Oscillatory solutions to equations in fluid dynamics

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November 7, 2018

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Abstract

We consider the phenomenon of oscillations in the solution families to partial differential equations. To begin, we briefly discuss mechanisms preventing oscillations/concentrations and make a short excursion in the theory of compensated compactness. Pursuing the philosophy "everything what is not forbidden is allowed" we show that certain problems in fluid dynamics admit oscillatory solutions. This fact gives rise to two rather unexpected and in a way contradictory results: **(i)**many problems describing inviscid fluid motion in several space dimensions admit global-in-time (weak solution); **(ii)** the solutions are not determined uniquely by their initial data. We examine the basic analytical tool behind these rather ground breaking results - the method of convex integration applied to problems in fluid mechanics

Keywords: Oscillations, concentrations, weak solution, Euler system, Navier Stokes system, convex integration

Contents

1 Oscillations, concentrations and how to handle them

Some of the arguments below are illustrative and may be not completely rigorous, in the sense that certain hypotheses are omitted to keep the presentation clear and concise.

1.1 Oscillations

Consider an a-periodic continuous function $g : R \to R$,

$$g(x + a) = g(x) \text{ for all } x \in R, \ \int_0^a g(x)\mathrm{d}x = 0,$$

and a sequence

$$g_n(x) = g(nx),\ n = 1, 2, \dots$$

Our goal is to describe the limit $\lim_{n\to\infty} g_n$. Apparently, the sequence $g(nx)$ does not converge pointwise, not even a.a. pointwise. To capture its asymptotic behavior, we have to consider its averaged values,

$$\int_R g_n(x)\varphi(x)\ \mathrm{d}x,\ \text{where } \varphi \in C_c^\infty(R).$$

Introducing the primite function G,

$$G(x) = \int_0^x g(z)\ \mathrm{d}z$$

we easily observe that G is also continuous and periodic. Consequently, by means of the by–parts–integration formula,

$$\int_R g_n(x)\varphi(x)\ \mathrm{d}x = \int_R g(nx)\varphi(x)\ \mathrm{d}x = -\frac{1}{n}\int_R G(nx)\partial_x\varphi(x)\ \mathrm{d}x \to 0$$

as $n \to \infty$ for any smooth function φ. We say that the sequence $\{g_n\}_{n=1}^\infty$ *converges weakly* to 0, $g_n \rightharpoonup 0$. As a straightforward consequence we deduce that a sequence

$$h_n(x) = h(nx),\ \text{where } h \text{ is } a-\text{periodic},$$

$$\text{converges weakly to the integral average} \int_0^a h(x)\ \mathrm{d}x,\ h_n \rightharpoonup \int_0^a h(x)\ \mathrm{d}x.$$

Thus the weak convergence does not, in general, commute with non–linear compositions, specifically

$$g_n \rightharpoonup g \text{ does not imply } H(g_n) \rightharpoonup H(g) \text{ if } H \text{ is not linear.}$$

Convex compositions are *weakly lower semi–continuous*,

$$g_n \rightharpoonup g \text{ implies } \int_R H(g)\varphi\,\mathrm{d}x \le \liminf_{n\to\infty}\int_R H(g_n)\varphi\ \mathrm{d}x \text{ for any } \varphi \ge 0$$

whenever H is convex. This can be easily seen from the Taylor decomposition,

$$H(g_n) = H(g) + H'(g)(g_n - g) + \frac{1}{2}H''(\xi)(g_n - g)^2. \tag{1.1}$$

Formula (1.1) also reveals the difference between the weak limit of $H(g_n)$ and $H(g)$ due to the quadratic term. Indeed one can deduce that g_n converges strongly to g only if there exists a *strictly convex* function H such that $H(g_n) \rightharpoonup H(g)$.

1.2 Concentrations

Consider a sequence

$$g_n(x) = ng(nx), \text{ where } g \in C_c^\infty(-1,1),\ g(-x) = g(x),\ g \geq 0,\ \int_R g(x)\ \mathrm{d}x = 1.$$

It is easy to check that

- $$g_n(x) \to 0 \text{ as } n \to \infty \text{ for any } x \neq 0, \text{ in particular } g_n \to 0 \text{ a.a. in } R;$$

- $$\|g_n\|_{L^1(R)} = \int_R g_n(x)\ \mathrm{d}x = \int_R g(x)\ \mathrm{d}x = 1 \text{ for any } n = 1,2,\dots.$$

Next, we observe that g_n *does not* converge weakly to 0. Indeed

$$\int_R g_n(x)\varphi(x)\ \mathrm{d}x = \int_{-1/n}^{1/n} g_n(x)\varphi(x)\ \mathrm{d}x \in \left[\min_{x\in[-1/n,1/n]} \varphi(x),\ \max_{x\in[-1/n,1/n]} \varphi(x)\right] \to \varphi(0)$$

as soon as φ is continuous. As a matter of fact, the limit object cannot be identified with any integral average, it is a *measure* - the Dirac mass δ_0 concentrated at 0.

2 Equations preventing oscillations, compactness and compensated compactness

We show that families of (hypothetical) solutions of certain problems cannot exhibit oscillations and/or concentrations.

2.1 Elliptic equations

A trivial examples of problems without oscillatory solutions are elliptic equations as

$$-\Delta_x u(x) = f(x) \text{ in a bounded domain } \Omega,\ u|_{\partial\Omega} = 0. \tag{2.1}$$

Indeed integrating by parts yields immediately

$$\int_\Omega |\nabla_x u|^2\ \mathrm{d}x = \int_\Omega fu\ \mathrm{d}x \leq \|f\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)}.$$

In view of the standard Poincaré inequality, we get

$$\|u\|_{L^2(\Omega)} \lesssim \|\nabla_x u\|_{L^2(\Omega)},$$

and, in accordance with the Rellich–Kondraschev theorem, the embedding

$$W^{1,2}(\Omega) \hookrightarrow\hookrightarrow L^2(\Omega)$$

is compact. Consequently, both oscillations and concentrations are prevented by constraint imposed by the elliptic equation (2.1).

2.2 Scalar conservation laws

A more subtle example of compactness is related to solution families of nonlinear scalar conservation laws. For the sake of simplicity, consider the Burgers equation

$$\partial_t u + \partial_x f(u) = 0, \ u(t,0) = u(t,1), \ t \in (0,T). \tag{2.2}$$

First observe that for *linear* f, the problem admits oscillatory solutions. Indeed any function of the form

$$u(t,x) = v(t-x), \ v(z+1) = v(z)$$

satisfies solves (2.2) with $f(u) = u$. General linear functions $f(u) = au$ can be handled by simple rescaling.

