INSTITUTE OF MATHEMATICS

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Amiran Gogatishvili
Júlio S. Neves

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WEIGHTED NORM INEQUALITIES FOR POSITIVE OPERATORS RESTRICTED ON THE CONE OF λ-QUASICONCAVE FUNCTIONS

AMIRAN GOGATISHVILI AND JÚLIO S. NEVES

Abstract. Let ρ be a monotone quasinorm defined on $\mathfrak{M}^+$, the set of all non-negative measurable functions on $[0, \infty)$. Let T be a monotone quasilinear operator on $\mathfrak{M}^+$. We show that the following inequality restricted on the cone of λ-quasiconcave functions

$$\rho(Tf) \leq C_1 \left(\int_0^\infty f^p v \right)^{\frac{1}{p}},$$

where $1 \leq p \leq \infty$ and v is a weighted function, is equivalent to slightly different inequalities consider for all non-negative measurable functions. The case $0 < p < 1$ is also studied for quasinorms and operators with additional properties. These results in turn enables us to establish necessary and sufficient conditions on the weights (u, v, w) for which the three weighted Hardy-type inequality

$$\left(\int_0^\infty \left(\int_0^x fu \right)^q w(x)dx \right)^{\frac{1}{q}} \leq C_1 \left(\int_0^\infty f^p v \right)^{\frac{1}{p}},$$

holds for all λ-quasiconcave functions and all $0 < p, q \leq \infty$.

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1. Introduction

Many papers were recently devoted to the study of weighted inequalities of classical operators restricted on the cones of quasi-monotone and quasiconcave functions. For the cone of quasi-monotone functions, which plays an important role in the study of Lorentz spaces, see for instance [1], [8], [33] and the recent survey [17], as well as the literature given there. The weighted inequalities restricted on the cones of quasi-concave functions were considered in the papers [9], [14], [15], [18], [19], [22], [25], [28], [32], [31] and [34], but some answers are not always satisfactory. Quasiconcave functions play an important role in real interpolation theory (see, for instance, the recent survey [30] and the literature given there). The weighted inequalities restricted on the cone of quasiconcave functions are closely related to the problems on optimal spaces in the embedding theory for Sobolev, Besov, and Calderón spaces, and Bessel and Riesz potentials, etc. (see, for example, [6], [7], [11], [12], [13], [20], [21], [23] and [24]).

Almost from the beginning, the method of reduction has been a fundamental tool in the study of the weighed inequalities in Lebesgue spaces. In this approach, a given inequality on monotone functions is reduced to some inequality on non-negative functions, which is more easily characterized than the original one. The

Sawyer duality principle [33], which applies for $0 < p \le \infty$ and $\rho(f) = \|f\|_{q,w}$ (weighted Lebesgue norm), $1 \le q \le \infty$, is one of the universal tools in the method of reduction for positive linear operators. The Sawyers duality theorem was extended for the first time to the cone of quasiconcave functions in [22] and [36], but this result was not satisfactory and more explicit formulas were obtained in [14] and [15] (see also [9] and [39]). Using this duality argument, weighted inequalities restricted on the cone of quasiconcave functions were reduced to some inequality on non-negative functions (see [9], [31]). This duality principle only applies to a linear operator T, $\rho(f) = \|f\|_{q,w}$ and $1 \le q \le \infty$. Recently, in [32], the weighted Hardy-type inequality, restricted on the cone of quasiconcave functions, was characterized by reducing them to iterated Hardy inequalities.

The main results of our paper are given in Sections 3, 4 and 5, where we propose a new method of reduction of an inequality for monotone quasilinear operators and for general monotone quasinorms, restricted on the cone of quasiconcave functions, to some inequality on the cone of non-negative functions. Our approach is somehow extension of the ideas from the paper [17] to the setting of λ-quasiconcave functions, where the cone of monotone functions was considered. Due to the new result of Kŕepela [26], we can avoid the technical part, which was the main difficult part in [17].

Using these reduction theorems we give, in Subsection 5.2, the complete characterization of the three weighted Hardy-type inequality

$$\left(\int_0^\infty \left(\int_0^x fu\right)^q w(x)dx\right)^{\frac{1}{q}} \le C_1 \left(\int_0^\infty f^p v\right)^{\frac{1}{p}},$$

restricted on the cone of λ-quasiconcave functions for all $0 < p, q \le \infty$. Our characterizations, in some cases, are more easily verifiable than the ones in the existing literature.

2. Preliminaries

We denote the set of all non-negative measurable functions on $[0,\infty)$ by $\mathfrak{M}^+$.

Throughout the paper, u, v and w are weights, which are non-negative measurable functions on $[0,\infty)$. $\|\cdot\|_{q,w}$ stands for the weighted Lebesgue quasinorm of mesurable functions on $[0,\infty)$. That is, $\|f\|_{q,w} = \left(\int_0^\infty |f(x)|^q w(x)\,dx\right)^{\frac{1}{q}}$, if $0 < q < \infty$, and $\|f\|_{q,w} = \operatorname{ess\,sup}_{x\in[0,\infty)} |f(x)|w(x)$, if $q = \infty$, for any measurable function f on $[0,\infty)$. When the weight w is the constant function equal to 1, we write $\|\cdot\|_q$ instead of $\|\cdot\|_{q,w}$. If $p = \infty$, the expression $\left(\int_0^\infty f^p v\right)^{\frac{1}{p}}$, $f \in \mathfrak{M}^+$, is understood as $\operatorname{ess\,sup}_{x\in[0,\infty)} f(x)v(x)$.

Expressions like $0 \cdot \infty$ are taken to be 0. The notation $A \lesssim B$ means the inequality $A \le cB$ with a constant c depending only on insignificant parameters. We shall write $A \approx B$ in place of $A \lesssim B \lesssim A$ or $A = cB$. We let $\mathbb{Z}$ denote the set of all integers and let χ_E denote the characteristic function (indicator) of a subset E of $[0,\infty)$. New quantities are defined using the symbols := and =: . We also set $p' := p/(p-1)$ for $1 < p < \infty$, $p' := 1$ for $p = \infty$, $p' := \infty$ for $p = 1$, and $r := pq/(p-q)$ for $0 < q < p < \infty$. By letters A, B, C with indices (say, C_1, C_2, $\cdots$) we denote constants, which may differ in different assertions even if they have the same indices.

Throughout the paper we sometimes refer to $f(t)$ as the function f itself and not to the image of t by f.

Definition 2.1. Let $\lambda > 0$. We say that a non-negative function h is λ-*quasiconcave* if h is equivalent to a non-decreasing function on $(0, \infty)$ and $\frac{h(t)}{t^\lambda}$ is equivalent to a non-increasing function on $(0, \infty)$. We denote by Ω_λ the family of λ-*quasiconcave* functions. We say that h is *quasiconcave* when $\lambda = 1$ and we write that $h \in \Omega$.

λ-quasiconcave functions have been treated, in one way or another, by several authors (cf. *e.g.* [4], [28] or [3]).

Remarks 2.2. (i) It will be useful to note that

$$h \in \Omega_\lambda \quad \text{if, and only if,} \quad \frac{t^\lambda}{h(t)} \in \Omega_\lambda.$$

(ii) Some authors add the restriction $h(t) = 0$ if, and only if, $t = 0$ to the definition of a quasiconcave function. However, the only difference is that our definition recognizes the zero function as quasiconcave.

(iii) Note that any λ-quasiconcave function is necessarily equivalent to a continuous function on $(0, \infty)$.

Examples 2.3. (i) Given a compatible couple (X_0, X_1) of Banach spaces and any $f \in X_0 + X_1$, $K(f, \cdot; X_0, X_1) \in \Omega$, where $K(\cdot, \cdot; X_0, X_1)$ is the Peetre $K-$functional defined for each $f \in X_0 + X_1$ and $t > 0$ by

$$K(f, t; X_0, X_1) = \inf\{\|f_0\|_{X_0} + t\|f_1\|_{X_1} : f = f_0 + f_1\},$$

where the infimum is taken over all representations $f = f_0 + f_1$ of f with $f_0 \in X_0$ and $f_1 \in X_1$.

(ii) Particular cases of the previous example are $K(f, t; L_1, L_\infty) = \int_0^t f^*(s)\, ds$, $t > 0$, where f^* is the non-increasing rearrangement of the measurable function $f \in L_1 + L_\infty$, and $K(f, t; L_p, W_p^k) \approx \min\{1, t^{1/k}\}\|f\|_p + \omega_k(f, t^{1/k})_p$, $t > 0$, where $\omega_k(f, \cdot)_p$ is the k order $p-$modulus of smoothness of f.

(iii) For a given $f \in L_p$ and $k \in \mathbb{N}$, $\omega_k(f, \cdot)_p \in \Omega_k$.

(iv) Let $0 < p_0 < p_1 < \infty$, $0 < q_0 < q_1 < \infty$. Then the Calderón operator (cf. [2, Chapter 3, Definition 5.1])

$$(Sf^*)(t) = \int_0^\infty f^*(s) \min\{\frac{s^{1/p_0}}{t^{1/q_0}}, \frac{s^{1/p_1}}{t^{1/q_1}}\} \frac{ds}{s}, \quad t > 0,$$

where f^* is the non-increasing rearrangement of the measurable function f, satisfies for any fixed (appropriate) f, $(Sf^*)(t)\, t^{1/q_0} \in \Omega_\lambda$ with $\lambda = \frac{1}{q_0} - \frac{1}{q_1}$.

Now we are going to define the the Stieltjes transform which plays an important role throughout the paper.

The *Stieltjes transform* S_λ, $\lambda > 0$, is defined for any $f \in \mathfrak{M}^+$ by

$$S_\lambda f(x) = \int_0^\infty \frac{f(t)}{(x+t)^\lambda}\, dt, \quad x > 0.$$

Let $\lambda > 0$ and let $f \in \mathfrak{M}^+$ be a fixed function. The function $S_\lambda f$ is non-increasing and the function $x^\lambda S_\lambda f(x)$ is non-decreasing, this means that the function $x^\lambda S_\lambda f(x)$

is λ-quasiconcave. We also have, for any $x \in (0,\infty)$,

$$
\begin{aligned}
S_\lambda f(x) &\approx \int_0^\infty \min\{x^{-\lambda}, t^{-\lambda}\}\, f(t)\, dt \\
&= x^{-\lambda} \int_0^x f(t)\, dt + \int_x^\infty t^{-\lambda} f(t)\, dt \\
&= \lambda\, x^{-\lambda} \int_0^x t^{\lambda-1} \int_t^\infty y^{-\lambda} f(y)\, dy\, dt \\
&= \lambda \int_x^\infty t^{-\lambda-1} \int_0^t f(y)\, dy\, dt. \qquad (2.1)
\end{aligned}
$$

To prove our results, we need some useful identities for the Stieltjes transform. This is given in the following lemma, which is of independent interest.

