon,p}(\Omega) \hookrightarrow W^{s,p}(\Omega)$, $B_{s+\epsilon}^{p,q}(\Omega) \hookrightarrow B_s^{p,r}(\Omega)$.

First we realize how smooth L^p -solutions of boundary value problems for the Laplace equation are:

Proposition 3.1. *Let $1 < p < \infty$. If $u \in C^\infty(\Omega)$, $\Delta u = 0$ and $M_a(u) \in L^p(\partial\Omega)$, then $u \in B_{1/p}^{p,q}(\Omega)$ with $q = \max(p, 2)$.*

(See [47], Corollary 4.4.)

Now we recall results about solvability of the Neumann problem and the Dirichlet problem for the Laplace equation:

Proposition 3.2. *Let $1 < q < \infty$, $h \in L^q(\partial\Omega)$. Suppose that one of the following conditions is satisfied:*

- $q \geq 2$,
- $\partial\Omega$ is of class C^1 ,
- Ω is convex.

Then there exists a unique L^q -solution ρ of the Dirichlet problem for the Laplace equation (2.1). Moreover,

$$\|M_a(\rho)\|_{L^q(\partial\Omega)} + \|\rho\|_{B_{1/q}^{q,\max(q,2)}(\Omega)} \leq C\|h\|_{L^q(\partial\Omega)}$$

with a constant C that does not depend on h .

Proof. According to [50, Theorem 5.1], [29, Theorem 2], [31, Theorem 5.8] and [47, Theorem 3.10] there exists a unique L^q -solution ρ_h of the Dirichlet problem for the Laplace equation (2.1), and

$$\|M_a(\rho_h)\|_{L^q(\partial\Omega)} \leq C_1\|h\|_{L^q(\partial\Omega)}$$

with a constant C_1 that does not depend on h . This inequality and Proposition 3.1 give that the mapping $Q_1 : h \mapsto \rho_h$ is a closed linear operator from $L^q(\partial\Omega)$ to $B_{1/q}^{q,\max(2,q)}(\Omega)$. According to the Closed graph theorem [19, Korollar 3.8] there exists a positive constant C_2 independent of h such that

$$\|\rho_h\|_{B_{1/q}^{q,\max(2,q)}(\Omega)} \leq C_2\|h\|_{L^q(\partial\Omega)}.$$

□

Proposition 3.3. *Let $1 < q < \infty$, $h \in W^{1,q}(\partial\Omega)$. Suppose that one of the following conditions is satisfied:*

- $q \leq 2$,
- $\partial\Omega$ is of class C^1 ,
- Ω is convex.

Then there exists a unique L^q -solution ρ of the Dirichlet problem for the Laplace equation (2.1) such that $M_a(\nabla\rho) \in L^q(\partial\Omega)$. Moreover,

$$\|M_a(\rho)\|_{L^q(\partial\Omega)} + \|M_a(\nabla\rho)\|_{L^q(\partial\Omega)} + \|\rho\|_{B_{1+1/q}^{q,\max(q,2)}(\Omega)} \leq C\|h\|_{W^{1,q}(\partial\Omega)}$$

with a constant C that does not depend on h .

Proof. According to [47, Theorem 3.11], [31, Theorem 5.8] and [50, Theorem 5.1] there exists a unique L^q -solution ρ_h of the Dirichlet problem for the Laplace equation (2.1) such that $M_a(\nabla\rho) \in L^q(\partial\Omega)$. Moreover,

$$\|M_a(\rho_h)\|_{L^q(\partial\Omega)} + \|M_a(\nabla\rho_h)\|_{L^q(\partial\Omega)} \leq C_1\|h\|_{L^q(\partial\Omega)}$$

with a constant C_1 that does not depend on h . This inequality and Proposition 3.1 give that the mapping $Q_1 : h \mapsto \rho_h$ is a closed linear operator from $W^{1,q}(\partial\Omega)$ to $B_{1+1/q}^{q,\max(2,q)}(\Omega)$. According to the Closed graph theorem [19, Korollar 3.8] there exists a positive constant C_2 independent of h such that

$$\|\rho_h\|_{B_{1+1/q}^{q,\max(2,q)}(\Omega)} \leq C_2\|h\|_{W^{1,q}(\partial\Omega)}.$$

□

Proposition 3.4. *Let $0 \leq \alpha < 1$, $h \in C^{0,\alpha}(\partial\Omega)$. Suppose that one of the following assumptions holds:*

- $\alpha \leq 1/2$.
- $\partial\Omega$ is of class C^1 ,
- Ω is convex.

Then there exists a unique solution $\rho \in C^{0,\alpha}(\overline{\Omega})$ of the Dirichlet problem for the Laplace equation (2.1). Moreover,

$$\|\rho\|_{C^{0,\alpha}(\overline{\Omega})} \leq \|h\|_{C^{0,\alpha}(\partial\Omega)}$$

where a constant C does not depend on h .

(See [10, Lemma 6.6.14 and Theorem 1.2.4], [31, Theorem 5.2], [47, Theorem 4.6], [65, §2.5.7, Theorem] and [4, Remark 2].)

Proposition 3.5. *Let $0 < \alpha < 1$ and $k \in \mathbb{N}$. Suppose that $\partial\Omega$ is of class $C^{k,\alpha}$. If $h \in C^{k,\alpha}(\partial\Omega)$, then there exist a unique solution $\rho \in C^{k,\alpha}(\overline{\Omega})$ of the Dirichlet problem for the Laplace equation (2.1).*

(See [26, Theorem 8.34, Lemma 6.38, Theorem 6.14 and Theorem 6.19].)

Proposition 3.6. *Let $1 < q, r < \infty$, $1/q < s < 1 + 1/q$. Suppose that one of the following conditions is satisfied:*

- $q = 2$,
- $\partial\Omega$ is of class C^1 ,
- Ω is convex.

Then the following holds:

- (1) There exists a solution $\rho \in B_s^{q,r}(\Omega)$ of the Dirichlet problem for the Laplace equation (2.1) (i.e. boundary condition is fulfilled in the sense of traces) if and only if $h \in B_{s-1/q}^{q,r}(\partial\Omega)$. This solution is unique and

$$\|\rho\|_{B_s^{q,r}(\Omega)} \leq C \|h\|_{B_{s-1/q}^{q,r}(\partial\Omega)}$$

with a constant C that does not depend on h . The function ρ is an L^q -solution of the problem.

- (2) There exists a solution $\rho \in W^{s,q}(\Omega)$ of the Dirichlet problem for the Laplace equation (2.1) (i.e. boundary condition is fulfilled in the sense of traces) if and only if $h \in W^{s-1/q,q}(\partial\Omega)$. This solution is unique and

$$\|\rho\|_{W^{s,q}(\Omega)} \leq C \|h\|_{W^{s-1/q,q}(\partial\Omega)}$$

with a constant C that does not depend on h . The function ρ is an L^q -solution of the problem.

Proof. For Ω convex see [47, Theorem 4.5]. The rest cases we obtain by the same way - i.e. by the interpolation:

$B_{s-1/q}^{q,r}(\partial\Omega)$ is the space of traces of $B_s^{q,r}(\Omega)$ by [51, Theorem 2.5.2]. The uniqueness of a solution of the Dirichlet problem in $B_s^{q,r}(\Omega)$ we get by [31, Proposition 5.17]. If $h \in L^q(\partial\Omega)$ then there exists a unique L^q -solution Lh of the Dirichlet problem (2.1). The operators $L : L^q(\partial\Omega) \rightarrow B_{1/q}^{q,\max(q,2)}(\Omega)$, $L : W^{1,q}(\partial\Omega) \rightarrow B_{1+1/q}^{q,\max(q,2)}(\Omega)$ are bounded. (See Proposition 3.2 and Proposition 3.3.) Using real interpolation we deduce that the operator $L : B_{s-1/q}^{q,r}(\partial\Omega) \rightarrow B_s^{q,r}(\Omega)$ is bounded. (See [63, Lemma 22.3], [54, Chapter 3, Corollary 3] and [67, Corollary 1.111].) If $h \in C^\infty(\partial\Omega)$ then h is the trace of Lh by Proposition 3.4. Continuity of the trace operator gives that h is the trace of Lh for arbitrary $h \in B_{s-1/q}^{q,r}(\partial\Omega)$.

