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# Complexity of distances between metric and Banach spaces

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# COMPLEXITY OF DISTANCES BETWEEN METRIC AND BANACH SPACES

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Abstract. We investigate the complexity and reducibility between analytic pseudometrics coming from functional analysis and metric geometry, such as Gromov-Hausdorff, Kadets, and Banach-Mazur distances. This leads us to introduce the notion of Borel reducibility between pseudometrics which generalizes the standard Borel reducibility between definable equivalence relations and is a quantitative version of it, and orbit pseudometrics, the continuous version of orbit equivalences. Our results include the mutual bi-reducibility between Gromov-Hausdorff, Banach-Mazur, Kadets, Hausdorff-Lipschitz, net and Lipschitz distances, and their reducibility to the uniform distance. We show that $E_1$ is not reducible to equivalences given by these pseudometrics. Among our applications are the proofs that the distance-zero classes in these pseudometrics are Borel, extending the results of Ben Yaacov, Doucha, Nies, and Tsankov, and answering their question in negative whether balls in these distances are Borel. Besides that, we provide many other examples and problem areas to be looked at, which suggests that there is enough further possible development in this field.

Contents



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## Introduction

One of the main active streams of the current descriptive set theory, often called *invariant descriptive set theory*, is concerned with the study of definable equivalence relations on standard Borel spaces and reductions between them. This is a subject that has turned out to be very helpful in many fields of mathematics. Indeed, most of the mathematics is concerned with classification of some sort. Researchers working in some particular mathematical category try to associate some relatively simple invariants to the objects they study that would, in the ideal case, completely distinguish the objects up to isomorphism. In other words, a common theme in mathematics is to study the isomorphism equivalence relation and to find some effective reduction from that equivalence to another (isomorphism) relation which is simpler and more understood. Invariant descriptive set theory provides a general framework for such investigations and can be viewed as a general classification theory. We refer to [23] for a reference to this subject.

However, it turns out that in many areas of mathematics, especially in those working with 'metric objects', such as functional analysis or metric geometry, it is often convenient and more accessible to replace the isomorphism relation with some approximations. These approximations usually come in the form of some metric or pseudometric which measures how close to being isomorphic two objects are. Prototypical examples are the Gromov-Haudorff distance between compact metric spaces introduced by Gromov ([28]) which measures how close two compact metric space are to being isometric, or the Banach-Mazur distance between finite-dimensional Banach spaces which measures how close two finite-dimensional Banach spaces are to being linearly isometric. In both these examples, when two spaces have distance zero they are isometric, resp. linearly isometric. However, in most more complicated examples this is not the case and the studied distance is in fact only a pseudometric. Thus it induces an equivalence relation that in general does not coincide with the standard isomorphism relation in the category and it is worth studying by its own. This happens e.g. when one considers the Gromov-Hausdorff, resp. the Banach-Mazur distances on general (complete) metric spaces, resp. general Banach spaces. We note that nowadays we have examples of such distances in many areas of mathematics, e.g. in metric space theory (see [28] or [15] for several other distances on metric spaces), in Banach space theory we mention the Gromov-Hausdorff distance analogue for Banach spaces, the Kadets distance (see [32]), or various distances introduced e.g. by Ostrovskii (see [45] and [47]), in operator algebras (see the Kadison-Kastler distance defined in [33]), in the theory of graph limits (see various distances defined on graph limits, e.g. graphons, in [42]), in measure theory (see a number of distances between measures in [25]).

Our goal in this paper is to view these pseudometrics as generalized equivalence relations. This follows the research from [9] where certain back-and-forth Borel equivalence relations approximating the isomorphism relation on a class of countable structures were replaced by pseudometrics measuring how close to being isomorphic two metric objects are. It is also in the spirit of the model theory for metric structures, which has been enjoying a lot of developments and applications recently, to generalize discrete notions by their continuous counterparts. See [8] for an introduction to that subject. In the case of equivalence relations, this is naturally the notion of a pseudometric. The innovation in our paper comes from the idea to generalize the standard notion of Borel reducibility between definable equivalence relations to a Borel reducibility between definable pseudometrics. We were naturally led to that notion when proving reductions between equivalences induced by the Gromov-Hausdorff, Kadets, and Banach-Mazur distances and realizing that our reductions are actually quantitative. Suppose we are given two standard Borel spaces of metric structures, each equipped with some definable pseudometric, and we want to effectively reduce the first pseudometric to the other. Again the natural choice, generalizing the standard theory, is that the reduction is Borel. However, now it must preserve the pseudometric in some sense. In any case, it should be a Borel reduction between the equivalence relations induced by the pseudometrics in the standard theory. Some obvious choices would be that the reduction is isometric, or bi-Lipschitz, which seems to be too strong though. The right notion that most often appears naturally in our considerations is that the reduction is a uniformly continuous embedding, and that is also sufficient for our applications. We call such a Borel reduction *Borel-uniformly continuous reduction*.

We also suggest a generalization of orbit equivalence relations. That is, something we called *orbit pseudometrics*. Under some natural restrictions, these seem to form an interesting class of analytic pseudometrics. Indeed, most of the analytic pseudometrics we consider in this paper are bi-reducible with such orbit pseudometrics. Moreover, the equivalence relation $E_1$ is not Borel reducible to equivalences given by these orbit pseudometrics.

We start where Section 8 of [9] finished, by focusing on the Gromov-Hausdorff and Kadets pseudometrics. However, we consider many other pseudometrics such as the Gromov-Hausdorff distances restricted on Banach spaces or several classes of metric spaces, the Hausdorff-Lipschitz distance of Gromov (from [28]), the net distance of Dutrieux and Kalton (from [20]), the Lipschitz distance on metric spaces and Banach spaces, or the standard Banach-Mazur distance.

We summarize our main results below. First we show several reductions between pseudometrics mentioned above.

**Theorem A.** (1) *The following pseudometrics are mutually Borel-uniformly continuous bi-reducible: the* **Gromov-Hausdorff distance** *when restricted to Polish metric spaces, to metric spaces bounded from above, from below, from both above and below, to Banach spaces; the* **Banach-Mazur distance** *on Banach spaces, the* **Lipschitz distance** *on Polish metric spaces and Banach spaces; the* **Kadets distance** *on Banach spaces; the* **Hausdorff-Lipschitz distance** *on Polish meric spaces; the* **net distance** *on Banach spaces.*

(2) *The pseudometrics above are Borel-uniformly continuous reducible to the* **uniform** *distance on Banach spaces.*

As mentioned above, the pseudometrics from the preceding theorem actually belong to a special class of analytic pseudometrics, which we call

*CTR orbit pseudometrics.* We refer the reader to Section 2.3 for a precise definition.

**Theorem B.** *(1) All the pseudometrics from Theorem A* (1) *are Borel-uniformly continuous bi-reducible with a CTR orbit pseudometric.*
*(2) The equivalence relation $E_1$ is not Borel reducible to any equivalence relation given by a CTR orbit pseudometric.*

Next we extend the results from [9] where it was shown that the equivalence classes of the Gromov-Hausdorff and Kadets distances are Borel.

**Theorem C.** *The pseudometrics from Theorem A, except the uniform distance for which we do not know the answer, have Borel classes of equivalence.*

Note that the equivalence relations induced by the pseudometrics from Theorem A are analytic non-Borel since a universal orbit equivalence relation is Borel reducible to them. It is well known that orbit equivalence relations are in general not Borel (see e.g. [23, Chapter 9]).

Finally, we answer Question 8.4 from [9] by proving the following.

**Theorem D.** *Let $\rho$ be any pseudometric to which the Kadets distance is Borel-uniformly continuous reducible (e.g. any of the pseudometrics from Theorem A). Then there are elements $A$ from the domain of $\rho$ such that the function $\rho(A, \cdot)$ is not Borel.*

Let us note that each pseudometric $\rho$ on a set $X$ induces also a cruder equivalence relation, which we denote here by $E^\rho$, where for $x, y \in X$ we set $xE^\rho y$ if and only if $\rho(x, y) < \infty$. The complexity of such relations, for the natural pseudometrics from functional analysis and metric geometry, has been studied recently rather extensively. For example, for the Lipschitz and Banach-Mazur distances, the complexity of such equivalences was determined in [22], where it was shown that they are complete analytic equivalence relations. For the Gromov-Hausdorff distance, it was studied recently in [1], where it was shown that this equivalence is not Borel reducible to an orbit equivalence relation.

The paper is organized as follows. In Section 1 we recall several basic notions from descriptive set theory, define the distances from our paper and prove basic facts about them. In Section 2 we introduce the new notions of our paper such as Borel reducibility between analytic pseudometrics and we provide several more examples. Moreover, we introduce there the continuous version of orbit equivalences, the CTR orbit pseudometrics, and we prove that the equivalence relation $E_1$ is not Borel reducible to the equivalence relations induced by them. The core of the paper is Section 3 where we concentrate the proofs of our reductions. Then in Section 4 we play certain metric games which provide an alternative, and probably more general, way to the methods from [9] how to show that these pseudometrics have Borel classes of equivalence, and in Section 5 we prove that $\rho$-balls are in general not Borel for any pseudometric $\rho$ to which the Kadets distance is reducible. Finally, in Section 6 we comment on our results and we present directions for further research.

## 1. Preliminaries and basic results

The goal of this section is to recall several basic notions from descriptive set theory, such as coding of Polish metric spaces or Banach spaces, and to introduce the distances we work with in this paper. We also prove here several basic results about these distances which will be needed in further sections. The notation and terminology is standard, for the undefined notions see [21] for Banach spaces and [37] for descriptive set theory.

1.1. **Coding of Polish metric spaces and Banach spaces.** We begin with formalizing the class of all infinite Polish metric spaces as a standard Borel space. In most situations it will not be important how we formalize this class, but whenever it does become important we shall use the following definition.

**Definition 1.** By $\mathcal{M}$ we denote the space of all metrics on $\mathbb{N}$. This gives $\mathcal{M}$ a Polish topology inherited from $\mathbb{R}^{\mathbb{N}\times\mathbb{N}}$.

If $p$ and $q$ are positive real numbers, by $\mathcal{M}_p$, $\mathcal{M}^q$ and $\mathcal{M}_p^q$ respectively, we denote the space of metrics with values in $\{0\}\cup[p,\infty)$, $[0,q]$, and $\{0\}\cup[p,q]$ (assuming that $p<q$), respectively.

