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Eduard Feireisl

Madalina Petcu

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Eduard Feireisl * Madalina Petcu

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Institute of Mathematics of the Academy of Sciences of the Czech Republic
Žitná 25, CZ-115 67 Praha 1, Czech Republic
and
Technical University Berlin, Institute of Mathematics
Strasse des 17. Juni 136, 10623 Berlin, Germany

Laboratoire de Mathématiques et Applications, UMR CNRS 7348 - SP2MI
Université de Poitiers, Boulevard Marie et Pierre Curie - Téléport 2
86962 Chasseneuil, Futuroscope Cedex, France
The Institute of Mathematics of the Romanian Academy, Bucharest, Romania
and
The Institute of Statistics and Applied Mathematics of the Romanian Academy, Bucharest, Romania

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Abstract

We consider a model of a two–phase flow based on the phase field approach, where the fluid bulk velocity obeys the standard Navier–Stokes system while the concentration difference of the two fluids plays a role of order parameter governed by the Allen–Cahn equations. Possible thermal fluctuations are incorporated through a random forcing term in the Allen–Cahn equation. We show that suitable *dissipative* martingale solutions satisfy a stochastic version of the relative energy inequality. This fact is used for showing the weak–strong uniqueness principle both pathwise and in law.

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1 Introduction

We consider a model of a two–phase flow based on the phase field approach, where the fluid bulk velocity obeys the standard Navier–Stokes system while the concentration difference of the two fluids plays a role of order parameter governed by the Allen–Cahn equations. Possible thermal fluctuations are incorporated through a random forcing term in the Allen–Cahn equation. Consistently, we impose a random forcing also in the momentum equation. The reader may consult the review papers by Anderson, McFadden, and Wheeler [1], Lowengrub and Truskinovski [13] for the general physical background, and Debussche, Goudenège [4], Goudenège [10], Goudenège and Manca [11], Gal and Tachim-Medjo [7], [8], Tachim-Medjo [14] or Scarpa [16] for the stochastic aspects of the problem.

1.1 Field equations

The basic field variables (unknowns) describing the mixture at a given time $t \in (0,T)$ and a spatial position $x \in Q \subset \mathbf{R}^3$ are the macroscopic fluid velocity $\mathbf{u} = \mathbf{u}(t,x)$, and the order parameter (concentration difference) $c = c(t,x)$ satisfying the following system of equations:

$$\begin{aligned}
&\mathrm{d}\mathbf{u} = (\nu\Delta\mathbf{u} - (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p - \varepsilon\mathrm{div}\,(\nabla c \otimes \nabla c))\mathrm{d}t + \sigma_1(\mathbf{u},c)\mathrm{d}\mathbf{W}^1,\\
&\mathrm{d}c = (\varepsilon\Delta c - \mathbf{u}\cdot\nabla c - \frac{1}{\varepsilon}f(c))\mathrm{d}t + \sigma_2(\mathbf{u},c)\mathrm{d}\mathbf{W}^2,\\
&\nabla\cdot\mathbf{u} = 0.
\end{aligned} \tag{1.1}$$

It standard to interpret the pressure p in the momentum equation as the Lagrange multiplier associated to the incompressibility constraint. Supplemented with the initial conditions

$$\mathbf{u}(0,\cdot) = \mathbf{u}_0,\ \ c(0,\cdot) = c_0, \tag{1.2}$$

and the boundary conditions

$$\mathbf{u} = 0,\ \frac{\partial c}{\partial \mathbf{n}} = 0 \text{ on } \partial Q \times (0,T),\ \text{ where } \mathbf{n} \text{ is the outer normal vector,} \tag{1.3}$$

the Navier–Stokes–Allen–Cahn (NSAC) system (1.1 – 1.3) is, at least formally, a well posed problem.

The function f is the derivative of a double-well potential $F \in \mathcal{C}^2(\mathbf{R})$, with two local minima ± 1. In addition, we suppose that F is positive and f globally Lipschitz. The constant $\nu > 0$ corresponds to the kinematic viscosity while $\varepsilon > 0$ is a constant proportional to the width of the interface.

Remark 1.1. As a matter of fact, the behavior of the potential F outside the natural physical range of the order parameter $-1 \leq c \leq 1$ is irrelevant as we show that this property is time–invariant, meaning, a suitable variant of the comparison principle holds for c.

1.2 Random forcing

The random forcing will be incorporated in the mathematical formulation as a stochastic integral of Itô's type. Accordingly, all quantities in (1.1) must be interpreted as *random variables* with respect to a stochastic basis $[\Omega, \mathfrak{B}, \mathcal{P}]$, where Ω is a probability space, $\mathfrak{B}$ a field of measurable sets, and $\mathcal{P}$ a probability measure. $\mathbf{W}^1$ and $\mathbf{W}^2$ are two cylindrical Wiener processes in a separable Hilbert space $\mathfrak{U}$ defined on the probability space $[\Omega, \mathfrak{B}, \mathcal{P}]$. We denote by $\{\mathfrak{F}_t\}_{t\geq 0}$ a complete right–continuous filtration in Ω, non–anticipative with respect to $\mathbf{W}^i$, $i = 1, 2$. We assume that $\mathbf{W}^i$, $i = 1, 2$ are formally given by the expansion

$$\mathbf{W}^i(t) = \sum_{k\geq 1} e_k W_k^i(t),$$

where $\{W_k^i\}_{k\geq 1}$ is a family of mutually independent real-valued Brownian motions and $\{e_k\}_{k\geq 1}$ is an orthonormal basis on $\mathfrak{U}$. We also define the auxiliary space $\mathfrak{U}_0$ containing $\mathfrak{U}$, that is defined by:

$$\mathfrak{U}_0 = \{v = \sum_{k=1}^{\infty} \alpha_k e_k \ : \ \sum_{k=1}^{\infty} \frac{\alpha_k^2}{k^2} < \infty\},$$

endowed with the scalar product:

$$(u, v)_{\mathfrak{U}_0} = \sum_{k=1}^{\infty} \frac{\alpha_k \beta_k}{k^2}, \text{ for } u = \sum_k \alpha_k e_k, \ u = \sum_k \beta_k e_k.$$

The stochastic forcing takes the following form:

$$\sigma^i(\mathbf{u}, c)\mathrm{d}\mathbf{W}^i = \sum_{k\geq 1} \sigma_k^i(\mathbf{u}, c)\mathrm{d}W_k^i, \ i = 1, 2,$$

with suitable restrictions on the growth of the diffusion coefficients σ_k^i specified below.

1.3 Main goals

The NSAC system (1.1), (1.3) admits a natural *energy functional*

$$\mathcal{E}(\mathbf{u}, c) = \int_Q \left(\frac{1}{2}|\mathbf{u}|^2 + \frac{1}{\varepsilon}F(c) + \frac{\varepsilon}{2}|\nabla c|^2 \right) \mathrm{d}x. \tag{1.4}$$

Suppose, for a moment, that $(\mathbf{u}, c)$ is a smooth solution. Then we can apply Itô's calculus and, after a bit tedious but straightforward manipulation of the equations in (1.1), we deduce the total energy balance

$$
\begin{aligned}
&\mathrm{d}\mathcal{E}(\mathbf{u},c) + (\nu|\nabla\mathbf{u}|_{L^2}^2 + \varepsilon|\Delta c - \frac{1}{\varepsilon}f(c)|_{L^2}^2)\mathrm{d}t = \sum_{k=1}^{\infty}\left(\int_Q \mathbf{u}\cdot\sigma_k^1(\mathbf{u},c)\ \mathrm{d}x\right)\mathrm{d}W_k^1 \\
&+ \frac{1}{\varepsilon}\sum_{k=1}^{\infty}\left(\int_Q (\frac{1}{\varepsilon}f(c) - \Delta c)\sigma_k^2(\mathbf{u},c)\ \mathrm{d}x\right) dW_k^2 + \frac{1}{2}\sum_{k=1}^{\infty}\int_Q |\sigma_k^1(\mathbf{u},c)|^2 \mathrm{d}x\ \mathrm{d}t \\
&+ \frac{\varepsilon}{2}\sum_{k=1}^{\infty}\int_Q |\nabla\sigma_k^2(\mathbf{u},c)|^2\mathrm{d}x\ \mathrm{d}t + \frac{1}{2\varepsilon}\sum_{k=1}^{\infty}\int_Q f'(c)|\sigma_k^2(\mathbf{u},c)|^2\mathrm{d}x\ \mathrm{d}t.
\end{aligned}
\tag{1.5}
$$

Similarly to [2], we focus on *dissipative* solutions of the stochastic system (1.1) satisfying a suitable form of (1.5). In particular, we introduce the concept of *dissipative martingale* (DM) solution, for which the associated energy inequality is incorporated as an inseparable part of its definition. The (DM) solutions can be seen as analogues of the martingale solutions introduced by Flandoli and Romito [6] defined as probability measures on the canonical trajectory spaces, whereas the total energy is considered as an a.a. supermartingale.

Our main goal is to show:

- a variant of the *relative energy inequality* representing a suitable "distance" between a (DM) solution and another stochastic process defined on the same probability space;
- as a corrollary: the weak–strong uniqueness principle. Specifically, a weak solution coincides with the strong solution emanating from the same initial data as long as the latter exists. In the case when the weak and strong solutions are defined on different probability spaces and their data coincide in law, the solutions coincide in law.

The paper is organized as follows. In Section 2, we collect the basic concepts and definitions, in particular, we introduce the dissipative martingale solutions. In Section 3, we introduce the relative energy inequality – the main tool used in the present paper. Finally, in Section 4, we introduce the strong solutions to the problem and show the weak–strong uniqueness principle.