We now show the second part of the proposition. For $s \neq 1$ the proposition follows from the fact that $W^{s,q}(\Omega) = B_s^{q,q}(\Omega)$ and $W^{s-1/q,q}(\partial\Omega) = B_{s-1/q}^{q,q}(\partial\Omega)$. Let now $s = 1$. Since $\{\rho \in W^{1,q}(\Omega); \Delta\rho = 0\} = \{\rho \in B_1^{q,q}(\Omega); \Delta\rho = 0\}$ by [31, Theorem 4.1, Theorem 4.2], L is a closed linear operator from $W^{1-1/q}(\partial\Omega) = B_{1-1/q}^{q,q}(\partial\Omega)$ to $W^{1,q}(\Omega)$. Therefore $L : W^{1-1/q}(\partial\Omega) \rightarrow W^{1,q}(\Omega)$ is a bounded operator by the Closed graph theorem ([19, Korollar 3.8]). \square

Proposition 3.7. *Let $\partial\Omega$ be of class $\mathcal{C}^{k,1}(\partial\Omega)$ where $k \in \mathbb{N}$, $1 < q < \infty$. If $h \in W^{k+1-1/q,q}(\partial\Omega)$ then there exists a unique solution $\rho \in W^{k+1,q}(\Omega)$ of the Dirichlet problem for the Laplace equation (2.1). Moreover,*

$$\|\rho\|_{W^{k+1,q}(\Omega)} \leq C \|h\|_{W^{k+1-1/q,q}(\partial\Omega)},$$

where a constant C does not depend on h .

(See [27, Theorem 2.4.2.5 and Theorem 2.5.1.1].)

Proposition 3.8. *Let $\partial\Omega$ be of class $\mathcal{C}^{k,1}(\partial\Omega)$ where $k \in \mathbb{N}$, $1 < q, r < \infty$, $1 < s < k+1$. If $h \in B_{s-1/q}^{q,r}(\partial\Omega)$ then there exists a unique solution $\rho \in B_s^{q,r}(\Omega)$ of the Dirichlet problem for the Laplace equation (2.1). Moreover,*

$$\|\rho\|_{B_s^{q,r}(\Omega)} \leq C \|h\|_{B_{s-1/q}^{q,r}(\partial\Omega)},$$

where a constant C does not depend on h .

Proof. For $h \in W^{1-1/q,q}(\partial\Omega)$ denote by ρ_h a solution of the Dirichlet problem for the Laplace equation (2.1). Then the mapping $h \mapsto \rho_h$ is a bounded linear mapping from $W^{1-1/q,q}(\partial\Omega)$ to $W^{1,q}(\Omega)$, and from $W^{k+1-1/q,q}(\partial\Omega)$ to $W^{k+1,q}(\Omega)$. (See Proposition 3.6 and Proposition 3.7). Using a real interpolation we deduce that $h \mapsto \rho_h$ is a bounded linear mapping from $B_{s-1/q}^{q,r}(\partial\Omega)$ to $B_s^{q,r}(\Omega)$. (See [63, Lemma 22.3], [65, §2.4.2, Theorem] and [67, Corollary 1.111].) Since $B_s^{p,q}(\Omega) \hookrightarrow B_1^{p,q}(\Omega)$ by [67, Theorem 1.97], Proposition 3.6 gives that a solution of the problem (2.1) in $B_s^{p,q}(\Omega)$ is unique and it is also an L^q -solution of the problem (2.1). \square

Proposition 3.9. *Let $1 < p < \infty$. Suppose that one of the following conditions is satisfied:*

- $1 < p \leq 2$,
- $\partial\Omega$ is of class \mathcal{C}^1 ,
- Ω is convex.

If $f \in L^p(\partial\Omega)$, then there exists an L^p -solution of the Neumann problem for the Laplace equation (2.2) if and only if

$$(3.1) \quad \int_{\partial\Omega} f \, d\sigma = 0.$$

The solution is unique up to an additive constant. If φ is an L^p -solution of the problem (2.2) then

$$(3.2) \quad \|M_a(\nabla\varphi)\|_{L^p(\partial\Omega)} \leq C\|f\|_{L^p(\partial\Omega)}$$

with a constant C that does not depend on f .

(See [50, Theorem 5.1], [36, Theorem 1.2], [33, Corollary 2.1.11], [33, Corollary 2.2.14] and [25, Theorem 1.1].)

Proposition 3.10. *Let $k \in \mathbb{N}$, $1 < p, r < \infty$, $\partial\Omega$ be of class \mathcal{C}^k , $f \in L^p(\partial\Omega)$ satisfying (3.1). Let φ be an L^p -solution of the Neumann problem for the Laplace equation (2.2).*

- *Then $\varphi \in B_{1+1/p}^{p,\max(p,2)}(\Omega) \subset W^{1,p}(\Omega) \cap \mathcal{C}(\bar{\Omega})$.*
- *If $\partial\Omega$ is of class $\mathcal{C}^{k,\alpha}$ with $k \in \mathbb{N}$, $0 < \alpha < 1$ and $f \in \mathcal{C}^{k-1,\alpha}(\partial\Omega)$, then $\varphi \in \mathcal{C}^{k,\alpha}(\bar{\Omega})$.*
- *If $\partial\Omega$ is of class $\mathcal{C}^{k,1}(\partial\Omega)$, $1/p < s < k$, $f \in B_{s-1/p}^{p,r}(\partial\Omega)$ then $\varphi \in B_{s+1}^{p,r}(\Omega)$.*
- *If $\partial\Omega$ is of class $\mathcal{C}^{k,1}(\partial\Omega)$, $1/p < s < k$, $s - 1/p$ is not a natural number, $f \in W^{s-1/p,p}(\partial\Omega)$, then $\varphi \in W^{s+1,p}(\Omega)$.*

Proof. The tangential derivative $\partial/\partial\tau$ is a continuous mapping $W^{1,p}(\partial\Omega)$ onto the set of all functions from $L^p(\partial\Omega)$ satisfying (3.1). So, we can choose $h \in W^{1,p}(\partial\Omega)$ such that $\partial h/\partial\tau = f$. According to Proposition 3.3 there exists an L^p -solution ρ of the Dirichlet problem for the Laplace equation (2.1) such that $M_a(\nabla\rho) \in L^p(\partial\Omega)$. Remark that $\rho \in B_{1+1/p}^{p,\max(p,2)}(\Omega) \subset W^{1,p}(\Omega)$ (see Proposition 3.1). According to [10, Theorem 1.1.3] there exists a harmonic function φ such that $\rho - i\varphi$ is a holomorphic function in Ω . Thus $\partial_1\varphi = \partial_2\rho \in L^p(\Omega)$, $\partial_2\varphi = -\partial_1\rho \in L^p(\Omega)$ by [12, Proposition 3.2]. So, $\varphi \in W^{1,p}(\Omega)$ by [41, §1.5.2–1.5.4]. Since $M_a(\nabla\rho), M_a(\nabla\varphi) \in L^p(\partial\Omega)$, there exist nontangential limits of $\nabla\rho$ and $\nabla\varphi$ at almost all points of $\partial\Omega$. (See [28] and [29, Theorem 1].) Clearly

$$\frac{\partial\varphi}{\partial n} = \frac{\partial\rho}{\partial\tau} = \frac{\partial h}{\partial\tau} = f.$$

So, φ is an L^p -solution of the Neumann problem for the Laplace equation (2.2). Other solutions of (2.2) differ from φ by a constant by Proposition 3.9.

Since $\partial_j \varphi \in B_{1/p}^{p, \max(p, 2)}(\Omega)$ by Proposition 3.1, [46, Proposition 7.6] forces that $\varphi \in B_{1+1/p}^{p, \max(p, 2)}(\Omega)$. According to [64, §2.3.3, Remark 4] and [66, §2.7.1, Remark 1] we have $B_{1+1/p}^{p, \max(p, 2)}(\Omega) \subset W^{1, p}(\Omega) \cap \mathcal{C}(\overline{\Omega})$.

Let now $\partial\Omega$ be of class $\mathcal{C}^{k, \alpha}$ with $0 < \alpha < 1$ and $f \in \mathcal{C}^{k-1, \alpha}(\partial\Omega)$. Proposition 3.5 gives that $\rho \in \mathcal{C}^{k, \alpha}(\overline{\Omega})$. Thus $\partial_j \varphi \in \mathcal{C}^{k-1, \alpha}(\overline{\Omega})$ and $\varphi \in \mathcal{C}^{k, \alpha}(\overline{\Omega})$.

Let now $\partial\Omega$ be of class $\mathcal{C}^{k, 1}(\partial\Omega)$, $1/p < s < k$, $f \in B_{s-1/p}^{p, r}(\partial\Omega)$. Then $h \in B_{s+1-1/p}^{p, r}(\partial\Omega)$. Proposition 3.8 gives that $\rho \in B_{s+1}^{p, r}(\Omega)$. Since $\partial_j \varphi \in B_s^{p, r}(\Omega)$, [46, Proposition 7.6] forces that $\varphi \in B_{s+1}^{p, r}(\Omega)$.