Lemma 2.4. *Let λ and α be positive numbers and let f be a measurable function positive a.e. in $(0,\infty)$ Then, for all $x > 0$,*

$$
(S_\lambda f(x))^\alpha \approx S_{\lambda\alpha} \left((S_\lambda f(t))^{\alpha-1} f(t) t^{\lambda(\alpha-1)} \right)(x), \qquad (2.2)
$$

$$
\begin{aligned}
x^{-\lambda\alpha} (S_\lambda f(x))^{-\alpha} &\approx S_{\lambda\alpha} \left((S_\lambda f(t))^{-\alpha-2}\, t^{-\lambda-1} \int_0^t f(y)\, dy \int_t^\infty y^{-\lambda} f(y)\, dy \right)(x) \\
&\quad + \frac{1}{(\int_0^\infty f(y)\, dy)^\alpha} + \frac{x^{-\lambda\alpha}}{(\int_0^\infty y^{-\lambda} f(y)\, dy)^\alpha} \qquad (2.3)
\end{aligned}
$$

Proof. [LHS (2.2) $\lesssim$ RHS (2.2)] As function $(\cdot)^\lambda S_\lambda f(\cdot)$ is non-decreasing and function $S_\lambda f(\cdot)$ is non-increasing, we have, for any $x > 0$,

$$
\begin{aligned}
&S_{\lambda\alpha} \left((S_\lambda f(t))^{\alpha-1} f(t) t^{\lambda(\alpha-1)} \right)(x) \\
&\quad \gtrsim \left(x^\lambda S_\lambda f(x) \right)^{-1} x^{-\lambda\alpha} \int_0^x \left(t^\lambda S_\lambda f(t) \right)^\alpha f(t)\, dt \\
&\qquad + (S_\lambda f(x))^{-1} \int_x^\infty (S_\lambda f(t))^\alpha\, t^{-\lambda} f(t)\, dt \\
&\quad \gtrsim (S_\lambda f(x))^{-1} x^{-\lambda(\alpha+1)} \int_0^x \left(\int_0^t f(y)\, dy \right)^\alpha f(t)\, dt \\
&\qquad + (S_\lambda f(x))^{-1} \int_x^\infty \left(\int_t^\infty y^{-\lambda} f(y)\, dy \right)^\alpha t^{-\lambda} f(t)\, dt \\
&\quad \gtrsim (S_\lambda f(x))^{-1} x^{-\lambda(\alpha+1)} \left(\int_0^x f(y)\, dy \right)^{\alpha+1} \\
&\qquad + (S_\lambda f(x))^{-1} \left(\int_x^\infty y^{-\lambda} f(y)\, dy \right)^{\alpha+1} \\
&\quad \approx (S_\lambda f(x))^{-1} (S_\lambda f(x))^{\alpha+1} \\
&\quad = (S_\lambda f(x))^\alpha. \qquad (2.4)
\end{aligned}
$$

[RHS (2.2) $\lesssim$ LHS (2.2)] Let $0 < \varepsilon < \min(\alpha, 1)$. Then, for any $x > 0$,

$$
S_{\lambda\alpha} \left((S_\lambda f(t))^{\alpha-1} f(t) t^{\lambda(\alpha-1)} \right)(x)
$$

$$
\begin{aligned}
&\lesssim \left(x^{\lambda} S_{\lambda} f(x)\right)^{\alpha-\varepsilon} x^{-\lambda\alpha} \int_0^x \left(t^{\lambda} S_{\lambda} f(t)\right)^{\varepsilon-1} f(t)\,dt \\
&\qquad + \left(S_{\lambda} f(x)\right)^{\alpha-\varepsilon} \int_x^{\infty} \left(S_{\lambda} f(t)\right)^{\varepsilon-1} t^{-\lambda} f(t)\,dt \\
&\lesssim \left(S_{\lambda} f(x)\right)^{\alpha-\varepsilon} x^{-\lambda\varepsilon} \int_0^x \left(\int_0^t f(y)\,dy\right)^{\varepsilon-1} f(t)\,dt \\
&\qquad + \left(S_{\lambda} f(x)\right)^{\alpha-\varepsilon} \int_x^{\infty} \left(\int_t^{\infty} y^{-\lambda} f(y)\,dy\right)^{\varepsilon-1} t^{-\lambda} f(t)\,dt \\
&\lesssim \left(S_{\lambda} f(x)\right)^{\alpha-\varepsilon} x^{-\lambda\varepsilon} \left(\int_0^x f(y)\,dy\right)^{\varepsilon} \\
&\qquad + \left(S_{\lambda} f(x)\right)^{\alpha-\varepsilon} \left(\int_x^{\infty} y^{-\lambda} f(y)\,dy\right)^{\varepsilon} \\
&\approx \left(S_{\lambda} f(x)\right)^{\alpha-ve} \left(S_{\lambda} f(x)\right)^{\varepsilon} \\
&= \left(S_{\lambda} f(x)\right)^{\alpha}. \qquad (2.5)
\end{aligned}
$$

Therefore, (2.2) it now follows from (2.4) and (2.5).
[RHS (2.3) $\lesssim$ LHS (2.3)] Again, as function $(\cdot)^{\lambda} S_{\lambda} f(\cdot)$ is non-decreasing and function $S_{\lambda} f(\cdot)$ is non-increasing, we have, for any $x > 0$,

$$
\frac{1}{\left(\int_0^{\infty} f(y)\,dy\right)^{\alpha}} = \lim_{\xi\to\infty} \frac{1}{\left(\xi^{\lambda} S_{\lambda} f(\xi)\right)^{\alpha}} \le x^{-\lambda\alpha} \left(S_{\lambda} f(x)\right)^{-\alpha}. \qquad (2.6)
$$

$$
\frac{x^{-\lambda\alpha}}{\left(\int_0^{\infty} y^{-\lambda} f(y)\,dy\right)^{\alpha}} = x^{-\lambda\alpha} \lim_{\xi\to 0} \frac{1}{\left(S_{\lambda} f(\xi)\right)^{\alpha}} \le x^{-\lambda\alpha} \left(S_{\lambda} f(x)\right)^{-\alpha}. \qquad (2.7)
$$

$$
\begin{aligned}
&S_{\lambda\alpha}\left(\left(S_{\lambda} f(t)\right)^{-\alpha-2} t^{-\lambda-1} \int_0^t f(y)\,dy \int_t^{\infty} y^{-\lambda} f(y)\,dy\right)(x) \\
&\approx x^{-\lambda\alpha} \int_0^x \frac{t^{-\lambda-1} \int_0^t f(y)\,dy \int_t^{\infty} y^{-\lambda} f(y)\,dy}{\left(t^{-\lambda} \int_0^t f(y)dy + \int_t^{\infty} y^{-\lambda} f(y)dy\right)^{\alpha+2}}\,dt \\
&\qquad + \int_x^{\infty} \frac{t^{\lambda-1} \int_0^t f(y)dy \int_t^{\infty} y^{-\lambda} f(y)\,dy}{\left(\int_0^t f(y)\,dy + t^{\lambda} \int_t^{\infty} y^{-\lambda} f(y)\,dy\right)^{\alpha+2}}\,dt \\
&\le x^{-\lambda\alpha} \int_0^x \frac{t^{-\lambda-1} \int_0^t f(y)\,dy}{\left(t^{-\lambda} \int_0^t f(y)\,dy + \int_t^{\infty} y^{-\lambda} f(y)\,dy\right)^{\alpha+1}}\,dt \\
&\qquad + \int_x^{\infty} \frac{t^{\lambda-1} \int_t^{\infty} y^{-\lambda} f(y)dy}{\left(\int_0^t f(y)\,dy + t^{\lambda} \int_t^{\infty} y^{-\lambda} f(y)\,dy\right)^{\alpha+1}}\,dt \\
&\lesssim \frac{x^{-\lambda\alpha}}{\left(x^{-\lambda} \int_0^x f(y)\,dy + \int_x^{\infty} y^{-\lambda} f(y)\,dy\right)^{\alpha}} \\
&\qquad + \frac{1}{\left(\int_0^x f(y)\,dy + x^{\lambda} \int_x^{\infty} y^{-\lambda} f(y)\,dy\right)^{\alpha}} \\
&\approx x^{-\lambda\alpha} \left(S_{\lambda} f(x)\right)^{-\alpha}. \qquad (2.8)
\end{aligned}
$$

From (2.6), (2.7) and (2.8) it now follows that RHS (2.3) $\lesssim$ LHS (2.3).
[LHS (2.3) $\lesssim$ RHS (2.3)] Let $x \in (0,\infty)$. Then,