2.2.1 Non–linear case, entropies

For any continuously differentiable function S, we easily derive the identity

$$\partial_t S(u) + \partial_x F(u) = 0, \ \text{where } F'(z) = f'(z)S'(z). \tag{2.3}$$

Although (2.3) can be formally derived from (2.2) on condition that the solutions is *smooth*, it may be viewed as an addition constraint imposed on the solution set. Convex S are called *entropies*, the associated function F_S is the entropy flux.

2.2.2 Maximum principle

Consider a convex function S such that

$$S(z) = \begin{cases} 0 \text{ for } z \leq L \\ \\ > 0 \text{ for } z > L \end{cases}$$

Integrating (2.3) over $(0,1)$ and using periodicity of u we easily obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_0^1 S(u(t,x)) \ \mathrm{d}x = 0.$$

In particular

$$\int_0^1 S(u(t,x)) \ \mathrm{d}x = 0 \text{ for all } t \geq t_0 \text{ as long as } \int_0^1 S(u(t_0,x)) \ \mathrm{d}x = 0.$$

This can be reformulated as the *maximum principle*:

$$u(t,x) \leq L \text{ for all } t \geq t_0 \text{ as soon as } u(t_0,x) \leq L.$$

In a similar fashion we conclude

$$\inf_{y\in[0,1]} u(t_0,y) \le u(t,x) \le \sup_{y\in[0,1]} u(t_0,y) \text{ for all } t \ge t_0,\ x\in[0,1]. \tag{2.4}$$

Thus concentrations in the solution set of (2.2) can be controlled by the initial data. Here and hereafter in this section we therefore assume that solutions of (2.2) are uniformly bounded.

Finally, we note that the same can be derived under a weaker stipulation than (2.3), namely

$$\partial_t S(u) + \partial_x F(u) \le 0,\ \text{where } S \text{ is convex},\ F'(z) = f'(z)S'(z). \tag{2.5}$$

2.2.3 Compensated compactness, Div–curl lemma

Our goal is to show that the family of constraints (2.3), or even (2.5) prevents oscillations in the family of solutions to (2.2) as long as f is a genuinely *non–linear* function. To this end, we recall the celebrated Div–Curl lemma of Murat and Tartar [2].

Lemma 2.1. *Let $\{\mathbf{U}_n\}_{n=1}^\infty$, $\{\mathbf{V}_n\}_{n=1}^\infty$ be two sequences of vector valued defined on a set $Q\subset R^N$ such that*

$$\mathbf{U}_n \to \mathbf{U} \text{ weakly in } L^p(Q;R^N),$$
$$\mathbf{V}_n \to \mathbf{V} \text{ weakly in } L^q(Q;R^N),$$

where

$$\frac{1}{p}+\frac{1}{q}<1.$$

In addition, let

$$\{\mathrm{div}\mathbf{U}_n\}_{n=1}^\infty \text{ be precompact in } W^{-1,s}(Q),$$
$$\{\mathbf{curl}\ \mathbf{V}_n\}_{n=1}^\infty \text{ be precompact in } W^{-1,s}(Q;R^{N\times N})$$

for some $s>1$.

Then

$$\mathbf{U}_n\cdot\mathbf{V}_n \to \mathbf{U}\cdot\mathbf{V} \text{ weakly in } L^r(Q),\ \frac{1}{r}=\frac{1}{p}+\frac{1}{q}.$$

We will not give the complete proof of this result but restrict ourselves to the situation

$$\mathrm{div}\mathbf{U}_n = 0,\ \mathbf{curl}\mathbf{V}_n = 0.$$

Moreover, as the result is local, we may assume that Q is a simply connected domain. Accordingly, we can write

$$\mathbf{V}_n = \nabla_x\Phi_n,\ \text{where } \|\Phi_n\|_{L^q(Q)} + \|\nabla_x\Phi_n\|_{L^q(Q;R^N)} \le c.$$

In particular, in accordance with the Rellich–Kondrachev compactness embedding theorem,

$$\Phi_n \to \Phi \text{ in } L^q(Q),\ \text{where } \nabla_x\Phi = \mathbf{V}.$$

Now, for a compactly supported function φ we get

$$\int_Q \mathbf{V}_n\cdot\mathbf{U}_n\varphi\ \mathrm{d}x = \int_Q \nabla_x\Phi_n\cdot\mathbf{U}_n\varphi\ \mathrm{d}x = -\int_Q \Phi_n\mathbf{U}_n\nabla_x\varphi\ \mathrm{d}x \to -\int_Q \Phi\mathbf{U}\cdot\nabla_x\varphi\ \mathrm{d}x = \int_Q \mathbf{U}\cdot\mathbf{V}\varphi\ \mathrm{d}x$$

2.2.4 Compactness for genuinely nonlinear scalar conservation law

Let $\{u_n\}_{n=1}^{\infty}$ be a sequence of solutions to (2.2) such that

$$|u_n(t,x)| \leq c \text{ for all } (t,x)$$

uniformly for $n \to \infty$.

Passing to suitable subsequences, we may suppose that

$$\begin{aligned} u_n &\to u \text{ weakly-(*) in } L^{\infty}, \\ S(u_n) &\to \overline{S(u)} \text{ weakly-(*) in } L^{\infty}, \\ F(u_n) &\to \overline{F(u)} \text{ weakly-(*) in } L^{\infty} \end{aligned}$$

for any convex entropy S with the corresponding flux F.