Suppose now $\partial\Omega$ is of class $\mathcal{C}^{k, 1}(\partial\Omega)$, $1/p < s < k$, $s-1/p$ is not a natural number and $f \in W^{s-1/p, p}(\partial\Omega)$. Suppose first that $s \in \mathbb{N}$. Then $h \in W^{s+1-1/p, p}(\partial\Omega)$. According to Proposition 3.7 one has $\rho \in W^{s+1, p}(\Omega)$. Since $\partial_j \varphi \in W^{s, p}(\Omega)$, the function $\varphi \in W^{s+1, p}(\Omega)$. If $s \notin \mathbb{N}$, then $f \in W^{s-1/p, p}(\partial\Omega) = B_{s-1/p}^{p, p}(\Omega)$ and thus $\varphi \in B_{s+1}^{p, p}(\Omega) = W^{s+1, p}(\Omega)$. \square

4. REDUCTION OF THE PROBLEM

In this section we show how to reduce the problem (2.3) to a Dirichlet and a Neumann problem for the Laplace equation.

If (\mathbf{u}, ρ) is an L^p - L^q solution of the problem (2.3), then $\rho \in \mathcal{C}^\infty(\Omega)$, $\Delta \rho = 0$ in Ω (see [38, p. 10]). So, ρ is an L^q -solution of the Dirichlet problem for the Laplace equation (2.1). First of all we solve this problem. Then we find $\Phi \in \mathcal{C}(\overline{\Omega}, \mathbb{R}^2)$ such that (Φ, ρ) is a solution of the Stokes system (2.3a) in Ω .

Lemma 4.1. *Let $1 < q < \infty$. Let ρ be an L^q -solution of the Dirichlet problem for the Laplace equation (2.1). Then $\rho \in B_{1/q}^{q, \max(q, 2)}(\Omega)$. Fix $s \in (0, \infty)$, $1 < r, t < \infty$, $0 < \alpha < 1$, $k \in \mathbb{N}$.*

- If $\rho \in B_s^{t, r}(\Omega)$ then there exists $\Phi \in \mathcal{C}^\infty(\Omega, \mathbb{R}^2) \cap B_{s+1}^{t, r}(\Omega, \mathbb{R}^2)$ such that (Φ, ρ) is a solution of the Stokes system (2.3a) in Ω . If, moreover, $B_{s+1}^{t, r}(\mathbb{R}^2) \hookrightarrow B_{1/q+1}^{q, \max(2, q)}(\mathbb{R}^2)$, then $\Phi \in \mathcal{C}(\overline{\Omega}, \mathbb{R}^2) \cap B_{1/q+1}^{q, \max(2, q)}(\Omega, \mathbb{R}^2)$.
- If $\rho \in W^{s, t}(\Omega)$ then there exists $\Phi \in \mathcal{C}^\infty(\Omega, \mathbb{R}^2) \cap W^{s+1, t}(\Omega, \mathbb{R}^2)$ such that (Φ, ρ) is a solution of the Stokes system (2.3a) in Ω . If, moreover, $W^{s+1, t}(\mathbb{R}^2) \hookrightarrow B_{1/q+1}^{q, \max(2, q)}(\mathbb{R}^2)$, then $\Phi \in \mathcal{C}(\overline{\Omega}, \mathbb{R}^2) \cap B_{1/q+1}^{q, \max(2, q)}(\Omega, \mathbb{R}^2)$.
- If $\partial\Omega$ is of class $\mathcal{C}^{k, \alpha}$ and $\rho \in \mathcal{C}^{k, \alpha}(\overline{\Omega})$ then there exists $\Phi \in \mathcal{C}^{k+1, \alpha}(\overline{\Omega}, \mathbb{R}^2)$ such that (Φ, ρ) is a solution of the Stokes system (2.3a) in Ω .

Proof. According to Proposition 3.1 we have $\rho \in B_{1/q}^{q, \max(q, 2)}(\Omega)$.

According to [10, Theorem 1.1.3] there exists a harmonic function ψ on Ω such that $\rho + i\psi$ is a holomorphic function in Ω . Thus $\partial_2 \psi = \partial_1 \rho$, $\partial_1 \psi = -\partial_2 \rho$ in Ω by [12, Proposition 3.1]. If $\rho \in B_s^{t, r}(\Omega)$ then $\partial_j \psi = (-1)^j \partial_{3-j} \rho \in B_{s-1}^{t, r}(\Omega)$, and therefore $\psi \in B_s^{t, r}(\Omega)$ by [46, Proposition 7.6]. We extend ψ as a function from $B_s^{t, r}(\mathbb{R}^2)$ with compact support. If $\rho \in W^{s, t}(\Omega)$ then $\partial_j \psi = (-1)^j \partial_{3-j} \rho \in W^{s-1, t}(\Omega)$, and therefore $\psi \in W^{s, t}(\Omega)$. According to [3, Theorem 5.24] we can extend ψ as a function from $W^{s-1, t}(\mathbb{R}^2)$ with compact support. If $\rho \in \mathcal{C}^{k, \alpha}(\overline{\Omega})$ then $\partial_j \psi \in \mathcal{C}^{k-1, \alpha}(\overline{\Omega})$. Since $\psi \in \mathcal{C}^\infty(\Omega)$, $\psi \in W^{1, 3}(\Omega)$ by [41, §1.5.2-§1.5.4]. Sobolev's embedding theorem gives that $\psi \in \mathcal{C}(\overline{\Omega})$. Thus $\psi \in \mathcal{C}^{k, \alpha}(\overline{\Omega})$. If $\partial\Omega$ is of

class $\mathcal{C}^{k,\alpha}$ then we can extend ψ as a function from $\mathcal{C}^{k,\alpha}(\mathbb{R}^2)$ with compact support. (See [26, Lemma 6.37].)

Remember that

$$h_\Delta(\mathbf{x}) = \frac{1}{2\pi} \ln \frac{1}{|\mathbf{x}|}$$

is a fundamental solution of the Laplace equation, i.e. $-\Delta h_\Delta = \delta_0$, where δ_0 is the unit mass concentrated in 0 (see [11, Theorem 2.4.1.2]). Since ψ has compact support we can define $\Phi = (\Phi_1, \Phi_2)$ by the convolution

$$\Phi_1 = -\psi * (\partial_2 h_\Delta), \quad \Phi_2 = \psi * (\partial_1 h_\Delta).$$

We have in Ω

$$\nabla \cdot \Phi = \partial_1(-\psi * \partial_2 h_\Delta) + \partial_2(\psi * \partial_1 h_\Delta) = -\partial_1 \partial_2(\psi * h_\Delta) + \partial_1 \partial_2(\psi * h_\Delta) = 0,$$

$$\Delta \Phi_1 = \Delta(-\psi * \partial_2 h_\Delta) = \partial_2[\psi * (-\Delta h_\Delta)] = \partial_2(\psi * \delta_0) = \partial_2 \psi = \partial_1 \rho,$$

$$\Delta \Phi_2 = \Delta(\psi * \partial_1 h_\Delta) = \partial_1[\psi * (\Delta h_\Delta)] = \partial_1[\psi * (-\delta_0)] = -\partial_1 \psi = \partial_2 \rho,$$

Since (Φ, ρ) is a solution of the Stokes system in Ω , we have $\Phi \in \mathcal{C}^\infty(\Omega, \mathbb{R}^2)$ by [55, §1.2].

If $\psi \in B_s^{t,r}(\mathbb{R}^2)$ then $\Phi_j \in B_{s+1}^{t,r}(\Omega)$ by [46, Theorem 3.3]. If, moreover, $B_{s+1}^{t,r}(\mathbb{R}^2) \hookrightarrow B_{1/q+1}^{q,\max(2,q)}(\mathbb{R}^2)$, then $\Phi \in \mathcal{C}(\bar{\Omega}, \mathbb{R}^2)$ by [64, §2.8.1, Theorem].

If $\psi \in W^{s,t}(\mathbb{R}^2)$ then $\Phi_j \in W^{s+1,t}(\Omega)$ by [46, Theorem 3.3]. If, moreover, $W^{s+1,t}(\Omega) \hookrightarrow B_{1/q+1}^{q,\max(2,q)}(\Omega)$, then $\Phi \in \mathcal{C}(\bar{\Omega}, \mathbb{R}^2)$ by [64, §2.8.1, Theorem].