*Remark* 2. Every $f\in\mathcal{M}$ is then a code for Polish metric space $M_f$ which is the completion of $(\mathbb{N},f)$. Hence, in this sense we may refer to the set $\mathcal{M}$ as to the standard Borel space of all infinite Polish metric spaces. This approach was used for the first time by Vershik [52] and further e.g. in [16], see also [23, page 324]. Another possible approach is to view all Polish metric spaces as the Effros-Borel space $F(\mathbb{U})$ of all closed subspaces of the Urysohn space $\mathbb{U}$. When one considers the space of all pseudometrics on $\mathbb{N}$ then these two approaches are equivalent, see e.g. [23, Theorem 14.1.3]. Similarly, one can get a Borel isomorphism $\Theta$ between $\mathcal{M}$ and $F(\mathbb{U})\setminus F_{fin}(\mathbb{U})$, where $F_{fin}(\mathbb{U})$ denotes the Borel set of finite subsets of $\mathbb{U}$, such that $\Theta(f)$ is isometric to $M_f$ for every $f\in\mathcal{M}$. Since the Borel set of finite metric spaces is not interesting from our point of view we will ignore it in the sequel.

*Remark* 3. Let $(M,d)$ be a separable metric space. If there is no danger of confusion, we write $M\in\mathcal{M}$ by which we mean that the metric $d$ restricted to a countable dense subset of $M$ induces a metric $d'\in\mathcal{M}$. Analogously, if there is no danger of confusion, we write $M\in\mathcal{M}_p$, $M\in\mathcal{M}^q$ or $M\in\mathcal{M}_p^q$.

Next, we formalize the class of all infinite-dimensional separable Banach spaces as a standard Borel space. As in the case of infinite Polish metric spaces, the concrete coding of this space is usually not important. However, when we compute that certain maps from or into this space are Borel we adopt a coding analogous to that one for $\mathcal{M}$ (and which is more similar to the general coding of metric structures from [9]).

**Definition 4.** Let us denote by $V$ the vector space over $\mathbb{Q}$ of all finitely supported sequences of rational numbers, that is, the unique infinite-dimensional vector space over $\mathbb{Q}$ with a countable Hamel basis $(e_n)_{n\in\mathbb{N}}$. By $\mathcal{B}_0$ we denote the space of all norms on the vector space $V$. This gives $\mathcal{B}_0$ a Polish topology inherited from $\mathbb{R}^V$. We shall consider only those norms for which its canonical extension to the real vector space $c_{00}$ is still a norm; that is, norms for which the elements $(e_n)_n$ are not only $\mathbb{Q}$-linearly independent, but also $\mathbb{R}$-linearly independent. Let us denote the subset of such norms by $\mathcal{B}$. It is a Borel subset of $\mathcal{B}_0$. Indeed, one can easily check that for $\|\cdot\| \in \mathcal{B}_0$ we have $\|\cdot\| \in \mathcal{B}$ if and only if for every $n \in \mathbb{N}$ there is $\varepsilon_n > 0$ such that for every $n$-tuple $(x_1, \ldots, x_n) \in \mathbb{Q}^n$ we have $\|x_1e_1 + \ldots + x_ne_n\| \geq \varepsilon_n(|x_1| + \ldots + |x_n|)$, which shows that $\mathcal{B}$ is a $F_{\sigma\delta}$ subset of $\mathcal{B}_0$.

*Remark* 5. Each norm $\nu \in \mathcal{B}$ is then a code for an infinite-dimensional Banach space $X_\nu$ which is the completion of $(V, \nu)$. The completion is naturally a complete normed space over $\mathbb{R}$. This is the same as taking the canonical extension of $\nu$ to $c_{00}$ and then taking the completion.

Hence, we may refer to the set $\mathcal{B}$ as to the standard Borel space of all infinite-dimensional separable Banach spaces. Another possible approach, introduced by Bossard [14], is to view all infinite-dimensional separable Banach spaces as the space $SB(X)$ of all closed linear infinite-dimensional subspaces of a universal separable Banach space $X$; then it is a Borel subset of the Effros-Borel space $F(X)$, the interested reader is referred to the monograph [19] for further information. Similarly as in the case of Polish metric spaces, those two approaches are equivalent which is witnessed by Theorem 7.

It would be possible to get a coding of all separable Banach spaces, i.e. even finite-dimensional, if we considered the space of all pseudonorms on $V$. As in the case of Polish metric spaces, the Borel set of all finite-dimensional Banach spaces is not interesting from our point of view, so we will ignore it in the sequel.

*Remark* 6. If there is no danger of confusion, we write $X \in \mathcal{B}$ as a shortcut for "$X$ is an infinite-dimensional separable Banach space".

**Theorem 7.** *For every universal separable Banach space $X$, there is a Borel isomorphism $\Theta$ between $\mathcal{B}$ and $SB(X)$ such that $\Theta(\nu)$ is isometric to $X_\nu$ for every $\nu \in \mathcal{B}$.*

*Proof.* First, let us observe that whenever $X$ and $Y$ are universal separable Banach spaces, there is a Borel isomorphism $\Phi$ between $SB(X)$ and $SB(Y)$ such that $\Phi(Z)$ is isometric to $Z$ for every $Z \in SB(X)$. Indeed, fix an isometry $i : X \to Y$. Then $SB(X) \ni Z \mapsto i(Z) \in SB(Y)$ defines a Borel injective map, let us call it $\Phi_1$, such that $Z$ is isometric to $\Phi_1(Z)$ for every $Z \in SB(X)$. Next, we find an analogous Borel injective map $\Phi_2 : SB(Y) \to SB(X)$. Finally, using the usual proof of the Cantor-Bernstein Theorem (see e.g. [37, Theorem 15.7]), we find a Borel isomorphism $\Phi$ between $SB(X)$ and $SB(Y)$ whose graph lies in the union of the graph of $\Phi_1$ and the inverse of the graph of $\Phi_2$.

Hence, we may without loss of generality assume that $X = C([0, 1]) \oplus_2 C([0, 1])$. Using the classical Kuratowski-Ryll-Nardzewski principle (see e.g. [19, Theorem 1.2]), we easily get a sequence of Borel maps $d_n : SB(X) \to X$ such that for every $Z \in SB(X)$ the sequence $(d_n(Z))_{n=1}^\infty$ is normalized, linearly independent and linearly dense in $Z$. Since all uncountable Polish metric spaces are Borel isomorphic, we may pick a Borel isomorphism $j$ between $SB(X)$ and the interval $[1, 2]$. Now, we define a Borel injective map $\Theta_1 : SB(X) \to \mathcal{B}$ by putting for every $Z \in SB(X)$

$$\Theta_1(Z)(\alpha) = \left\| j(Z)\alpha_1 d_1(Z) + \sum_{i=2}^{\infty} \alpha_i d_i(Z) \right\|, \quad \alpha \in V.$$

Then $\Theta_1$ is an injective Borel map from $SB(X)$ into $\mathcal{B}$ such that $X_{\Theta_1(Z)}$ is isometric to $Z$ for every $Z \in SB(X)$.

Next, by [40, Lemma 2.4], there is a Borel map $\widetilde{\Theta_2} : \mathcal{B} \to SB(C([0,1]))$ such that $\widetilde{\Theta_2}(\nu)$ is isometric to $X_\nu$ for every $\nu \in \mathcal{B}$. Pick a Borel isomorphism $j$ between $\mathcal{B}$ and the interval $[0,1]$ and for every $\nu \in \mathcal{B}$ define $\Theta_2(\nu)$ as the Banach space of all $(j(\nu)f, \sqrt{1-j^2(\nu)}f) \in X$ where $f \in \widetilde{\Theta_2}(\nu)$. Then $\Theta_2$ is an injective Borel map from $\mathcal{B}$ into $SB(X)$ such that $\Theta_2(\nu)$ is isometric to $X_\nu$ for every $\nu \in \mathcal{B}$.

Finally, using the usual proof of the Cantor-Bernstein Theorem (see e.g. [37, Theorem 15.7]), we find a Borel isomorphism $\Theta$ between $\mathcal{B}$ and $SB(X)$ whose graph lies in the union of the graph of $\Theta_2$ and the inverse of the graph of $\Theta_1$. $\square$

## 1.2. **Distances between metric spaces and Banach spaces.**

### 1.2.1. *Gromov-Hausdorff distance.*

**Definition 8** (Gromov-Hausdorff distance)**.** Let $(M, d_M)$ be a metric space and $A, B \subseteq M$ two non-empty subsets. The *Hausdorff distance* between $A$ and $B$ in $M$, $\rho_H^M(A,B)$, is defined as

$$\max\left\{ \sup_{a\in A} d_M(a,B), \sup_{b\in B} d_M(b,A) \right\},$$

where for an element $a \in M$ and a subset $B \subseteq M$, $d_M(a,B) = \inf_{b\in B} d_M(a,b)$.

Suppose now that $M$ and $N$ are two metric spaces. Their *Gromov-Hausdorff distance*, $\rho_{GH}(M,N)$, is defined as the infimum of the Hausdorff distances of their isometric copies contained in a single metric space, that is

$$\rho_{GH}(M,N) = \inf_{\substack{\iota_M : M \hookrightarrow X \\ \iota_N : N \hookrightarrow X}} \rho_H^X(\iota_M(M), \iota_N(N)),$$

where $\iota_M$ and $\iota_N$ are isometric embeddings into a metric space $X$.

For two metrics $f, g \in \mathcal{M}$ we denote by $\rho_{GH}(f,g)$ the Gromov-Hausdorff distance between $(\mathbb{N}, f)$ and $(\mathbb{N}, g)$, which is easily seen to be equal to the Gromov-Hausdorff distance between their completions $M_f$ and $M_g$.

Let $A$ and $B$ be two sets. A *correspondence* between $A$ and $B$ is a binary relation $\mathcal{R} \subseteq A \times B$ such that for every $a \in A$ there is $b \in B$ such that $a\mathcal{R}b$, and for every $b \in B$ there is $a \in A$ such that $a\mathcal{R}b$.

**Fact 9** (see e.g. Theorem 7.3.25. in [15])**.** *Let $M$ and $N$ be two metric spaces. For every $r > 0$ we have $\rho_{GH}(M,N) < r$ if and only if there exists a correspondence $\mathcal{R}$ between $M$ and $N$ such that* $\sup |d_M(m,m') - d_N(n,n')| < 2r$*, where the supremum is taken over all $m, m' \in M$ and $n, n' \in N$ with $m\mathcal{R}n$ and $m'\mathcal{R}n'$.*

It is easier to work with bijections instead of correspondences. One may wonder in which situations we may do so. Let us define the corresponding concept and prove some results in this direction. Those will be used later.

**Definition 10.** By $S_\infty$ we denote the set of all bijections from $\mathbb{N}$ to $\mathbb{N}$. For two metrics on natural numbers $f, g \in \mathcal{M}$ and $\varepsilon > 0$, we consider the relation

$$f \simeq_\varepsilon g \quad \Leftrightarrow \quad \exists \pi \in S_\infty \, \forall \{n, m\} \in [\mathbb{N}]^2 : |f(\pi(n), \pi(m)) - g(n, m)| \leq \varepsilon.$$

We write $f \simeq g$ if $f \simeq_\varepsilon g$ for every $\varepsilon > 0$.