2 Preliminaries, dissipative martingale solutions

We start by introducing the function spaces related to the NSAC model (see [12], [18]). Let $\mathcal{C}^{\infty}_{0,div}(Q)$ be the space of all divergence free vectors in $(\mathcal{C}_0^{\infty}(Q)^3)$. We denote by H the closure of $\mathcal{C}^{\infty}_{0,div}(Q)$ in $(L^2(Q))^3$ and we set $V_1 = (H_0^1(Q))^3 \cap H$, $V_2 = (H^2(Q))^3 \cap V_1$. Next, let $\mathcal{W} = \{\phi \in \mathcal{C}^{\infty}(Q), \partial_{\mathbf{n}}\phi = 0 \ \text{ on } \partial Q\}$, and $\mathcal{W}_s$ the closure of $\mathcal{W}$ in $H^s(Q)$ for $s \in \mathbf{N}^+$. Finally, we set

$$\mathcal{H} = H \times \mathcal{W}_1,\ \mathcal{V} = V_1 \times \mathcal{W}_2.$$

2.1 Diffusion coefficients

We impose the following conditions on $\sigma = (\sigma^1, \sigma^2)$:

$$\sigma(\mathbf{u}, c, t) = (\sigma^1(\mathbf{u}, c, t), \sigma^2(\mathbf{u}, c, t)) : \mathcal{H} \times [0, T] \to \mathcal{L}_2(\mathfrak{U}, \mathcal{H}), \tag{2.1}$$

is $\mathcal{B}(\mathcal{H} \times [0, T], \mathcal{B}(\mathcal{L}_2(\mathfrak{U}, H)))$-measurable, essentially bounded in time and continuous in $(\mathbf{u}, c)$:

$$\sum_k \|\sigma_k(\mathbf{u}^*, c^*) - \sigma_k(\mathbf{u}, c)\|^2_{\mathcal{H}} \lesssim \|(\mathbf{u}^*, c^*) - (\mathbf{u}, c)\|^2_{\mathcal{H}}, \ \sum_k \|\sigma_k(\mathbf{u}, c)\|^2_{\mathcal{H}} \lesssim 1, \tag{2.2}$$

uniformly in $t \in [0, T]$ for all $(\mathbf{u}^*, c^*), (\mathbf{u}, c) \in \mathcal{H}$. Moreover,

$$\sigma(\mathbf{u}, c, t) = (\sigma^1(\mathbf{u}, c, t), \sigma^2(\mathbf{u}, c, t)) : \mathcal{V} \times [0, T] \to \mathcal{L}_2(\mathfrak{U}, \mathcal{V}), \tag{2.3}$$

is $\mathcal{B}(\mathcal{V} \times [0, T], \mathcal{B}(\mathcal{L}_2(\mathfrak{U}, H)))$-measurable essentially bounded in time and continuous in $(\mathbf{u}, c)$:

$$\sum_k \|\sigma_k(\mathbf{u}^*, c^*) - \sigma_k(\mathbf{u}, c)\|^2_{\mathcal{V}} \lesssim \|(\mathbf{u}^*, c^*) - (\mathbf{u}, c)\|^2_{\mathcal{V}}, \ \sum_k \|\sigma_k(\mathbf{u}, c)\|^2_{\mathcal{V}} \lesssim 1, \tag{2.4}$$

uniformly in $t \in [0, T]$ for all $(\mathbf{u}^*, c^*), (\mathbf{u}, c) \in \mathcal{V}$. In addition, σ^2 satisfies

$$\sum_k |\sigma^2_k(\mathbf{u}, c, t)|_{L^2(Q)} < \infty, \ \sum_k |\sigma^2_k(\mathbf{u}, c, t) - \sigma^2_k(\mathbf{u}, c^*, t)|_{L^2(Q)} \lesssim |c - c^*|_{L^2(Q)}, \tag{2.5}$$

uniformly in $t \in [0, T]$ for all c, $c^* \in L^2(Q)$, $\mathbf{u} \in V_1$.

Here and hereafter, $A \lesssim B$ means there is a positive constant C such that $A \le CB$, similarly $A \gtrsim B$ stands for $A \ge CB$, and $A \approx B$ means $A \lesssim B$ and $B \lesssim A$.

Finally, we suppose that the noise $\sigma^2 \mathrm{d}\mathbf{W}^2$ is multiplicative with respect to the constant states -1 and 1, which translates into

$$\sigma^2_k(\mathbf{u}, -1, t) = \sigma^2_k(\mathbf{u}, 1, t) = 0 \text{ for all } \mathbf{u}, \ t \in [0, T], \text{for all } k. \tag{2.6}$$

As we shall see below, hypothesis (2.6) forces the order parameter c to remain in the physically relevant range $c \in [-1, 1]$.

2.2 Dissipative martingale solutions

Martingale solutions, in general, may live in a different probability space than Ω, whereas the process W is considered as an integral part of the solution. Dissipative martingale solutions satisfy, in addition, a suitable form of the energy balance (inequality).

Definition 2.1. Let μ_0 be a Borel probability measure on the space $\mathcal{H}$ such that

$$\int_{\mathcal{H}} \Big| \int_Q \frac{1}{2}|\mathbf{u}|^2 + \frac{1}{\varepsilon}F(c) + \frac{\varepsilon}{2}|\nabla c|^2 dx \Big|^\beta d\mu_0(\mathbf{u}, c) < \infty$$

for all $\beta \ge 1$.

The quantity $((\Omega, \mathfrak{B}, \{\mathfrak{F}_t\}_{t\ge0}, \mathcal{P}), \mathbf{u}, c, \mathbf{W} = (\mathbf{W}^1, \mathbf{W}^2))$ is called a *dissipative martingale (DM) solution* to the Navier-Stokes-Allen-Cahn system (1.1), (1.3) with the initial law μ_0 if:

1. $(\Omega, \mathfrak{B}, \{\mathfrak{F}_t\}_{t\geq 0}, \mathcal{P})$ is a stochastic basis with a complete right-continous filtration;
2. $\mathbf{W}^i$ is a cylindrical Wiener process, with $\{\mathfrak{F}_t\}_{t\geq 0}$ non–anticipative with respect $\mathbf{W}^i$, $i = 1, 2$;
3. $\mathbf{u} \in \mathcal{C}_w([0, T], H)$, $c \in \mathcal{C}_w([0, T], \mathcal{W}_1)$ are $\mathfrak{F}_t$-adapted random processes;
4. $(\mathbf{u}(0, \cdot), c(0, \cdot))$ are $\mathfrak{F}_0$-measurable random variables on $\mathcal{H}$ such that $\mu_0 = \mathcal{L}[\mathbf{u}(0, \cdot), c(0, \cdot)]$;
5. the momentum balance holds $\mathcal{P}-$a.s.:

$$\begin{aligned}
&-\int_0^\tau \phi_t \int_Q \mathbf{u} \cdot \varphi \mathrm{d}x\mathrm{d}t + \Big[\int_Q \phi \mathbf{u} \cdot \varphi \mathrm{d}x\Big]_{t=0}^{t=\tau} + \int_0^\tau \int_Q \nu \nabla \mathbf{u} \cdot \nabla \varphi \mathrm{d}x\mathrm{d}t = \\
&-\int_0^\tau \phi \int_Q [(\mathbf{u} \cdot \nabla)\mathbf{u}] \cdot \varphi \mathrm{d}x\mathrm{d}t + \varepsilon \int_0^\tau \phi \int_Q (\nabla c \otimes \nabla c) : \nabla \varphi \mathrm{d}x\mathrm{d}t \\
&+ \int_0^\tau \phi \sum_{k=1}^\infty \left(\int_Q \sigma_k^1(\mathbf{u}, c) \cdot \varphi \mathrm{d}x \right) \mathrm{d}W_k^1,
\end{aligned} \tag{2.7}$$

   for any $0 \leq \tau \leq T$, and all deterministic test functions $\phi \in \mathcal{C}_c^\infty([0, T))$, $\varphi \in \mathcal{C}_c^\infty(Q)$, $\nabla \cdot \varphi = 0$;
6. the equation for the order parameter holds $\mathcal{P}-$a.s.:

$$\mathrm{d}c - \varepsilon \Delta c \, \mathrm{d}t = \left(-\mathbf{u} \cdot \nabla c + \frac{1}{\varepsilon} f(c) \right) \mathrm{d}t + \sigma_2(\mathbf{u}, c) \; \mathrm{d}\mathbf{W}^2, \; \frac{\partial c}{\partial \mathbf{n}}|_{\partial\Omega} = 0, \; c(0, \cdot) = c_0; \tag{2.8}$$

7. the energy inequality holds $\mathcal{P}$-a.s.:

$$\begin{aligned}
&\Big[\phi(t)\mathcal{E}(\mathbf{u}, c)(t)\Big]_{t=0}^{t=\tau} - \int_0^\tau \phi_t \mathcal{E}(\mathbf{u}, c)(t) dt + \int_0^\tau \phi(\nu |\nabla \mathbf{u}|^2_{L^2(Q)} + |\varepsilon \Delta c - \frac{1}{\varepsilon} f(c)|^2_{L^2(Q)}) \mathrm{d}t \\
&\leq \frac{1}{2} \int_0^\tau \phi \sum_{k=1}^\infty \int_Q |\sigma_k^1(\mathbf{u}, c)|^2 \mathrm{d}x\mathrm{d}t + \frac{\varepsilon}{2} \int_0^\tau \phi \sum_{k=1}^\infty \int_Q |\nabla \sigma_k^2(\mathbf{u}, c)|^2 \mathrm{d}x\mathrm{d}t \\
&+ \frac{1}{2\varepsilon} \int_0^\tau \phi \sum_{k=1}^\infty \int_Q f'(c) |\sigma_k^2(\mathbf{u}, c)|^2 \mathrm{d}x\mathrm{d}t + \int_0^\tau \phi \sum_{k=1}^\infty \left(\int_Q \mathbf{u} \sigma_k^1(\mathbf{u}, c) \mathrm{d}x \right) \mathrm{d}W_k^1 \\
&+ \int_0^\tau \phi \sum_{k=1}^\infty \left(\int_Q (\frac{1}{\varepsilon} f(c) - \Delta c) \sigma_k^2(\mathbf{u}, c) \mathrm{d}x \right) \mathrm{d}W_k^2
\end{aligned} \tag{2.9}$$

   for a.a. $0 \leq \tau \leq T$ and for all deterministic test functions $\phi \geq 0$, $\phi \in \mathcal{C}_c^\infty([0, T))$.