Let now $\psi \in \mathcal{C}^{k,\alpha}(\mathbb{R}^2)$. If β is a multiindex with $|\beta| \leq k$ then $\partial^\beta \Phi_j = (\partial^\beta \psi) * (-1)^j (\partial_{3-j} h_\Delta) \in \mathcal{C}^{1,\alpha}(\bar{\Omega})$ by [32, Theorem 10.1.1]. Hence $\Phi \in \mathcal{C}^{k+1,\alpha}(\bar{\Omega})$. \square

Let now Φ be a vector function from Lemma 4.1. Then (\mathbf{u}, ρ) is an L^p - L^q -solution of the problem (2.3) if and only if for $\tilde{\mathbf{v}} = \mathbf{u} - \Phi$, $t \equiv 0$ the couple $(\tilde{\mathbf{v}}, t)$ is an L^p - L^q -solution of the problem

$$(4.1) \quad -\Delta \tilde{\mathbf{v}} + \nabla t = 0, \quad \nabla \cdot \tilde{\mathbf{v}} = 0 \quad \text{in } \Omega, \quad \tilde{\mathbf{v}} \cdot \boldsymbol{\tau} = \tilde{f} := g - \Phi \cdot \boldsymbol{\tau}, \quad t = 0 \quad \text{on } \partial\Omega.$$

We want to represent \tilde{v}_j as derivatives of an L^p -solution of the Neumann problem for the Laplace equation with boundary condition \tilde{f} . But the Neumann problem is solvable only for a boundary condition \tilde{f} with $\int_{\partial\Omega} \tilde{f} \, d\sigma = 0$. So, first we must show that there exists a solution of the problem (4.1) also for some \tilde{f} with $\int_{\partial\Omega} \tilde{f} \, d\sigma \neq 0$.

Lemma 4.2. *Define $\mathbf{w}(\mathbf{x}) = (-x_2, x_1)[\int_\Omega 2 \, d\mathbf{y}]^{-1}$. Then*

$$\Delta \mathbf{w} = 0, \quad \nabla \cdot \mathbf{w} = 0 \quad \text{in } \Omega,$$

$$\int_{\partial\Omega} \boldsymbol{\tau} \cdot \mathbf{w} \, d\sigma = 1.$$

Proof. Easy calculation yields $\Delta \mathbf{w} = 0$, $\nabla \cdot \mathbf{w} = 0$ in Ω . The Divergence theorem gives

$$\int_{\partial\Omega} \boldsymbol{\tau} \cdot \mathbf{w} \, d\sigma = \int_\Omega [\partial_2(x_2) + \partial_1(x_1)] \left[\int_\Omega 2 \, d\mathbf{y} \right]^{-1} \, d\mathbf{x} = 1.$$

\square

Let now \mathbf{w} be from Lemma 4.2, $t \equiv 0$. Then $(\tilde{\mathbf{v}}, t)$ is an L^p - L^q -solution of the problem (4.1) if and only if (\mathbf{v}, t) is an L^p - L^q -solution of the problem

$$(4.2) \quad -\Delta \mathbf{v} + \nabla t = 0, \quad \nabla \cdot \mathbf{v} = 0 \quad \text{in } \Omega, \quad \mathbf{v} \cdot \boldsymbol{\tau} = f, \quad t = 0 \quad \text{on } \partial\Omega,$$

where

$$(4.3) \quad \mathbf{v} = \tilde{\mathbf{v}} - \mathbf{w} \int_{\partial\Omega} \tilde{f} \, d\sigma \left[\int_{\partial\Omega} 1 \, d\sigma \right]^{-1}, \quad f = \tilde{f} - \int_{\partial\Omega} \tilde{f} \, d\sigma \left[\int_{\partial\Omega} 1 \, d\sigma \right]^{-1}.$$

Lemma 4.3. *Let $1 < p < \infty$, $f \in L^p(\partial\Omega)$, and φ be an L^p -solution of the Neumann problem for the Laplace equation (2.2). Define $v_1 = -\partial_2\varphi$, $v_2 = \partial_1\varphi$, $\mathbf{v} = (v_1, v_2)$, $t \equiv 0$. Then (\mathbf{v}, t) is an L^p - L^q -solution of the problem (4.2).*

Proof. We have $-\Delta \mathbf{v} + \nabla t = (\partial_2\Delta\varphi, -\partial_1\Delta\varphi) = 0$, $\nabla \cdot \mathbf{v} = -\partial_1\partial_2\varphi + \partial_2\partial_1\varphi = 0$,

$$f = n_1\partial_1\varphi + n_2\partial_2\varphi = \boldsymbol{\tau} \cdot (-\partial_2\varphi, \partial_1\varphi) = \boldsymbol{\tau} \cdot \mathbf{v}.$$

□

5. UNIQUE SOLVABILITY OF THE PROBLEM

Proposition 5.1. *Let $1 < p, q < \infty$, (\mathbf{u}, ρ) be an L^p - L^q -solution of the problem (2.3) with $h \equiv 0$. Suppose that one of the following conditions is satisfied:*

- $q \geq 2$,
- $\partial\Omega$ is of class C^1 ,
- Ω is convex,
- $M_\alpha(\nabla\rho) \in L^q(\partial\Omega)$.

Then $\rho \equiv 0$. If moreover $\mathbf{g} \equiv 0$, then $\mathbf{u} \equiv 0$.

Proof. Since (\mathbf{u}, ρ) is a solution of the Stokes system (2.3a), we have $\rho \in C^\infty(\Omega)$, $\Delta\rho = 0$ in Ω (see [38, p. 10]). So, ρ is an L^q -solution of the Dirichlet problem for the Laplace equation $\Delta\rho = 0$ in Ω , $\rho = 0$ on $\partial\Omega$. Hence, $\rho \equiv 0$ by Proposition 3.2 and Proposition 3.3.

Let now $\mathbf{g} \equiv 0$. First we show that $u_2 + iu_1$ is a holomorphic function in Ω . Since $\rho \equiv 0$, we have $\Delta \mathbf{u} = 0$. According to [10, Theorem 1.1.3] there exists a harmonic function v_2 on Ω such that $u_2 + iv_2$ is a holomorphic function in Ω . We have $\partial_2 u_2 = -\partial_1 v_2$ by [12, Proposition 3.1]. Since $\nabla \cdot \mathbf{u} = 0$, we have $\partial_2 u_2 = -\partial_1 u_1$. Hence $\partial_1(u_1 - v_2) = 0$. Thus there exists a function $w(x_2)$ such that $u_1(\mathbf{x}) - v_2(\mathbf{x}) = w(x_2)$. Since $w'' = \Delta u_1 - \Delta v_2 = 0$, there exist constants c_1, c_2 such that $w(x_2) = c_1 x_2 + c_2$. Therefore there exists a constant c such that the function $u_2 + i(u_1 + cx_2)$ is holomorphic in Ω . According to [68, Theorem 1.12] there is a sequence of domains Ω_j with boundaries of class C^∞ such that

- $\overline{\Omega}_j \subset \Omega$.
- There are $a > 0$ and homeomorphisms $\Lambda_j : \partial\Omega \rightarrow \partial\Omega_j$, such that $\Lambda_j(y) \in \Gamma_a(y)$ for each j and each $y \in \partial\Omega$ and $\sup\{|y - \Lambda_j(y)|; y \in \partial\Omega\} \rightarrow 0$ as $j \rightarrow \infty$.
- There are positive functions ω_j on $\partial\Omega$ bounded away from zero and infinity uniformly in j such that for any measurable set $E \subset \partial\Omega$, $\int_E \omega_j \, d\sigma = \sigma(\Lambda_j(E))$, and so that $\omega_j \rightarrow 1$ pointwise a.e. and in every $L^s(\partial\Omega)$, $1 \leq s < \infty$.
- The normal vectors to Ω_j , $n(\Lambda_j(y))$, converge pointwise a.e. and in every $L^s(\partial\Omega)$, $1 \leq s < \infty$, to $n(y)$.

Since $u_2 + i(u_1 + cx_2)$ is holomorphic, the Cauchy integral

$$\int_{\partial\Omega_j} [u_2 + i(u_1 + cx_2)] d(x_1 + ix_2) = 0,$$

i.e.

$$\int_{\partial\Omega_j} (u_1 + cx_2, u_2) \cdot \mathbf{n} \, d\sigma = 0, \quad \int_{\partial\Omega_j} (u_1 + cx_2, u_2) \cdot \tau \, d\sigma = 0.$$

Letting $j \rightarrow \infty$ we obtain by virtue the Lebesgue lemma

$$0 = \int_{\partial\Omega} (u_1 + cx_2, u_2) \cdot \tau \, d\sigma.$$

Since $\mathbf{u} \cdot \tau = 0$ on $\partial\Omega$, we obtain by the Green formula

$$0 = \int_{\partial\Omega} (cx_2, 0) \cdot \tau \, d\sigma = - \int_{\Omega} \partial_2(cx_2) \, dx = -c \int_{\Omega} 1 \, dx.$$

Thus $c = 0$ and $u_2 + iu_1$ is a holomorphic function.