**Lemma 11.** *For any two metrics on natural numbers $f, g \in \mathcal{M}$ and any $\varepsilon > 0$ we have $\rho_{GH}(f, g) \leq \varepsilon$ whenever $f \simeq_{2\varepsilon} g$.*

*Proof.* Suppose that $f \simeq_{2\varepsilon} g$. The permutation $\pi \in S_\infty$ witnessing that $f \simeq_{2\varepsilon} g$ induces a correspondence $\mathcal{R}$ between the metric spaces $(\mathbb{N}, f)$ and $(\mathbb{N}, g)$ which, by Fact 9, shows that $\rho_{GH}(f, g) \leq \varepsilon$. $\square$

**Lemma 12.** *Let $p > 0$ be a real number. For any two metrics on natural numbers $f, g \in \mathcal{M}_p$ we have $\rho_{GH}(f, g) = \inf\{r : f \simeq_{2r} g\}$ provided that $\rho_{GH}(f, g) < p/2$.*

*Proof.* By Lemma 11, $\rho_{GH}(f, g) \leq r$ whenever $f \simeq_{2r} g$. Conversely, suppose that $\rho_{GH}(f, g) < r$, where $r < p/2$. By Fact 9, there exists a correspondence $\mathcal{R}$ between the metric spaces $(\mathbb{N}, f)$ and $(\mathbb{N}, g)$ such that $\sup |f(m, m') - g(n, n')| < 2r$, where the supremum is over all $m, m', n, n' \in \mathbb{N}$ with $m\mathcal{R}n$ and $m'\mathcal{R}n'$. We claim that $\mathcal{R}$ is the graph of some permutation $\pi \in S_\infty$. That will, by definition of $\simeq_{2r}$, show that $f \simeq_{2r} g$. Suppose that $\mathcal{R}$ is not such graph. Say e.g. that for some $m$ there are $n \neq n'$ such that $m\mathcal{R}n$ and $m\mathcal{R}n'$. Then we have $g(n, n') = |f(m, m) - g(n, n')| < 2r < p$, which contradicts that $g \in \mathcal{M}_p$. Analogously we can show that for no $m \neq m'$ there is $n$ such that $m\mathcal{R}n$ and $m'\mathcal{R}n$. $\square$

**Lemma 13.** *Let $f, g \in \mathcal{M}$ define two perfect metric spaces, that is, spaces without isolated points. Then $\rho_{GH}(f, g) = \inf\{r : f \simeq_{2r} g\}$.*

*Proof.* By Lemma 11, $\rho_{GH}(f, g) \leq r$ whenever $f \simeq_{2r} g$. For the other inequality, suppose $\rho_{GH}(f, g) < r$ and fix $s$ with $\rho_{GH}(f, g) < s < r$. By Fact 9, there is a correspondence $\mathcal{R} \subseteq \mathbb{N} \times \mathbb{N}$ witnessing that $\rho_{GH}(f, g) < s$. Now we recursively define a permutation $\pi \in S_\infty$. During the $(2n-1)$-th step of the recursion we ensure that $n$ is in the domain of $\pi$ and during the $2n$-th step we ensure that $n$ is in the range of $\pi$.

Pick an arbitrary $n \in \mathbb{N}$ such that $1\mathcal{R}n$ and set $\pi(1) = n$. If $n = 1$ then we have ensured that 1 is both in the domain and the range of $\pi$. If $n \neq 1$, then pick some $m \in \mathbb{N}$ such that $m\mathcal{R}1$ and set $\pi(m) = 1$. If the only integer $m$ with the property that $m\mathcal{R}1$ is equal to 1, which has been already used, we pick an arbitrary $m' \in \mathbb{N}$ that has not been used yet and such that $f(m, m') < r - s$. The existence of such $m'$ follows since $f$ is perfect. We set $\pi(m') = 1$. In the general $(2n-1)$-th step we proceed analogously. If $n$ has not been added to the domain of $\pi$ yet we pick some $m$ that has not been added to the range of $\pi$ yet and such that $n\mathcal{R}m$. Then we may set $\pi(n) = m$. If there is no such $m$, we pick an arbitrary $m$ such that $n\mathcal{R}m$ and take an arbitrary $m'$ with $g(m, m') < r - s$ that has not been added to the range of $\pi$ yet and set $\pi(n) = m'$. The $2n$-th step is done analogously.

When the recursion is finished we claim that for every $n, m$ we have $|f(m, n) - g(\pi(m), \pi(n))| \leq 2r$ which is what we should prove. Suppose e.g.

that $\pi(m)$, resp. $\pi(n)$ are such that there are $m'$, resp. $n'$ with $g(m',\pi(m)) < r-s$ and $g(n',\pi(n)) < r-s$, and $m\mathcal{R}m'$ and $n\mathcal{R}n'$. The other cases are treated analogously. Then by the choice of $\mathcal{R}$ we have

$$\begin{aligned}|f(m,n)-g(\pi(m),\pi(n))| &\leq |f(m,n)-g(m',n')|+|g(m',n')-g(\pi(m),n')|\\ &\quad+|g(\pi(m),n')-g(\pi(m),\pi(n))|\\ &< 2s+g(m',\pi(m))+g(n',\pi(n)) < 2r.\end{aligned}$$

□

*Remark* 14. If $f,g\in\mathcal{M}$ define neither perfect metric spaces, nor do they belong to $\mathcal{M}_p$, for some $p>0$, then $\simeq_\varepsilon$ does not give good estimates for the Gromov-Hausdorff distance between $f$ and $g$. Consider e.g. $\mathbb{N}$ as a metric space with its standard metric and a metric space $C_k=\{m+1/n : m\in\mathbb{N}, n\geq k\}\subseteq\mathbb{R}$, $k\geq 2$, with a metric inherited from $\mathbb{R}$. We have $\rho_{GH}(\mathbb{N},C_k)\to 0$ as $k\to\infty$, but clearly there are no bijections between $\mathbb{N}$ and $C_k$ witnessing the convergence.

1.2.2. *Kadets distance.*

**Definition 15** (Kadets distance)**.** Suppose that $X$ and $Y$ are two Banach spaces. Their *Kadets distance*, $\rho_K(X,Y)$, is defined as the infimum of the Hausdorff distances of their unit balls over all isometric linear embeddings of $X$ and $Y$ into a common Banach space $Z$. That is

$$\rho_K(X,Y)=\inf_{\substack{\iota_X:X\hookrightarrow Z\\ \iota_Y:Y\hookrightarrow Z}}\rho_H^Z(\iota_X(B_X),\iota_Y(B_Y)),$$

where $\iota_X$ and $\iota_Y$ are linear isometric embeddings into a Banach space $Z$.

Similarly as the Gromov-Hausdorff distance, the Kadets distance may be expressed in terms of correspondences. First, call a subset $A\subseteq X$ of a real vector space $\mathbb{Q}$-*homogeneous* if it is closed under scalar multiplication by rationals. The following lemma generalizes [35, Theorem 2.3], which uses homogeneous maps. The proof is however very similar.

**Lemma 16.** *Let $X$ and $Y$ be Banach spaces and $E$ and $F$ be some dense $\mathbb{Q}$-homogeneous subsets of $X$ and $Y$ respectively. Then we have $\rho_K(X,Y)<\varepsilon$ if and only if there exist $\delta\in(0,\varepsilon)$ and a $\mathbb{Q}$-homogeneous correspondence $\mathcal{R}\subseteq E\times F$ with the property that for every $x\in E$ there is $y\in F$ with $x\mathcal{R}y$ and $\|y\|_Y\leq\|x\|_X$, for every $y\in F$ there is $x\in E$ with $x\mathcal{R}y$ and $\|x\|_X\leq\|y\|_Y$, and*

$$\left|\Big\|\sum_{i\leq n}x_i\Big\|_X-\Big\|\sum_{i\leq n}y_i\Big\|_Y\right|\leq(\varepsilon-\delta)\Big(\sum_{i\leq n}\max\{\|x_i\|_X,\|y_i\|_Y\}\Big)$$

*for all $(x_i)_i\subseteq E$ and $(y_i)_i\subseteq F$, where for all $i$, $x_i\mathcal{R}y_i$.*

*Proof.* If $\rho_K(X,Y)<\varepsilon$, then fix some $\delta\in(0,\varepsilon-\rho_K(X,Y))$ and some isometric embeddings of $X$ and $Y$ into a Banach space $Z$ such that $\rho_H^Z(B_X,B_Y)<\varepsilon-\delta$. Then set $x\mathcal{R}y$, for $x\in E$ and $y\in F$, if and only if $\|x-y\|_Z\leq\max\{(\varepsilon-\delta)\|x\|_X,(\varepsilon-\delta)\|y\|_Y\}$.

Suppose conversely that we have such $\delta \in (0, \varepsilon)$ and $\mathcal{R} \subseteq E \times F$. Set $E'$ to be the linear span of $E$, analogously $F'$ to be the linear span of $F$. Then set $Z = E' \oplus F'$ and define a norm $\|\cdot\|_Z$ on $Z$ as follows: for $(x, y) \in Z$ set

$$\|(x,y)\|_Z = \inf \Big\{ \|x_0\|_X + \|y_0\|_Y + (\varepsilon - \delta)\Big(\sum_{i \leq n} \max\{\|x_i\|_X, \|y_i\|_Y\}\Big) : \\ x = x_0 + \sum_{i \leq n} x_i,\ y = y_0 - \sum_{i \leq n} y_i, x_0 \in E', y_0 \in F',\ x_i \mathcal{R} y_i \Big\}.$$

It is clear that $\|\cdot\|_Z$ satisfies the triangle inequality. Moreover, it is easy to check, using the $\mathbb{Q}$-homogeneity of $\mathcal{R}$, that $\|\cdot\|_Z$ is $\mathbb{Q}$-homogeneous, i.e. for every $z \in Z$ and $q \in \mathbb{Q}$, $\|qz\|_Z = |q|\|z\|_Z$. By continuity, we also get the full homogeneity for all real scalars.