$$\begin{aligned}
x^{-\lambda\alpha}(S_\lambda f(x))^{-\alpha} &\approx \int_0^x f(y)\,dy\,\left(x^\lambda S_\lambda f(x)\right)^{-\alpha-1} \\
&\quad + x^{-\lambda\alpha}\int_x^\infty y^{-\lambda} f(y)\,dy\,(S_\lambda f(x))^{-\alpha-1} \\
&= \int_0^x f(y)\,dy\left[\left(x^\lambda S_\lambda f(x)\right)^{-\alpha-1} - \lim_{\xi\to\infty}\left(\xi^\lambda S_\lambda f(\xi)\right)^{-\alpha-1}\right] \\
&\quad + x^{-\lambda\alpha}\int_x^\infty y^{-\lambda} f(y)\,dy\left[(S_\lambda f(x))^{-\alpha-1} - \lim_{\xi\to 0}(S_\lambda f(\xi))^{-\alpha-1}\right] \\
&\quad + \int_0^x f(y)\,dy\,\lim_{\xi\to\infty}\left(\xi^\lambda S_\lambda f(\xi)\right)^{-\alpha-1} \\
&\quad + x^{-\lambda\alpha}\int_x^\infty y^{-\lambda} f(y)\,dy\,\lim_{\xi\to 0}(S_\lambda f(\xi))^{-\alpha-1} \\
&\lesssim \int_0^x f(y)\,dy\int_x^\infty \frac{t^{\lambda-1}\int_t^\infty y^{-\lambda}f(y)\,dy}{\left(\int_0^t f(y)\,dy + t^\lambda\int_t^\infty y^{-\lambda}f(y)\,dy\right)^{\alpha+2}}\,dt \\
&\quad + x^{-\lambda\alpha}\int_x^\infty y^{-\lambda}f(y)\,dy\int_0^x \frac{t^{-\lambda-1}\int_0^t f(y)\,dy}{\left(t^{-\lambda}\int_0^t f(y)\,dy + \int_t^\infty y^{-\lambda}f(y)\,dy\right)^{\alpha+2}}\,dt \\
&\quad + \frac{1}{\left(\int_0^\infty f(y)\,dy\right)^\alpha} + \frac{x^{-\lambda\alpha}}{\left(\int_0^\infty y^{-\lambda}f(y)\,dy\right)^\alpha} \\
&\lesssim \int_x^\infty \frac{t^{\lambda-1}\int_0^t f(y)\,dy\int_t^\infty y^{-\lambda}f(y)\,dy}{\left(\int_0^t f(y)\,dy + t^\lambda\int_t^\infty y^{-\lambda}f(y)\,dy\right)^{\alpha+2}}\,dt \\
&\quad + x^{-\lambda\alpha}\int_0^x \frac{t^{-\lambda-1}\int_0^t f(y)\,dy\int_t^\infty y^{-\lambda}f(y)\,dy}{\left(t^{-\lambda}\int_0^t f(y)\,dy + \int_t^\infty y^{-\lambda}f(y)\,dy\right)^{\alpha+2}}\,dt \\
&\quad + \frac{1}{\left(\int_0^\infty f(y)\,dy\right)^\alpha} + \frac{x^{-\lambda\alpha}}{\left(\int_0^\infty y^{-\lambda}f(y)\,dy\right)^\alpha} \\
&\approx S_{\lambda\alpha}\left((S_\lambda f(t))^{-\alpha-2}\,t^{-\lambda-1}\int_0^t f(y)\,dy\int_t^\infty y^{-\lambda}f(y)\,dy\right)(x) \\
&\quad + \frac{1}{\left(\int_0^\infty f(y)\,dy\right)^\alpha} + \frac{x^{-\lambda\alpha}}{\left(\int_0^\infty y^{-\lambda}f(y)\,dy\right)^\alpha}.
\end{aligned}$$

□

3. Monotone quasilinear operators and Reduction Throrems

An operator $T : \mathfrak{M}^+ \to \mathfrak{M}^+$ is called a monotone quasilinear operator if

(i) $T(\lambda f) = \lambda T f$ for all $\lambda \geq 0$ and $f \in \mathfrak{M}^+$;

(ii) $T(f+g) \le c(Tf+Tg)$ for all $f,g \in \mathfrak{M}^+$, where c is a positive constant independent of f and g;

(iii) $Tf(x) \le cTg(x)$ for almost every $x \in [0,\infty)$, if $f(x) \le g(x)$ for almost every $x \in [0,\infty)$, where c is a positive constant independent of f and g.

We refer to [16] for examples of such operators.

A mapping $\rho : \mathfrak{M}^+ \to [0,\infty)$ is called a monotone quasinorm if

(a) $\rho(\lambda f) = \lambda \rho(f)$ for all $\lambda \ge 0$ and $f \in \mathfrak{M}^+$;

(b) $\rho(f+g) \le c(\rho(f)+\rho(g))$ for all $f,g \in \mathfrak{M}^+$, where c is a positive constant independent of f and g;

(c) $\rho(f) \le c\rho(g)$ for almost every $x \in [0,\infty)$, if $f(x) \le g(x)$ for almost every $x \in [0,\infty)$, where c is a positive constant independent of f and g.

In the next theorem and in what follows, $\mathbf{1}$ denotes the constant function equal to 1 in $[0,\infty)$.

Theorem 3.1. *Let $\lambda > 0$ and $1 \le p < \infty$. Let ρ be any monotone quasinorm and let $T : \mathfrak{M}^+ \to \mathfrak{M}^+$ be a monotone quasilinear operator. Then the inequality*

$$\rho(Tf) \le C_1 \left(\int_0^\infty (f(t))^p v(t)\, dt \right)^{\frac{1}{p}}, \quad f \in \Omega_\lambda, \tag{3.1}$$

holds if, and only if, the following three inequalities are valid:

$$\rho\left(T\left(\int_0^x h + x^\lambda \int_x^\infty t^{-\lambda} h\right)\right) \le C_2 \left(\int_0^\infty h^p(x) \frac{x^{\lambda p(1-p)} \left(\int_0^x t^{\lambda p} v\right)^{1-p} \left(\int_x^\infty v\right)^{1-p}}{\left(\int_0^x t^{\lambda p} v + x^{\lambda p} \int_x^\infty v\right)^{1-2p}}\, dx \right)^{\frac{1}{p}}, \quad h \in \mathfrak{M}^+; \tag{3.2}$$

$$\rho\left(T(\mathbf{1})\right) \le C_3 \left(\int_0^\infty v\right)^{\frac{1}{p}}; \tag{3.3}$$

$$\rho\left(T\left(x^{\lambda p}\right)\right) \le C_4 \left(\int_0^\infty x^{\lambda p} v(x)\, dx\right)^{\frac{1}{p}}. \tag{3.4}$$

Proof. Let $1 \le p < \infty$. *Necessity.* Let $h \in \mathfrak{M}^+$ be such that $\int_0^\infty \frac{h(x)}{(1+x)^\lambda}\, dx < \infty$. Then $f(\cdot) = (\cdot)^\lambda S_\lambda h(\cdot) \in \Omega_\lambda$. Using (3.1), (2.1), Lemma 2.4 and Stieltjes inequalities of [10, Proposition 4.6], when $1 < p < \infty$, and Fubini's Theorem, when $p = 1$, we obtain

$$\begin{aligned} &\rho\left(T\left(\int_0^x h + x^\lambda \int_x^\infty t^{-\lambda} h\right)\right) \\ &\qquad \le C \left(\int_0^\infty \left(\int_0^x h + x^\lambda \int_x^\infty t^{-\lambda} h \right)^p v(x)\, dx \right)^{\frac{1}{p}} \\ &\qquad \le C \left(\int_0^\infty h^p(x) \frac{x^{\lambda p(1-p)} \left(\int_0^x t^{\lambda p} v\right)^{1-p} \left(\int_x^\infty v\right)^{1-p}}{\left(\int_0^x t^{\lambda p} v + x^{\lambda p} \int_x^\infty v\right)^{1-2p}}\, dx \right)^{\frac{1}{p}}. \end{aligned}$$

Now, (3.2) and (3.3) follow from (3.1) with $f = \mathbf{1}$ and $f(x) = x^\lambda$, respectively.

Sufficiency. Suppose that $f \in \Omega_\lambda$. Using Lemma 2.4, we obtain

$$f(x) = f(x) \left(S_{\lambda p}(t^{\lambda p} v)(x)\right)^{\frac{2}{p}} \left(S_{\lambda p}(t^{\lambda p} v)(x)\right)^{-\frac{2}{p}}$$

$$\approx f(x)\left(S_{\lambda p}(t^{\lambda p}v)(x)\right)^{\frac{2}{p}}$$
$$\times x^{2\lambda}S_{2\lambda}\left(\left(S_{\lambda p}\left(t^{\lambda p}v\right)(y)\right)^{-\frac{2}{p}-2}y^{-\lambda p-1}\int_0^y t^{\lambda p}v(t)dt\int_y^\infty v(t)dt\right)(x)$$
$$+\frac{f(x)\left(S_{\lambda p}(t^{\lambda p}v)(x)\right)^{\frac{2}{p}}x^{2\lambda}}{\left(\int_0^\infty t^{\lambda p}v(t)dt\right)^{\frac{2}{p}}}+\frac{f(x)\left(S_{\lambda p}(t^{\lambda p}v)(x)\right)^{\frac{2}{p}}}{\left(\int_0^\infty v(t)dt\right)^{\frac{2}{p}}}$$
$$\approx f(x)\left(S_{2\lambda p}\left(S_{\lambda p}\left(t^{\lambda p}v\right)(y)y^{2\lambda p}v(y)\right)(x)\right)^{\frac{1}{p}}\times$$
$$\times x^{2\lambda}S_{2\lambda}\left(\left(S_{\lambda p}\left(t^{\lambda p}v\right)(y)\right)^{-\frac{2}{p}-2}y^{-\lambda p-1}\int_0^y t^{\lambda p}v(t)dt\int_y^\infty v(t)dt\right)(x)$$
$$+\frac{f(x)\left(S_{\lambda p}(t^{\lambda p}v)(x)\right)^{\frac{2}{p}}x^{2\lambda}}{\left(\int_0^\infty t^{\lambda p}v(t)dt\right)^{\frac{2}{p}}}+\frac{f(x)\left(S_{\lambda p}(t^{\lambda p}v)(x)\right)^{\frac{2}{p}}}{\left(\int_0^\infty v(t)dt\right)^{\frac{2}{p}}}$$
$$\lesssim\left(S_{\lambda p}\left(f(y)^p y^{-\lambda p}S_{\lambda p}\left(t^{\lambda p}v\right)(y)y^{2\lambda p}v(y)\right)(x)\right)^{\frac{1}{p}}\times$$
$$\times x^{2\lambda}S_{2\lambda}\left(\left(S_{\lambda p}\left(t^{\lambda p}v\right)(y)\right)^{-\frac{2}{p}-2}y^{-\lambda p-1}\int_0^y t^{\lambda p}v(t)dt\int_y^\infty v(t)dt\right)(x)$$
$$+\frac{f(x)\left(S_{\lambda p}(t^{\lambda p}v)(x)\right)^{\frac{1}{p}}x^{\lambda}}{\left(\int_0^\infty t^{\lambda p}v(t)dt\right)^{\frac{1}{p}}}+\frac{f(x)\left(S_{\lambda p}(t^{\lambda p}v)(x)\right)^{\frac{1}{p}}}{\left(\int_0^\infty v(t)dt\right)^{\frac{1}{p}}}$$
$$\lesssim x^\lambda S_\lambda\left(\left(S_{\lambda p}\left(f(z)^p z^{-\lambda p}S_{\lambda p}\left(t^{\lambda p}v\right)(z)z^{2\lambda p}v(z)\right)(y)\right)^{\frac{1}{p}}\times\right.$$
$$\left.\times\left(S_{\lambda p}\left(t^{\lambda p}v\right)(y)\right)^{-\frac{2}{p}-2}y^{-\lambda p-1}\int_0^y t^{\lambda p}v(t)dt\int_y^\infty v(t)dt\right)(x)$$
$$+\frac{\left(S_{\lambda p}(f(t)^p t^{\lambda p}v)(x)\right)^{\frac{1}{p}}x^{\lambda}}{\left(\int_0^\infty t^{\lambda p}v(t)dt\right)^{\frac{1}{p}}}+\frac{\left(S_{\lambda p}(f(t)^p t^{\lambda p}v)(x)\right)^{\frac{1}{p}}}{\left(\int_0^\infty v(t)dt\right)^{\frac{1}{p}}}$$
$$\lesssim x^\lambda S_\lambda\left(\left(S_{\lambda p}\left(f(z)^p z^{-\lambda p}S_{\lambda p}\left(t^{\lambda p}v\right)(z)z^{2\lambda p}v(z)\right)(y)\right)^{\frac{1}{p}}\times\right.$$
$$\left.\times\left(S_{\lambda p}\left(t^{\lambda p}v\right)(y)\right)^{-\frac{2}{p}-2}y^{-\lambda p-1}\int_0^y t^{\lambda p}v(t)dt\int_y^\infty v(t)dt\right)(x)$$
$$+\frac{x^\lambda\left(\int_0^\infty f(t)^p v(t)dt\right)^{\frac{1}{p}}}{\left(\int_0^\infty t^{\lambda p}v(t)dt\right)^{\frac{1}{p}}}+\frac{\left(\int_0^\infty f(t)^p v(t)dt\right)^{\frac{1}{p}}}{\left(\int_0^\infty v(t)dt\right)^{\frac{1}{p}}}$$