Since $u_2 + iu_1$ is a holomorphic function, there exists a holomorphic function $\psi_1 + i\psi_2$ such that $(\psi_1 + i\psi_2)' = u_2 + iu_1$ (see [12, Theorem 8.5]). Thus $u_2 + iu_1 = \partial_1(\psi_1 + i\psi_2)$. Since $\partial_1\psi_2 = -\partial_2\psi_1$, we have $\nabla\psi_1 = (u_2, -u_1)$. Hence ψ_1 is an L^p -solution of the Neumann problem $\Delta\psi_1 = 0$ in Ω , $\partial\psi_1/\partial n = 0$ on $\partial\Omega$. Proposition 3.9 gives that ψ_1 is constant. Since $\nabla\psi_1 = (u_2, -u_1)$, we infer that $\mathbf{u} \equiv 0$. \square

Theorem 5.2. *Let $1 < p, q < \infty$. Suppose that one of the following conditions is satisfied:*

- $p \leq 2 \leq q$,
- $\partial\Omega$ is of class \mathcal{C}^1 ,
- Ω is convex.

If $g \in L^p(\partial\Omega)$, $h \in L^q(\partial\Omega)$, then there exists a unique L^p - L^q -solution (\mathbf{u}, ρ) of the problem (2.3). Moreover, $\mathbf{u} \in B_{1/p}^{\max(p,2),p}(\Omega, \mathbb{R}^2)$ and $\rho \in B_{1/q}^{\max(q,2),q}(\Omega)$.

Proof. The uniqueness follows from Proposition 5.1.

According to Proposition 3.2 there exists a unique L^q -solution ρ of the Dirichlet problem for the Laplace equation (2.1), $\rho \in B_{1/q}^{\max(q,2),q}(\Omega)$. According to Lemma 4.1 there exists $\Phi \in \mathcal{C}^\infty(\Omega, \mathbb{R}^2) \cap \mathcal{C}(\bar{\Omega}, \mathbb{R}^2)$ such that (Φ, ρ) is a solution of the Stokes system (2.3a) in Ω and $\Phi \in \mathcal{C}(\bar{\Omega}) \cap B_{1+1/q}^{q,\max(q,2)}(\Omega)$. Let $\mathbf{w} = (x_2, x_1)[\int_{\Omega} 1 \, dy]^{-1}$ be the vector function from Lemma 4.2. Define

$$\tilde{\mathbf{f}} := g - \Phi \cdot \tau, \quad f = \tilde{f} - \int_{\partial\Omega} \tilde{f} \, d\sigma \left[\int_{\partial\Omega} 1 \, d\sigma \right]^{-1}.$$

According to Proposition 3.9 there exists an L^p -solution φ of the Neumann problem for the Laplace equation (2.2). Proposition 3.1 gives that $\partial_j\varphi \in B_{1/p}^{p,\max(p,2)}(\Omega)$. Define $v_1 = -\partial_2\varphi$, $v_2 = \partial_1\varphi$, $\mathbf{v} = (v_1, v_2)$, $t \equiv 0$. Then (\mathbf{v}, t) is an L^p - L^q -solution of the problem (4.2) by Lemma 4.3. Define

$$\tilde{\mathbf{v}} = \mathbf{v} + \mathbf{w} \int_{\partial\Omega} \tilde{f} \, d\sigma \left[\int_{\partial\Omega} 1 \, d\sigma \right]^{-1}.$$

Then $(\tilde{\mathbf{v}}, t)$ is an L^p - L^q -solution of the problem (4.1). Define $\mathbf{u} = \tilde{\mathbf{v}} + \Phi$. Then (\mathbf{u}, ρ) is an L^p - L^q -solution of the Navier problem (2.3). \square

Theorem 5.3. *Let $1 < p, q < \infty$. Suppose that one of the following conditions is satisfied:*

- $p, q \leq 2$,
- $\partial\Omega$ is of class \mathcal{C}^1 ,
- Ω is convex.

If $g \in L^p(\partial\Omega)$, $h \in W^{1,q}(\partial\Omega)$, then there exists a unique L^p - L^q -solution (\mathbf{u}, ρ) of the problem (2.3) such that $M_a(\nabla\rho) \in L^q(\partial\Omega)$. Moreover, $\mathbf{u} \in B_{1/p}^{\max(p,2),p}(\Omega, \mathbb{R}^2)$ and $\rho \in B_{1+1/q}^{\max(q,2),q}(\Omega) \cap \mathcal{C}^{0,(q-1)/q}(\overline{\Omega})$.

Proof. The uniqueness follows from Proposition 5.1.

According to the Sobolev embedding theorem [41, Chapter I, §1.8.1] we have $h \in \mathcal{C}^{0,(p-1)/p}(\partial\Omega)$. Put $r = \max(q, 2)$. According to Theorem 5.2 there exists an L^p - L^r -solution (\mathbf{u}, ρ) of the problem (2.3) and $\mathbf{u} \in B_{1/p}^{\max(p,2),p}(\Omega, \mathbb{R}^2)$. The function ρ is an L^r -solution of the Dirichlet problem for the Laplace equation $\Delta\rho = 0$ in Ω , $\rho = h$ on $\partial\Omega$. Proposition 3.3 and Proposition 3.4 give that $M_a(\nabla\rho) \in L^q(\partial\Omega)$ and $\rho \in B_{1+1/q}^{\max(q,2),q}(\Omega) \cap \mathcal{C}^{0,(q-1)/q}(\overline{\Omega})$. \square

Lemma 5.4. *Let (\mathbf{u}, ρ) be an L^p - L^q solution of the problem (2.3) with $1 < p, q, r < \infty$.*

- *Let $0 \leq \alpha < 1$, $h \in \mathcal{C}^{0,\alpha}(\partial\Omega)$. If $\alpha \leq 1/2$ or $\partial\Omega$ is of class \mathcal{C}^1 or Ω is convex, then $\rho \in \mathcal{C}^{0,\alpha}(\overline{\Omega})$.*
- *Let $k \in \mathbb{N}$, $0 < \alpha < 1$. If $\partial\Omega$ is of class $\mathcal{C}^{k,\alpha}$ and $h \in \mathcal{C}^{k,\alpha}(\partial\Omega)$, then $\rho \in \mathcal{C}^{k,\alpha}(\overline{\Omega})$.*
- *Let $1/q < s < 1 + 1/q$. Suppose that $q = 2$ or $\partial\Omega$ is of class \mathcal{C}^1 or Ω is convex. If $h \in B_{s-1/q}^{q,r}(\partial\Omega)$ then $\rho \in B_s^{q,r}(\Omega)$. If $h \in W^{s-1/q,q}(\partial\Omega)$ then $\rho \in W^{s,q}(\Omega)$.*
- *Let $\partial\Omega$ be of class $\mathcal{C}^{k,1}(\partial\Omega)$ where $k \in \mathbb{N}$, $1 < q < \infty$. If $1 < s \leq k + 1$, $s - 1/q \notin \mathbb{N}_0$, $h \in W^{s-1/q,q}(\partial\Omega)$ then $\rho \in W^{s,q}(\Omega)$.*
- *Let $\partial\Omega$ be of class $\mathcal{C}^{k,1}(\partial\Omega)$ where $k \in \mathbb{N}$, $1 < q, r < \infty$, $1 < s < k + 1$. If $h \in B_{s-1/q}^{q,r}(\partial\Omega)$ then $\rho \in B_s^{q,r}(\Omega)$.*

Proof. ρ is an L^q -solution of the Dirichlet problem $\Delta\rho = 0$ in Ω , $\rho = h$ on $\partial\Omega$. The rest is a consequence of Proposition 3.4, Proposition 3.5, Proposition 3.6, Proposition 3.7, Proposition 3.8 and the fact that $W^{s,p}(\Omega) = B_s^{p,p}(\Omega)$, $W^{s,p}(\partial\Omega) = B_s^{p,p}(\partial\Omega)$ for non-integer s . \square

Remark 5.5. There exists $h \in \mathcal{C}^{0,1}(\partial\Omega)$ such that for each L^p - L^q -solution (\mathbf{u}, ρ) of the problem (2.3) we have $\rho \notin \mathcal{C}^{0,1}(\overline{\Omega})$.