Seeing that $[S(u), F(u)]$ satisfy the equality (2.3), or even the inequality (2.5), we may use Div–Curl lemma to deduce

$$S_1(u_n)F_2(u_n) - F_1(u_n)S_2(u_n) \to \overline{S_1(u)}\ \overline{F_2(u)} - \overline{F_1(u)}\ \overline{S_2(u)} \tag{2.6}$$

passing again to subsequences as the case may be. Indeed we may apply Lemma 2.1, with $N = 2$,

$$\mathrm{div}_{t,x}[S_1, F_1] = \partial_t S_1 + \partial_x F_1,\ \mathrm{curl}_{t,x}[F_2, -S_2] = \partial_t S_2 + \partial_x F_2.$$

It turns out that validity of (2.6) for any convex entropies S_1, S_2 with the corresponding fluxes F_1, F_2 implies *strong convergence* of $\{u_n\}_{n=1}^{\infty}$ as soon as the flux f is a strictly convex function. Taking

$$S_1(u) = u,\ F_1(u) = f(u),\ S_2(u) = |u - U|,\ F_2(u) = \mathrm{sgn}(u - U)(f(u) - f(U)),\ U - \text{constant},$$

we deduce from (2.6)

$$\begin{aligned} &\lim_{n\to\infty} \Big[u_n \mathrm{sgn}(u_n - U)(f(u_n) - f(U)) - |u_n - U| f(u_n) \Big] \\ &\quad = \lim_{n\to\infty} u_n \lim_{n\to\infty} \Big[\mathrm{sgn}(u_n - U)(f(u_n) - f(U)) \Big] - \lim_{n\to\infty} |u_n - U| \lim_{n\to\infty} f(u_n), \end{aligned}$$

where the limits are understood in the weak sense. This relation can be rewritten as

$$\lim_{n\to\infty} \Big[|u_n - U| \Big(\overline{f(u)} - f(U) \Big) \Big] = (u - U)\overline{\mathrm{sgn}(u - U)(f(u) - f(U))} \tag{2.7}$$

Let $U = u(\tau, y)$, where (τ, y) is a Lebesgue point of u, specifically

$$\lim_{r\to 0} \frac{1}{|B_r(\tau, y)|} \int_{B_r(\tau,y)} |u - u(\tau, y)|\ \mathrm{d}x\,\mathrm{d}t = 0. \tag{2.8}$$

Now, relation (2.7) yields

$$\lim_{r\to 0}\frac{1}{|B_r(\tau,y)|}\int_{B_r(\tau,y)}\left[\lim_{n\to\infty}\left[|u_n-u(\tau,y)|\left(\overline{f(u)}-f(u(\tau,y))\right)\right]\right]\,{\rm d}x\,{\rm d}t$$
$$=\lim_{r\to 0}\frac{1}{|B_r(\tau,y)|}\int_{B_r(\tau,y)}(u-u(\tau,y))\overline{{\rm sgn}(u-u(\tau,y))(f(u)-f(u(\tau,y)))}\,{\rm d}x\,{\rm d}t=0.$$

Combining this with (2.8) we replace $u(\tau,y)$ by u in the integral on the left hand side to conclude

$$\lim_{r\to 0}\frac{1}{|B_r(\tau,y)|}\int_{B_r(\tau,y)}\left[\lim_{n\to\infty}|u_n-u|\left(\overline{f(u)}-f(u)\right)\right]\,{\rm d}x\,{\rm d}t=0$$

This relation can be interpreted that for any Lebesgue point of u

- either
$$\lim_{n\to\infty}|u_n-u|=0,$$
- or
$$\overline{f(u)}=f(u).$$

For strictly convex f this implies strong convergence of $\{u_n\}_{n=1}^{\infty}$.

3 Oscillatory solutions

Our model problem is the compressible (barotropic) Euler system describing the time evolution of the density ϱ and the velocity $\mathbf{u}$ of a compressible inviscid fluid:

$$\begin{aligned}\partial_t\varrho+{\rm div}_x(\varrho\mathbf{u})&=0,\\ \partial_t(\varrho\mathbf{u})+{\rm div}_x(\varrho\mathbf{u}\otimes\mathbf{u})+\nabla_x p(\varrho)&=0.\end{aligned}\tag{3.1}$$

For the sake of simplicity, we consider the periodic boundary conditions,

$$x\in\Omega=\left([-1,1]|_{\{-1,1\}}\right)^N.\tag{3.2}$$

The problem is supplemented by the initial conditions,

$$\varrho(0,\cdot)=\varrho_0,\ \ \mathbf{u}(0,\cdot)=\mathbf{u}_0.\tag{3.3}$$

3.1 Weak solutions

The weak formulation of problem (3.1–3.3) reads:

- $$\left[\int_\Omega \varrho\varphi \ \mathrm{d}x\right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega [\varrho\partial_t\varphi + \varrho\mathbf{u}\cdot\nabla_x\varphi] \ \mathrm{d}x\,\mathrm{d}t \tag{3.4}$$
  for any $0 \leq \tau \leq T$, and any $\varphi \in C^1([0,T]\times\Omega)$;
- $$\left[\int_\Omega \varrho\mathbf{u}\cdot\boldsymbol{\varphi} \ \mathrm{d}x\right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega [\varrho\mathbf{u}\cdot\partial_t\boldsymbol{\varphi} + \varrho\mathbf{u}\otimes\mathbf{u} : \nabla_x\varphi + p(\varrho)\mathrm{div}_x\boldsymbol{\varphi}] \ \mathrm{d}x\,\mathrm{d}t \tag{3.5}$$
  for any $0 \leq \tau \leq T$, and any $\varphi \in C^1([0,T]\times\Omega; R^N)$.

3.2 Reformulation via Helmholtz decomposition

We start by introducing the momentum $\mathbf{m}$ along with its Helmholtz decomposition

$$\mathbf{m} = \mathbf{v} + \nabla_x\Phi,\ \mathrm{div}_x\mathbf{v} = 0,\ \int_\Omega \Phi \ \mathrm{d}x = 0.$$

Accordingly, the equation of continuity reads

$$\partial_t\varrho + \Delta_x\Phi = 0,\ \varrho(0,\cdot) = \varrho_0, \tag{3.6}$$

while the momentum equation takes the form

$$\partial_t\mathbf{v} + \mathrm{div}_x\left(\frac{(\mathbf{v}+\nabla_x\Phi)\otimes(\mathbf{v}+\nabla_x\Phi)}{\varrho}\right) + \nabla_x\left(\partial_t\Phi + p(\varrho)\right) = 0,\ \mathbf{v}(0,\cdot) = \mathbf{v}_0. \tag{3.7}$$

3.2.1 Density ansatz

At this stage we can choosed and *arbitrary* (smooth) density profile $\varrho(t,x)$, $t \in [0,T]$, $x \in \Omega$ satisfying only

$$\varrho(0,\cdot) = \varrho_0,\ \partial_t\varrho(0,\cdot) = -\Delta_x\Phi_0,\ \varrho_0\mathbf{u}_0 = \mathbf{v}_0 + \nabla_x\Phi_0,$$

and

$$0 < \underline{\varrho} \leq \varrho(t,x) \leq \overline{\varrho},\ t \in [0,T],\ x \in \Omega.$$