Applying (i) -(iii), (a)-(c) and (3.2) with

$$h(y)=\left(S_{\lambda p}\left(f(z)^p z^{-\lambda p}S_{\lambda p}\left(t^{\lambda p}v\right)(z)z^{2\lambda p}v(z)\right)(y)\right)^{\frac{1}{p}}\times$$
$$\times\left(S_{\lambda p}\left(t^{\lambda p}v\right)(y)\right)^{-\frac{2}{p}-2}y^{-\lambda p-1}\int_0^y t^{\lambda p}v(t)dt\int_y^\infty v(t)dt,$$

and also (3.3) and (3.4), we find that

$$\text{LHS (3.1)}\lesssim\left(\int_0^\infty S_{\lambda p}\left(f(z)^p z^{-\lambda p}S_{\lambda p}\left(t^{\lambda p}v\right)(z)z^{2\lambda p}v(z)\right)(y)\times\right.$$
$$\times\left(S_{\lambda p}\left(t^{\lambda p}v\right)(y)\right)^{-2-2p}y^{-\lambda p^2-p}\left(\int_0^y t^{\lambda p}v(t)dt\right)^p\left(\int_y^\infty v(t)dt\right)^p\times$$

$$
\times \frac{y^{\lambda p(1-p)} \left(\int_0^y t^{\lambda p} v\right)^{1-p} \left(\int_y^\infty v\right)^{1-p}}{\left(\int_0^y t^{\lambda p} v + y^{\lambda p} \int_y^\infty v\right)^{1-2p}} \, dy \Bigg)^{\frac{1}{p}}
$$

$$
+ \left(\int_0^\infty f(t)^p v(t)\right)^{\frac{1}{p}}
$$

$$
\approx \Bigg(\int_0^\infty f(z)^p z^{-\lambda p} S_{\lambda p} \left(t^{\lambda p} v\right)(z) z^{2\lambda p} v(z) \times
$$

$$
\times S_{\lambda p} \left(\left(S_{\lambda p}\left(t^{\lambda p} v\right)(y)\right)^{-3} y^{-\lambda p - 1} \int_0^y t^{\lambda p} v(t) dt \int_y^\infty v(t) dt \right)(z) dz \Bigg)^{\frac{1}{p}}
$$

$$
+ \left(\int_0^\infty f(t)^p v\right)^{\frac{1}{p}}
$$

$$
\approx \left(\int_0^\infty f(t)^p v\right)^{\frac{1}{p}}.
$$

□

We can consider other values of p in the previous theorem provided (3.2) is replaced by (3.5), as can be seen in the next result.

Theorem 3.2. *Let $\lambda > 0$ and $0 < s \le p < \infty$. Let ρ be any monotone quasinorm, and let $T : \mathfrak{M}^+ \to \mathfrak{M}^+$ be a monotone quasilinear operator. Then the inequality (3.1) holds if, and only if, (3.3), (3.4) and the inequality*

$$
\rho\left(T\left(\left(\int_0^x h + x^\lambda \int_x^\infty t^{-\lambda} h\right)^{\frac{1}{s}}\right)\right)
$$

$$
\le C_4 \left(\int_0^\infty h^{\frac{p}{s}}(x) \frac{x^{\lambda \frac{p}{s}(1-\frac{p}{s})} \left(\int_0^x t^{\lambda \frac{p}{s}} v\right)^{1-\frac{p}{s}} \left(\int_x^\infty v\right)^{1-\frac{p}{s}}}{\left(\int_0^x t^{\lambda \frac{p}{s}} v + x^{\lambda \frac{p}{s}} \int_x^\infty v\right)^{1-2\frac{p}{s}}} \, dx\right)^{\frac{1}{p}}, \quad h \in \mathfrak{M}^+, \tag{3.5}
$$

are valid.

Proof. As $f \in \Omega_\lambda$ if, and only if, $f^{\frac{p}{s}} \in \Omega_{\frac{\lambda p}{s}}$, the inequality (3.1) is equivalent to

$$
\rho\left(T\left(f^{\frac{1}{s}}\right)\right)^s \le C_1^s \left(\int_0^\infty (f(y))^{\frac{p}{s}} v(y)\, dy\right)^{\frac{s}{p}}, \; f \in \Omega_{\frac{\lambda p}{s}}. \tag{3.6}
$$

By using Theorem 3.1 for the operator $Tf(x) = T\left(f^{\frac{1}{s}}\right)$ and the monotone quasinorm ρ^s, it results that (3.6) holds if, and only if, (3.3), (3.4) and (3.5) are valid. □

For completeness, the next result deals with the case $p = \infty$.

Theorem 3.3. *Let $\lambda > 0$. Let ρ be any monotone quasinorm and let $T : \mathfrak{M}^+ \to \mathfrak{M}^+$ be a monotone quasilinear operator. Then the inequality*

$$
\rho(T(f)) \le C_1 \operatorname*{ess\,sup}_{x \in [0,\infty)} f(x) v(x)
$$

holds for all $f \in \Omega_\lambda$ if, and only, if

$$
C_5 := \rho\left(T\left(\frac{x^\lambda}{\operatorname{ess\,sup}_{y \in [0,\infty)} v(y) \min(y^\lambda, x^\lambda)}\right)\right) < \infty.
$$

Proof. The proof easily follows from the following identity

$$\operatorname*{ess\,sup}_{x\in[0,\infty)} f(x)v(x) = \operatorname*{ess\,sup}_{x\in[0,\infty)} f(x)x^{-\lambda}\left(\operatorname*{ess\,sup}_{y\in[0,\infty)} v(y)\min(y^{\lambda},x^{\lambda})\right)$$

and the fact that f defined by $f(x) := \frac{x^{\lambda}}{\operatorname{ess\,sup}_{y\in[0,\infty)} v(y)\min(y^{\lambda},x^{\lambda})}$, $x\in(0,\infty)$, belongs to Ω_λ. Remark that φ defined by $\varphi(x) := \operatorname{ess\,sup}_{y\in[0,\infty)} v(y)\min(y^{\lambda},x^{\lambda}) = \operatorname{ess\,sup}_{y\in[0,\infty)} v(y)y^{\lambda}\min(1,\frac{x^{\lambda}}{y^{\lambda}})$, $x\in(0,\infty)$, belongs to Ω_λ.Therefore, $f(x) = \frac{x^{\lambda}}{\varphi(x)} \in \Omega_\lambda$.

□

4. p-convex (p-concave) monotone quasilinear operators

In this section we characterize inequality (3.1) when the operator T has additional properties, that is T is a p-convex monotone quasilinear operator. In this situation, the conditions that characterize inequality (3.1) are much simpler.

When T is a p-concave monotone quasilinear operator, we are able to characterize the converse inequality of (3.1).

Let $T : \mathfrak{M}^+ \to \mathfrak{M}^+$ be a monotone quasilinear operator. T is called a p-convex monotone quasilinear operator if there exists a constant M_1 such that

$$T\left(\sum_{n\in\mathbb{N}} f_n\right) \le M_1\left(\sum_{n\in\mathbb{N}} (Tf_n)^p\right)^{\frac{1}{p}} \tag{4.1}$$

for any sequence $\{f_n\}_{n=1}^{\infty}$ in $\mathfrak{M}^+$.

A monotone quasinorm ρ is called a p-convex monotone quasilinear norm if there exists a constant M_2 such that

$$\rho\left(\left(\sum_{n\in\mathbb{N}} (f_n)^p\right)^{\frac{1}{p}}\right) \le M_2\left(\sum_{n\in\mathbb{N}} \rho\,(f_n)^p\right)^{\frac{1}{p}} \tag{4.2}$$

for any sequence $\{f_n\}_{n=1}^{\infty}$ in $\mathfrak{M}^+$.

It is easy to see that (4.1) and (4.2) are true if we consider sums in $\mathbb{Z}$ and sequences $\{f_n\}_{n\in\mathbb{Z}}$ in $\mathfrak{M}^+$.

Let $f\in\Omega_\lambda$. Then there exist a sequence $\{x_n\}_{n\in\mathbb{Z}}\subset[0,\infty)$ such that

$$f(x) \approx \sum_{n\in\mathbb{Z}} \min\left(f(x_n), x^{\lambda}\frac{f(x_n)}{x_n^{\lambda}}\right). \tag{4.3}$$

This estimate follows from [5, Proposition 3.5] by using the simple observation that $f\in\Omega_\lambda$ if, and only if, $f(x^{\frac{1}{\lambda}})\in\Omega$.