Proof. According to [4, Theorem 2] there is $h \in \mathcal{C}^{0,1}(\partial\Omega)$ such that it does not exist $\rho \in \mathcal{C}^{0,1}(\overline{\Omega}) \cap \mathcal{C}^2(\Omega)$ with $\Delta\rho = 0$ in Ω , $\rho = h$ on $\partial\Omega$. Let now (\mathbf{u}, ρ) be an L^p - L^q -solution of the problem (2.3). Then ρ is an L^q -solution of the Dirichlet problem for the Laplace equation (2.1) (see [38, p. 10]). Thus $\rho \notin \mathcal{C}^{0,1}(\overline{\Omega})$. \square

6. SOLUTIONS IN SOBOLEV AND BESOV SPACES

Let $1 < p, q, r, \beta < \infty$, $1/p < t < \infty$, $1/q < s < \infty$. In this section we study the problem

$$(6.1a) \quad -\Delta\mathbf{u} + \nabla\rho = \mathbf{F}, \quad \nabla \cdot \mathbf{u} = G \quad \text{in } \Omega,$$

$$(6.1b) \quad \mathbf{u} \cdot \boldsymbol{\tau} = g, \quad \rho = h \quad \text{on } \partial\Omega,$$

where $\mathbf{u} \in W^{t,p}(\Omega, \mathbb{R}^2) \cup B_t^{p,\beta}(\Omega, \mathbb{R}^2)$ and $\rho \in W^{s,q}(\Omega) \cup B_s^{q,r}(\Omega)$. The nonhomogeneous Stokes system (6.1a) is fulfilled in the sense of distributions and the boundary conditions (6.1b) are fulfilled in the sense of traces.

Proposition 6.1. *Let $\partial\Omega$ be of class \mathcal{C}^1 , $1 < p, q, r, \beta < \infty$, $1/p < t < \infty$, $1/q < s < \infty$. Suppose that $\mathbf{u} \in W^{t,p}(\Omega, \mathbb{R}^2) \cup B_t^{p,\beta}(\Omega, \mathbb{R}^2)$ and $\rho \in W^{s,q}(\Omega) \cup B_s^{q,r}(\Omega)$. If (\mathbf{u}, ρ) is a solution of the problem (2.3) with $g \equiv 0$, $h \equiv 0$ then $\mathbf{u} \equiv 0$, $\rho \equiv 0$.*

Proof. Since ρ is a solution of the problem $\Delta\rho = 0$ in Ω , $\rho = 0$ on $\partial\Omega$, Proposition 3.6 gives that $\rho \equiv 0$. Therefore $\Delta\mathbf{u} = \nabla\rho = 0$. Denote by $\boldsymbol{\Psi}$ the trace of \mathbf{u} . Proposition 3.6 forces that \mathbf{u} is an L^p -solution of the problem $\Delta\mathbf{u} = 0$ in Ω , $\mathbf{u} = \boldsymbol{\Psi}$ on $\partial\Omega$. Hence (\mathbf{u}, ρ) is an L^p - L^q solution of the Navier problem (2.3). According to Proposition 5.1 we have $\mathbf{u} \equiv 0$. \square

Theorem 6.2. *Let $k \in \mathbb{N}$, $1 < p, q < \infty$, $\partial\Omega$ be of class $\mathcal{C}^{k,1}$, $1/q < s < k + 1$, $s - 1/q \notin \mathbb{N}_0$, $1/p < t < k$, $t - 1/p \notin \mathbb{N}_0$, and $t \leq s + 1$, $p \leq q$. If $t = s + 1$ suppose moreover that $p = q$. If $g \in W^{t-1/p,p}(\partial\Omega)$, $h \in W^{s-1/q,q}(\partial\Omega)$, then there exists a unique solution $(\mathbf{u}, \rho) \in W^{t,p}(\Omega, \mathbb{R}^2) \times W^{s,q}(\Omega)$ of the Navier problem (2.3). Remark that (\mathbf{u}, ρ) is a unique L^p - L^q solution of (2.3).*

Proof. Uniqueness follows from Proposition 6.1.

According to Theorem 5.2 there exists a unique L^p - L^q solution (\mathbf{u}, ρ) of (2.3). Lemma 5.4 gives that $\rho \in W^{s,q}(\Omega)$. For the regularity of \mathbf{u} we follow the construction of a solution in §4. Let $\mathbf{w}(\mathbf{x}) = (-x_2, x_1)[\int_{\Omega} 2 \, d\mathbf{y}]^{-1}$, and $\boldsymbol{\Phi}$, $\tilde{\mathbf{v}}$, \mathbf{v} , f , \tilde{f} , φ have meaning from §4. Since $W^{s,q}(\mathbb{R}^2) \hookrightarrow B_{1/q}^{q,\max(2,q)}(\mathbb{R}^2)$ (see [64, §2.3.3, Remark 4]), Lemma 4.1 gives that $\boldsymbol{\Phi} \in W^{s+1,q}(\Omega)$. Hence $\boldsymbol{\Phi} \in W^{t,p}(\Omega)$ by [53, Chap. 2, §5.4] and Hölder's inequality. Moreover, $W^{t,p}(\Omega) \hookrightarrow W^{t-1/p,p}(\partial\Omega)$ by [27, Theorem 1.5.1.2]. Thus $\tilde{f} = g - \boldsymbol{\Phi} \cdot \boldsymbol{\tau} \in W^{t-1/p,p}(\partial\Omega)$ and therefore $f \in W^{t-1/p,p}(\partial\Omega)$. Since φ is a solution of the Neumann problem for the Laplace equation with the boundary condition f , Proposition 3.10 gives that $\varphi \in W^{t+1,p}(\Omega)$. Hence $\mathbf{v} = (-\partial_2\varphi, \partial_1\varphi) \in W^{t,p}(\Omega, \mathbb{R}^2)$. Since \mathbf{u} is a linear combination of $\boldsymbol{\Phi}$, \mathbf{w} and \mathbf{v} , we deduce that $\mathbf{u} \in W^{t,p}(\Omega)$. \square

Theorem 6.3. *Let $k \in \mathbb{N}$, $1 < p, q, r, \beta < \infty$, $\partial\Omega$ be of class $\mathcal{C}^{k,1}$, $1/q < s < k + 1$, $1/p < t < k$, and $t \leq s + 1$, $p \leq q$. If $t = s + 1$ suppose moreover that $p = q$ and $r \leq \beta$. If $g \in B_{t-1/p}^{p,\beta}(\partial\Omega)$, $h \in B_{s-1/q}^{q,r}(\partial\Omega)$, then there exists a unique solution $(\mathbf{u}, \rho) \in B_t^{p,\beta}(\Omega, \mathbb{R}^2) \times B_s^{q,r}(\Omega)$ of the Navier problem (2.3). Remark that (\mathbf{u}, ρ) is a unique L^p - L^q solution of (2.3).*

Proof. Define the operator U by

$$U[\mathbf{u}, p] = [\mathbf{u} \cdot \boldsymbol{\tau}|_{\partial\Omega}, p|_{\partial\Omega}].$$

By virtue of Theorem 6.2 there exists $\epsilon > 0$ such that

$$\begin{aligned} U : \{[\mathbf{u}, p] \in W^{t+\epsilon,p}(\Omega, \mathbb{R}^2) \times W^{s+\epsilon,q}(\Omega); \nabla\mathbf{u} = \nabla p, \nabla \cdot \mathbf{u} = 0\} \\ \rightarrow W^{t+\epsilon-1/p}(\partial\Omega) \times W^{s+\epsilon-1/q}(\partial\Omega), \\ U : \{[\mathbf{u}, p] \in W^{t-\epsilon,p}(\Omega, \mathbb{R}^2) \times W^{s-\epsilon,q}(\Omega); \nabla\mathbf{u} = \nabla p, \nabla \cdot \mathbf{u} = 0\} \\ \rightarrow W^{t-\epsilon-1/p}(\partial\Omega) \times W^{s-\epsilon-1/q}(\partial\Omega) \end{aligned}$$

are isomorphisms. The real interpolation gives the theorem. (See [63, Lemma 22.3], [65, §2.4.2] and [67, Corollary 1.111].) \square