Remark 2.2. As a matter of fact, the energy functional $\mathcal{E}(\mathbf{u}, c)$ is convex, thus weakly lower semi–continuous, in particular the energy inequality (2.9) holds for *any* $\tau \in [0, T]$.

The existence of weak solutions to the deterministic version of the system was shown in [18]. Tachim-Medjo [17] proved the existence of weak martingale solutions. The existence of *dissipative martingale solutions* requires incorporating the energy inequality in the approximate system. This can be achieved by modifying [17] or by using the mixed approximation scheme introduced in [5].

2.3 Maximum principle for the order parameter

We conclude this preliminary part by observing that the order parameter c remains confined to its natural range

$$-1 \leq c \leq 1$$

as long as the same holds for the initial data and σ^2 satisfies (2.6).

For the deterministic Allen-Cahn model, it is well-known that the maximum principle holds when the potential is regular. For the deterministic Navier-Stokes-Allen-Cahn model the weak maximum principle is also known, one can easily prove that both strong and weak solutions for the convective Allen-Cahn equation satisfy the maximum principle (see [12]). The maximum principle for certain types of stochastic partial differential equations with multiplicative noise was proved by [3]. Under the hypothesis (2.5)-(2.6), the results from [3], Section 5.5 applied to (2.8) give rise to:

$$\begin{aligned}\mathbb{E}|(c+1)^-(t)|^2_{L^2(Q)} + \frac{2}{\varepsilon}\mathbb{E}\int_0^t\int_Q (c+1)^- f(c)dxds \lesssim \mathbb{E}|(c_0+1)^-|^2_{L^2(Q)}\\ - 2\mathbb{E}\int_0^t\int_Q (c+1)^-(\mathbf{u}\cdot\nabla)c\mathrm{d}x\mathrm{d}s - 2\varepsilon\mathbb{E}\int_0^t\int_Q |\nabla(c+1)^-|^2\mathrm{d}x\mathrm{d}s + \mathbb{E}\int_0^t\int_Q |(c+1)^-|^2\mathrm{d}x\mathrm{d}s.\end{aligned} \tag{2.10}$$

The second term on the left hand side is non–negative since F is monotone on $(-\infty,-1)\cup(1,\infty)$. Furthermore, if

$$-1 \leq c_0 \leq 1,$$

then $\mathbb{E}|(c_0+1)^-|^2_{L^2(Q)} = 0$. Finally, the second integral on the right hand side vanishes,

$$\mathbb{E}\int_0^t\int_Q (c+1)^-(\mathbf{u}\cdot\nabla)cdxds = \mathbb{E}\int_0^t\int_Q (\mathbf{u}\cdot\nabla)|(c+1)^-|^2\mathrm{d}x\mathrm{d}s = 0,$$

after integration by parts and using the divergence-free condition on the velocity. We therefore obtain

$$\mathbb{E}|(c+1)^-(t)|^2_{L^2(Q)} \lesssim \mathbb{E}\int_0^t\int_Q |(c+1)^-|^2\mathrm{d}x\mathrm{d}s;$$

whence, by means of Gronwall's lemma, $\mathbb{E}|(c+1)^-(t)|^2_{L^2(Q)} = 0$, which implies $c(x,t) \geq -1$ a.e. t and x and a.s. Similarly we can also prove that $c(x,t) \leq 1$ a.e. t and x and a.s.

3 Relative energy

Following Hošek and Mácha [12] we introduce the relative energy functional:

$$\mathcal{E}(\mathbf{u},c|\mathbf{U},C) = \int_Q \frac{1}{2}|\mathbf{u}-U|^2 + \frac{\varepsilon}{2}|\nabla c - \nabla C|^2\mathrm{d}x. \tag{3.1}$$

Next, motivated by [2], we consider “test functions” $[\mathbf{U}, C]$ – continuous stochastic processes adapted to $\{\mathfrak{F}_t\}_{t\geq 0}$ – that can be written in the form

$$\mathrm{d}\mathbf{U} = D^d\mathbf{U}\mathrm{d}t + D^s\mathbf{U}\mathrm{d}\mathbf{W}^1,\ \mathrm{d}C = D^dC\mathrm{d}t + D^sC\mathrm{d}\mathbf{W}^2, \tag{3.2}$$

Here, the deterministic components $D^d\mathbf{U}$, D^dc as well as the martingale components $D^s\mathbf{U}$, D^sC of the time derivative are functions of (ω, t, x).

3.1 Relative energy inequality

We claim the following result describing the time evolution of the relative energy associated to a dissipative martingale solution of problem (1.1), (1.3).

Theorem 3.1. *Let* $((\Omega, \mathfrak{B}, \{\mathfrak{F}_t\}_{t\geq 0}, \mathcal{P}), \mathbf{u}, c, \mathbf{W} = (\mathbf{W}^1, \mathbf{W}^2))$ *be a dissipative martingale solution of problem* (1.1), (1.3) *in the sense of Definition* (2.1). *Suppose that* $\mathbf{U}, C$ *are stochastic processes adapted to* $\{\mathfrak{F}_t\}_{t\geq 0}$*,*

$$\mathbf{U} \in \mathcal{C}([0,T], V_1),\ c \in \mathcal{C}([0,T], \mathcal{W}_2) \quad \mathcal{P} - a.s.$$

satisfying (3.2)*, where* D^dC*,* $D^d\mathbf{U}$*,* D^sC*,* $D^s\mathbf{U}$ *are* $\{\mathfrak{F}_t\}-$*progressively measurable,*

$$\begin{aligned} &(D^dC, D^d\mathbf{U}) \in L^2(\Omega, L^2(0,T,\mathcal{W}_2)) \times L^2(\Omega, L^2(0,T,V_1)),\\ &(D^s\mathbf{U}, D^sC) \in L^2(\Omega, L^2(0,T,\mathcal{L}_2(\mathfrak{U}, \mathcal{V}))). \end{aligned} \tag{3.3}$$

Then the relative energy inequality holds $\mathcal{P}-$*a.s.:*

$$\begin{aligned} &-\int_0^T \partial_t\psi\mathcal{E}(\mathbf{u}, c|\mathbf{U}, C)\mathrm{d}t + \int_0^T \psi|\varepsilon\Delta c - \frac{1}{\varepsilon}f(c) - \varepsilon\Delta C + \frac{1}{\varepsilon}f(C)|^2_{L^2(Q)}\mathrm{d}t\\ &+\nu\int_0^T \psi|\nabla\mathbf{u} - \nabla\mathbf{U}|^2_{L^2(Q)}\mathrm{d}t \leq \psi(0)\mathcal{E}(\mathbf{u}, c|\mathbf{U}, C)(0) + \int_0^T \psi\mathcal{R}(\mathbf{u}, c|\mathbf{U}, C)\mathrm{d}t + \int_0^T \psi\mathrm{d}M_{RE}, \end{aligned} \tag{3.4}$$

for all $\psi \in \mathcal{C}_c^\infty([0,T))$*,* $\psi \geq 0$*. Here* M_{RE} *is a real-valued square integrable martingale,*

$$\begin{aligned} M_{RE} = &\int_0^t \left(\int_Q (\mathbf{u} - \mathbf{U}) \cdot (\sigma^1(\mathbf{u}, c) - D^s\mathbf{U})\,\mathrm{d}x\right)\mathrm{d}\mathbf{W}^1\\ &+\varepsilon\int_0^t \left(\int_Q (\nabla(c - C)) \cdot (\nabla(\sigma^2(\mathbf{u}, c) - D^sC))\,\mathrm{d}x\right)\mathrm{d}\mathbf{W}^2, \end{aligned} \tag{3.5}$$

and the reminder term is given by:

$$\mathcal{R}(\mathbf{u},c|\mathbf{U},C) = \frac{1}{2}\sum_{k\geq 1}\int_Q |\sigma_k^1(\mathbf{u},c)|^2\mathrm{d}x + \frac{\varepsilon}{2}\sum_{k\geq 1}\int_Q |\nabla\sigma_k^2(\mathbf{u},c)|^2\mathrm{d}x + \frac{1}{2}\sum_{k\geq 1}|D^s\mathbf{U}\cdot e_k|^2_{L^2(Q)}$$
$$+ \frac{\varepsilon}{2}\sum_{k\geq 1}|\nabla D^s C\cdot e_k|^2_{L^2(Q)} - \varepsilon\sum_{k\geq 1}\int_Q \nabla\sigma_k^2(\mathbf{u},c)\cdot\nabla(D^s C\cdot e_k)\mathrm{d}x + \nu\int_Q \nabla\mathbf{U} : \nabla(\mathbf{u}-\mathbf{U})\mathrm{d}x$$
$$|\varepsilon\Delta C - \frac{1}{\varepsilon}f(C)|^2_{L^2(Q)} - \int_Q \frac{1}{\varepsilon}f(c)(\varepsilon\Delta c - \mathbf{u}\cdot\nabla c - \frac{1}{\varepsilon}f(c))\mathrm{d}x + \varepsilon\int_Q \nabla(C-c)\cdot\nabla D^d C\mathrm{d}x \tag{3.6}$$
$$+ \int_Q \varepsilon\Delta C(\varepsilon\Delta c - \mathbf{u}\cdot\nabla c - \frac{1}{\varepsilon}f(c))\mathrm{d}x - \varepsilon\int_Q \nabla c\otimes\nabla c : \nabla\mathbf{U}\mathrm{d}x + \int_Q (\mathbf{U}-\mathbf{u})\cdot D^d\mathbf{U}\mathrm{d}x$$
$$- \int_Q [(\mathbf{u}\cdot\nabla)\mathbf{U}]\cdot\mathbf{u}\mathrm{d}x - \sum_{k\geq 1}\int_Q \sigma_k^1(\mathbf{u},c)D^s\mathbf{U}\cdot e_k\mathrm{d}x.$$