Let us check that for any $x \in E'$ we have $\|(x, 0)\|_Z = \|x\|_X$. Clearly, $\|(x, 0)\|_Z \leq \|x\|_X$. Suppose there is a strict inequality. Then we have

$$\|x_0\|_X + \|y_0\|_Y + (\varepsilon - \delta)\Big(\sum_{i \leq n} \max\{\|x_i\|_X, \|y_i\|_Y\}\Big) < \|x\|_X,$$

where $x_0 \in E'$, $x_1, \dots, x_n \in E$, $y_0 \in F'$, $y_1, \dots, y_n \in F$, $x = x_0 + \sum_{i \leq n} x_i$, $y = 0 = y_0 - \sum_{i \leq n} y_i$ and $x_i \mathcal{R} y_i$. However, by our assumption we have

$$\begin{aligned} \|x_0\|_X + \|y_0\|_Y + (\varepsilon - \delta)\Big(\sum_{i \leq n} \max\{\|x_i\|_X, \|y_i\|_Y\}\Big) = \\ = \|x_0\|_X + \Big\|\sum_{i \leq n} y_i\Big\|_Y + (\varepsilon - \delta)\Big(\sum_{i \leq n} \max\{\|x_i\|_X, \|y_i\|_Y\}\Big) \geq \\ \geq \|x_0\|_X + \Big\|\sum_{i \leq n} x_i\Big\|_X \geq \|x\|_X, \end{aligned}$$

a contradiction. Analogously, we show that for every $y \in F'$ we have $\|y\|_Y = \|(0, y)\|_Z$. So $E'$ and $F'$ are isometrically embedded into $Z$. Now for any $x \in B_X \cap E$ by the assumption there is $y \in F$ such that $\|y\|_Y \leq \|x\|_X$ and $x \mathcal{R} y$. So

$$\|(x, -y)\|_Z \leq (\varepsilon - \delta)\|x\|_X$$

since $x$ can be written as $x_0 + x$, where $x_0 = 0$, and $-y$ as $y_0 - y$, where $y_0 = 0$. Analogously, for every $y \in B_Y \cap F$ there is $x \in E$ such that $\|x\|_X \leq \|y\|_Y$ and $\|(x, -y)\|_Z \leq (\varepsilon - \delta)\|y\|_Y$. Finally we take the completion of $Z$ and get a Banach space $Z'$ to which $X$ and $Y$ linearly isometrically embed so that $\rho_H^{Z'}(B_X, B_Y) < \varepsilon$. $\square$

1.2.3. *Lipschitz distance.*

**Definition 17** (Lipschitz distance)**.** Let $M$ and $N$ be two metric spaces. Their *Lipschitz distance* is defined as

$$\rho_L(M, N) = \inf \big\{ \log \max\{\mathrm{Lip}(T), \mathrm{Lip}(T^{-1})\} : T : M \to N \text{ is bi-Lipschitz bijection} \big\},$$

where

$$\mathrm{Lip}(T) = \sup_{m \neq n \in M} \frac{d_N(T(m), T(n))}{d_M(m, n)}$$

is the Lipschitz norm of $T$.

*Remark* 18. The previous definition of the Lipschitz distance is from [15, Definition 7.2.1]. We note that Gromov in [28, Definition 3.1] defines the Lipschitz distance (between $M$ and $N$) as

$$\inf\big\{|\log \mathrm{Lip}(T)| + |\log \mathrm{Lip}(T^{-1})| :\ T : M \to N \text{ is bi-Lipschitz}\big\}.$$

Nevertheless, one can easily check that these two definitions give equivalent distances. Indeed, if we denote by $\rho'_L$ the Lipschitz distance in the sense of Gromov, then we easily see that

$$\rho_L \leq \rho'_L \leq 2\rho_L.$$

More differently, Dutrieux and Kalton in [20] define the Lipschitz distance analogously to the definition of the Banach-Mazur distance, which we recall later, as [1]

$$\inf\big\{\log \mathrm{Lip}(T)\,\mathrm{Lip}(T^{-1}) :\ T : M \to N \text{ is bi-Lipschitz}\big\}.$$

Denote this distance by $\rho''_L$. Clearly, $\rho''_L$ is not equivalent with $\rho_L$ since for example intervals $[0,1]$ and $[0,2]$ have distance zero only in $\rho''_L$. However, in [20] the authors work mainly with Banach spaces and if $M$ and $N$ are Banach spaces, it is easy to see that we have $\rho''_L(M,N) = \rho'_L(M,N)$. That follows from the fact that we may consider only those bi-Lipschitz maps such that both $\log \mathrm{Lip}(T)$ and $\log \mathrm{Lip}(T^{-1})$ are non-negative. Indeed, if say $\mathrm{Lip}(T) < 1$, then we define $T' = T/\mathrm{Lip}(T)$ and we get $\log \mathrm{Lip}(T) + \log \mathrm{Lip}(T^{-1}) = |\log \mathrm{Lip}(T')| + |\log \mathrm{Lip}((T')^{-1})|$.

However, for Banach spaces we have $\rho'_L(M,N) = 2\rho_L(M,N)$. This again follows after the appropriate rescaling of the maps $T : M \to N$.

One of the differences between the Gromov-Hausdorff distance and the Lipschitz distance on metric spaces is that for the former if $M$ and $N$ are metric spaces and $M'$, resp. $N'$ their dense subsets, then $\rho_{GH}(M,N) = \rho_{GH}(M',N')$. That an analogous equality does not hold for the Lipschitz distance is witnessed by the following fact. We thank to Benjamin Vejnar for providing us an example on which it is based.

**Fact 19.** *There exist metrics $d_M, d_N \in \mathcal{M}$ on $\mathbb{N}$ such that their completions are isometric, however there is no bi-Lipschitz map between $(\mathbb{N}, d_M)$ and $(\mathbb{N}, d_N)$.*

*Proof.* Let $M$ be a Polish metric space, let $G$ be the group of bi-Lipschitz autohomeomorphisms of $M$, and suppose there exists $m \in M$ such that $M \setminus G \cdot m$ is dense in $M$, where $G \cdot m$ is the orbit of $m$ under the action of $G$ on $M$. Let $(x_i)_i$ be some countable dense subset of $M$ such that $\{x_i :\ i \in \mathbb{N}\} \cap G \cdot m = \emptyset$, and let $(y_j)_j$ be another countable dense subset of $M$ such that $y_1 = m$. Then there is no bi-Lipschitz map between $(x_i)_i$ and $(y_j)_j$. Indeed, otherwise such a bi-Lipschitz map would extend to some bi-Lipschitz autohomeomorphism $g \in G$ and we would have $g \cdot y_1 = g \cdot m = x_k$, for some $k \in \mathbb{N}$, which is a contradiction.

To give a simple concrete example, consider $M = [0,1]$ and $m = 0$. $\square$

[1]More precisely, they define it without the logarithm which we add in order to satisfy the triangle inequality.

It follows that we cannot in general for $d, p \in \mathcal{M}$ decide whether $\rho_L(M_d, M_p) < \varepsilon$ just by computing $\rho_L((\mathbb{N}, d), (\mathbb{N}, p))$. For a correspondence $\mathcal{R} \subseteq \mathbb{N}^2$ and $n \in \mathbb{N}$ we denote by $n\mathcal{R}$ the set $\{m \in \mathbb{N} : \; n\mathcal{R}m\}$ and by $\mathcal{R}n$ the set $\{m \in \mathbb{N} : \; m\mathcal{R}n\}$.

**Lemma 20.** *Let $d, p \in \mathcal{M}$. Then $\rho_L(M_d, M_p) < r$ if and only if there exists $r' < r$ and a sequence of correspondences $\mathcal{R}_i \subseteq \mathbb{N} \times \mathbb{N}$ decreasing in inclusion such that*

*(1) for every $\varepsilon > 0$ there exists $i \in \mathbb{N}$ such that for every $n \in \mathbb{N}$ we have $p\text{-}\operatorname{diam}(n\mathcal{R}_i) < \varepsilon$ and $d\text{-}\operatorname{diam}(\mathcal{R}_i n) < \varepsilon$;*
*(2) for every $i \in \mathbb{N}$ and every $n, m, n', m' \in \mathbb{N}$ such that $d(n, m) \geq 2^{-i}$ and $n\mathcal{R}_i n'$ and $m\mathcal{R}_i m'$ we have $p(n', m') \leq \exp(r')d(n, m)$;*
*(3) for every $i \in \mathbb{N}$ and every $n, m, n', m' \in \mathbb{N}$ such that $p(n, m) \geq 2^{-i}$ and $n'\mathcal{R}_i n$ and $m'\mathcal{R}_i m$ we have $d(n', m') \leq \exp(r')p(n, m)$.*

*Proof.* For the implication from the right to the left, for every $n \in \mathbb{N}$ we define $\phi(n) \in M_p$ and $\psi(n) \in M_d$ as the unique element of $\bigcap_i \overline{n\mathcal{R}_i}$ and $\bigcap_i \overline{\mathcal{R}_i n}$, respectively. We leave to the reader to verify the simple fact that $\phi : \mathbb{N} \to M_p$ is a Lipschitz map with Lipschitz constant less than $\exp(r)$, which therefore extends to a Lipschitz map $\bar{\phi} : M_d \to M_p$ with the same Lischitz constant, and if $\bar{\psi}$ is defined analogously, then $\bar{\phi} = (\bar{\psi})^{-1}$

For the other implication, suppose that we are given a bi-Lipschitz map $\phi : M_d \to M_p$ such that $L := \max\{\operatorname{Lip}(\phi), \operatorname{Lip}(\phi^{-1})\} < \exp(r)$ and pick $\varepsilon > 0$ with $L + \varepsilon < \exp(r)$. For every $i \in \mathbb{N}$, put $\varepsilon_i := \frac{\varepsilon}{i2^{i+1}(1+L)}$ and define correspondence $\mathcal{R}_i$ by

$$\mathcal{R}_i := \big\{(n, n') \in \mathbb{N} \times \mathbb{N} : \; \exists \tilde{n} \in \mathbb{N} \quad d(n, \tilde{n}) < \varepsilon_i \;\;\&\;\; p(\phi(\tilde{n}), n') < \varepsilon_i\big\}.$$

We claim that the correspondences $(\mathcal{R}_i)_i$ are as desired. It is easy to see that $\mathcal{R}_i \subseteq \mathbb{N} \times \mathbb{N}$ are correspondences decreasing in inclusion and that (1) is satisfied. We check condition (2) and find the number $r'$, the condition (3) is checked similarly. Fix some $i \in \mathbb{N}$ and $n, m, n', m' \in \mathbb{N}$ with $n\mathcal{R}_i n'$, $m\mathcal{R}_i m'$ and $d(n, m) \geq 2^{-i}$. Let $\tilde{n}$ and $\tilde{m}$ be natural numbers witnessing that $n\mathcal{R}_i n'$ and $m\mathcal{R}_i m'$, respectively. Then we have

$$\begin{aligned} p(n', m') &\leq 2\varepsilon_i + p(\phi(\tilde{n}), \phi(\tilde{m})) \leq 2\varepsilon_i + Ld(\tilde{n}, \tilde{m}) \\ &\leq 2\varepsilon_i + L(2\varepsilon_i + d(n, m)) = d(n, m)\left(L + \frac{2\varepsilon_i(1 + L)}{d(n, m)}\right) \\ &\leq d(n, m)(L + 2^{i+1}\varepsilon_i(1 + L)) = d(n, m)(L + \tfrac{\varepsilon}{i}), \end{aligned}$$

so if we put $r' = \log(L + \varepsilon)$ we get that (2) holds and $r' < r$. □

1.2.4. *Banach-Mazur distance.*

**Definition 21** (Banach-Mazur distance)**.** We recall that if $X$ and $Y$ are Banach spaces, their (logarithmic) *Banach-Mazur distance* is defined as

$$\rho_{BM}(X, Y) = \inf\big\{\log \|T\|\|T^{-1}\| : \; T : X \to Y \text{ is a linear isomorphism}\big\}.$$

In contrast to the Lipschitz distance, Banach-Mazur distance can be verified just by looking at isomorphisms that are defined on some fixed countable dense linear subspaces over $\mathbb{Q}$. That is made precise in the following lemma.[2]

**Lemma 22.** *Let $X$ and $Y$ be separable Banach spaces, let $(e_n)_{n\in\mathbb{N}}$ and $(f_n)_{n\in\mathbb{N}}$ be linearly independent and linearly dense sequences in $X$ and $Y$, respectively, and put $V = \mathbb{Q}\operatorname{span}\{e_n : n \in \mathbb{N}\}$, $W = \mathbb{Q}\operatorname{span}\{f_n : n \in \mathbb{N}\}$.*

*Then $\rho_{BM}(X,Y) < r$ if and only if there exists a surjective linear isomorphism $T : X \to Y$ with $\log \|T\|\|T^{-1}\| < r$ and $T(V) = W$.*

Throughout the proof of the lemma (including the following claim), by an isomorphism we mean a surjective linear isomorphism.