Now, the potential Φ is uniquely determined by the elliptic problem

$$\Delta_x\Phi = -\partial_t\varrho.$$

3.2.2 Reformulation as abstract "Euler" system

From now on, we shall assume that $\mathbf{h} = \nabla_x\Phi$ and $H = \partial_t\Phi + p(\varrho)$ are given functions and look for solution of (3.7) that reads:

$$\partial_t\mathbf{v} + \mathrm{div}_x\left(\frac{(\mathbf{v}+\mathbf{h})\otimes(\mathbf{v}+\mathbf{h})}{\varrho} - \frac{1}{N}\frac{|\mathbf{v}+\mathbf{h}|^2}{\varrho}\mathbb{I}\right) + \nabla_x\left(H + \frac{1}{N}\frac{|\mathbf{v}+\mathbf{h}|^2}{\varrho}\right) = 0, \tag{3.8}$$

$$\mathrm{div}_x\mathbf{v} = 0, \tag{3.9}$$

$$\mathbf{v}(0,\cdot) = \mathbf{v}_0. \tag{3.10}$$

3.2.3 Kinetic energy

We will look for solutions with prescribed kinetic energy

$$\frac{1}{2}\frac{|\mathbf{v}+\mathbf{h}|^2}{\varrho} = E,$$

where $E = E(t,x)$ is a given non–negative function. Going back to (3.8) we get

$$H + \frac{1}{N}\frac{|\mathbf{v}+\mathbf{h}|^2}{\varrho} = H + \frac{2}{N}E;$$

whence the choice

$$E = \Lambda - \frac{N}{2}H,$$

where Λ is constant, reduces (3.8) to the "pressureless" equation

$$\partial_t \mathbf{v} + \mathrm{div}_x \left(\frac{(\mathbf{v}+\mathbf{h}) \otimes (\mathbf{v}+\mathbf{h})}{\varrho} - \frac{1}{N}\frac{|\mathbf{v}+\mathbf{h}|^2}{\varrho}\mathbb{I} \right) = 0.$$

Finally, we impose the impermeability conditions

$$\mathbf{v}\cdot\mathbf{n}|_{\partial\Omega} = 0.$$

3.3 Abstract problem

For given functions h, $\varrho \geq 0$, and $E \geq 0$, our goal is to find a weak solution for the problem:

$$\partial_t \mathbf{v} + \mathrm{div}_x \left(\frac{(\mathbf{v}+\mathbf{h}) \otimes (\mathbf{v}+\mathbf{h})}{\varrho} - \frac{1}{N}\frac{|\mathbf{v}+\mathbf{h}|^2}{\varrho}\mathbb{I} \right) = 0, \tag{3.11}$$

$$\mathrm{div}_x \mathbf{v} = 0, \tag{3.12}$$

$$\mathbf{v}\cdot\mathbf{n}|_{\partial\Omega} = 0, \tag{3.13}$$

$$\mathbf{v}(0,\cdot) = \mathbf{v}_0, \tag{3.14}$$

$$\frac{1}{2}\frac{|\mathbf{v}+\mathbf{h}|^2}{\varrho} = E. \tag{3.15}$$

The weak formulation of (3.11–3.15) reads:

- $$\mathbf{v} \in L^\infty((0,T)\times\Omega) \cap C_{\mathrm{weak}}([0,T]; L^2(\Omega; R^N)),\ \ \mathbf{v}(0,\cdot) = \mathbf{v}_0; \tag{3.16}$$

- $$\int_0^T \int_\Omega \mathbf{v} \cdot \partial_t \boldsymbol{\varphi} + \left(\frac{(\mathbf{v}+\mathbf{h}) \otimes (\mathbf{v}+\mathbf{h})}{\varrho} - \frac{1}{N} \frac{|\mathbf{v}+\mathbf{h}|^2}{\varrho} \mathbb{I} \right) : \nabla_x \boldsymbol{\varphi} \ \mathrm{d}x \, \mathrm{d}t = 0 \tag{3.17}$$

  for all $\boldsymbol{\varphi} \in C^1_c((0,T) \times \Omega)$;

- $$\int_0^T \int_\Omega \mathbf{v} \cdot \nabla_x \varphi \ \mathrm{d}x \, \mathrm{d}t = 0 \tag{3.18}$$

  for any $\varphi \in C^1([0,T] \times \overline{\Omega})$;

- $$\frac{1}{2} \frac{|\mathbf{v}+\mathbf{h}|^2}{\varrho} = E \text{ a.a. in } (0,T) \times \Omega. \tag{3.19}$$

4 Oscillatory lemma

We reproduce the basic tool of the $L^\infty-$approach in fluid mechanics called *oscillatory lemma* due to DeLellis and Székelyhidi [1].

4.1 Formulation with constant coefficients

We consider a very particular case of problem (3.11–3.15) with a specific choice of parameters: $\Omega = [-1,1]^N$, $N = 2,3$, $\mathbf{h} = 0$, $\varrho = 1$, $E > 0$ - a positive constant. In addition, we consider the "do nothing" boundary conditions, meaning there is no restriction imposed on the test functions on $\partial\Omega$. The relevant weak formulation reads:

- $$\mathbf{v} \in L^\infty((0,T) \times \Omega) \cap C_{\rm weak}([0,T]; L^2(\Omega; R^N)), \ \mathbf{v}(0,\cdot) = \mathbf{v}_0; \tag{4.1}$$

- $$\int_0^T \int_\Omega \mathbf{v} \cdot \partial_t \boldsymbol{\varphi} + \left(\mathbf{v} \otimes \mathbf{v} - \frac{1}{N} |\mathbf{v}|^2 \mathbb{I} \right) : \nabla_x \boldsymbol{\varphi} \ \mathrm{d}x \, \mathrm{d}t = 0 \tag{4.2}$$

  for all $\boldsymbol{\varphi} \in C^1_c([0,T] \times R^N)$;

- $$\int_0^T \int_\Omega \mathbf{v} \cdot \nabla_x \varphi \ \mathrm{d}x \, \mathrm{d}t = 0 \tag{4.3}$$

  for any $\varphi \in C^1_c([0,T] \times R^N)$;