3.2 Proof of Theorem 3.1

The proof of Theorem 3.1 is rather technical based on the idea of [2] that the time increments of the relative energy can be evaluated by means of the weak formulation (2.7–2.9).

3.2.1 Relative energy decomposition

We first notice that

$$\begin{aligned}\mathcal{E}(\mathbf{u},c|\mathbf{U},C) &= \int_Q \frac{1}{2}|\mathbf{u}|^2 + \frac{\varepsilon}{2}|\nabla c|^2\mathrm{d}x + \int_Q \frac{1}{2}|\mathbf{U}|^2 + \frac{\varepsilon}{2}|\nabla C|^2\mathrm{d}x - \int_Q \mathbf{u}\cdot\mathbf{U}\mathrm{d}x - \varepsilon\int_Q \nabla c\cdot\nabla C\mathrm{d}x \\ &= \mathcal{E}(\mathbf{u},c) + \underbrace{\int_Q \frac{1}{2}|\mathbf{U}|^2 + \frac{\varepsilon}{2}|\nabla C|^2\mathrm{d}x}_{I_1} + \underbrace{\varepsilon\int_Q c\cdot\Delta C\mathrm{d}x - \frac{1}{\varepsilon}\int_Q F(c)\mathrm{d}x}_{I_2} - \underbrace{\int_Q \mathbf{u}\cdot\mathbf{U}\mathrm{d}x}_{I_3}.\end{aligned} \tag{3.7}$$

3.2.2 Energy increments

The first term on the right–hand side of (3.7) can be expressed by means of the energy inequality (2.9):

$$
\begin{aligned}
&-\psi(0)\mathcal{E}(\mathbf{u},c)(0)-\int_0^T\psi_t\mathcal{E}(\mathbf{u},c)(t)dt+\int_0^T\psi(\nu|\nabla\mathbf{u}|^2_{L^2(Q)}+|\varepsilon\Delta c-\frac{1}{\varepsilon}f(c)|^2_{L^2(Q)})\mathrm{d}t\\
&\leq\frac{1}{2}\int_0^t\psi\sum_{k=1}^\infty\int_Q|\sigma^1_k(\mathbf{u},c)|^2\mathrm{d}x\mathrm{d}t+\frac{\varepsilon}{2}\int_0^T\psi\sum_{k=1}^\infty\int_Q|\nabla\sigma^2_k(\mathbf{u},c)|^2\mathrm{d}x\mathrm{d}t\\
&+\frac{1}{2\varepsilon}\int_0^T\psi\sum_{k=1}^\infty\int_Q f'(c)|\sigma^2_k(\mathbf{u},c)|^2\mathrm{d}x\mathrm{d}t+\int_0^T\psi\sum_{k=1}^\infty\left(\int_Q\mathbf{u}\sigma^1_k(\mathbf{u},c)\mathrm{d}x\right)\mathrm{d}W^1_k\\
&+\int_0^T\psi\sum_{k=1}^\infty\left(\int_Q(\frac{1}{\varepsilon}f(c)-\Delta c)\sigma^2_k(\mathbf{u},c)\mathrm{d}x\right)\mathrm{d}W^2_k
\end{aligned}
$$

for all $\psi\in\mathcal{C}^\infty_c([0,T))$, $\psi\geq 0$.

3.2.3 Time increment of test functions

As the test functions C and $\mathbf{u}$ are smooth, the time increment $\mathrm{d}I_1$ can be computed directly using (3.2) and Itô's chain rule.

3.3 Time increment of $c-$dependent integrals

As for I_2, we have observed that c solves the Allen–Cahn equation in the strong sense (2.8). Thus Itô's calculus adapted to the infinite–dimensional setting can be used to obtain:

$$
\begin{aligned}
-\mathrm{d}\big(\int_Q c\Delta C\mathrm{d}x\big)&=-\varepsilon\int_Q\Delta C\Delta c\mathrm{d}x\mathrm{d}t+\int_Q\Delta C(\mathbf{u}\cdot\nabla c+\frac{1}{\varepsilon}F(c))\mathrm{d}x\mathrm{d}t\\
&+\varepsilon\int_Q\nabla c\cdot\nabla D^dC\mathrm{d}x\mathrm{d}t+\big(\sum_{k\geq1}\int_Q\sigma^1_k(\mathbf{u},c)D^s\mathbf{U}\cdot e_k\mathrm{d}x\big)\mathrm{d}t+\int_Q\mathbf{u}\cdot D^d\mathbf{U}\mathrm{d}x\mathrm{d}t+\mathrm{d}M_1,
\end{aligned}
$$

where M_1 is given by:

$$
M_1=\int_0^t\left(\int_Q\nabla c\cdot\nabla(D^sC\cdot e_k)+\nabla C\cdot\nabla\sigma^2_k(\mathbf{u},c)\mathrm{d}x\right)\mathrm{d}W^2_k.
$$

Similarly, we have

$$
\begin{aligned}
\mathrm{d}\big(\frac{1}{\varepsilon}\int_Q F(c)\mathrm{d}x\big)&=\int_Q\frac{1}{\varepsilon}f(c)(\varepsilon\Delta c-\mathbf{u}\cdot\nabla c-\frac{1}{\varepsilon}f(c))\mathrm{d}x\mathrm{d}t+\frac{1}{2\varepsilon}\sum_{k\geq1}\int_Q f'(c)|\sigma^2_k(\mathbf{u},c)|^2\mathrm{d}x\mathrm{d}t\\
&+\int_Q\mathbf{U}\cdot D^d\mathbf{U}\mathrm{d}x\mathrm{d}t+\frac{1}{2}\sum_{k\geq1}\int_Q|D^s\mathbf{U}\cdot e_k|^2\mathrm{d}x\mathrm{d}t+\mathrm{d}M_2,
\end{aligned}
\tag{3.8}
$$

with M_2 given by:

$$M_2 = \int_0^t \left(\frac{1}{\varepsilon} \sum_{k\geq 1} \int_Q f(c)\sigma_k^2(\mathbf{u}, c) + \mathbf{U} \cdot (D^s\mathbf{U} \cdot e_k)\mathrm{d}x \right) \mathrm{d}W_k^2$$

3.3.1 Time increment of u−dependent integrals

As the momentum balance (2.7) holds only in the weak (PDE) sense, we are not allowed to apply Itô's calculus directly to I_3. Instead we first regularize (2.7) by means of spatial convolutions, apply Itô's chain rule to the regularized system, and pass to the limit with the regularization, see [2, Section 3.2]. Thus, exactly as in [2, Lemma 3.1], we deduce

$$\begin{aligned}\mathrm{d}\big(\int_Q \mathbf{u}\cdot\mathbf{U}\mathrm{d}x\big) &= \int_Q \big((\mathbf{u}\cdot\nabla)\mathbf{U}\cdot\mathbf{u} - \nu\nabla\mathbf{u} : \nabla\mathbf{U} + \varepsilon\nabla c \otimes \nabla c : \nabla\mathbf{U}\big)\mathrm{d}x\mathrm{d}t \\ &\quad + \big(\sum_{k\geq 1}\int_Q \nabla\sigma_k^2(\mathbf{u}, c)\cdot\nabla(D^sC\cdot e_k)\mathrm{d}x\big)\mathrm{d}t + \int_Q \nabla c\cdot\nabla D^dC\mathrm{d}x\mathrm{d}t + \mathrm{d}M_3,\end{aligned}$$

where M_3 is given by:

$$M_3 = \int_0^t \left(\int_Q \mathbf{u}\cdot(D^sU\cdot e_k) + \mathbf{U}\cdot\sigma_k^1(\mathbf{u}, c)\mathrm{d}x \right) \mathrm{d}W_k^1.$$

Gathering the above relations we obtain (3.4), which completes the proof of Theorem 3.1.

4 Weak-strong uniqueness

Our ultimate goal is to show the weak–strong uniqueness principle for the problem (1.1–1.3). The obvious idea is to use the strong solution $(\mathbf{U}, C)$ as a test function in the relative energy inequality (3.4). To this end, the strong solutions must belong to the regularity class specified in Theorem 3.1.