Observe that, for any $r\in(0,\infty)$,

$$f(x)^r \approx \sum_{n\in\mathbb{Z}} \min\left(f(x_n), x^{\lambda}\frac{f(x_n)}{x_n^{\lambda}}\right)^r. \tag{4.4}$$

Theorem 4.1. *Let $0 < p,\lambda < \infty$. Let $T : \mathfrak{M}^+ \to \mathfrak{M}^+$ be a p-convex monotone quasilinear operator and let ρ be a p-convex monotone quasinorm. Then the* *inequality* (3.1) *is equivalent to the validity of the inequality*

$$\mathbf{D} := \sup_{t>0} \frac{\rho\left(T\left(\frac{(\cdot)^\lambda}{t^\lambda}\chi_{[0,t]}(\cdot)+\chi_{[t,\infty]}(\cdot)\right)\right)}{\left(t^{-\lambda p}\int_0^t s^{\lambda p}v(s)ds+\int_t^\infty v(s)ds\right)^{\frac{1}{p}}} < \infty. \tag{4.5}$$

Moreover,

$$C \approx \mathbf{D}.$$

Proof. The implication (3.1)⇒(4.5) follows by applying (3.1) to the test function $f_t(s) := \frac{s^\lambda}{t^\lambda}\chi_{[0,t]}(s)+\chi_{[t,\infty)}(s),\ t>0$. Let us now show that (4.5)⇒ (3.1). It follows from (4.3) and (4.1), that

$$\begin{aligned}
(Tf)(x) &\approx T\left(\sum_{n\in\mathbb{Z}}\min\left(f(x_n),(\cdot)^\lambda\frac{f(x_n)}{x_n^\lambda}\right)\right)(x)\\
&\lesssim \left(\sum_{n\in\mathbb{Z}}\left(T\left(\min\left(f(x_n),(\cdot)^\lambda\frac{f(x_n)}{x_n^\lambda}\right)\right)\right)^p(x)\right)^{\frac{1}{p}}\\
&\approx \left(\sum_{n\in\mathbb{Z}} f(x_n)^p\left(T\left(\frac{(\cdot)^\lambda}{x_n^\lambda}\chi_{[0,x_n]}(\cdot)+\chi_{[x_n,\infty)}(\cdot)\right)(x)\right)^p\right)^{\frac{1}{p}}.
\end{aligned} \tag{4.6}$$

Now, using (4.6), (4.2) and (4.5), we find

$$\begin{aligned}
\rho(Tf) &\lesssim \rho\left(\left(\sum_{n\in\mathbb{Z}} f(x_n)^p\left(T\left(\frac{(\cdot)^\lambda}{x_n^\lambda}\chi_{[0,x_n]}(\cdot)+\chi_{[x_n,\infty)}(\cdot)\right)\right)^p\right)^{\frac{1}{p}}\right)\\
&\lesssim \left(\sum_{n\in\mathbb{Z}} f(x_n)^p\rho\left(T\left(\frac{(\cdot)^\lambda}{x_n^\lambda}\chi_{[0,x_n]}(\cdot)+\chi_{[x_n,\infty)}(\cdot)\right)\right)^p\right)^{\frac{1}{p}}\\
&\lesssim D\left(\sum_{n\in\mathbb{Z}} f(x_n)^p\left(x_n^{-\lambda p}\int_0^{x_n} s^{\lambda p}v(s)ds+\int_{x_n}^\infty v(s)ds\right)\right)^{\frac{1}{p}}\\
&\lesssim D\left(\sum_{n\in\mathbb{Z}}\left(\int_{x_{n-1}}^{x_n} f(s)^p v(s)ds+\int_{x_n}^{x_{n+1}} f(s)^p v(s)ds\right)\right)^{\frac{1}{p}}\\
&\approx D\left(\int_0^\infty f^p v\right)^{\frac{1}{p}}.
\end{aligned}$$

Consequently, $C \lesssim D$ and (3.1) follows. □

We are able to study the converse inequality of (3.1), *i.e.* inequality (4.9), when T has additional properties. This will be done in what follows.

Let $T : \mathfrak{M}^+ \to \mathfrak{M}^+$ be a monotone quasilinear operator. T is called a p-concave monotone quasilinear operator if there exists a constant M_3 such that

$$\left(\sum_{n\in\mathbb{N}}(Tf_n)^p\right)^{\frac{1}{p}} \le M_3 T\left(\sum_{n\in\mathbb{N}} f_n\right) \tag{4.7}$$

for any sequence $\{f_n\}_{n=1}^\infty$ in $\mathfrak{M}^+$.

A monotone quasinorm ρ is called a p-concave monotone quasinorm if there exists a constant M_4 so that

$$\left(\sum_{n\in\mathbb{N}}\rho\left(f_n\right)^p\right)^{\frac{1}{p}}\leq M_4\rho\left(\left(\sum_{n\in\mathbb{N}}\left(f_n\right)^p\right)^{\frac{1}{p}}\right) \tag{4.8}$$

for any sequence $\{f_n\}_{n=1}^{\infty}$ in $\mathfrak{M}^+$.

Again, it is easy to see that (4.7) and (4.8) are true if we consider sums in $\mathbb{Z}$ and sequences $\{f_n\}_{n\in\mathbb{Z}}$ in $\mathfrak{M}^+$.

Now we study the converse inequality of (3.1), *i.e.*, inequality

$$\left(\int_0^{\infty}(f)^p\,v\right)^{\frac{1}{p}}\leq C\rho\left(Tf\right),\ f\in\Omega_\lambda. \tag{4.9}$$

Theorem 4.2. *Let $0<p,\lambda<\infty$. Let $T:\mathfrak{M}^+\to\mathfrak{M}^+$ be a p-concave monotone quasilinear operator and let ρ be a p-concave monotone quasinorm. Then the inequality (4.9) is equivalent to the validity of the inequality*

$$\mathfrak{D}:=\sup_{t>0}\frac{\left(t^{-\lambda p}\int_0^t s^{\lambda p}v(s)ds+\int_t^{\infty}v(s)ds\right)^{\frac{1}{p}}}{\rho\left(T\left(\frac{(\cdot)^\lambda}{t^\lambda}\chi_{[0,t]}(\cdot)+\chi_{[t,\infty]}(\cdot)\right)\right)}<\infty. \tag{4.10}$$

Moreover,

$$C\approx\mathfrak{D}. \tag{4.11}$$

Proof. The implication (4.9)⇒(4.10) is clear. Let us now show (4.10)⇒(4.9). Using (4.3), (4.4), (4.10), (4.8) and (4.7), we have

$$\begin{aligned}
\left(\int_0^{\infty}f^p(x)v(x)\,dx\right)^{\frac{1}{p}} &\approx \left(\int_0^{\infty}\left(\sum_{n\in\mathbb{Z}}\min\left(f(x_n),x^{\lambda}\frac{f(x_n)}{x_n^{\lambda}}\right)\right)^p dx\right)^{\frac{1}{p}}\\
&= \left(\sum_{n\in\mathbb{Z}}f(x_n)^p\left(x_n^{-\lambda p}\int_0^{x_n}x^{\lambda p}w(x)\,dx+\int_{x_n}^{\infty}w(x)\,dx\right)\right)^{\frac{1}{p}}\\
&\lesssim \left(\sum_{n\in\mathbb{Z}}f(x_n)^p\rho\left(T\left(\frac{(\cdot)^\lambda}{x_n^\lambda}\chi_{(0,x_n]}(\cdot)+\chi_{[x_n,\infty)}(\cdot)\right)\right)^p\right)^{\frac{1}{p}}\\
&= \left(\sum_{n\in\mathbb{Z}}\rho\left(T\left(\min\left(f(x_n),\frac{(\cdot)^\lambda f(x_n)}{x_n^\lambda}\right)\right)\right)^p\right)^{\frac{1}{p}}\\
&\lesssim \rho\left(\left(\sum_{n\in\mathbb{Z}}\left(T\left(\min\left(f(x_n),\frac{(\cdot)^\lambda f(x_n)}{x_n^\lambda}\right)\right)\right)^p\right)^{\frac{1}{p}}\right)\\
&\lesssim \rho\left(T\left(\sum_{n\in\mathbb{Z}}\left(\min\left(f(x_n),\frac{(\cdot)^\lambda f(x_n)}{x_n^\lambda}\right)\right)\right)\right)\\
&\approx \rho\left(Tf\right),
\end{aligned}$$

and (4.11) follows. □

Remark 4.3. Let $T : \mathfrak{M}^+ \to \mathfrak{M}^+$ be the operator defined by

$$Tf(x) = \int_0^x f(y)u(y)\,dy, \quad x \in [0,\infty), \quad f \in \mathfrak{M}^+.$$

Then T is a p-convex monotone quasilinear operator for $p \in (0,1]$ and it is a p-concave monotone quasilinear operator for $p \geq 1$.

5. Characterization of the three weighted Hardy-type inequality restricted on the cone of λ-quasiconcave functions

Let u, v and w be weights and let $\lambda > 0$. In this section we fully characterize the three weighted Hardy-type inequality restricted on the cone of λ-quasiconcave functions, *i.e.*, we give the characterization of the following inequality

$$\left(\int_0^\infty \left(\int_0^x fu\right)^q w(x)dx\right)^{\frac{1}{q}} \leq C_1 \left(\int_0^\infty f^p v\right)^{\frac{1}{p}}, \tag{5.1}$$

for all $f \in \Omega_\lambda$, for all the values of the parameters $p, q \in (0,\infty]$.

We consider in what follows $U(x) := \int_0^x u(z)dz$, $U_\lambda(x) := \int_0^x z^\lambda u(z)dz$, $U(x,y) := \int_y^x u(z)dz$, $V_*(x) := \int_x^\infty v(z)dz$, $V_{\lambda p}(x) := \int_0^x z^{\lambda p} v(z)dz$, $\mathcal{V}_{\lambda p}(x) := x^{-\lambda p} V_{\lambda p}(x) + V_*(x)$ and $W_*(x) := \int_x^\infty w(z)dz$,

Firstly, we start with a reduction theorem in the case $0 < q < p \leq 1$ for the three weighted Hardy-type inequality restricted on the cone of λ-quasiconcave functions.