- $$\frac{1}{2} |\mathbf{v}|^2 = E \text{ a.a. in } (0,T) \times \Omega. \tag{4.4}$$

4.2 Subsolutions

We introduce a the set X of subsolutions $[\mathbf{v}, \mathbb{U}]$ satisfying

- $$\mathbf{v} \in C^1([0,T] \times \overline{\Omega}; R^N),\ \mathbb{U} \in C^1\left([0,T) \times \overline{\Omega}; R^{N\times N}_{\mathrm{sym},0}\right); \tag{4.5}$$

- $$\int_0^T \int_\Omega \mathbf{v} \cdot \partial_t \boldsymbol{\varphi} + \mathbb{U} : \nabla_x \boldsymbol{\varphi}\ \mathrm{d}x\,\mathrm{d}t = 0,\ \mathbf{v}(0,\cdot) = \mathbf{v}_0 \tag{4.6}$$

  for all $\boldsymbol{\varphi} \in C^1_c([0,T] \times R^N)$;

- $$\int_0^T \int_\Omega \mathbf{v} \cdot \nabla_x \varphi\ \mathrm{d}x\,\mathrm{d}t = 0 \tag{4.7}$$

  for any $\varphi \in C^1_c([0,T] \times R^N)$;

- $$\frac{1}{2}|\mathbf{v}|^2 \leq \frac{N}{2}\lambda_{\max}\left[\mathbf{v} \otimes \mathbf{v} - \mathbb{U}\right] < E \text{ in } [0,T] \times \overline{\Omega}. \tag{4.8}$$

4.3 Oscillatory lemma

The basic result reads as follows:

Lemma 4.1. *Let $Q = (-1,1) \times (-1,1)^N$, $N = 2,3$, and let $\mathbf{h} \in R^N$, $\mathbb{V} \in R^{N\times N}_{\mathrm{sym},0}$, $E > 0$ be constant satisfying*

$$\frac{1}{2}|\mathbf{h}|^2 \leq \frac{N}{2}\lambda_{\max}\left[\mathbf{h} \otimes \mathbf{h} - \mathbb{V}\right] < E \leq \overline{E}.$$

Then there are sequences

$$\mathbf{v}_n \in C^\infty_c(Q; R^N),\ \mathbb{U}_n \in C^\infty_c(Q; R^N),\ n = 1,2,\dots$$

such that

$$\partial_t \mathbf{v}_n + \mathrm{div}_x \mathbb{U}_n = 0,$$

$$\frac{1}{2}|\mathbf{h} + \mathbf{v}_n|^2 \leq \frac{N}{2}\lambda_{\max}\left[(\mathbf{h} + \mathbf{v}_n) \otimes (\mathbf{h} + \mathbf{v}_n) - \mathbb{V} - \mathbb{U}_n\right] < E,\ n = 1,2,\dots, \tag{4.9}$$

$$\mathbf{v}_n \to 0 \text{ weakly in } L^2(Q; R^N) \text{ as } n \to \infty,$$

and

$$\liminf_{n\to\infty} \frac{1}{|Q|}\int_Q |\mathbf{v}_n|^2\ \mathrm{d}y \geq c(\overline{E})\left(E - \frac{1}{2}|\mathbf{h}|^2\right)^2. \tag{4.10}$$

Remark 4.2. As the constant fields $\mathbf{h}$ and $\mathbb{V}$ obviously solve the exact problem, the oscillatory lemma asserts the existence of compactly supported perturbation such that:

- the perturbed solution remain in the set of subsolutions;
- the oscillatory increment of the L^2-norm of the field $\mathbf{v}_n$ expressed through (4.10) "shifts" the subsolution to the neighborhood of the boundary of the set

$$\frac{1}{2}|\mathbf{v}|^2 = \frac{N}{2}\lambda_{\max}\left[\mathbf{v}\otimes\mathbf{v} - \mathbb{U}\right] = E.$$

The rest of this section is devoted to the proof of Lemma 4.1. We follow the arguments of De Lellis and Székelyhidi [1].

4.3.1 Geometric setting

We introduce the set

$$\mathfrak{C}(E) = \left\{ \mathbf{h} \in R^N,\ \mathbb{V} \in R^{N\times N}_{\mathrm{sym},0} \ \middle|\ \frac{N}{2}\lambda_{\max}\left[\mathbf{h}\otimes\mathbf{h} - \mathbb{V}\right] < E \right\}.$$

We list several facts that can be verified by direct computation:

- $$[\mathbf{h},\mathbb{V}] \mapsto \lambda_{\max}\left[\mathbf{h}\otimes\mathbf{h} - \mathbb{V}\right] \tag{4.11}$$
  is a convex function on $R^N \times R^{N\times N}_{\mathrm{sym},0}$;
- $$\frac{N}{2}\lambda_{\max}\left[\mathbf{h}\otimes\mathbf{h} - \mathbb{V}\right] \geq \frac{1}{2}|\mathbf{h}|^2 \tag{4.12}$$
  $$\frac{N-1}{2}\lambda_{\max}\left[\mathbf{h}\otimes\mathbf{h} - \mathbb{V}\right] \geq \frac{1}{2}\|\mathbb{V}\|^2 \tag{4.13}$$
  for any $\mathbf{v} \in R^N$, $\mathbb{V} \in R^{N\times N}_{\mathrm{sym},0}$, moreover,
  $$\frac{N}{2}\lambda_{\max}\left[\mathbf{h}\otimes\mathbf{h} - \mathbb{V}\right] = \frac{1}{2}|\mathbf{h}|^2 \ \Leftrightarrow\ \mathbf{h}\otimes\mathbf{h} - \frac{1}{N}|\mathbf{h}|^2\mathbb{I} = \mathbb{V}. \tag{4.14}$$

In accordance with (4.11–4.13), the set $\mathfrak{C}(E)$ is an open bounded convex set that is non–void as soon as $E > 0$. Moreover, it can be shown that

$$\partial\mathfrak{C}(E) = \left\{ \mathbf{h} \in R^N,\ \mathbb{V} = \mathbf{h}\otimes\mathbf{h} - \frac{1}{N}|\mathbf{h}|^2\mathbb{I} \ \middle|\ \frac{1}{2}|\mathbf{h}|^2 = E \right\}. \tag{4.15}$$

Our next goal is to show that for any $[\mathbf{h},\mathbb{V}]$ lying in the interior of $\mathfrak{C}(E)$, there is a sufficiently long segment parallel to a difference of two boundary elements with the center at $[\mathbf{h},\mathbb{V}]$.