4.1 Strong solutions

Definition 4.1. Let $(\Omega, \mathfrak{B}, \{\mathfrak{F}_t\}_{t\geq 0}, \mathcal{P})$ be a stochastic basis with a complete right-continous filtration and let $\mathbf{W}^1, \mathbf{W}^2$ be $\{\mathfrak{F}_t\}$–cylindrical Wiener processes. Let $(\mathbf{U}_0, C_0)$ be an $\mathfrak{F}_0$-measurable random variable in the space $\mathcal{V}$. A pair $(\mathbf{U}, C)$ and an $\{\mathfrak{F}_t\}$–stopping time τ is called a local strong pathwise solution for system (1.1) if $(\mathbf{U}, C)(t\wedge\tau)$ is an $\mathfrak{F}_t$-adapted process in $\mathcal{V}$ such that:

$$\begin{aligned}&(\mathbf{U}, C) \in L^2(\Omega, L^\infty(0, T, \mathcal{V})), \\ &(\mathbf{U}, C)\mathbf{1}_{t\leq\tau} \in L^2(\Omega, L^2(0, T, V_2) \times L^2(0, T, \mathcal{W}_3);\end{aligned} \tag{4.1}$$

it holds $\mathcal{P}$-a.s.:

$$\begin{aligned}\mathbf{U}(t\wedge\tau)+\int_0^{t\wedge\tau}(-\nu\Delta\mathbf{U}+(\mathbf{U}\cdot\nabla)\mathbf{U}+\varepsilon\mathrm{div}(\nabla C\otimes\nabla C)\mathrm{d}t'&=\mathbf{U}_0+\int_0^{t\wedge\tau}\sigma^1(\mathbf{U},C)\mathrm{d}\mathbf{W}^1,\\ C(t\wedge\tau)+\int_0^{t\wedge\tau}(-\varepsilon\Delta C+\mathbf{U}\cdot\nabla C+\frac{1}{\varepsilon}f(C))\mathrm{d}t'&=C_0+\int_0^{t\wedge\tau}\sigma^2(\mathbf{U},C)\mathrm{d}\mathbf{W}^2\end{aligned}\tag{4.2}$$

for any $t\geq 0$.

In what follows we give some brief details concerning the existence of a local strong pathwise solution. Let

$$G_N(r)=\min\left\{1;\frac{N}{r}\right\},\ r>0.$$

Tachim–Medjo [17] considered the following regularized system

$$\begin{aligned}\mathrm{d}\mathbf{U}+\nabla p&=\nu\Delta\mathbf{U}-\Big[G_N\left(\|\mathbf{U}\|_{H^1}\right)(\mathbf{U}\cdot\nabla)\mathbf{U}-\varepsilon G_N\left(\|(\mathbf{U},C)\|_{\mathcal{V}}\right)\mathrm{div}\left(\nabla C\otimes\nabla C\right)\Big]\mathrm{d}t\\ &\quad+\sigma_1(\mathbf{U},C)\mathrm{d}\mathbf{W}^1,\\ \mathrm{d}C&=\varepsilon\Delta C-\Big[G_N\left(\|(\mathbf{U},C)\|_{\mathcal{V}}\right)\mathbf{U}\cdot\nabla C-\frac{1}{\varepsilon}f(C)\Big]\mathrm{d}t+\sigma_2(\mathbf{U},C)\mathrm{d}\mathbf{W}^2,\\ \nabla\cdot\mathbf{U}&=0,\end{aligned}\tag{4.3}$$

$$\mathbf{U}=0,\ \frac{\partial C}{\partial\mathbf{n}}=0 \text{ on } \partial Q,\tag{4.4}$$

$$\mathbf{U}(0,\cdot)=\mathbf{U}_0,\ \ C(0,\cdot)=C_0.\tag{4.5}$$

As shown in [17, Theorem 1], the regularized system (4.3–4.5) admits a global-in-time strong solution $(\mathbf{U}^N,C^N)$, unique in the regularity class (4.1), whenever

$$(\mathbf{U}_0,C_0)\in L^2(\Omega,\mathcal{V}).$$

Obviously, any such solution $(\mathbf{U}^N,C^N)$, with the stopping time

$$\tau^N=\inf\left\{t\in[0,T]\ \Big|\ \|(\mathbf{U}(t,\cdot),C(t,\cdot))\|_{\mathcal{V}}\geq\frac{N}{2}\right\},\ \inf\{\emptyset\}\equiv T,$$

represents a strong solution of (4.2). Moreover,

$$\tau^N>0 \text{ whenever } \|(\mathbf{U}_0,C_0)\|_{\mathcal{V}}<\frac{N}{2}\ \mathcal{P}-\text{a.s.}$$

4.2 Weak-strong pathwise uniqueness

We are ready to establish the pathwise weak–strong uniqueness principle. In order to use the relative energy inequality and prove that the two solutions coincide, we need to show that, in a certain sense, the relative energy represents a distance between them. Similar result in the deterministic context was obtained in [12]. We claim the following lemma:

Lemma 4.2. *Let* $c_0 \in \mathcal{W}_1$ *satisfy*

$$-1 \leq c_0 \leq 1 \ \mathcal{P} - a.s.$$

Let $(\mathbf{u}, c)$, $(\mathbf{U}, C)$ *satisfy* (2.8) *(with the same* $\mathbf{W^2}$*),*

$$c, C \in L^2(\Omega, L^\infty(0, T, \mathcal{W}_1) \cap L^2(0, T, \mathcal{W}_2)),$$

$$C(0, \cdot) = c(0, \cdot) = c_0 \ \mathcal{P} - a.s.$$

Then

$$\mathbb{E}\big(\sup_{r\in[0,t]} |c(r) - C(r)|^2_{L^2(Q)}\big) \lesssim \mathbb{E}\Big(\int_0^t \mathcal{E}(\mathbf{u}, c|\mathbf{U}, C)\mathrm{d}r\Big). \tag{4.6}$$

Proof. By definition, the functions c, C satisfy (2.8). Therefore we can use directly Itô's calculus to obtain

$$\begin{aligned} &|(c - C)(r)|^2_{L^2(Q)} + \varepsilon \int_0^r |\nabla(c - C)(s)|^2 \mathrm{d}s \leq |\int_0^r (\mathbf{u} \cdot \nabla c - \mathbf{U} \cdot \nabla C, c - C)_{L^2(Q)} \mathrm{d}s| \\ &\frac{1}{\varepsilon} \int_0^r |(f(c) - f(C), c - C)_{L^2(Q)}| \mathrm{d}s + |\int_0^r \sum_{k\geq 1} (c - C, \sigma_k^2(\mathbf{u}, c) - \sigma_k^2(\mathbf{U}, C))_{L^2(Q)} \mathrm{d}W_k^2| \\ &+ \frac{1}{2} \int_0^r \sum_{k\geq 1} |\sigma_k^2(\mathbf{u}, c) - \sigma_k^2(\mathbf{U}, C)|^2_{L^2(Q)} \mathrm{d}s. \end{aligned} \tag{4.7}$$

Applying the Burkholder-Davis-Gundy inequality and using the properties of σ^2, we have:

$$\begin{aligned} &\mathbb{E}\big(\sup_{r\in[0,t]} |\int_0^r \sum_{k\geq 1} (c - C, \sigma_k^2(\mathbf{u}, c) - \sigma_k^2(\mathbf{U}, C))_{L^2(Q)} \mathrm{d}W_k^2|\big) \\ &\lesssim \mathbb{E}\Big(\int_0^t \sum_{k\geq 1} (c - C, \sigma_k^2(\mathbf{u}, c) - \sigma_k^2(\mathbf{U}, C))^2_{L^2(Q)} \mathrm{d}s\Big)^{1/2} \\ &\leq \frac{1}{2}\mathbb{E}\big(\sup_{r\in[0,t]} |c - C|^2_{L^2(Q)}\big) + k\mathbb{E}\Big(\int_0^t |c - C|^2_{L^2(Q)} \mathrm{d}r\Big), \end{aligned} \tag{4.8}$$

where k is a positive constant.

Integrating by parts and using solenoidality of the velocity, we have (see [12] for more details):

$$\int_Q (\mathbf{u}\cdot\nabla c - \mathbf{U}\cdot\nabla C)(c-C)\mathrm{d}x = -\int_Q C\nabla(c-C)\cdot(\mathbf{u}-\mathbf{U})\mathrm{d}x, \tag{4.9}$$

which implies:

$$\begin{aligned}&\mathbb{E}\big(\sup_{r\in[0,t]}|\int_0^r\int_Q(\mathbf{u}\cdot\nabla c-\mathbf{U}\cdot\nabla C)(c-C)\mathrm{d}x|\big)\\ &\le \mathbb{E}\big(\int_0^t |C|_{L^\infty(Q)}|\nabla(c-C)|_{L^2(Q)}|\mathbf{u}-\mathbf{U}|_{L^2(Q)}\mathrm{d}r\big)\le \mathbb{E}\int_0^r \mathcal{E}(\mathbf{u},c|\mathbf{U},C)\mathrm{d}s,\end{aligned} \tag{4.10}$$

where we have used the fact that the concentration difference remains in $[-1,1]$.