**Claim 23.** *Let $T : X \to Y$ be an isomorphism and $v_1, \dots, v_n, v \in V$ be such that $Tv_j \in W$ for $1 \le j \le n$. Then, given $\eta > 0$, there is an isomorphism $S : X \to Y$ such that*

- *$\|S - T\| \le \eta$ and $\|S^{-1} - T^{-1}\| \le \eta$,*
- *$Sv_j = Tv_j$ for $1 \le j \le n$,*
- *$Sv \in W$.*

*Proof.* We consider two cases.

(1) Assume that $v$ does not belong to the linear span of $v_1, \dots, v_n$. In this case, there is $x^* \in X^*$ such that $x^*(v) = 1$ and $x^*(v_j) = 0$ for $1 \le j \le n$. Let $\varepsilon > 0$ be such that $\varepsilon \le \eta$, $\varepsilon < \|T^{-1}\|^{-1}$, $(\|T^{-1}\|^{-1} - \varepsilon)^{-1} \cdot \|T^{-1}\| \cdot \varepsilon \le \eta$ and every linear operator $S : X \to Y$ with $\|S - T\| \le \varepsilon$ is an isomorphism (which is possible, because the set of isomorphisms is open). Let $w \in W$ be such that $\|w - Tv\| \le \varepsilon / \|x^*\|$, and let

$$Sx = Tx + x^*(x) \cdot (w - Tv), \quad x \in X.$$

Clearly, $Sv_j = Tv_j$ for $j \le n$ and $Sv = w \in W$. At the same time, $\|S - T\| \le \|x^*\|\|w - Tv\| \le \varepsilon \le \eta$. Note that $\|Sx\| \ge \|Tx\| - \varepsilon\|x\| \ge (\|T^{-1}\|^{-1} - \varepsilon)\|x\|$ for $x \in X$, and that $S$ is an isomorphism with $\|S^{-1}\| \le (\|T^{-1}\|^{-1} - \varepsilon)^{-1}$ in particular. Finally, we obtain $\|S^{-1} - T^{-1}\| = \|S^{-1}(T - S)T^{-1}\| \le \|S^{-1}\|\|T - S\|\|T^{-1}\| \le (\|T^{-1}\|^{-1} - \varepsilon)^{-1} \cdot \varepsilon \cdot \|T^{-1}\| \le \eta$.

(2) Assume that, on the other hand, $v$ belongs to the linear span of $v_1, \dots, v_n$. We just need to check that $v$ belongs to the $\mathbb{Q}$-linear span of $v_1, \dots, v_n$ as well, since then clearly $Tv \in W$ and the choice $S = T$ works. There are a large enough $m \in \mathbb{N}$ and rational numbers $q^i, q^i_j$ such that

$$v = \sum_{i=1}^m q^i e_i, \quad v_j = \sum_{i=1}^m q^i_j e_i.$$

For some real numbers $\alpha_1, \dots, \alpha_n$, we have $v = \sum_{j=1}^n \alpha_j v_j$. That is,

$$\sum_{i=1}^m q^i e_i = \sum_{j=1}^n \alpha_j \sum_{i=1}^m q^i_j e_i = \sum_{i=1}^m \Big( \sum_{j=1}^n q^i_j \alpha_j \Big) e_i.$$

[2] After proving the lemma, we were told by Gilles Godefroy that a similar statement is already in [27]

As $e_1, \dots, e_n$ are assumed to be linearly independent, we obtain

$$\sum_{j=1}^{n} q_j^i \alpha_j = q^i, \quad i = 1, \dots, m.$$

Hence, the system of linear equations $\sum_{j=1}^{n} q_j^i x_j = q^i, i = 1, \dots, m$, has a solution. It follows from the methods of solving systems of linear equations that it has a solution $\beta_1, \dots, \beta_n$ consisting of rational numbers. By a similar computation as above, we can obtain $v = \sum_{j=1}^{n} \beta_j v_j$. $\square$

*Proof of Lemma 22.* Let $T_0 : X \to Y$ be an isomorphism with $\|T_0\|\|T_0^{-1}\| < e^r$. Let us pick a small enough $\varepsilon > 0$ such that $(\|T_0\| + \varepsilon)(\|T_0^{-1}\| + \varepsilon) < e^r$. We are going to find sequences $T_1, T_2, \dots$ of isomorphisms, $x_1, x_2, \dots$ of points in $V$ and $y_1, y_2, \dots$ of points in $W$ such that

- $\|T_k - T_{k-1}\| \le 2^{-k}\varepsilon$ and $\|T_k^{-1} - T_{k-1}^{-1}\| \le 2^{-k}\varepsilon$,
- $T_k e_j = y_j$ and $T_k^{-1} f_j = x_j$ for $j \le k$.

Let us assume that $k \in \mathbb{N}$ and that we have already found $T_j$, $x_j$ and $y_j$ for $j < k$. Applying Claim 23, we obtain an isomorphism $\tilde{T}_{k-1} : X \to Y$ such that

- $\|\tilde{T}_{k-1} - T_{k-1}\| \le 2^{-k-1}\varepsilon$ and $\|\tilde{T}_{k-1}^{-1} - T_{k-1}^{-1}\| \le 2^{-k-1}\varepsilon$,
- $\tilde{T}_{k-1} e_j = T_{k-1} e_j$ for $j < k$ and $\tilde{T}_{k-1} x_j = T_{k-1} x_j$ for $j < k$,
- $\tilde{T}_{k-1} e_k \in W$.

Let us put $y_k = \tilde{T}_{k-1} e_k$. Applying Claim 23 once more, we obtain an isomorphism $S_k : Y \to X$ such that

- $\|S_k - \tilde{T}_{k-1}^{-1}\| \le 2^{-k-1}\varepsilon$ and $\|S_k^{-1} - \tilde{T}_{k-1}\| \le 2^{-k-1}\varepsilon$,
- $S_k f_j = \tilde{T}_{k-1}^{-1} f_j$ for $j < k$ and $S_k y_j = \tilde{T}_{k-1}^{-1} y_j$ for $j \le k$,
- $S_k f_k \in V$.

Let us put $x_k = S_k f_k$ and $T_k = S_k^{-1}$. Let us check that the choice works. We have $\|T_k - T_{k-1}\| = \|S_k^{-1} - T_{k-1}\| \le \|S_k^{-1} - \tilde{T}_{k-1}\| + \|\tilde{T}_{k-1} - T_{k-1}\| \le 2^{-k-1}\varepsilon + 2^{-k-1}\varepsilon = 2^{-k}\varepsilon$ and $\|T_k^{-1} - T_{k-1}^{-1}\| = \|S_k - T_{k-1}^{-1}\| \le \|S_k - \tilde{T}_{k-1}^{-1}\| + \|\tilde{T}_{k-1}^{-1} - T_{k-1}^{-1}\| \le 2^{-k-1}\varepsilon + 2^{-k-1}\varepsilon = 2^{-k}\varepsilon$. For $j < k$, we have $T_k e_j = S_k^{-1}\tilde{T}_{k-1}^{-1}\tilde{T}_{k-1} e_j = S_k^{-1}\tilde{T}_{k-1}^{-1} T_{k-1} e_j = S_k^{-1}\tilde{T}_{k-1}^{-1} y_j = S_k^{-1} S_k y_j = y_j$ and $T_k^{-1} f_j = S_k f_j = \tilde{T}_{k-1}^{-1} f_j = \tilde{T}_{k-1}^{-1} T_{k-1} T_{k-1}^{-1} f_j = \tilde{T}_{k-1}^{-1} T_{k-1} x_j = \tilde{T}_{k-1}^{-1}\tilde{T}_{k-1} x_j = x_j$. Finally, $T_k e_k = S_k^{-1}\tilde{T}_{k-1}^{-1}\tilde{T}_{k-1} e_k = S_k^{-1}\tilde{T}_{k-1}^{-1} y_k = S_k^{-1} S_k y_k = y_k$ and $T_k^{-1} f_k = S_k f_k = x_k$.

So, the sequences $T_k$, $x_k$ and $y_k$ are found. Clearly, the sequence $T_0, T_1, \dots$ is Cauchy and has a limit $T$ with $\|T - T_0\| \le \sum_{k=1}^{\infty} 2^{-k}\varepsilon = \varepsilon$. Similarly, the sequence $T_0^{-1}, T_1^{-1}, \dots$ has a limit $S$ with $\|S - T_0^{-1}\| \le \varepsilon$. Moreover, $TS = \lim_{k\to\infty} T_k T_k^{-1} = \lim_{k\to\infty} I = I$, and so $T$ is an isomorphism with $T^{-1} = S$. It follows that

$$\|T\|\|T^{-1}\| \le (\|T_0\| + \varepsilon)(\|T_0^{-1}\| + \varepsilon) < e^r.$$

At the same time, $Te_j = y_j \in W$ and $T^{-1} f_j = x_j \in V$ for every $j$. Hence, we arrive at $T(V) = W$. $\square$

The last three distances we shall present are all related to the coarse (or large scale) geometry of metric (and Banach) spaces. We refer the reader to [15, Chapter 8] or the monograph [44] for an introduction into this subject.

1.2.5. *Hausdorff-Lipschitz and net distances.* Gromov defines in [28, Definition 3.19] a distance defined as some variation of both the Gromov-Hausdorff and Lipschitz distances.