Lemma 4.3. *Let $E > 0$ and $[\mathbf{h},\mathbb{V}] \in \mathfrak{C}(E)$.*

Then there exists $\mathbf{a}$, $\mathbf{b}$ *enjoying the following properties:*

- $$\frac{1}{2}|\mathbf{a}|^2 = \frac{1}{2}|\mathbf{b}|^2 = E, \ \ |\mathbf{a} \pm \mathbf{b}| > 0; \tag{4.16}$$

- *there is $L > 0$ such that*

$$[\mathbf{h} + \lambda(\mathbf{a} - \mathbf{b}), \mathbb{V} + \lambda\left(\mathbf{a} \otimes \mathbf{a} - \mathbf{b} \otimes \mathbf{b}\right)] \in \mathfrak{C}(A),$$

$$\mathrm{dist}\left[\ [\mathbf{h} + \lambda(\mathbf{a} - \mathbf{b}), \mathbb{V} + \lambda\left(\mathbf{a} \otimes \mathbf{a} - \mathbf{b} \otimes \mathbf{b}\right)]\ ; \partial\mathfrak{C}(A)\ \right] \geq \frac{1}{2}\mathrm{dist}\left[\ [\mathbf{h}, \mathbb{V}]\ ; \partial\mathfrak{C}(A)\ \right] \tag{4.17}$$

for all $-L \leq \lambda \leq L$;

- $$L|\mathbf{a} - \mathbf{b}| \geq C(N)\frac{1}{\sqrt{E}}\left(E - \frac{1}{2}|\mathbf{h}|^2\right). \tag{4.18}$$

Proof. The dimension of the space $R^N \times R^{N\times N}_{\mathrm{sym},0}$ is

$$n = \frac{N(N+3)}{2} - 1.$$

As the set $\mathfrak{C}(A)$ is convex with the boundary given by (4.15), Caratheodory's theorem yields

$$[\mathbf{h}, \mathbb{V}] = \sum_{i=1}^{n}\alpha_i\left[\mathbf{a}_i, \mathbf{a}_i \otimes \mathbf{a}_i - \frac{1}{N}E\mathbb{I}\right], \ \frac{1}{2}|\mathbf{a}_i| = E, \ \sum_{i=1}^{n}\alpha_i = 1, \ \alpha_1 \geq \alpha_2 \ldots \alpha_n \geq 0.$$

Moreover, shifting $[\mathbf{h}, \mathbb{V}]$ slightly as the case may be, we may assume

$$|\mathbf{a}_i \pm \mathbf{a}_j| > 0 \text{ whenever } i \neq j.$$

Now, consider the segment

$$[\mathbf{h} + \lambda[\mathbf{a}_j - \mathbf{a}_1], \mathbf{V} + \lambda[\mathbf{a}_j \otimes \mathbf{a}_j - \mathbf{a}_1 \otimes \mathbf{a}_1]] \ \ \lambda \in [-\alpha_j, \alpha_j], \ j > 1. \tag{4.19}$$

Since

$$[\mathbf{h}, \mathbb{V}] = \sum_{i=1}^{n}\alpha_i\left[\mathbf{a}_i, \mathbf{a}_i \otimes \mathbf{a}_i - \frac{1}{N}E\mathbb{I}\right],$$

and $\alpha_1 \geq \alpha_j$, $j > 1$, the endpoints of the segment represent a convex combination of $[\mathbf{a}_i, \mathbf{a}_i \otimes \mathbf{a}_i - \frac{1}{N}E\mathbb{I}]$; whence they belong to $\mathfrak{C}(A)$, and, in view of convexity of this set, the whole segment (4.19) is contained in $\mathfrak{C}(A)$.

Finally, we chose $\mathbf{a}_j$, $j > 1$ such that

$$\alpha_j|\mathbf{a}_j - \mathbf{a}_1| \geq \alpha_k|\mathbf{a}_k - \mathbf{a}_1| \text{ for all } k > 1, \tag{4.20}$$

and set

$$L = \frac{1}{2}\alpha_j,\ \ \mathbf{a} = \mathbf{a}_1,\ \ \mathbf{b} = \mathbf{a}_j.$$

One can easily check, using convexity, that (4.17) holds. Thus it remains to verify (4.18). To see (4.18) we realize that

$$\mathbf{h} = \sum_{i=1}^{n} \alpha_i \mathbf{a}_i;$$

whence, by virtue of (4.20),

$$|\mathbf{h} - \mathbf{a}_1| \leq \left| \sum_{i=2}^{n} \alpha_i (\mathbf{a}_i - \mathbf{a}_1) \leq n \right| \leq n\alpha_j |\mathbf{a}_j - \mathbf{a}_1|.$$

Finally,

$$2E - |\mathbf{h}|^2 = \left(\sqrt{2E} + |\mathbf{h}|\right)\left(\sqrt{2E} - |\mathbf{h}|\right) \leq 2\sqrt{2E}\left(\sqrt{2E} - |\mathbf{h}|\right) = 2\sqrt{2E}\left(|\mathbf{a}_1| - |\mathbf{h}|\right) \leq 4\sqrt{2E}nL.$$

□

4.3.2 PDE setting

Following De Lellis and Székelyhidi [1], we introduce a mapping:

$$\xi = [\xi_0, \xi_1, \dots, \xi_n] \mapsto \mathbb{A}_{\mathbf{a},\mathbf{b}}(\xi) \in R^{(N+1)\times(N+1)}_{0,\mathrm{sym}},$$

$$\mathbb{A}_{\mathbf{a},\mathbf{b}}(\xi) = \frac{1}{2}\left((\mathbb{R}\cdot\xi) \otimes (\mathbb{Q}(\xi)\cdot\xi) + (\mathbb{Q}(\xi)\cdot\xi) \otimes (\mathbb{R}\cdot\xi) \right) \tag{4.21}$$

where

$$\mathbb{Q} = \xi \otimes e_0 - e_0 \otimes \xi,\ \ \mathbb{R} = ([0,\mathbf{a}] \otimes [0,\mathbf{b}]) - ([0,\mathbf{b}] \otimes [0,\mathbf{a}]),$$

and

$$e_0 = [1, 0, \dots, 0],\ \ \mathbf{a}, \mathbf{b} \in R^N,\ \ \frac{1}{2}|\mathbf{a}|^2 = \frac{1}{2}|\mathbf{b}|^2 = E > 0,\ \ |\mathbf{a} \pm \mathbf{b}| > 0.$$

$\mathbb{A}_{\mathbf{a},\mathbf{b}}$ can be seen as a Fourier symbol of a pseudo–differential operator, where $\xi = (\xi_0, \xi_1, \dots, \xi_N)$ corresponds to $\partial = [\partial_t, \partial_{x_1}, \dots, \partial_{x_N}]$.