Theorem 6.4. *Let $k \in \mathbb{N}$, $1 < p, q, r, \beta < \infty$, $\partial\Omega$ be of class $\mathcal{C}^{k,1}$, $1/q < s < k + 1$, $1/p < t < k$, and $t \leq s + 1$, $p \leq q$. If $t = s + 1$ suppose moreover that $p = q$ and $r \leq \beta$. If $g \in B_{t-1/p}^{p,\beta}(\partial\Omega)$, $h \in B_{s-1/q}^{q,r}(\partial\Omega)$, $\mathbf{F} \in B_{s-1}^{q,r}(\Omega, \mathbb{R}^2)$, $G \in B_s^{q,r}(\Omega)$, then there exists a unique solution $(\mathbf{u}, \rho) \in B_t^{p,\beta}(\Omega, \mathbb{R}^2) \times B_s^{q,r}(\Omega)$ of the Navier problem (6.1).*

Proof. Fix $R > 0$ such that $\bar{\Omega} \subset B(0; R) = \{x; |x| < R\}$. If $l \in \mathbb{N}$, $\tilde{G} \in W^{l-1,q}(B(0; R))$ and $\tilde{\mathbf{F}} \in W^{l-2,q}(B(0; R), \mathbb{R}^2)$, $\int_{B(0; R)} \tilde{G} \, dx = 0$, then there exists a unique $\mathbf{v} \in W^{l,q}(B(0; R), \mathbb{R}^2)$, $\phi \in W^{l-1,q}(B(0; R))$ such that

$$\begin{aligned} -\Delta \mathbf{v} + \nabla \phi &= \tilde{\mathbf{F}}, & \nabla \cdot \tilde{\mathbf{v}} &= G \quad \text{in } B(0; R), \\ \mathbf{v} &= 0 \quad \text{on } \partial B(0; R), & \int_{B(0; R)} \phi \, dx &= 0. \end{aligned}$$

Moreover,

$$\|\mathbf{v}\|_{W^{l,q}(B(0; R))} + \|\phi\|_{W^{l-1,q}(B(0; R))} \leq C \left(\|\tilde{\mathbf{F}}\|_{W^{l-2,q}(B(0; R))} + \|\tilde{G}\|_{W^{l-1,q}(B(0; R))} \right)$$

where a constant C does not depend on $\tilde{\mathbf{F}}$ and \tilde{G} . (See [24, Theorem 2.1] and [23, Theorem 2.1].) Using the interpolation we infer that for $\tilde{\mathbf{F}} \in B_{s-1}^{q,r}(B(0; R); \mathbb{R}^2)$ and $\tilde{G} \in B_s^{q,r}(B(0; R))$ we have $\mathbf{v} \in B_{s+1}^{q,r}(B(0; R); \mathbb{R}^2)$, $\phi \in B_s^{q,r}(B(0; R))$. (See [63, Lemma 22.3], [67, Corollary 1.111] and [67, Theorem 1.122].)

We can choose $\tilde{\mathbf{F}} \in B_{s-1}^{q,r}(B(0; R); \mathbb{R}^2)$, $\tilde{G} \in B_s^{q,r}(B(0; R))$ such that $\int_{B(0; R)} \tilde{G} \, dx = 0$, and $\tilde{\mathbf{F}} = \mathbf{F}$, $\tilde{G} = G$ in Ω . We have proved that there exist $\mathbf{v} \in B_{s+1}^{q,r}(B(0; R), \mathbb{R}^2)$, $\phi \in B_s^{q,r}(B(0; R))$ such that

$$-\Delta \mathbf{v} + \nabla \phi = \tilde{\mathbf{F}}, \quad \nabla \cdot \mathbf{v} = \tilde{G} \quad \text{in } B(0; R).$$

One has $\mathbf{v} \in B_{s+1}^{q,r}(\Omega, \mathbb{R}^2) \subset B_{s+1}^{q,\beta}(\Omega, \mathbb{R}^2) \subset B_t^{p,\beta}(\Omega, \mathbb{R}^2)$ by [66, §2.3.2, Proposition 2] and [67, Theorem 1.97].

According to [27, Theorem 1.5.1.2] there exist $s(1)$, $s(2)$ such that $s(1) < s < s(2)$ and the trace γ_Ω is a bounded linear operator from $W^{s(j),q}(\Omega) = B_{s(j)}^{q,q}(\Omega)$ to $W^{s(j)-1/q,q}(\partial\Omega) = B_{s(j)-1/q}^{q,q}(\partial\Omega)$. By virtue of the interpolation we deduce that $\gamma_\Omega : B_s^{q,r}(\Omega) \rightarrow B_{s-1/q}^{q,r}(\partial\Omega)$ is a bounded linear operator. (See [63, Lemma 22.3] and [67, Corollary 1.111].) By the same way we prove that $\gamma_\Omega : B_t^{p,\beta}(\Omega) \rightarrow B_{t-1/p}^{p,\beta}(\partial\Omega)$ is a bounded linear operator. So, $\mathbf{v} \in B_{t-1/p}^{p,\beta}(\partial\Omega, \mathbb{R}^2)$, $\phi \in B_{s-1/q}^{q,r}(\partial\Omega)$.

According to Theorem 6.3 there exists a solution $(\mathbf{w}, \psi) \in B_t^{p,\beta}(\Omega, \mathbb{R}^2) \times B_s^{q,r}(\Omega)$ of the Navier problem

$$\begin{aligned} -\Delta \mathbf{w} + \nabla \psi &= 0, & \nabla \cdot \mathbf{w} &= 0 \quad \text{in } \Omega, \\ \mathbf{w} \cdot \boldsymbol{\tau} &= g - \mathbf{v} \cdot \boldsymbol{\tau}, & \psi &= h - \phi \quad \text{on } \partial\Omega. \end{aligned}$$

Clearly, $(\mathbf{u}, \rho) := (\mathbf{w} + \mathbf{v}, \psi + \phi) \in B_t^{p,\beta}(\Omega, \mathbb{R}^2) \times B_s^{q,r}(\Omega)$ is a solution of the Navier problem (6.1). The uniqueness of a solution follows from Proposition 6.1. \square

Theorem 6.5. *Let $k \in \mathbb{N}$, $1 < p, q < \infty$, $\partial\Omega$ be of class $\mathcal{C}^{k,1}$, $1/q < s < k + 1$, $s - 1/q \notin \mathbb{N}_0$, $1/p < t < k$, $t - 1/p \notin \mathbb{N}_0$, and $t \leq s + 1$, $p \leq q$. If $t = s + 1$ suppose moreover that $p = q$. If $g \in W^{t-1/p,p}(\partial\Omega)$, $h \in W^{s-1/q,q}(\partial\Omega)$, $\mathbf{F} \in W^{s-1,q}(\Omega, \mathbb{R}^2)$,*

$G \in W^{s,q}(\Omega)$, then there exists a unique solution $(\mathbf{u}, \rho) \in W^{t,p}(\Omega, \mathbb{R}^2) \times W^{s,q}(\Omega)$ of the Navier problem (6.1).

Proof. According to Theorem 6.4, [24, Theorem 2.1] and [23, Theorem 2.1] there exist $(\mathbf{v}, \rho_1) \in W^{s+1,q}(\Omega, \mathbb{R}^2) \times W^{s,q}(\Omega)$ such that $-\Delta \mathbf{v} + \nabla \rho_1 = \mathbf{F}$, $\nabla \cdot \mathbf{v} = G$ in Ω . Since $p \leq q$, and $t \leq s+1$ we have $\mathbf{v} \in W^{t,p}(\Omega, \mathbb{R}^2)$. So, $\mathbf{v} \in W^{t-1/p,p}(\Omega, \mathbb{R}^2)$, $\rho_1 \in W^{s-1/q,q}(\partial\Omega)$ by [27, Theorem 1.5.1.2].