**Definition 24** (Hausdorff-Lipschitz distance)**.** For metric spaces $M$ and $N$, their *Hausdorff-Lipschitz distance* is defined as

$$\rho_{HL}(M,N)=\inf\big\{\rho_{GH}(M,M')+\rho_L(M',N')+\rho_{GH}(N',N):\ M',N' \text{ metric spaces}\big\}.$$

The Hausdorff-Lipschitz distance corresponds to the notion of *quasi-isometry* or *coarse Lipschitz equivalence*, because for metric spaces $M$ and $N$ we have $\rho_{HL}(M,N)<\infty$ if and only if the spaces $M$ and $N$ are quasi-isometric, or coarse Lipschitz equivalent (see e.g. [15, Section 8.3] for further information). For information about coarse geometry of Banach spaces we refer to the survey [41] or the monograph [46].

Following [11, Definition 10.18], by an $(a,b)$*-net* in a metric space $M$, where $a,b$ are positive reals, we mean a subset $\mathcal{N}\subseteq M$ such that for every $m\neq n\in\mathcal{N}$ we have $d(m,n)\geq a$, and for every $x\in M$ there exists $n\in\mathcal{N}$ with $d(x,n)<b$. If the constants $a$ and $b$ are not important, we just call the subset $\mathcal{N}$ *a net*. Observe that a maximal $\varepsilon$-separated subset $\mathcal{N}\subseteq M$ (which exists by Zorn's lemma) is an $(\varepsilon,\varepsilon)$-net. Dutrieux and Kalton [20] consider the net distance which we define as follows (let us note that a slightly different definition of $\rho_L$ is used in [20]).

**Definition 25** (Net distance)**.** The *net distance* between two Banach spaces $X$ and $Y$ is defined as

$$\rho_N(X,Y)=\inf\big\{\rho_L(\mathcal{N}_X,\mathcal{N}_Y):\ \mathcal{N}_X,\mathcal{N}_Y \text{ are nets in } X,Y \text{ respectively}\big\}.$$

The next observation is in a sense quantitative version of [15, Proposition 8.3.4], where it is proved that two metric spaces are quasi-isometric if and only if they have Lipschitz equivalent nets.

**Proposition 26.** *For Banach spaces $X$ and $Y$ we have $\rho_N(X,Y)=\rho_{HL}(X,Y)$.*

*Proof.* Fix Banach spaces $X$ and $Y$ and a positive real $K$. Suppose that $\rho_N(X,Y)<K$. So there exist $(a,b)$-net $\mathcal{N}_X\subseteq X$ and $(a',b')$-net $\mathcal{N}_Y\subseteq Y$ and a bi-Lipschitz map $T:\mathcal{N}_X\to\mathcal{N}_Y$ with $\log\max\{\mathrm{Lip}(T),\mathrm{Lip}(T^{-1})\}<K$. Take any $\varepsilon>0$. By rescaling the nets $\mathcal{N}_X$ and $\mathcal{N}_Y$ by a sufficiently large constant $C$ if necessary, that is, taking $\mathcal{N}_X/C=\{x/C:\ x\in\mathcal{N}_X\}$ and $\mathcal{N}_Y/C$, we may suppose that the nets $\mathcal{N}_X$ and $\mathcal{N}_Y$ are $(a,\varepsilon)$-net, resp. $(a',\varepsilon)$-net. Then we clearly have $\rho_{GH}(X,\mathcal{N}_X)\leq\varepsilon$ and $\rho_{GH}(Y,\mathcal{N}_Y)\leq\varepsilon$, so

$$\rho_{HL}(X,Y)\leq\rho_{GH}(X,\mathcal{N}_X)+\rho_L(\mathcal{N}_X,\mathcal{N}_Y)+\rho_{GH}(\mathcal{N}_Y,Y)<K+2\varepsilon.$$

Since $\varepsilon>0$ was arbitrary, it shows that $\rho_{HL}(X,Y)\leq K$.

Conversely, suppose that $\rho_{HL}(X,Y)<K$. So there exist metric spaces $X'$ and $Y'$ such that $\rho_{GH}(X,X')+\rho_L(X',Y')+\rho_{GH}(Y',Y)<K$. By Fact 9 there are correspondences $\mathcal{R}_X\subseteq X\times X'$ and $\mathcal{R}_Y\subseteq Y'\times Y$ witnessing that $\rho_{GH}(X,X')<K$ and $\rho_{GH}(Y',Y)<K$. Let $C>0$ be a sufficiently large constant, more precisely specified later, and find some $C$-maximal separated set $\mathcal{N}_X$ in $X$, which is therefore a $(C,C)$-net. Since $C$ is large, for every $n \neq m \in \mathcal{N}_X$ we have that $\{x \in X' : \ n\mathcal{R}_X x\} \cap \{x \in X' : \ m\mathcal{R}_X x\} = \emptyset$, so we pick some injective map $f_1 : \mathcal{N}_X \to X'$ such that for every $n \in \mathcal{N}_X$ we have $n\mathcal{R}_X f_1(n)$. Since $\rho_L(X', Y') < K$ there exists a bi-Lipschitz map $T : X' \to Y'$ with $\max\{\mathrm{Lip}(T), \mathrm{Lip}(T^{-1})\} < \exp(K)$. Again since $C$ is large enough it follows that for every $n \neq m \in \mathcal{N}_X$ we have that $\{y \in Y : (T \circ f_1)(n)\mathcal{R}_Y y\} \cap \{y \in Y : \ (T \circ f_1)(m)\mathcal{R}_Y y\} = \emptyset$, so we pick some injective map $f_2 : (T \circ f_1)[\mathcal{N}_X] \to Y$ such that for every $z \in (T \circ f_1)[\mathcal{N}_X]$ we have $z\mathcal{R}_Y f_2(z)$. Set $\phi = f_2 \circ T \circ f_1 : \mathcal{N}_X \to Y$. It follows the range of $\phi$ is a net $\mathcal{N}_Y$ in $Y$. Let us compute the Lipschitz constant of $\phi$ and $\phi^{-1}$. For any $n \neq m \in \mathcal{N}_X$ we have

$$\begin{aligned}
\|\phi(n) - \phi(m)\|_Y &\leq d_{Y'}((T \circ f_1)(n), (T \circ f_1)(m)) + 2K \\
&< \exp(K) d_{X'}(f_1(n), f_1(m)) + 2K \\
&\leq \exp(K)(\|n - m\|_X + 2K) + 2K \\
&\leq \Big( \exp(K) + \tfrac{2K(\exp(K)+1)}{C} \Big) \|n - m\|_X.
\end{aligned}$$

However, $\frac{2K(\exp(K)+1)}{C} \to 0$ as $C \to \infty$. The computation of $\mathrm{Lip}(\phi^{-1})$ is analogous, so we get that $\rho_N(X, Y) \leq K$, and we are done. □

*Remark* 27. Note that in Proposition 26 the only geometric property of Banach spaces that we used in the proof is that any rescaling of a Banach space $X$ is isometric to $X$. Spaces with this property are called *cones* [15, Definition 8.2.1]. So we have proved that if $\rho_N$ was defined in an obvious way on metric spaces, it would coincide with $\rho_{HL}$ on cones.

Our next result shows it is possible to express the Hausdorff-Lipschitz distance, up to uniform equivalence, in terms of correspondences. This observation will be used further.

**Definition 28.** Let $d, e \in \mathcal{M}$ and $\varepsilon > 0$. We say that $d$ and $e$ are *$HL(\varepsilon)$-close* if there exists a correspondence $\mathcal{R} \subseteq \mathbb{N} \times \mathbb{N}$ such that for every $i, i', j, j' \in \mathbb{N}$ with $i\mathcal{R}j$ and $i'\mathcal{R}j'$ we have

$$e(j, j') \leq d(i, i') + \varepsilon \cdot \max\{1, d(i, i')\}, \tag{1}$$

$$d(i, i') \leq e(j, j') + \varepsilon \cdot \max\{1, e(j, j')\}. \tag{2}$$

**Lemma 29.** *There are continuous functions $\varphi_i : (0, \infty) \to (0, \infty)$, $i \in \{1, 2\}$, such that $\lim_{\varepsilon \to 0} \varphi_i(\varepsilon) = 0$ and, whenever $d, e \in \mathcal{M}$ and $\varepsilon > 0$ are given, we have*

$$\rho_{HL}(d, e) < \varepsilon \Rightarrow d \textit{ and } e \textit{ are } HL(\varphi_1(\varepsilon))\textit{-close};$$

$$d \textit{ and } e \textit{ are } HL(\varepsilon)\textit{-close} \Rightarrow \rho_{HL}(d, e) < \varphi_2(\varepsilon).$$

*Proof.* First, let us assume that $\rho_{HL}(d, e) < \varepsilon$, that is, there are $d', e' \in \mathcal{M}$ with $\rho_{GH}(d, d') + \rho_L(d', e') + \rho_{GH}(e', e) < \varepsilon$. By Fact 9, there are correspondences $\mathcal{R}_1 \subseteq \mathbb{N} \times \mathbb{N}$ and $\mathcal{R}_3 \subseteq \mathbb{N} \times \mathbb{N}$ witnessing that $\rho_{GH}(d, d') < \varepsilon$ and $\rho_{GH}(e', e) < \varepsilon$. Further, let $f : M_{d'} \to M_{e'}$ be a bi-Lipschitz bijection witnessing that $\rho_L(d',e')<\varepsilon$. Consider now the correspondence

$$\mathcal{R}:=\big\{(i,j)\in\mathbb{N}\times\mathbb{N}:\text{there are }k,l\in\mathbb{N}\text{ such that }(i,l)\in\mathcal{R}_1,(k,j)\in\mathcal{R}_3,\\ d'(l,f^{-1}(k))<\varepsilon\text{ and }e'(f(l),k)<\varepsilon\big\}.$$

This is indeed a correspondence since given $i\in\mathbb{N}$ we find $l$ with $(i,l)\in\mathcal{R}_1$, pick $k\in\mathbb{N}$ with $e'(f(l),k)<\min\{\varepsilon,\frac{\varepsilon}{\mathrm{Lip}(f^{-1})}\}$ and find $j$ with $(k,j)\in\mathcal{R}_3$; thus, we have $(i,j)\in\mathcal{R}$ and similarly for every $j\in\mathbb{N}$ there is $i$ with $(i,j)\in\mathcal{R}$.