5.1. Reduction Theorem in the case $0 < q < p \leq 1$.

Theorem 5.1. *Let $\lambda > 0$, $0 < q < p \leq 1$. The following are equivalent:*

(i) *Inequality* (5.1)*, with the best constant C_1, holds for all $f \in \Omega_\lambda$.*

(ii) *The following five inequalities are valid:*

$$\left(\int_0^\infty \left(\int_0^x U^p(x,y)h(y)\,dy\right)^{\frac{q}{p}} w(x)\,dx\right)^{\frac{p}{q}} \leq C_2^p \int_0^\infty h(x)\mathcal{V}_{\lambda p}(x)\,dx, \; h \in \mathfrak{M}^+, \tag{5.2}$$

$$\left(\int_0^\infty \left(\int_x^\infty h(y)\,dy\right)^{\frac{q}{p}} U_\lambda^q(x)w(x)\,dx\right)^{\frac{p}{q}} \leq C_3^p \int_0^\infty h(x)\mathcal{V}_{\lambda p}(x)\,dx, \; h \in \mathfrak{M}^+, \tag{5.3}$$

$$\left(\int_0^\infty \left(\int_0^x U_\lambda^p(y)y^{-\lambda p}h(y)\,dy\right)^{\frac{q}{p}} w(x)\,dx\right)^{\frac{p}{q}} \leq C_4^p \int_0^\infty h(x)\mathcal{V}_{\lambda p}(x)\,dx, \; h \in \mathfrak{M}^+, \tag{5.4}$$

$$\left(\int_0^\infty \left(\int_0^x u\right)^q w(x)\,dx\right)^{\frac{1}{q}} \leq C_5 \left(\int_0^\infty v\right)^{\frac{1}{p}}, \tag{5.5}$$

$$\left(\int_0^\infty \left(\int_0^x y^\lambda u(y)\,dy\right)^q w(x)\,dx\right)^{\frac{1}{q}} \leq C_6 \left(\int_0^\infty y^{\lambda p} v(y)\,dy\right)^{\frac{1}{p}}. \tag{5.6}$$

(iii) *The following two inequalities together with* (5.5) *and* (5.6) *are valid:*

$$\left(\int_0^\infty \left(\sup_{0<y<x} U^p(x,y)\int_0^y h(z)\,dz\right)^{\frac{q}{p}} w(x)\,dx\right)^{\frac{p}{q}} \leq C_7^p \int_0^\infty h(x)\mathcal{V}_{\lambda p}(x)\,dx, \quad h \in \mathfrak{M}^+, \tag{5.7}$$

$$\left(\int_0^\infty \left(\sup_{0<y<x} U_\lambda^p(y) \int_y^\infty z^{-\lambda p} h(z)\, dz\right)^{\frac{q}{p}} w(x)\, dx\right)^{\frac{p}{q}} \leq C_8^p \int_0^\infty h(x) \mathcal{V}_{\lambda p}(x)\, dx, \quad h \in \mathfrak{M}^+. \tag{5.8}$$

Moreover,

$$C_1 \approx C_2 + C_3 + C_4 + C_5 + C_6 \approx C_5 + C_6 + C_7 + C_8.$$

Proof. By Theorem 3.2, for the operator $Tf(x) = \int_0^x fu$, the quasinorm $\rho(f) = \|f\|_{q,w}$ (weighted Lebesgue norm), for any $f \in \mathfrak{M}^+$, and $s = p$, we have that (5.1) holds if, and only if, the following three inequalities are valid

$$\left(\int_0^\infty \left(\int_0^x \left(\int_0^y h + y^{\lambda p} \int_y^\infty z^{-\lambda p} h\right)^{\frac{1}{p}} u(y) dy\right)^q w\right)^{\frac{p}{q}} \leq C \int_0^\infty h(x) \left(\int_0^x y^{\lambda p} v(y) dy + x^{\lambda p} \int_x^\infty v(y)\, dy\right) dx, \quad h \in \mathfrak{M}^+, \tag{5.9}$$

$$\left(\int_0^\infty \left(\int_0^x u(y) dy\right)^q w(x)\, dx\right)^{\frac{1}{q}} \leq C \left(\int_0^\infty v\right)^{\frac{1}{p}}, \tag{5.10}$$

$$\left(\int_0^\infty \left(\int_0^x y^\lambda u(y)\, dy\right)^q w\right)^{\frac{1}{q}} \leq C \left(\int_0^\infty x^{\lambda p} v(x)\, dx\right)^{\frac{1}{p}}. \tag{5.11}$$

Using Minkowski inequality we obtain

$$\int_0^x \left(\int_0^y h + y^{\lambda p} \int_y^\infty z^{-\lambda p} h\right)^{\frac{1}{p}} u(y)\, dy \leq \left(\int_0^x U^p(x,y) h(y)\, dy\right)^{\frac{1}{p}} + \left(\int_x^\infty h(y)\, dy\right)^{\frac{1}{p}} U_{\lambda p}(x) + \left(\int_0^x U_\lambda^p(y) y^{-\lambda p} h(y)\, dy\right)^{\frac{1}{p}}. \tag{5.12}$$

We also have

$$\int_0^x \left(\int_0^y h + y^{\lambda p} \int_y^\infty z^{-\lambda p} h\right)^{\frac{1}{p}} u(y)\, dy \geq \left(\sup_{0<y<x} U^p(x,y) \int_0^y h(z)\, dz\right)^{\frac{1}{p}} + \left(\sup_{0<y<x} U_\lambda^p(y) \int_y^\infty z^{-\lambda p} h(z)\, dz\right)^{\frac{1}{p}}. \tag{5.13}$$

As we know that (5.1) is equivalent with (5.9), (5.10) and (5.11), by inequality (5.12) we have that (5.1) follows from (5.2), (5.3), (5.4), (5.5) and (5.6). Moreover, $C_1 \lesssim C_2 + C_3 + C_4 + C_5 + C_6$.

By inequality (5.13) we have that (5.7), (5.8), (5.5) and now (5.6) follows from (5.1). Moreover, $C_1 \gtrsim C_7 + C_8 + C_5 + C_6$. By [16, Theorem 4.2] we have $C_1 \approx C_7$ and by [16, Theorem 4.5] we obtain that $C_2 + C_3 \approx C_8$. □

5.2. **Full characterization.** We now obtain the complete characterization of inequality (5.1).

Theorem 5.2. *Let $0 < \lambda, q, p < \infty$. Then the inequality (5.1), with the best constant C_1, holds for every $f \in \Omega_\lambda$ if, and only if, one of the following is satisfied:*

(i) *$0 < p \le 1$, $p \le q < \infty$ and $B_1 < \infty$, where*

$$B_1 := \sup_{x\in(0,\infty)} \frac{\left(\int_0^x U_\lambda^q(y)w(y)dy + U_\lambda^q(x)\int_x^\infty w(y)dy\right)^{\frac{1}{q}}}{\left(\int_0^x y^{\lambda p}v(y)dy + x^{\lambda p}\int_x^\infty v(y)dy\right)^{\frac{1}{p}}}.$$

Moreover, in this case, $C_1 \approx B_1$.

(ii) *$0 < q < p \le 1$, $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$ and $B_2 + B_3 + B_4 + B_5 + B_6 + B_7 < \infty$, where*

$$B_2 := \left(\int_0^\infty U_\lambda^q(y)w(y)dy\right)^{\frac{1}{q}} \left(\int_0^\infty y^{\lambda p}v(y)dy\right)^{-\frac{1}{p}},$$

$$B_3 := \left(\int_0^\infty U^q(y)w(y)dy\right)^{\frac{1}{q}} \left(\int_0^\infty v(y)dy\right)^{-\frac{1}{p}},$$

$$B_4 := \left(\int_0^\infty W_*(x)^{\frac{r}{p}} w(x) \sup_{0<y<x}\left(U_\lambda^r(y)y^{\lambda r}\mathcal{V}_{\lambda p}^{-\frac{r}{p}}(y)\right)dx\right)^{\frac{1}{r}},$$

$$B_5 := \left(\int_0^\infty \left(\int_0^x U_\lambda^q(y)w(y)dy\right)^{\frac{r}{p}} U_\lambda^q(x)w(x)\mathcal{V}_{\lambda p}^{-\frac{r}{p}}(x)dx\right)^{\frac{1}{r}},$$

$$B_6 := \left(\int_0^\infty W_*(x)^{\frac{r}{p}} w(x) \sup_{0<y<x}\left(U^{\frac{r}{p}}(x,y)V_{\lambda p}^{-\frac{r}{p}}(y)\right)dx\right)^{\frac{1}{r}},$$

and

$$B_7 := \left(\int_0^\infty\left(\int_x^\infty U^q(y,x)w(y)dy\right)^{\frac{r}{p}} w(x)\sup_{0<y<x}\left(U^q(x,y)\mathcal{V}_{\lambda p}^{-\frac{r}{p}}(y)\right)dx\right)^{\frac{1}{r}}.$$

Moreover, in this case, $C_1 \approx B_2 + B_3 + B_4 + B_5 + B_6 + B_7$.

(iii) *$1 < p \le q < \infty$, $1/p' := 1 - 1/p$ and $B_2 + B_3 + B_8 + B_9 + B_{10} + B_{11} < \infty$, where*

$$B_8 := \sup_{x\in(0,\infty)} W_*^{\frac{1}{q}}(x)\left(\int_0^x U_\lambda^{p'}(y)y^{\lambda p}V_{\lambda p}(y)V_*(y)\mathcal{V}_{\lambda p}^{1-p'}(y)dy\right)^{\frac{1}{p'}},$$

$$B_9 := \sup_{x\in(0,\infty)}\left(\int_0^x U_\lambda^q(y)w(y)dy\right)^{\frac{1}{q}}\left(\int_x^\infty y^{\lambda p}V_{\lambda p}(y)V_*(y)\mathcal{V}_{\lambda p}^{1-p'}(y)dy\right)^{\frac{1}{p'}},$$

$$B_{10} := \sup_{x\in(0,\infty)} W_*^{\frac{1}{q}}(x)\left(\int_0^x U^{p'}(x,y)y^{\lambda p}V_{\lambda p}(y)V_*(y)\mathcal{V}_{\lambda p}^{1-p'}(y)dy\right)^{\frac{1}{p'}},$$

and

$$B_{11} := \sup_{x\in(0,\infty)}\left(\int_x^\infty U^q(y,x)w(y)dy\right)^{\frac{1}{q}}\left(\int_0^x y^{\lambda p}V_{\lambda p}(y)V_*(y)\mathcal{V}_{\lambda p}^{1-p'}(y)dy\right)^{\frac{1}{p'}}.$$

Moreover, in this case, $C_1 \approx B_2 + B_3 + B_8 + B_9 + B_{10} + B_{11}$.