Since f is globally Lipschitz, we have:

$$\mathbb{E}\big(\frac{1}{\varepsilon}\int_0^r |(f(c)-f(C),c-C)_{L^2(Q)}|\mathrm{d}s\big)\lesssim \mathbb{E}\big(\int_0^r |c-C|^2_{L^2(Q)}\mathrm{d}s\big).$$

Finally, we obtain

$$\mathbb{E}(\sup_{r\in[0,t]}|c-C|^2_{L^2(Q)})\lesssim \mathbb{E}\int_0^t \mathcal{E}(\mathbf{u},c|\mathbf{U},C)\mathrm{d}s+\mathbb{E}\big(\int_0^t \sup_{r\in[0,s]}|c-C|^2_{L^2(Q)}\mathrm{d}s\big); \tag{4.11}$$

therefore the (deterministic) Gronwall lemma yields the desired conclusion:

$$\mathbb{E}(\sup_{r\in[0,t]}|c-C|^2_{L^2(Q)})\lesssim \mathbb{E}\int_0^t \mathcal{E}(\mathbf{u},c|\mathbf{U},C)\mathrm{d}s. \tag{4.12}$$

□

We are now able to prove the pathwise weak-strong uniqueness:

Theorem 4.3. *Let* $((\Omega,\mathfrak{B},\{\mathfrak{F}_t\}_{t\ge0},\mathcal{P}),\mathbf{u},c,\mathbf{W}=(\mathbf{W}^1,\mathbf{W}^2))$ *be a dissipative martingale solution to the Navier-Stokes-Allen-Cahn system (1.1 - 1.3), and let* $(\mathbf{U},C)$*, with a stopping time* τ*, be a strong solution of the same problem in the sense of Definition 4.1, defined on the same stochastic basis with the same Wiener processes, and with the same initial data:*

$$\mathbf{u}(0,\cdot)=\mathbf{U}(0,\cdot),\ c(0,\cdot)=C(0,\cdot)\,\mathcal{P}-a.s.$$

and such that $-1\le C(0,\cdot)\le 1\,\mathcal{P}-a.s.$

Then

$$\mathbf{u}(\cdot\wedge\tau)=\mathbf{U}(\cdot\wedge\tau),\ c(\cdot\wedge\tau)=C(\cdot\wedge\tau)\,\mathcal{P}-a.s.$$

Proof. Let us introduce the following stopping times:

$$\tau_L = \tau_L^1 \wedge \tau_L^2 \wedge \tau,$$
$$\tau_L^1 = \inf\left\{ t \in [0,T] \;\middle|\; \|(\mathbf{U}, C)(t \wedge \tau)\|_{\mathcal{V}} > L \right\},$$
$$\tau_L^2 = \inf\left\{ t \in [0,T] \;\middle|\; \|\mathbf{1}_{t \le \tau}(\mathbf{U}, C)\|_{L^2(0,t;H^2(Q)\times H^3(Q))} > L \right\}$$

Since $(\mathbf{U}, C)$ is a strong solution, we have $\mathcal{P}(\lim_{L\to\infty} \tau_L = \tau) = 1$.

As $(\mathbf{U}, C)$ solves (4.2), the remainder term in the relative energy inequality (3.4) takes the form:

$$\begin{aligned} \mathcal{R}(\mathbf{u}, c|\mathbf{U}, C) =& \frac{1}{2}\sum_{k\ge 1}\int_Q |\sigma_k^1(\mathbf{u}, c) - \sigma_k^1(\mathbf{U}, C)|^2 \mathrm{d}x \\ &+ \frac{\varepsilon}{2}\sum_{k\ge 1}\int_Q |\nabla(\sigma_k^2(\mathbf{u}, c) - \sigma_k^2(\mathbf{U}, C))|^2 \mathrm{d}x + \sum_{i=1}^{3}\mathcal{R}_i, \end{aligned} \tag{4.13}$$

where

$$\mathcal{R}_1 = \int_Q (\mathbf{U}\cdot\nabla)\mathbf{U}\cdot\mathbf{u} - (\mathbf{u}\cdot\nabla)\mathbf{U}\cdot\mathbf{u}\mathrm{d}x,$$

$$\begin{aligned} \mathcal{R}_2 =& -2(\varepsilon\Delta c - \frac{1}{\varepsilon}f(c), \varepsilon\Delta C - \frac{1}{\varepsilon}f(C))_{L^2(Q)} + |\varepsilon\Delta C - \frac{1}{\varepsilon}f(C)|^2_{L^2(Q)} \\ &- (\varepsilon\Delta C, \varepsilon\Delta C - \frac{1}{\varepsilon}f(C))_{L^2(Q)} - (\frac{1}{\varepsilon}f(c), \varepsilon\Delta c - \frac{1}{\varepsilon}f(c))_{L^2(Q)} \\ &+ (\varepsilon\Delta C, \varepsilon\Delta c - \frac{1}{\varepsilon}f(c))_{L^2(Q)} + (\varepsilon\Delta c, \varepsilon\Delta C - \frac{1}{\varepsilon}f(C))_{L^2(Q)}, \end{aligned} \tag{4.14}$$

and

$$\begin{aligned} \mathcal{R}_3 =& \varepsilon\int_Q \nabla(\mathbf{U}-\mathbf{u}) : (\nabla C\otimes\nabla C)\mathrm{d}x + \varepsilon\int_Q \nabla c\cdot\nabla(\mathbf{U}\cdot\nabla C)\mathrm{d}x \\ &- \varepsilon\int_Q (\nabla c\otimes\nabla c) : \nabla\mathbf{U}\mathrm{d}x + \varepsilon\int_Q (\mathbf{U}\cdot\nabla C - \mathbf{u}\cdot\nabla c)\Delta C\mathrm{d}x. \end{aligned} \tag{4.15}$$

Integrating by parts and using $\nabla\cdot\mathbf{u} = \nabla\cdot\mathbf{U} = 0$, we get:

$$\mathcal{R}_1 = \int_Q (\mathbf{U}-\mathbf{u})\otimes\mathbf{U} : \nabla(\mathbf{U}-\mathbf{u})\mathrm{d}x. \tag{4.16}$$

As for $\mathcal{R}_2$, after some immediate manipulations, we obtain:

$$\mathcal{R}_2 = \int_Q \frac{1}{\varepsilon}(f(C) - f(c), \varepsilon\Delta c - \frac{1}{\varepsilon}f(c) - \varepsilon\Delta C + \frac{1}{\varepsilon}f(C))\mathrm{d}x. \tag{4.17}$$

The last term is treated in exactly as in the determinist case (see [12] for more details):

$$\mathcal{R}_3 = \varepsilon \int_Q \Delta(c-C)\mathbf{U}\cdot\nabla(c-C)\mathrm{d}x - \varepsilon\int_Q \Delta C(\mathbf{u}-\mathbf{U})\cdot\nabla(C-c)\mathrm{d}x. \tag{4.18}$$

After several successive integrations by parts, the last integral in (4.18) reads

$$\begin{aligned}
&\varepsilon\int_Q \Delta C(\mathbf{u}-\mathbf{U})\cdot\nabla(C-c)\mathrm{d}x\\
&= -\varepsilon\int_Q \nabla C\otimes\nabla(C-c):\nabla(\mathbf{u}-\mathbf{U})\mathrm{d}x - \varepsilon\int_Q \nabla C\otimes(\mathbf{u}-\mathbf{U}):\nabla\otimes\nabla(C-c)\mathrm{d}x\\
&= -\varepsilon\int_Q \nabla C\otimes\nabla(C-c):\nabla(\mathbf{u}-\mathbf{U})\mathrm{d}x + \varepsilon\int_Q \nabla(C-c)\otimes(\mathbf{u}-\mathbf{U}):\nabla\otimes\nabla C\mathrm{d}x\\
&= -\varepsilon\int_Q \nabla C\otimes\nabla(C-c):\nabla(\mathbf{u}-\mathbf{U})\mathrm{d}x - \varepsilon\int_Q \nabla(C-c)\otimes\nabla C:\nabla(\mathbf{u}-\mathbf{U})\mathrm{d}x\\
&- \varepsilon\int_Q \Delta(C-c)(\mathbf{u}-\mathbf{U})\cdot\nabla C\mathrm{d}x.
\end{aligned} \tag{4.19}$$

Thus we finally obtain

$$\begin{aligned}
\mathcal{R}_3 =& \varepsilon\int_Q \Delta(c-C)\mathbf{U}\cdot\nabla(c-C)\mathrm{d}x + \varepsilon\int_Q \nabla C\otimes\nabla(c-C):\nabla(\mathbf{u}-\mathbf{U})\mathrm{d}x\\
&+\varepsilon\int_Q \nabla(C-c)\otimes\nabla C:\nabla(\mathbf{u}-\mathbf{U})\mathrm{d}x + \varepsilon\int_Q \Delta(C-c)(\mathbf{u}-\mathbf{U})\cdot\nabla C\mathrm{d}x\\
=&\sum_{i=1}^{4}\mathcal{R}_{3,i}.
\end{aligned} \tag{4.20}$$

Our goal is to control each integral by means of $\mathcal{R}(\mathbf{u},c|\mathbf{U},C)$. Using the Lipschitz continuity of σ^1 and σ^2, we get:

$$\begin{aligned}
&\frac{1}{2}\sum_{k\geq 1}\int_Q |\sigma_k^1(\mathbf{u},c)-\sigma_k^1(\mathbf{U},C)|^2\mathrm{d}x + \frac{\varepsilon}{2}\sum_{k\geq 1}\int_Q |\nabla(\sigma_k^2(\mathbf{u},c)-\sigma_k^2(\mathbf{U},C))|^2\mathrm{d}x\\
&\lesssim \int_Q |\mathbf{U}-\mathbf{u}|^2\mathrm{d}x + \int_Q |c-C|^2\mathrm{d}x + \int_Q |\nabla(c-C)|^2\mathrm{d}x.
\end{aligned} \tag{4.21}$$

For the convective part of the remainder, we get

$$|\mathcal{R}_1| \leq \frac{\nu}{4}\int_Q |\nabla(\mathbf{U}-\mathbf{u})|^2\mathrm{d}x + k_1|\mathbf{U}|^2_{L^\infty(Q)}\int_Q |\mathbf{U}-\mathbf{u}|^2\mathrm{d}x. \tag{4.22}$$