Fix $i,i',j,j'\in\mathbb{N}$ with $i\mathcal{R}j$ and $i'\mathcal{R}j'$. Then there are $l,l'\in\mathbb{N}$ and $k,k'\in\mathbb{N}$ with $i\mathcal{R}_1l$, $i'\mathcal{R}_1l'$, $k\mathcal{R}_3j$, $k'\mathcal{R}_3j'$, $d'(l,f^{-1}(k))<\varepsilon$, $e'(f(l),k)<\varepsilon$, $d'(l',f^{-1}(k'))<\varepsilon$ and $e'(f(l'),k')<\varepsilon$. We have

$$\begin{aligned}d(i,i')&\le d'(l,l')+2\varepsilon\le d'(f^{-1}(k),f^{-1}(k'))+4\varepsilon\le \mathrm{Lip}(f^{-1})e'(k,k')+4\varepsilon\\ &\le\exp(\varepsilon)(e(j,j')+2\varepsilon)+4\varepsilon\\ &=e(j,j')+\big(\exp(\varepsilon)-1\big)e(j,j')+2\varepsilon\big(\exp(\varepsilon)+2\big).\end{aligned}$$

By symmetry, similar inequality holds when the roles of $d$ and $e$ are changed. Hence, if $\varphi_1(\varepsilon)=\exp(\varepsilon)-1+2\varepsilon\exp(\varepsilon)+4\varepsilon$, then $d$ and $e$ are $HL(\varphi_1(\varepsilon))$-close.

Conversely, let $\mathcal{R}\subseteq\mathbb{N}\times\mathbb{N}$ be a correspondence witnessing that $d$ and $e$ are $HL(\varepsilon)$-close. Put $\delta=\varepsilon+\sqrt{\varepsilon}$. Let $\mathcal{N}_d$ be a maximal $\delta$-separated set in $(\mathbb{N},d)$. For every $i\in\mathcal{N}_d$, we pick some $r(i)\in\mathbb{N}$ such that $i\mathcal{R}r(i)$. Then we put $\mathcal{N}_e:=\{r(i):\ i\in\mathcal{N}_d\}$. Clearly,

$$\rho_{GH}((\mathbb{N},d),\mathcal{N}_d)\le\delta.$$

We claim that for every $j\in\mathbb{N}$ there is $j'\in\mathcal{N}_e$ with $e(j,j')<\delta+\varepsilon\cdot\max\{1,\delta\}$, which gives

$$\rho_{GH}((\mathbb{N},e),\mathcal{N}_e)\le\delta+\varepsilon\cdot\max\{1,\delta\}.$$

Indeed, if $j\in\mathbb{N}$ is given, there is $i$ with $i\mathcal{R}j$. Pick $i'\in\mathcal{N}_d$ with $d(i,i')<\delta$. Using (1), we obtain $e(j,r(i'))<\delta+\varepsilon\cdot\max\{1,\delta\}$.

Now, let us compute the Lipschitz constant for $r$ and $r^{-1}$. Consider $i,i'\in\mathcal{N}_d$, $i\neq i'$. If $d(i,i')\ge1$, by (1), we get $e(r(i),r(i'))\le(1+\varepsilon)d(i,i')$. If $d(i,i')\le1$, by (1) and using that $\mathcal{N}_d$ is $\delta$-separated, we get $e(r(i),r(i'))\le d(i,i')+\varepsilon\le(1+\frac{\varepsilon}{\delta})d(i,i')$. Hence, $\mathrm{Lip}(r)\le\max\{1+\varepsilon,1+\frac{\varepsilon}{\delta}\}$. Note that for every $k,k'\in\mathcal{N}_d$, $k\neq k'$, with $e(r(k),r(k'))\le1$, by (2), we have $e(r(k),r(k'))\ge d(k,k')-\varepsilon\ge\delta-\varepsilon$; hence, similar computation gives $\mathrm{Lip}(r^{-1})\le\max\{1+\varepsilon,1+\frac{\varepsilon}{\delta-\varepsilon}\}=1+\max\{\varepsilon,\sqrt{\varepsilon}\}$. Thus, we have $\rho_L(\mathcal{N}_d,\mathcal{N}_e)\le\log(1+\max\{\varepsilon,\sqrt{\varepsilon}\})$. Finally, if

$$\varphi_2(\varepsilon)=2\varepsilon+2\sqrt{\varepsilon}+\log(1+\max\{\varepsilon,\sqrt{\varepsilon}\})+\varepsilon\cdot\max\{1,\varepsilon+\sqrt{\varepsilon}\},$$

we get

$$\rho_{HL}(d,e)\le\delta+\log(1+\max\{\varepsilon,\sqrt{\varepsilon}\})+\delta+\varepsilon\cdot\max\{1,\delta\}=\varphi_2(\varepsilon).$$

$\square$

1.2.6. *Uniform distance.* The following definition comes from [20][3].

**Definition 30** (Uniform distance)**.** Let $X$ and $Y$ be Banach spaces. If $u : X \to Y$ is uniformly continuous, we put

$$\operatorname{Lip}_\infty u := \inf_{\eta>0} \sup \left\{ \frac{\|u(x)-u(y)\|}{\|x-y\|} :\ \|x-y\| \geq \eta \right\}.$$

The *uniform distance* between $X$ and $Y$ is defined as

$$\rho_U(X,Y) = \inf \big\{ \log((\operatorname{Lip}_\infty u)(\operatorname{Lip}_\infty u^{-1})) :\ u : X \to Y \text{ is uniform homeomorphism} \big\}.$$

Let us note the easy fact that we have

$$\operatorname{Lip}_\infty u = \inf \big\{ A > 0 :\ \exists B > 0\ \forall x, y \in X :\ \|u(x) - u(y)\| \leq A\|x-y\| + B \big\}.$$

The following is an analogue of Lemma 20.

**Lemma 31.** *Let $\mu, \nu \in \mathcal{B}$. Then $\rho_U(X_\mu, X_\nu) < r$ if and only if there exist $B > 0$, $r' \in (0, r)$ and a sequence of correspondences $\mathcal{R}_i \subseteq V \times V$ decreasing in inclusion such that*

*(1) for every $i \in \mathbb{N}$ and every $v, w, v', w' \in V$ such that $v\mathcal{R}_i w$ and $v'\mathcal{R}_i w'$ we have $\nu(w - w') \leq \exp(r')\mu(v - v') + B$;*
*(2) for $i \in \mathbb{N}$ and every $v, w, v', w' \in V$ such that $v\mathcal{R}_i w$ and $v'\mathcal{R}_i w'$ we have $\mu(v - v') \leq \nu(w - w') + B$;*
*(3) for every $\varepsilon > 0$ there exist $\delta > 0$ and $i \in \mathbb{N}$ such that for every $v, v' \in V$ with $\mu(v - v') < \delta$ we have $\nu(w - w') < \varepsilon$ whenever $v\mathcal{R}_i w$ and $v'\mathcal{R}_i w'$;*
*(4) for every $\varepsilon > 0$ there exist $\delta > 0$ and $i \in \mathbb{N}$ such that for every $w, w' \in V$ with $\nu(w - w') < \delta$ we have $\mu(v - v') < \varepsilon$ whenever $v\mathcal{R}_i w$ and $v'\mathcal{R}_i w'$.*

*Proof.* For the implication from the right to the left, for every $n \in V$ we define $\phi(n) \in X_\nu$ and $\psi(n) \in X_\mu$ as the unique element of $\bigcap_i \overline{n\mathcal{R}_i}$ and $\bigcap_i \overline{\mathcal{R}_i n}$, respectively. We leave to the reader to verify the simple fact that $\phi : (V, \mu) \to X_\nu$ is a uniformly continuous map with $\operatorname{Lip}_\infty \phi \leq \exp(r')$, which therefore extends to a uniformly continuous map $\bar{\phi} : X_\mu \to X_\nu$ and if $\bar{\psi}$ is defined analogously, then $\bar{\phi} = (\bar{\psi})^{-1}$ and $\operatorname{Lip}_\infty \phi \operatorname{Lip}_\infty \psi < \exp(r)$.

For the other implication, suppose that we are given a uniform homeomorphism $u : X_\mu \to X_\nu$ such that $\operatorname{Lip}_\infty u^{-1} = 1$ and $\operatorname{Lip}_\infty u < \exp(r')$ for some $r' < r$. For every $i \in \mathbb{N}$ define correspondence $\mathcal{R}_i$ by

$$\mathcal{R}_i := \big\{ (v, w) \in V \times V :\ \exists \tilde{v} \in V \quad \mu(v - \tilde{v}) < \tfrac{1}{i}\ \&\ \nu(u(\tilde{v}) - w) < \tfrac{1}{i} \big\}.$$

It is straightforward to check that $\mathcal{R}_i \subseteq \mathbb{N}\times\mathbb{N}$ are correspondences decreasing in inclusion satisfying all the conditions from the lemma. We omit further details, because this is similar to the proof of Lemma 20. $\square$

[3]More precisely, they define it without the logarithm which we add in order to satisfy the triangle inequality.

## 2. Analytic pseudometrics and reductions between them

In this section, we introduce several new concepts that generalize the standard theory of Borel/analytic equivalence relations on Polish and standard Borel spaces and the reductions between them, as well as the theory of orbit equivalence relations, i.e. equivalence relations given by actions of Polish groups.

Recall that a Borel (analytic) equivalence relation $E$ on a Polish (or more generally standard Borel) space $X$ is a subset $E \subseteq X^2$ that is an equivalence relation and is a Borel (analytic) subset of the space $X^2$. If $E$ and $F$ are two equivalence relations, Borel or analytic, on spaces $X$, resp. $Y$, then we say that $E$ is Borel reducible to $F$, $E \leq_B F$ in symbols, if there exists a Borel function $f : X \to Y$ such that for every $x, y \in X$ we have $xEy$ if and only if $f(x)Ff(y)$. Our reference for invariant descriptive set theory dealing with these notions is [23].

Below we introduce the notions of Borel/analytic pseudometrics, generalizing the Borel/analytic equivalence relations, and the Borel reductions between them. We provide few general results about them. Clearly, there is no hope of having the same basic relations that appear in the bottom levels of the Borel reducibility diagram for equivalence relations. Notice that as soon as the standard Borel space is countably infinite the reducibility between definable Borel pseudometrics becomes complicated in contrast to plain equivalence relations. Instead, we focus our attention on pseudometrics that naturally appear in various areas of functional analysis and metric geometry, as well as other fields of mathematics. An important part of this section is also a list of such examples that demonstrates there is enough space for further investigations in this area.

### 2.1. Analytic pseudometrics on standard Borel spaces.

**Definition 32.** Let $X$ be a standard Borel space. A pseudometric $\rho : X \times X \to [0, \infty]$ is called *an analytic pseudometric*, resp. *a Borel pseudometric*, if for every $r > 0$ the set $\{(x, y) \in X^2 : \ \rho(x, y) < r\}$ is analytic, resp. Borel.

Note that in the Borel case, this is equivalent to saying that $\rho$ is a Borel function. We emphasize that pseudometrics in our definition may attain $\infty$ as a value.