(iv) $1 < q < p < \infty$, $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$ *and* $B_2 + B_3 + B_{12} + B_{13} + B_{14} + B_{15} < \infty$, *where*

$$B_{12} := \left(\int_0^\infty W_*^{\frac{r}{p}}(x) w(x) \left(\int_0^x U_\lambda^{p'}(y) y^{\lambda p} V_{\lambda p}(y) V_*(y) \mathcal{V}_{\lambda p}^{1-p'}(y) dy \right)^{\frac{r}{p'}} dx \right)^{\frac{1}{r}},$$

$$B_{13} := \left(\int_0^\infty \left(\int_0^x U_\lambda^q(y) w(y) dy \right)^{\frac{r}{p}} U_\lambda^q(x) w(x) \times \right.$$

$$\left. \times \left(\int_x^\infty y^{\lambda p} V_{\lambda p}(y) V_*(y) \mathcal{V}_{\lambda p}^{1-p'}(y) dy \right)^{\frac{r}{p'}} dx \right)^{\frac{1}{r}},$$

$$B_{14} := \left(\int_0^\infty W_*^{\frac{r}{p}}(x) w(x) \left(\int_0^x U^{p'}(x,y) y^{\lambda p} V_{\lambda p}(y) V_*(y) \mathcal{V}_{\lambda p}^{1-p'}(y) dy \right)^{\frac{r}{p'}} dx \right)^{\frac{1}{r}},$$

and

$$B_{15} := \left(\int_0^\infty \left(\int_x^\infty U^q(y,x) w(y) dy \right)^{\frac{r}{q}} \left(\int_0^x y^{\lambda p} V_{\lambda p}(y) V_*(y) \mathcal{V}_{\lambda p}^{1-p'}(y) dy \right)^{\frac{r}{q'}} \times \right.$$

$$\left. \times\, x^{\lambda p} V_{\lambda p}(x) V_*(x) \mathcal{V}_{\lambda p}^{1-p'}(x) dx \right)^{\frac{1}{r}}.$$

Moreover, in this case, $C_1 \approx B_2 + B_3 + B_{12} + B_{13} + B_{14} + B_{15}$.

(v) $q = 1 < p < \infty$ *and* $B_2 + B_3 + B_{16} + B_{17} < \infty$, *where*

$$B_{16} := \left(\int_0^\infty \left(\int_0^x y^\lambda W_*(y) u(y) dy \right)^{p'} x^{\lambda p} V_{\lambda p}(x) V_*(x) \mathcal{V}_{\lambda p}^{1-p'}(x) dx \right)^{\frac{1}{p'}},$$

and

$$B_{17} := \left(\int_0^\infty \left(x^\lambda \int_x^\infty W_*(y) u(y) dy \right)^{p'} x^{\lambda p} V_{\lambda p}(x) V_*(x) \mathcal{V}_{\lambda p}^{1-p'}(x) dx \right)^{\frac{1}{p'}}.$$

Moreover, in this case, $C_1 \approx B_2 + B_3 + B_{16} + B_{17}$.

(vi) $0 < q < 1 < p < \infty$ *and* $B_2 + B_3 + B_{12} + B_{13} + B_{14} + B_{18} < \infty$, *where*

$$B_{18} := \left(\int_0^\infty \left(\int_x^\infty U^q(y,x) w(y) dy \right)^{\frac{r}{p}} w(x) \times \right.$$

$$\left. \times \sup_{0<y<x} \left(U^q(x,y) \left(\int_0^y z^{\lambda p} V_{\lambda p}(z) V_*(z) \mathcal{V}_{\lambda p}^{1-p'}(z) dz \right)^{\frac{r}{p'}} \right) dx \right)^{\frac{1}{r}}.$$

Moreover, in this case, $C_1 \approx B_2 + B_3 + B_{12} + B_{13} + B_{14} + B_{18}$.

Proof. The part (i) follows by Theorem 4.1, applied to the operator $Tf(x) = \int_0^x f(y)u(y)dy$ and to the quasinorm $\rho(f) = \rho_q(f) := \|fw\|_q$, for any $f \in \mathfrak{M}^+$, and by using the fact that T satisfies (4.1) for every $0 < p \leq 1$, and that ρ_q is a p-convex monotone quasinorm for $p \leq q$ by Minkowski inequality. The part (ii) follows by Theorem 5.1 and by applying [35, Theorem 3.3] and [26, Corollary 9]. Using Theorem 3.1, we reduce (5.1) to the inequality for the integral operator with Oinarov's kernel. Then parts (iii) - (vi) follow by using the results of [29] or [38] when $q>1$, and the results of [35] and [26] when $0<q<1$, and the reverse Hölder inequality when $q=1$. □

Remark 5.3. (i) The case $\lambda=1$, $p=q$, $w=v$ and $u(t)=1/t$, $t\in(0,\infty)$, in the previous theorem, was already obtained in [37, Theorem 5.1].

(ii) The case $0<p\le 1$, $p\le q<\infty$, in the previous theorem, can also be obtained as a special case of Theorem 6.14 in [27], which was obtained in a different way (see [27, 6.6.7, p. 333] for the contribution of several authors to such result).

Theorem 5.4. *Let $0<\lambda,p<\infty$ Then the inequality*

$$\operatorname*{ess\,sup}_{x\in(0,\infty)}\left(\int_0^x f(y)u(y)dy\right)w(x)\le C_1\left(\int_0^\infty (f(t))^p v(t)dt\right)^{\frac{1}{p}},$$

with the best constant C_1, holds for every $f\in\Omega_\lambda$ if, and only if, one of the following is satisfied:

(i) $0<p\le 1$ *and* $B_{19}<\infty$, *where*

$$B_{19}:=\sup_{x\in(0,\infty)}\frac{\operatorname{ess\,sup}_{y\in(0,x)}U_\lambda(y)w(y)+U_\lambda(x)\operatorname{ess\,sup}_{y\in(x,\infty)}w(y)}{\left(\int_0^x y^{\lambda p}v(y)dy+x^{\lambda p}\int_x^\infty v(y)dy\right)^{\frac{1}{p}}}.$$

Moreover, in this case, $C_1\approx B_{19}$

(ii) $1\le p$, $1/p':=1-1/p$ *and* $B_{20}+B_{21}+B_{22}+B_{23}+B_{24}+B_{25}<\infty$, *where*

$$B_{20}:=\operatorname*{ess\,sup}_{y\in(0,\infty)}\Big(U_\lambda(y)w(y)\Big)\left(\int_0^\infty z^{\lambda p}v(z)dz\right)^{-\frac{1}{p}},$$

$$B_{21}:=\operatorname*{ess\,sup}_{y\in(0,\infty)}\Big(U(y)w(y)\Big)\left(\int_0^\infty v(z)dz\right)^{-\frac{1}{p}},$$

$$B_{22}:=\sup_{x\in(0,\infty)}\operatorname*{ess\,sup}_{y\in(x,\infty)}w(y)\left(\int_0^x U_\lambda^{p'}(z)z^{\lambda p}V_{\lambda p}(z)V_*(z)\mathcal{V}_{\lambda p}^{1-p'}(z)dz\right)^{\frac{1}{p'}},$$

$$B_{23}:=\sup_{x\in(0,\infty)}\operatorname*{ess\,sup}_{y\in(0,x)}\Big(U_\lambda(y)w(y)\Big)\left(\int_x^\infty z^{\lambda p}V_{\lambda p}(z)V_*(z)\mathcal{V}_{\lambda p}^{1-p'}(z)dz\right)^{\frac{1}{p'}},$$

$$B_{24}:=\sup_{x\in(0,\infty)}\operatorname*{ess\,sup}_{y\in(x,\infty)}w(y)\left(\int_0^x U^{p'}(x,z)z^{\lambda p}V_{\lambda p}(z)V_*(z)\mathcal{V}_{\lambda p}^{1-p'}(z)dz\right)^{\frac{1}{p'}},$$

and

$$B_{25}:=\sup_{x\in(0,\infty)}\operatorname*{ess\,sup}_{y\in(x,\infty)}\Big(U(y,x)w(y)\Big)\left(\int_0^x z^{\lambda p}V_{\lambda p}(z)V_*(z)\mathcal{V}_{\lambda p}^{1-p'}(z)dz\right)^{\frac{1}{p'}}.$$

Moreover, in this case, $C_1\approx B_{20}+B_{21}+B_{22}+B_{23}+B_{24}+B_{25}$.

Proof. The part (i) follows by Theorem 4.1, applied to the operator $Tf(x)=\int_0^x f(y)u(y)dy$ and to the quasinorm $\rho(f)=\rho_\infty(f):=\|fw\|_\infty$, for any $f\in\mathfrak{M}^+$, and using the fact that T satisfies (4.1) for every $0<p\le 1$, and ρ_∞ is a p-convex monotone quasinorm. Using Theorem 3.1, we reduce (5.1) to the inequality for the integral operator with Oinarov's kernel. Then part (ii) follows by using the results of [29] or [38]. □

From Theorem 3.3 immediately follows.

Theorem 5.5. *Let $\lambda > 0$ and $0 < q \leq \infty$. Then,*

(i) *if $0 < q < \infty$, the inequality*

$$\left(\int_0^\infty \left(\int_0^x f(y)u(y)\,dy\right)^q w(x)\,dx\right)^{\frac{1}{q}} \leq C_1 \operatorname*{ess\,sup}_{x\in[0,\infty)} f(x)v(x),$$

with the best constant C_1, holds for all $f \in \Omega_\lambda$ if, and only if, the following is valid:

$$B_{26} := \left(\int_0^\infty \left(\int_0^x \frac{y^\lambda u(y)}{\operatorname{ess\,sup}_{z\in(0,\infty)} v(z)\min(z^\lambda, y^\lambda)}\,dy\right)^q w(x)dx\right)^{\frac{1}{q}} < \infty.$$

Moreover, $C_1 = B_{26}$.

(ii) *if $q = \infty$, the inequality*

$$\operatorname*{ess\,sup}_{x\in[0,\infty)} \int_0^x f(y)u(y)dy\, w(x) \leq C_1 \operatorname*{ess\,sup}_{x\in[0,\infty)} f(x)v(x),$$

with the best constant C_1, holds for all $f \in \Omega_\lambda$ if, and only if, the following is valid:

$$B_{27} := \operatorname*{ess\,sup}_{x\in[0,\infty)} \int_0^x \frac{y^\lambda u(y)}{\operatorname{ess\,sup}_{z\in(0,\infty)} v(z)\min(z^\lambda, y^\lambda)}\,dy\; w(x) < \infty.$$

Moreover, $C_1 = B_{27}$.