The following was shown in [1]:

- if $\phi \in C_c^\infty(R \times R^N)$, then

$$\mathbb{A}_{\mathbf{a},\mathbf{b}}(\partial)[\phi] = \left[\begin{array}{cc} 0 & \mathbf{w} \\ \mathbf{w} & \mathbb{H} \end{array} \right] \text{ satisfies } \partial_t \mathbf{w} + \mathrm{div}_x \mathbb{H} = 0,\ \mathrm{div}_x \mathbf{w} = 0; \tag{4.22}$$

- for

$$\eta_{\mathbf{a},\mathbf{b}} = -\frac{1}{(|\mathbf{a}||\mathbf{b}| + \mathbf{a}\cdot\mathbf{b})^{2/3}} \Big[[0,\mathbf{a}] + [0,\mathbf{b}] - (|\mathbf{a}||\mathbf{b}| + \mathbf{a}\cdot\mathbf{b})\, e_0 \Big],\ \ \psi \in C^\infty(R), \tag{4.23}$$

  we have

$$\mathbb{A}_{\mathbf{a},\mathbf{b}}(\partial)[\psi([t,\mathbf{x}]\cdot\eta_{\mathbf{a},\mathbf{b}})] = \psi'''([t,\mathbf{x}]\cdot\eta_{\mathbf{a},\mathbf{b}}) \left[\begin{array}{cc} 0 & \mathbf{a}-\mathbf{b} \\ \mathbf{a}-\mathbf{b} & \mathbf{a}\otimes\mathbf{a} - \mathbf{b}\otimes\mathbf{b} \end{array} \right]. \tag{4.24}$$

4.3.3 Proof of oscillatory lemma

We are ready to prove Lemma 4.1.

Step 1
Given $[\mathbf{h}, \mathbb{V}] \in \mathfrak{C}(E)$, we identify the vectors $\mathbf{a}$, $\mathbf{b}$ satisfying (4.16–4.18).

Step 2
$\mathbf{a}$, $\mathbf{b}$ being fixed, we consider the operator $\mathbb{A}_{\mathbf{a},\mathbf{b}}$ and the vector $\eta_{\mathbf{a},\mathbf{b}}$ as in (4.21–4.23).

Step 3
Let $\varphi \in C_c^\infty(Q)$ such that

$$0 \leq \varphi \leq 1,\ \varphi(t,x) = 1 \text{ whenever } -\frac{1}{2} \leq t \leq \frac{1}{2},\ -\frac{1}{2} \leq x_j \leq \frac{1}{2},\ j = 1, \dots, N.$$

The vectors $\mathbf{v}_n$, $\mathbb{U}_n$ can be taken in the form

$$\begin{bmatrix} 0 & \mathbf{v}_n \\ \mathbf{v}_n & \mathbb{U}_n \end{bmatrix} = \mathbb{A}_{\mathbf{a},\mathbf{b}}(\partial) \left[\varphi \frac{L}{n^3} \cos\left(n[t,x] \cdot \eta_{\mathbf{a},\mathbf{b}}\right) \right],$$

where L is the constant form (4.17), (4.18). In accordance with (4.22), the functions $\mathbf{v}_n$, $\mathbb{U}_n$ satisfy

$$\partial_t \mathbf{v}_n + \mathrm{div}_x \mathbb{U}_n = 0,\ \mathrm{div}_x \mathbf{v}_n = 0$$

and are compactly supported in Q. Moreover, in view of (4.24) and the fact that $\mathbb{A}_{\mathbf{a},\mathbf{b}}$ is a differential operator of third order, we get

$$\mathbb{A}_{\mathbf{a},\mathbf{b}}(\partial) \left[\varphi \frac{L}{n^3} \cos\left(n[t,x] \cdot \eta_{\mathbf{a},\mathbf{b}}\right) \right] = \varphi \sin\left(n[t,x] \cdot \eta_{\mathbf{a},\mathbf{b}}\right) L \begin{bmatrix} 0 & (\mathbf{a}-\mathbf{b}) \\ (\mathbf{a}-\mathbf{b}) & \mathbf{a} \otimes \mathbf{a} - \mathbf{b} \otimes \mathbf{b} \end{bmatrix} + \frac{1}{n} R_n$$

with $|R_n|$ uniformly bounded for $n \to \infty$. This yields (4.9) for n large enough, and, in view of (4.18), relation (4.10) follows. We have proved Lemma 4.1.

5 Ill posedness of the Euler system in the space dimension $N = 2, 3$

As a corollary of the existence of oscillatory subsolutions we show that the compressible Euler system is basically ill posed in the class of weak solutions. The basic result in this direction concerns the "pressureless" incompressible Euler system:

$$\begin{aligned} \mathrm{div}_x \mathbf{v} &= 0, \\ \partial_t \mathbf{v} + \mathrm{div}_x \left(\mathbf{v} \otimes \mathbf{v} - \frac{1}{N} |\mathbf{v}|^2 \right) &= 0, \end{aligned} \tag{5.1}$$

with the boundary conditions

$$\mathbf{v} \cdot \mathbf{n}|_{\partial\Omega} = 0, \tag{5.2}$$

and prescribed energy

$$\frac{1}{2}|\mathbf{v}|^2 = E \text{ a.a. in } (0,T) \times \Omega. \tag{5.3}$$

Note that if $E > 0$ is constant, then the equation of continuity reduces to

$$\partial_t \mathbf{v} + \mathrm{div}_x (\mathbf{v} \otimes \mathbf{v}) = 0;$$

whence $\mathbf{v}$ solves the *compressible* Euler system (3.1) with $\varrho \equiv 1$!