According to Theorem 6.2 there exists a solution $(\mathbf{w}, \rho_2) \in W^{t,p}(\Omega, \mathbb{R}^2) \times W^{s,q}(\Omega)$ of the problem

$$\begin{aligned} -\Delta \mathbf{w} + \nabla \rho_2 &= 0, \quad \nabla \cdot \mathbf{w} = 0 \quad \text{in } \Omega, \\ \mathbf{w} \cdot \boldsymbol{\tau} &= g - \mathbf{v} \cdot \boldsymbol{\tau}, \quad \rho_2 = h - \rho_1 \quad \text{on } \partial\Omega. \end{aligned}$$

Clearly, $(\mathbf{u}, \rho) := (\mathbf{v} + \mathbf{w}, \rho_1 + \rho_2) \in W^{t,p}(\Omega, \mathbb{R}^2) \times W^{s,q}(\Omega)$ is a solution of (6.1). The uniqueness of a solution follows from Proposition 6.1. \square

7. CLASSICAL SOLUTIONS

Theorem 7.1. *Let $k \in N$, $0 < \alpha, \gamma < 1$, $\partial\Omega$ be of class $\mathcal{C}^{k,\gamma}$. Let $h \in \mathcal{C}^0(\partial\Omega)$, $g \in \mathcal{C}^{0,\alpha}(\partial\Omega)$. Then there exist unique $\mathbf{u} \in \mathcal{C}^0(\bar{\Omega}, \mathbb{R}^2) \cap \mathcal{C}^\infty(\Omega, \mathbb{R}^2)$, $\rho \in \mathcal{C}^0(\bar{\Omega}) \cap \mathcal{C}^\infty(\Omega)$ such that (\mathbf{u}, ρ) is a classical solution of the problem (2.3).*

- If $\alpha \leq \gamma$ then $\mathbf{u} \in \mathcal{C}^{0,\alpha}(\bar{\Omega}, \mathbb{R}^2)$.
- If $h \in \mathcal{C}^{0,\gamma}(\partial\Omega)$, then $\rho \in \mathcal{C}^{0,\gamma}(\bar{\Omega})$.
- If $h \in \mathcal{C}^{k,\gamma}(\partial\Omega)$ then $\rho \in \mathcal{C}^{k,\gamma}(\bar{\Omega})$.
- If $g \in \mathcal{C}^{k-1,\gamma}(\partial\Omega)$, $h \in \mathcal{C}^{\max(k-2,1),\gamma}(\partial\Omega)$ then $\mathbf{u} \in \mathcal{C}^{k-1,\gamma}(\bar{\Omega}, \mathbb{R}^2)$.

Proof. According to Theorem 5.2 there exist a unique $L^2 - L^2$ -solution (\mathbf{u}, ρ) of the problem (2.3). Lemma 5.4 gives that $\rho \in \mathcal{C}^0(\bar{\Omega})$. The regularity of \mathbf{u} we get from the construction of \mathbf{u} in §4. Let $\mathbf{w}(\mathbf{x}) = (-x_2, x_1)[\int_{\Omega} 2 \, d\mathbf{y}]^{-1}$, and $\Phi, \tilde{\mathbf{v}}, \mathbf{v}, \tilde{f}, \varphi$ have meaning from §4.

Suppose that $\alpha \leq \gamma$. Let $2 < q < \infty$. Since ρ is a classical solution of the Dirichlet problem for the Laplace equation, Lemma 4.1 gives that $\Phi \in B_{1+1/q}^{q,q}(\Omega, \mathbb{R}^2)$. According to [66, §2.7.1, Remark 1] we have $B_{1+1/q}^{q,q}(\Omega) \subset \mathcal{C}^{0,(q-1)/q}(\bar{\Omega})$. For sufficiently large q we have $\Phi \in \mathcal{C}^{0,\alpha}(\bar{\Omega}, \mathbb{R}^2)$. Thus $\tilde{f} = g - \Phi \cdot \boldsymbol{\tau} \in \mathcal{C}^{0,\alpha}(\bar{\Omega})$ and hence $f \in \mathcal{C}^{0,\alpha}(\bar{\Omega})$. Since φ is an L^p -solution of the Neumann problem for the Laplace equation (2.2), $\varphi \in \mathcal{C}^{1,\alpha}(\bar{\Omega})$ by Proposition 3.10. Thus $v_1 = -\partial_2 \varphi \in \mathcal{C}^{0,\alpha}(\bar{\Omega})$, $v_2 = \partial_1 \varphi \in \mathcal{C}^{0,\alpha}(\bar{\Omega})$. So, $\tilde{\mathbf{v}} \in \mathcal{C}^{0,\alpha}(\bar{\Omega}, \mathbb{R}^2)$ and $\mathbf{u} = \Phi + \tilde{\mathbf{v}} \in \mathcal{C}^{0,\alpha}(\bar{\Omega}, \mathbb{R}^2)$.

If $h \in \mathcal{C}^{0,\gamma}(\partial\Omega)$, then $\rho \in \mathcal{C}^{0,\gamma}(\bar{\Omega})$ by Lemma 5.4. If $h \in \mathcal{C}^{k,\gamma}(\partial\Omega)$ then $\rho \in \mathcal{C}^{k,\gamma}(\bar{\Omega})$ (see Lemma 5.4).

Let now $g \in \mathcal{C}^{k-1,\gamma}(\partial\Omega)$, $h \in \mathcal{C}^{l,\gamma}(\partial\Omega)$ with $l = \max(k-2, 1)$. Since $\rho \in \mathcal{C}^{l,\gamma}(\bar{\Omega})$, $\Phi \in \mathcal{C}^{l+1,\gamma}(\bar{\Omega})$ by Lemma 4.1. So $f \in \mathcal{C}^{k-1,\gamma}(\partial\Omega)$. Proposition 3.10 gives $\rho \in \mathcal{C}^{k,\gamma}(\bar{\Omega})$. Therefore $\mathbf{u} \in \mathcal{C}^{k-1,\gamma}(\bar{\Omega}, \mathbb{R}^2)$. \square

Theorem 7.2. *Let $k \in N$, $0 < \gamma < 1$, $\partial\Omega$ be of class $\mathcal{C}^{k+2,\gamma}$. Let $h \in \mathcal{C}^{k,\gamma}(\partial\Omega)$, $g \in \mathcal{C}^{k+1,\gamma}(\partial\Omega)$, $\mathbf{F} \in \mathcal{C}^{k-1,\gamma}(\bar{\Omega}, \mathbb{R}^2)$, $G \in \mathcal{C}^{k,\gamma}(\bar{\Omega})$. Then there exists a unique solution $(\mathbf{u}, \rho) \in \mathcal{C}^{k+1,\gamma}(\bar{\Omega}, \mathbb{R}^2) \times \mathcal{C}^{k,\gamma}(\bar{\Omega})$ of the problem (6.1).*

Proof. According to [26, Theorem 6.19] there exists a solution $\omega \in \mathcal{C}^{k+2,\gamma}(\bar{\Omega})$ of the problem $\Delta \omega = G$ in Ω , $\omega = 0$ on $\partial\Omega$. Define $\mathbf{w} = \nabla \omega$, $\tilde{g} = g - \mathbf{w} \cdot \boldsymbol{\tau}$, $\tilde{\mathbf{F}} = \mathbf{F} + \Delta \mathbf{w}$. Then $\mathbf{w} \in \mathcal{C}^{k+1,\gamma}(\bar{\Omega}, \mathbb{R}^2)$, $\tilde{g} \in \mathcal{C}^{k+1,\gamma}(\partial\Omega)$, $\tilde{\mathbf{F}} \in \mathcal{C}^{k-1,\gamma}(\bar{\Omega}, \mathbb{R}^2)$, and $\nabla \cdot \mathbf{w} = G$ in

Ω . According to [22, Theorem IV.7.2 and Remark IV.7.1] there exists a solution $(\mathbf{v}, p) \in \mathcal{C}^{k+1,\gamma}(\overline{\Omega}, \mathbb{R}^2) \times \mathcal{C}^{k,\gamma}(\overline{\Omega})$ of the problem

$$-\Delta \mathbf{v} + \nabla p = \tilde{\mathbf{F}}, \quad \nabla \cdot \mathbf{v} = 0 \text{ in } \Omega, \quad \mathbf{v} = 0 \text{ on } \partial\Omega.$$

Define $\tilde{h} = h - p$. Then $\tilde{h} \in \mathcal{C}^{k,\gamma}(\partial\Omega)$. According to Theorem 7.1 there exists a solution $(\tilde{\mathbf{v}}, \tilde{p}) \in \mathcal{C}^{k+1,\gamma}(\overline{\Omega}, \mathbb{R}^2) \times \mathcal{C}^{k,\gamma}(\overline{\Omega})$ of the problem

$$-\Delta \tilde{\mathbf{v}} + \nabla \tilde{p} =, \quad \nabla \cdot \tilde{\mathbf{v}} = 0 \text{ in } \Omega, \quad \tau \cdot \tilde{\mathbf{v}} = \tilde{g}, \quad \tilde{p} = \tilde{h} \text{ on } \partial\Omega.$$

Put $\mathbf{u} = \mathbf{w} + \mathbf{v} + \tilde{\mathbf{v}}$, $\rho = p + \tilde{p}$. Then $(\mathbf{u}, \rho) \in \mathcal{C}^{k+1,\gamma}(\overline{\Omega}, \mathbb{R}^2) \times \mathcal{C}^{k,\gamma}(\overline{\Omega})$ is a solution of the problem (6.1).

The uniqueness follows from Theorem 7.1. \square

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INSTITUTE OF MATHEMATICS OF THE CZECH ACADEMY OF SCIENCES, ŽITNÁ 25, 115 67 PRAHA 1, CZECH REPUBLIC

E-mail address: medkova@math.cas.cz