Remark 5.6. Using the results of Subsection 5.2, we can extend the embedding theorems for Besov spaces consider in the Subsections 2.4 and 3.3. of [13]. We will consider this in a future paper dedicated to the study of optimal embeddings.

6. A remark on the symetric version of the three weighted Hardy-type inequality

Using a change of variable $t \mapsto y^{-1}$ twice and using the fact that $f \in \Omega_\lambda$ if, and only if, $t^\lambda f(t^{-1}) \in \Omega_\lambda$, we get that the symmetric inequality of (5.1)

$$\left(\int_0^\infty \left(\int_x^\infty fu\right)^q w(x)dx\right)^{\frac{1}{q}} \leq C_1 \left(\int_0^\infty f^p v\right)^{\frac{1}{p}}, \tag{6.1}$$

holds for all $f \in \Omega_\lambda$ if, and only if, the following inequality holds

$$\left(\int_0^\infty \left(\int_0^x f\widetilde{u}\right)^q \widetilde{w}(x)dx\right)^{\frac{1}{q}} \leq C_2 \left(\int_0^\infty f^p \widetilde{v}\right)^{\frac{1}{p}},$$

for every $f \in \Omega_\lambda$, where $\widetilde{u}(x) = x^{-\lambda-2}u(x^{-1})$, $\widetilde{w}(x) = x^{-2}w(x^{-1})$, $\widetilde{v}(x) = x^{-\lambda p-2}v(x^{-1})$ and $C_1 \approx C_2$. Therefore, we can easily obtain the complete characterization of the inequality (6.1) from Theorems 5.2, 5.4 and 5.5.

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References

[1] M. A. Ariño and B. Muckenhoupt. Maximal functions on classical Lorentz spaces and Hardy's inequality with weights for nonincreasing functions. *Trans. Amer. Math. Soc.*, 320(2):727–735, 1990.

[2] C. Bennett and R. Sharpley. *Interpolation of operators*, volume 129 of *Pure and Applied Mathematics*. Academic Press, Inc., Boston, MA, 1988.

[3] J. Bergh, V. Burenkov, and L.-E. Persson. Best constants in reversed Hardy's inequalities for quasimonotone functions. *Acta Sci. Math. (Szeged)*, 59(1-2):221–239, 1994.

[4] J. Bergh and J. Löfström. *Interpolation spaces. An introduction*. Springer-Verlag, Berlin-New York, 1976. Grundlehren der Mathematischen Wissenschaften, No. 223.

[5] Y. Brudnyi and A. Shteinberg. Calderón couples of Lipschitz spaces. *J. Funct. Anal.*, 131(2):459–498, 1995.

[6] A. M. Caetano, A. Gogatishvili, and B. Opic. Sharp embeddings of Besov spaces involving only logarithmic smoothness. *J. Approx. Theory*, 152(2):188–214, 2008.

[7] A. M. Caetano, A. Gogatishvili, and B. Opic. Embeddings and the growth envelope of Besov spaces involving only slowly varying smoothness. *J. Approx. Theory*, 163(10):1373–1399, 2011.

[8] M. J. Carro, J. A. Raposo, and J. Soria. Recent developments in the theory of Lorentz spaces and weighted inequalities. *Mem. Amer. Math. Soc.*, 187(877):xii+128, 2007.

[9] A. Gogatishvili and R. Kerman. The rearrangement-invariant space $\Gamma_{p,\psi}$. *Positivity*, 18(2):319–345, 2014.

[10] A. Gogatishvili, A. Kufner, and L.-E. Persson. Some new scales of weight characterizations of the class B_p. *Acta Math. Hungar.*, 123(4):365–377, 2009.

[11] A. Gogatishvili, J. S. Neves, and B. Opic. Optimality of embeddings of Bessel-potential-type spaces into Lorentz-Karamata spaces. *Proc. Roy. Soc. Edinburgh Sect. A*, 134(6):1127–1147, 2004.

[12] A. Gogatishvili, J. S. Neves, and B. Opic. Optimal embeddings and compact embeddings of Bessel-potential-type spaces. *Math. Z.*, 262(3):645–682, 2009.

[13] A. Gogatishvili, B. Opic, S. Tikhonov, and W. Trebels. Ulyanov-type inequalities between Lorentz-Zygmund spaces. *J. Fourier Anal. Appl.*, 20(5):1020–1049, 2014.

[14] A. Gogatishvili and L. Pick. Discretization and anti-discretization of rearrangement-invariant norms. *Publ. Mat.*, 47(2):311–358, 2003.

[15] A. Gogatishvili and L. Pick. Embeddings and duality theorems for weak classical Lorentz spaces. *Canad. Math. Bull.*, 49(1):82–95, 2006.

[16] A. Gogatishvili and V. D. Stepanov. Reduction theorems for operators on the cones of monotone functions. *J. Math. Anal. Appl.*, 405(1):156–172, 2013.

[17] A. Gogatishvili and V. D. Stepanov. Reduction theorems for weighted integral inequalities on the cone of monotone functions. *Uspekhi Mat. Nauk*, 68(4(412)):3–68, 2013. [translation in Russian Math. Surveys 68 (2013), no. 4, 597–664].

[18] M. L. Gol′dman. Integral inequalities on a cone of functions with monotonicity properties. *Dokl. Akad. Nauk SSSR*, 320(5):1037–1042, 1991. [translation in Soviet Math. Dokl. 44 (1992), no. 2, 581–587].

[19] M. L. Gol′dman. On integral inequalities on the set of functions with some properties of monotonicity. In *Function spaces, differential operators and nonlinear analysis (Friedrichroda, 1992)*, volume 133 of *Teubner-Texte Math.*, pages 274–279. Teubner, Stuttgart, 1993.

[20] M. L. Gol′dman. On optimal embeddings of generalized Bessel and Riesz potentials. *Tr. Mat. Inst. Steklova*, 269:91–111, 2010. [translation in Proc. Steklov Inst. Math. 269 (2010), no. 1, 85–105].

[21] M. L. Gol′dman and D. D. Haroske. Estimates for continuity envelopes and approximation numbers of Bessel potentials. *J. Approx. Theory*, 172:58–85, 2013.

[22] M. L. Gol′dman, H. P. Heinig, and V. D. Stepanov. On the principle of duality in Lorentz spaces. *Canad. J. Math.*, 48(5):959–979, 1996.

[23] M. L. Gol′dman and R. A. Kerman. On the optimal embedding of Calderón spaces and of generalized Besov spaces. In *Function Spaces, Approximations, and Differential Equations, Collected papers dedicated to Oleg Vladimirovich Besov on his 70th birthday*, volume 243,

pages 161–193. (Nauka, Moscow), 2003. [translation in Proc. Steklov Inst. Math. 2003, no. 4 (243), 154–184].

[24] D. D. Haroske. *Envelopes and sharp embeddings of function spaces*, volume 437 of *Chapman & Hall/CRC Research Notes in Mathematics*. Chapman & Hall/CRC, Boca Raton, FL, 2007.

[25] H. Heinig and L. Maligranda. Weighted inequalities for monotone and concave functions. *Studia Math.*, 116(2):133–165, 1995.

[26] A. Kufner and L.-E. Persson. Weighted inequalities of Hardy type. World Scientific Publishing Co., Inc., River Edge, NJ, 2003.

[27] M. Křepela. Boundedness of Hardy-type operators with a kernel: integral weighted conditions for the case $0 < q < 1 \le p < \infty$. *Rev. Mat. Complut.*, 30:547–587, 2017.

[28] L. Maligranda. Weighted inequalities for quasi-monotone functions. *J. London Math. Soc. (2)*, 57(2):363–370, 1998.

[29] R. Oĭnarov. Two-sided estimates for the norm of some classes of integral operators. *Trudy Mat. Inst. Steklov.*, 204(Issled. po Teor. Differ. Funktsii Mnogikh Peremen. i ee Prilozh. 16):240–250, 1993. [translation in Proc. Steklov Inst. Math. 1994, no. 3 (204), 205–214].

[30] V. I. Ovchinnikov. Interpolation functions and the Lions-Peetre interpolation construction. *Uspekhi Mat. Nauk*, 69(4(418)):103–168, 2014. [translation in Russian Math. Surveys 69 (2014), no. 4, 681–741].

[31] L.-E. Persson, O. V. Popova, and V. D. Stepanov. Weighted Hardy-type inequalities on the cone of quasi-concave functions. *Math. Inequal. Appl.*, 17(3):879–898, 2014.

[32] L.-E. Persson, G. E. Shambilova, and V. D. Stepanov. Hardy-type inequalities on the weighted cones of quasi-concave functions. *Banach J. Math. Anal.*, 9(2):21–34, 2015.

[33] E. Sawyer. Boundedness of classical operators on classical Lorentz spaces. *Studia Math.*, 96(2):145–158, 1990.

[34] G. Sinnamon. Embeddings of concave functions and duals of Lorentz spaces. *Publ. Mat.*, 46(2):489–515, 2002.

[35] G. Sinnamon and V. D. Stepanov. The weighted Hardy inequality: new proofs and the case $p = 1$. *J. London Math. Soc. (2)*, 54(1):89–101, 1996.

[36] V. D. Stepanov. Weighted inequalities for a class of Volterra convolution operators. *J. London Math. Soc. (2)*, 45(2):232–242, 1992.

[37] V. D. Stepanov. Integral operators on the cone of monotone functions. *J. London Math. Soc. (2)*, 48(3):465–487, 1993.

[38] V. D. Stepanov. Weighted norm inequalities of Hardy type for a class of integral operators. *J. London Math. Soc. (2)*, 50(1):105–120, 1994.

[39] V. D. Stepanov. On optimal Banach spaces containing a weight cone of monotone or quasi-concave functions. *Mat. Zametki*, 98(6):907–922, 2015. [translation in Math. Notes 98 (2015), no. 5-6, 957–970].

Amiran Gogatishvili, Institute of Mathematics of the Academy of Sciences of the Czech Republic, Žitna 25, 11567 Praha 1, Czech Republic gogatish@math.caz.cz

Júlio S. Neves, CMUC, Department of Mathematics, University of Coimbra, Apartado 3008, EC Santa Cruz, 3001-454 Coimbra, Portugal jsn@mat.uc.pt