As for $\mathcal{R}_2$, we use the fact that f is globally Lipschitz to deduce

$$|\mathcal{R}_2| \leq \frac{1}{6}\int_Q |\varepsilon\Delta c - \frac{1}{\varepsilon}f(c) - \varepsilon\Delta C + \frac{1}{\varepsilon}f(C)|^2\mathrm{d}x + k_2\int_Q |c-C|^2\mathrm{d}x. \tag{4.23}$$

Finally, each term in $\mathcal{R}_3$ is bounded as follows:

$$
\begin{aligned}
|\mathcal{R}_{3,1}| &\leq \frac{1}{12}\int_Q |\varepsilon\Delta(c-C)|^2\mathrm{d}x + k_3|\mathbf{U}|^2_{L^\infty(Q)}\int_Q |\nabla(c-C)|^2\mathrm{d}x \\
&\leq \frac{1}{6}\int_Q |\varepsilon\Delta c - \frac{1}{\varepsilon}f(c) - \varepsilon\Delta C + \frac{1}{\varepsilon}f(C)|^2\mathrm{d}x + k_4\int_Q |c-C|^2\mathrm{d}x \\
&\quad + k_3|\mathbf{U}|^2_{L^\infty(Q)}\int_Q |\nabla(c-C)|^2\mathrm{d}x,
\end{aligned}
\tag{4.24}
$$

$$
|\mathcal{R}_{3,2} + \mathcal{R}_{3,3}| \leq \frac{\nu}{4}\int_Q |\nabla(\mathbf{U}-\mathbf{u})|^2\mathrm{d}x + k_5|\nabla C|^2_{L^\infty(Q)}\int_Q |\nabla(c-C)|^2\mathrm{d}x,
\tag{4.25}
$$

and,

$$
\begin{aligned}
|\mathcal{R}_{3,4}| &\leq \frac{1}{12}\int_Q |\varepsilon\Delta(c-C)|^2\mathrm{d}x + k_6|\nabla C|^2_{L^\infty(Q)}\int_Q |\mathbf{u}-\mathbf{U}|^2\mathrm{d}x \\
&\leq \frac{1}{6}\int_Q |\varepsilon\Delta c - \frac{1}{\varepsilon}f(c) - \varepsilon\Delta C + \frac{1}{\varepsilon}f(C)|^2\mathrm{d}x + k_4\int_Q |c-C|^2\mathrm{d}x \\
&\quad + k_6|\nabla C|^2_{L^\infty(Q)}\int_Q |\mathbf{u}-\mathbf{U}|^2\mathrm{d}x.
\end{aligned}
\tag{4.26}
$$

Now, consider two arbitrary stopping times such that $0 \leq \tau_a \leq \tau_b \leq \tau_L$. Let t be such that $\tau_a \leq t \leq \tau_b$. From the relative energy inequality (3.4) we deduce that:

$$
\begin{aligned}
&\mathcal{E}(\mathbf{u}, c|\mathbf{U}, C)(t) + \int_{\tau_a}^t\int_Q |\nabla(\mathbf{u}-\mathbf{U})|^2\mathrm{d}x\mathrm{d}t' + \int_{\tau_a}^t\int_Q |\varepsilon\Delta c - \frac{1}{\varepsilon}f(c) - \varepsilon\Delta C + \frac{1}{\varepsilon}f(C)|^2\mathrm{d}x\mathrm{d}t' \\
&\quad\lesssim \mathcal{E}(\mathbf{u}, c|\mathbf{U}, C)(\tau_a) + \int_{\tau_a}^t\int_Q |c-C|^2\mathrm{d}x\mathrm{d}t' \\
&\quad + \int_{\tau_a}^t (1 + |U|^2_{L^\infty(Q)} + |\nabla C|^2_{L^\infty(Q)})\mathcal{E}(\mathbf{u}, c|\mathbf{U}, C)(t')\mathrm{d}t' \\
&\quad + |M_{RE}(t) - M_{RE}(\tau_a)|.
\end{aligned}
\tag{4.27}
$$

Passing expectations in (4.27) we get

$$
\begin{aligned}
&\mathbb{E}(\sup_{t\in[\tau_a,\tau_b]}\mathcal{E}(\mathbf{u}, c|\mathbf{U}, C)(t)) + \mathbb{E}(\int_{\tau_a}^{\tau_b}\int_Q |\nabla(\mathbf{u}-\mathbf{U})|^2\mathrm{d}x\mathrm{d}t') \\
&\quad + \mathbb{E}(\int_{\tau_a}^{\tau_b}\int_Q |\varepsilon\Delta c - \frac{1}{\varepsilon}f(c) - \varepsilon\Delta C + \frac{1}{\varepsilon}f(C)|^2\mathrm{d}x\mathrm{d}t') \\
&\quad\lesssim \mathbb{E}(\mathcal{E}(\mathbf{u}, c|\mathbf{U}, C)(\tau_a)) + \mathbb{E}(\int_{\tau_a}^{\tau_b}\int_Q |c-C|^2\mathrm{d}x\mathrm{d}t') \\
&\quad + \mathbb{E}(\int_{\tau_a}^t (1 + |U|^2_{L^\infty(Q)} + |\nabla C|^2_{L^\infty(Q)})\mathcal{E}(\mathbf{u}, c|\mathbf{U}, C)(t')\mathrm{d}t') \\
&\quad + \mathbb{E}(\sup_{t\in[\tau_a,\tau_b]}|M_{RE}(t) - M_{RE}(\tau_a)|).
\end{aligned}
\tag{4.28}
$$

The last term in (4.28) is estimated using the Burkholder-Davis-Gundy inequality:

$$
\begin{aligned}
&\mathbb{E}(\sup_{t\in[\tau_a,\tau_b]}|\int_{\tau_a}^{t}\int_Q(\mathbf{u}-\mathbf{U})\cdot(\sigma^1(\mathbf{u},c)-\sigma^2(\mathbf{U},C))\mathrm{d}x\mathrm{d}W^1|)\\
&\lesssim \mathbb{E}(\int_{\tau_a}^{\tau_b}|\mathbf{u}-\mathbf{U}|^2_{L^2(Q)}(|\mathbf{u}-\mathbf{U}|^2_{L^2(Q)}+|c-C|^2_{H^1(Q)})\mathrm{d}t)^{1/2}\\
&\leq \frac{1}{4}\mathbb{E}(\sup_{t\in[\tau_a,\tau_b]}\mathcal{E}(\mathbf{u},c|\mathbf{U},C)(t))+\mathbb{E}(\int_{\tau_a}^{\tau_b}|c-C|^2_{L^2(Q)}\mathrm{d}t)+k_7\mathbb{E}(\int_{\tau_a}^{\tau_b}\mathcal{E}(\mathbf{u},c|\mathbf{U},C)(t)\mathrm{d}t),
\end{aligned}
\tag{4.29}
$$

and similarly:

$$
\begin{aligned}
&\mathbb{E}(\sup_{t\in[\tau_a,\tau_b]}|\varepsilon\int_{\tau_a}^{t}\int_Q\nabla(c-C)\cdot\nabla(\sigma^2(\mathbf{u},c)-\sigma^2(\mathbf{U},C))\mathrm{d}x\mathrm{d}W^2|)\\
&\leq \frac{1}{4}\mathbb{E}(\sup_{t\in[\tau_a,\tau_b]}\mathcal{E}(\mathbf{u},c|\mathbf{U},C)(t))+\mathbb{E}(\int_{\tau_a}^{\tau_b}|c-C|^2_{L^2(Q)}\mathrm{d}t)+k_8\mathbb{E}(\int_{\tau_a}^{\tau_b}\mathcal{E}(\mathbf{u},c|\mathbf{U},C)(t)\mathrm{d}t).
\end{aligned}
\tag{4.30}
$$

Inserting these estimates into (4.28), we obtain the following inequality:

$$
\begin{aligned}
\mathbb{E}(\sup_{t\in[\tau_a,\tau_b]}\mathcal{E}(\mathbf{u},c|\mathbf{U},C)(t))&+\mathbb{E}(\int_{\tau_a}^{\tau_b}\int_Q|\nabla(\mathbf{u}-\mathbf{U})|^2\mathrm{d}x\mathrm{d}t')\\
&+\mathbb{E}(\int_{\tau_a}^{\tau_b}\int_Q|\varepsilon\Delta c-\frac{1}{\varepsilon}f(c)-\varepsilon\Delta C+\frac{1}{\varepsilon}f(C)|^2\mathrm{d}x\mathrm{d}t')\\
&\lesssim \mathbb{E}(\mathcal{E}(\mathbf{u},c|\mathbf{U},C)(\tau_a))+\mathbb{E}(\int_{\tau_a}^{\tau_b}|c-C|^2_{L^2(Q)}\mathrm{d}t)\\
&+\mathbb{E}(\int_{\tau_a}^{\tau_b}(1+|U|^2_{L^\infty(Q)}+|\nabla C|^2_{L^\infty(Q)})\mathcal{E}(\mathbf{u},c|\mathbf{U},C)(t')\mathrm{d}t').
\end{aligned}
\tag{4.31}
$$

Taking into account the Sobolev embedding $H^2(Q)$ into $L^\infty(Q)$, we can easily check that since $(U,C)\in L^2(\Omega,L^2(0,\tau;H^2(Q)\times H^3(Q)))$, we have

$$
\int_0^{\tau_L}(1+|U|^2_{L^\infty(Q)}+|\nabla C|^2_{L^\infty(Q)})\mathrm{d}t<\kappa(L),\ a.s.,
$$

where $\kappa(L)$ is a positive constant depending on L.