5.1 Infinitely many weak solutions to problem (5.1–5.3)

Our goal will be to show that given a continuously differentiable solenoidal field $\mathbf{v}_0$, problem (5.1–5.3) possesses infinitely many weak solutions starting from $\mathbf{v}_0$ provided $E > 0$ is chosen large enough. The main tool is the oscillatory lemma proved in the previous section.

5.1.1 The set of subsolutions

We consider the set of subsolutions introduced in (4.5–4.8) with *constant* energy $E > 0$. Let the initial velocity $\mathbf{v}_0$ be a smooth vector field, $\mathrm{div}_x \mathbf{v}_0 = 0$, $\mathbf{v}_0 \cdot \mathbf{n}|_{\partial\Omega} = 0$. Taking $\mathbf{v} \equiv \mathbf{v}_0$ and $\mathbb{U} \equiv = 0$ we observe easily that (4.5–4.7) are satisfied. In addition, (4.8) holds provided E is large enough. We conclude that the subsolution set is non–empty.

Next, we observe that for E bounded, the set of function (subsolutions) satisfying (4.5–4.7) is **(i)** bounded in $L^\infty((0,T) \times \Omega)$, **(ii)** a subset of $C_{\mathrm{weak}}([0,T]; L^2(\Omega; R^N))$. Consequently, it is metrizable by the metrics

$$d(\mathbf{u}, \mathbf{v}) = \sum_{k=1}^{\infty} \frac{1}{k^2} \frac{| < \mathbf{u} - \mathbf{v}; \phi_k > |}{1 + | < \mathbf{u} - \mathbf{v}; \phi_k > |}, \ \phi_k \text{ a basis of } L^2(\Omega; R^N).$$

We consider the topological metric space X - the completion of the set of subsolutions with respect to the metrics d. It is easy to check that:

- for any $\mathbf{v} \in X$, there is $\mathbb{U} \in L^\infty((0,T) \times \Omega; R^{N \times N}_{0,\mathrm{sym}})$ such that (4.5–4.7) are satisfied;
- $$\frac{1}{2}|\mathbf{v}|^2 \leq \frac{N}{2} \lambda_{\max} [\mathbf{v} \otimes \mathbf{v} - \mathbb{U}] \leq E. \tag{5.4}$$

Note that (5.4) follows from (4.8) and the fact that

$$(\mathbf{v}, \mathbb{U}) \mapsto \frac{N}{2} \lambda_{\max} [\mathbf{v} \otimes \mathbf{v} - \mathbb{U}]$$

is convex.

5.1.2 A Baire category argument

We introduce a concave functional

$$I(\mathbf{v}) = \int_0^T \int_\Omega \left(E - \frac{|\mathbf{v}|^2}{2} \right) \ \mathrm{d}x \, \mathrm{d}t$$

defined on the space X. As X is a complete metric space, the points of continuity of I form a residual set, in particular they are dense in X.

Our final goal is to show that if $\mathbf{v}$ is a point of continuity of I, then $I(\mathbf{v}) = 0$, or, equivalently,

$$\frac{1}{2}|\mathbf{v}|^2 = E \text{ a.a. in } (0,T) \times \Omega, \ \mathbb{U} = \mathbf{v} \otimes \mathbf{v} - \frac{1}{N}|\mathbf{v}|^2 \mathbb{I}.$$

Suppose that $I(\mathbf{v}) > 0$ for some $\mathbf{v} \in X$. Then there is a sequence of subsolutions $\{\mathbf{v}_n\}_{n=1}^\infty$, with the corresponding fluxes $\{\mathbb{U}_n\}_{n=1}^\infty$ such that:

$$\mathbf{v}_n \to \mathbf{v} \text{ in } X, \ I(\mathbf{v}_n) \to I(\mathbf{v}) \geq \delta > 0 \text{ as } n \to \infty.$$

By virtue of Oscillatory Lemma (Lemma 4.1), we may construct a sequence

$$\mathbf{v}_{m,n} \to \mathbf{v}_n \text{ in } X \text{ as } m \to \infty$$

satisfying

$$\liminf_{m \to \infty} \int_0^T \int_\Omega \frac{1}{2} \frac{|\mathbf{v}_n + \mathbf{v}_{m,n}|^2}{2} \ \mathrm{d}x \, \mathrm{d}t \geq \int_0^T \int_\Omega \frac{|\mathbf{v}_n|^2}{2} \ \mathrm{d}x \, \mathrm{d}t + c_1(\Omega, E) \int_0^T \int_\Omega \left(E - \frac{1}{2}|\mathbf{v}_n|^2 \right)^2 \ \mathrm{d}x \, \mathrm{d}t$$
$$\geq \int_0^T \int_\Omega \frac{|\mathbf{v}_n|^2}{2} \ \mathrm{d}x \, \mathrm{d}t + c_2(\Omega, E) I(\mathbf{v}_n)^2 \geq \int_0^T \int_\Omega \frac{|\mathbf{v}_n|^2}{2} \ \mathrm{d}x \, \mathrm{d}t + c_3(\Omega, E)\delta^2.$$

Consequently, if $\delta > 0$, we can construct a sequence $\{\mathbf{w}_n\}_{n=1}^\infty$,

$$\mathbf{w}_n \to \mathbf{v} \text{ in } X, \ \limsup_{n \to \infty} I(\mathbf{w}_m) < I(\mathbf{v}),$$

meaning $\mathbf{v}$ cannot be a point of continuity of I.

As each point of continuity of I represent a weak solution of (5.1), we obtain the following result.

Theorem 5.1. *Let $\Omega \subset R^N$, $N = 2, 3$ be a bounded smooth domain. Let $\mathbf{v}_0 \in C^1(\overline{\Omega}; R^N)$ be given such that*

$$\mathrm{div}_x \mathbf{v}_0 = 0, \ \mathbf{v}_0 \cdot \mathbf{n}|_{\partial\Omega} = 0.$$

Then there exists $E_0 > 0$ such that for any $E \geq E_0$, the problem (5.1–5.3) admits inifinitely many weak solutions satisfying

$$\frac{1}{2}|\mathbf{v}|^2 = E \textit{ for a.a. } (t,x) \in (0,T) \times \Omega. \tag{5.5}$$

Note carefully that solutions $\mathbf{v}$ obtained in Theorem 5.1 solve the compressible Euler system (3.1 – 3.3) with constant density $\varrho \equiv 1$.

References

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