We are now able to apply the stochastic Gronwall lemma (Lemma 5.1 stated in the Appendix, we obtain:

$$
\mathbb{E}(\sup_{t\in[0,\tau]}\mathcal{E}(\mathbf{u},c|\mathbf{U},C)(t))\lesssim\mathbb{E}(\mathcal{E}(\mathbf{u},c|\mathbf{U},C)(0)+\int_0^\tau|c-C|^2_{L^2(Q)}).
\tag{4.32}
$$

Using the fact that the two solutions $(\mathbf{u},c)$ and $(\mathbf{U},C)$ coincide at origin, as well as (4.6), we end up with:

$$
\mathbb{E}(\sup_{t\in[0,\tau]}\mathcal{E}(\mathbf{u},c|\mathbf{U},C)(t))\lesssim\mathbb{E}(\int_0^\tau\mathcal{E}(\mathbf{u},c|\mathbf{U},C)(t)\mathrm{d}t),
\tag{4.33}
$$

and by use of deterministic Gronwall lemma, we finally obtain that the two solutions $(\mathbf{u}, c)$ and $(\mathbf{U}, C)$ coincide on $(0, \tau_L)$. $\square$

4.3 Weak-strong uniqueness in law

Previously we proved that the strong solution and the dissipative weak martingale solution defined on the same probability space and emanating from the same initial data coincide. Typically, however, the martingale solution is defined on a different probability basis with different Wiener processes $(\mathbf{W}^1, \mathbf{W}^2)$, whereas the only relevant piece of information is that *the law* of their initial data coincide. To handle such a situation, we restrict ourselves to the case when the strong solution is stopped when leaving a Borel set. More specifically, for a Borel set $B \subset \mathcal{V}$ and an $\mathfrak{F}_t-$adapted process $(\mathbf{U}, C) \in C([0,T]; \mathcal{V})$, we denote

$$\tau_B = \tau_B[\mathbf{U}, C] = \inf \left\{ t \in [0,T] \;\middle|\; (\mathbf{U}, C) \in B^c \right\}, \; \inf\{\emptyset\} = T.$$

We claim the following result:

Theorem 4.4. *Let $(c, \mathbf{u})$ be a dissipative martingale solution of the Navier–Stokes–Allen–Cahn system (1.1–1.3) defined on a probability basis $(\Omega^m, \mathfrak{B}^m, \mathcal{P}^m, \{\mathfrak{F}_t^m\}_{t\geq 0})$, with the associated Wiener processes $\mathbf{W}^m = (\mathbf{W}^{1,m}, \mathbf{W}^{2,m})$. Let $(\mathbf{U}, C)$ be a strong solution of the same problem on a probability basis $(\Omega^s, \mathfrak{B}^s, \mathcal{P}^s, \{\mathfrak{F}_t^s\}_{t\geq 0})$ with the Wiener processes $\mathbf{W}^s = (\mathbf{W}^{1,s}, \mathbf{W}^{2,s})$, associated to the initial condition $(\mathbf{U}_0, C_0)$, and a stopping time τ^s,*

$$\tau^s = \tau_B[(\mathbf{U}, C)] \text{ for some Borel set } B \subset \mathcal{V} \text{containing the initial condition} (\mathbf{U}_0, C_0).$$

Finally, let

$$\mathcal{L}_{\mathcal{V}}[\mathbf{u}(0,\cdot), c(0,\cdot)] = \mathcal{L}_{\mathcal{V}}[\mathbf{U}(0,\cdot), C(0,\cdot)] = \mu_0, \;\; \mu_0\left\{-1 \leq c_0 \leq 1\right\} = 1.$$

Then there exists an $\{\mathfrak{F}_t^m\}$-stopping time $\tau^m > 0$ such that

$$(\mathbf{u}, c)(\cdot \wedge \tau^m) \in C([0,T]; \mathcal{V}) \; \mathcal{P}^m - a.s.,$$

and

$$\mathcal{L}_{C([0,T];\mathcal{V})}\left[\mathbf{u}(\cdot \wedge \tau^m), c(\cdot \wedge \tau^m)\right] = \mathcal{L}_{C([0,T];\mathcal{V})}\left[\mathbf{U}(\cdot \wedge \tau^s), C(\cdot \wedge \tau^s)\right].$$

Proof. In accordance with [17, Theorem 1], the regularized problem (4.3–4.5), solved with the initial data $(\mathbf{U}(0,\cdot), C(0,\cdot))$ admits a unique solution $(\mathbf{U}^N, C^N)$ defined on $(\Omega^s, \mathfrak{B}^s, \mathcal{P}^s, \{\mathfrak{F}_t^s\}_{t\geq 0})$ that represents a strong solution with a stopping time

$$\tau^N = \inf \left\{ t \in [0,T] \;\middle|\; \|(\mathbf{U}(t,\cdot), C(t,\cdot))\|_{\mathcal{V}} \geq \frac{N}{2} \right\}, \; \inf\{\emptyset\} \equiv T.$$

Moreover, using the arguments of the proof of Theorem 4.3, we can show that

$$(\mathbf{U}^N, C^N)(t \wedge (\tau^N \wedge \tau^s)) = (\mathbf{U}, C)(t \wedge (\tau^N \wedge \tau^s)),\ t \in [0,T],\ \mathcal{P}^s - \text{a.s.},$$

in particular,

$$\tau^N \wedge \tau^s = \tau^N \wedge \tau_B[\mathbf{U}^N, C^N].$$

Now, we can solve the regularized problem on the space $(\Omega^m, \mathfrak{B}^m, \mathcal{P}^m, \{\mathfrak{F}_t^m\}_{t\geq 0})$ obtaining the strong solution $(\mathbf{u}^N, c^N)$. By virtue of the infinite dimensional version of the Yamada–Watanabe theorem, see Roeckner, Scmuland, Zhang [15], we have

$$\mathcal{L}_{C([0,T];\mathcal{V})}[(\mathbf{u}^N, c^N)] = \mathcal{L}_{C([0,T];\mathcal{V})}[(\mathbf{U}^N, C^N)].$$

Moreover, for

$$\tau_B = \tau_B[\mathbf{u}^N, c^N]$$

we get the equality of laws

$$\begin{aligned} \mathcal{L}_{C([0,T];\mathcal{V})}[(\mathbf{u}^N, c^N)(\cdot \wedge (\tau^N \wedge \tau_B))] &= \mathcal{L}_{C([0,T];\mathcal{V})}[(\mathbf{U}^N, C^N)(\cdot \wedge (\tau^N \wedge \tau_B[(\mathbf{u}^N, C^N)]))] \\ = \mathcal{L}_{C([0,T];\mathcal{V})}[(\mathbf{U}^N, C^N)(\cdot \wedge (\tau^N \wedge \tau^s))] &= \mathcal{L}_{C([0,T];\mathcal{V})}[(\mathbf{U}, C)(\cdot \wedge (\tau^N \wedge \tau^s))]. \end{aligned} \tag{4.34}$$

On the other hand $(\mathbf{u}^N, c^N)$ is a strong solution with the stopping time τ^N. Consequently, in accordance with the pathwise weak–strong uniqueness principle established in Theorem 4.3, we have

$$(\mathbf{u}^N, c^N)(t \wedge \tau^N) = (\mathbf{u}, c)(t \wedge \tau^N).$$

As $(\mathbf{U}, C)$ is a strong solution, we have

$$\mathcal{P}^s \left\{ \lim_{N\to\infty} (\tau^N \wedge \tau^s) = \tau^s \right\} = 1;$$

whence, letting $N \to \infty$ in $\mathcal{L}_{C([0,T];\mathcal{V})}[(\mathbf{u}, c)(\cdot \wedge (\tau^N \wedge \tau_B))] = \mathcal{L}_{C([0,T];\mathcal{V})}[(\mathbf{U}, C)(\cdot \wedge (\tau^N \wedge \tau^s))]$, we get the desired conclusion with

$$\tau^m = \lim_{N\to\infty} \left(\tau^N \wedge \tau_B[\mathbf{u}^N, c^N] \right).$$

□

5 Appendix

We used the following stochastic Gronwall lemma (see e.g. [9] for the proof of this result):

Lemma 5.1. *Let us fix $T > 0$ and assume that $X, Y, Z, R : \Omega \times [0, T) \to \mathbb{R}$ are non-negative stochastic processes. Let $\tau \leq T$ be a stopping time such that:*

$$\mathbb{E}\big(\int_0^\tau (RX + Z)\mathrm{d}s \big) < \infty, \quad \textit{and} \; \int_0^\tau R < \kappa, \; \textit{a.s.}, \tag{5.1}$$

for some fixed positive constant κ.

Suppose that for all stopping times $0 \leq \tau_a \leq \tau_b \leq \tau$

$$\mathbb{E}\big(\sup_{t \in [\tau_a, \tau_b]} X + \int_{\tau_a}^{\tau_b} Y \mathrm{d}s \big) \leq \kappa_0 \mathbb{E}\big(X(\tau_a) + \int_{\tau_a}^{\tau_b} (RX + Z)\mathrm{d}s \big), \tag{5.2}$$

where κ_0 is a positive constant independent of τ_a and τ_b. Then:

$$\mathbb{E}\big(\sup_{t \in [0, \tau]} X + \int_0^\tau Y \mathrm{d}s \big) \leq \kappa_1 \mathbb{E}\big(X(0) + \int_0^\tau Z \mathrm{d}s \big), \tag{5.3}$$

where κ_1 is a constant depending only on κ_0, T and κ.

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