The trivial examples of analytic (or Borel) pseudometrics come from analytic equivalence relations. Conversely, every analytic pseudometric induces an analytic equivalence relation.

**Definition 33.** Let $X$ be a set, $\rho$ a pseudometric on $X$ and $E$ an equivalence relation on $X$. By $E_\rho$ we denote the equivalence relation on $X$ defined by $E_\rho := \{(x, y) \in X \times X : \ \rho(x, y) = 0\}$. By $\rho_E$ we denote the pseudometric on $X$ with values in $\{0, 1\}$ defined by $\rho_E(x, y) = 0$ iff $(x, y) \in E$.

It is easy to check the following.

**Fact 34.** *For every analytic (Borel) pseudometric $\rho$ on a standard Borel space $X$, the induced equivalence relation $E_\rho$ on $X$ is analytic (Borel). Conversely, for every analytic (Borel) equivalence relation $E$, the pseudometric $\rho_E$ is analytic (Borel).*

We shall discuss some more involved examples below.

**Examples.**

**1. Gromov-Hausdorff distance** Equip the Polish space $\mathcal{M}$ with the Gromov-Hausdorff distance $\rho_{GH}$ defined in Definition 8. Let us check that $\rho_{GH}$ is analytic. Fix some $r > 0$. We claim that the set $D_r = \{(d,p) \in \mathcal{M}^2 : \rho_{GH}(d,p) < r\}$ is analytic. Note that by Fact 9, $(d,p) \in D_r$ if and only if there exist a correspondence $\mathcal{R} \subseteq \mathbb{N} \times \mathbb{N}$ and $k \in \mathbb{N}$ such that $\forall i,j,m,n \in \mathbb{N}$ $(i\mathcal{R}j, m\mathcal{R}n \Rightarrow |d(i,m) - p(j,n)| \leq 2r - 1/k)$. Moreover, it is easy to see that the set

$$E_r = \bigcup_{k\in\mathbb{N}} \Big\{(\mathcal{R},d,p) \in \mathcal{P}(\mathbb{N}\times\mathbb{N}) \times \mathcal{M}^2 : \forall i,j,m,n \in \mathbb{N} \\ \big(i\mathcal{R}j, m\mathcal{R}n \Rightarrow |d(i,m) - p(j,n)| \leq 2r - 1/k\big)\Big\}$$

is Borel for every $r > 0$. Since one can view the space of all correspondences $\mathcal{C}$ as a $G_\delta$ subspace of $\mathcal{P}(\mathbb{N}\times\mathbb{N})$ and $(p,d) \in D_r$ if and only if $\exists \mathcal{R} \in \mathcal{C}$ $((\mathcal{R},p,d) \in E_r)$, we get that $D_r$ is an analytic subset of $\mathcal{M}^2$.

We also consider the pseudometric $\rho_{GH}$ on the space $\mathcal{B}$ of codes for separable Banach spaces, denoted there by $\rho_{GH}^{\mathcal{B}}$. Note that for Banach spaces $X$ and $Y$, $\rho_{GH}^{\mathcal{B}}(X,Y)$ is defined as the Gromov-Hausdorff distance of the unit balls $B_X$ and $B_Y$ (see e.g. the introduction in [35]). We leave to the reader to check that $\rho_{GH}^{\mathcal{B}}$ is still analytic in such a case.

**2. Kadets distance** Equip the Polish space $\mathcal{B}$ with the Kadets distance $\rho_K$ defined in Definition 15. Similarly as in the case of the Gromov-Hausdorff distance, using Lemma 16, it is not difficult to check that $\rho_K$ is analytic on $\mathcal{B}$. We leave the details to the reader.

**3. Lipschitz distance** Equip the Polish spaces $\mathcal{M}$ and $\mathcal{B}$ with the Lipschitz distance $\rho_L$ introduced in Definition 17, where for $d,p \in \mathcal{M}$ and $\mu,\nu \in \mathcal{B}$ by $\rho_L(d,p)$ and $\rho_L(\mu,\nu)$ we understand $\rho_L(M_d,M_p)$ and $\rho_L(X_\mu,X_\nu)$, respectively. We leave it to the reader to verify, using Lemma 20, that $\rho_L$ is analytic on $\mathcal{M}$ as well as on $\mathcal{B}$. Whenever we consider the pseudometric $\rho_L$ on $\mathcal{B}$ and we want to emphasize it, we write $\rho_L^{\mathcal{B}}$ instead of just $\rho_L$.

**4. Banach-Mazur distance** Equip the Polish space $\mathcal{B}$ by the Banach-Mazur distance $\rho_{BM}$ defined in Definition 21. We leave it to the reader to verify, using Lemma 22, that $\rho_{BM}$ is an analytic pseudometric on $\mathcal{B}$.

**5. Hausdorff-Lipschitz and net distances** Equip the Polish spaces $\mathcal{M}$ and $\mathcal{B}$ with the Hausdorff-Lipschitz distance $\rho_{HL}$ from Definition 24. It is easy to check that for $d,p \in \mathcal{M}$ we then have

$$\rho_{HL}(d,p) = \inf\{\rho_{GH}(d,e_1) + \rho_L(e_1,e_2) + \rho_{GH}(e_2,p) : e_1,e_2 \in \mathcal{M}\}.$$

Analogously, for elements from $\mathcal{B}$. It therefore follows from the fact that $\rho_{GH}$ and $\rho_L$ are analytic that $\rho_{HL}$ is analytic as well.

Moreover, equip the Polish space $\mathcal{B}$ with the net distance $\rho_N$ from Definition 25. It is clearly analytic as it coincides there with $\rho_{HL}$.

**6. Uniform distance** Equip the Polish space $\mathcal{B}$ with the uniform distance $\rho_U$ from Definition 30. We leave it to the reader to verify, using Lemma 31, that $\rho_U$ is an analytic pseudometric on $\mathcal{B}$.

**7. Completely bounded Banach-Mazur distance** Recall that an operator space is a closed linear subspace of a C*-algebra. The natural type of a morphism between operator spaces is a *completely bounded* isomorphism (cb-isomorphism). In [48], Pisier introduced the *Banach-Mazur cb-distance* between two operator spaces $E$ and $F$:

$$\rho_{CB}(E,F) = \inf\big\{ \log \|u\|_{cb}\|u^{-1}\|_{cb} : \ u : E \to F \text{ is a cb-isomorphism}\big\},$$

where $\|u\|_{cb}$ is the completely bounded norm of $u$. For a background on completely bounded maps and operator spaces the reader is referred to [49]. The standard Borel space of operator spaces was considered and described in [3, Section 2.3]. We leave to the reader to verify that $\rho_{CB}$ is an analytic distance.

For more examples from e.g. Banach space theory we refer the reader to articles [45] and [47] of Ostrovskii where various distances between subspaces of a given Banach space are considered. See for example the *Kadets path distance* in [45] or the *operator opening distance* in [47]. A good source of examples is also the Encyclopedia of distances [18].

We shall also see more examples in the subsection about pseudometrics given by actions of Polish groups.

2.2. **Borel-uniformly continuous reductions.** Now we introduce the main new definition of the paper.

**Definition 35.** Let $X$, resp. $Y$ be standard Borel spaces and let $\rho_X$, resp. $\rho_Y$ be analytic pseudometrics on $X$, resp. on $Y$. We say that $\rho_X$ is *Borel-uniformly continuous reducible* to $\rho_Y$, $\rho_X \leq_{B,u} \rho_Y$ in symbols, if there exists a Borel function $f : X \to Y$ such that, for every $\varepsilon > 0$ there are $\delta_X > 0$ and $\delta_Y > 0$ satisfying

$$\forall x, y \in X : \quad \rho_X(x,y) < \delta_X \Rightarrow \rho_Y(f(x), f(y)) < \varepsilon$$

and

$$\forall x, y \in X : \quad \rho_Y(f(x), f(y)) < \delta_Y \Rightarrow \rho_X(x,y) < \varepsilon.$$

In this case we say that $f$ is a *Borel-uniformly continuous reduction.* If $\rho_X \leq_{B,u} \rho_Y$ and $\rho_Y \leq_{B,u} \rho_X$, we say that $\rho_X$ is *Borel-uniformly continuous bi-reducible* with $\rho_Y$ and write $\rho_X \sim_{B,u} \rho_Y$.

Moreover, if $f$ is injective we say it is an *injective Borel-uniformly continuous reduction.*

If $f$ is an isometry from the pseudometric space $(X, \rho_X)$ into $(Y, \rho_Y)$, we say it is a *Borel-isometric reduction.*

If there is $C > 0$ such that for every $x, y \in X$ we have

$$\rho_Y(f(x), f(y)) \leq C\rho_X(x,y) \quad \text{ and } \quad \rho_X(x,y) \leq C\rho_Y(f(x), f(y)),$$

we say that $f$ is a *Borel-Lipschitz reduction.*

If there are $\varepsilon > 0$ and $C > 0$ such that for every $x, y \in X$ we have

$$\begin{aligned} \rho_X(x,y) < \varepsilon &\implies \rho_Y(f(x), f(y)) \leq C\rho_X(x,y) \\ \text{and} \quad \rho_Y(f(x), f(y)) < \varepsilon &\implies \rho_X(x,y) \leq C\rho_Y(f(x), f(y)), \end{aligned}$$

we say that $f$ is a *Borel-Lipschitz on small distances reduction.*

The definition of a Borel-uniformly continuous reduction seems to be the most useful one in the sense that it is strong enough for our applications, yet it naturally arises in our examples. Sometimes we are able to demonstrate the reducibility between some pseudometrics by maps with stronger properties and this is the reason why we mentioned the remaining notions above.

*Remark* 36. Note that in particular $\rho_X \leq_{B,u} \rho_Y$ implies the reducibility between the corresponding equivalence relations, i.e. $E_{\rho_X} \leq_B E_{\rho_Y}$ and the same Borel function $f$ is a witness. So Borel-uniform reducibility between pseudometrics is a stronger notion than the Borel reducibility between the corresponding equivalence relations.

Moreover, $E_X \leq_B E_Y$ is the same as $\rho_{E_X} \leq_{B,u} \rho_{E_Y}$. So Borel-uniform reducibility between pseudometrics is a generalization of the notion of Borel reducibility between equivalence relations.

**Theorem 37.** *There exists a universal analytic pseudometric. That is, for any analytic pseudometric there is a Borel-isometric reduction into the universal one.*

*Proof.* Let $\mathcal{U} \subseteq \mathbb{N}^{\mathbb{N}} \times ((\mathbb{N}^{\mathbb{N}})^2) \times \mathbb{Q}^+$ be a universal analytic subset for $(\mathbb{N}^{\mathbb{N}})^2 \times \mathbb{Q}^+$ (see e.g. [37, Theorem 14.2] for the existence). That is, for every analytic subset $A \subseteq (\mathbb{N}^{\mathbb{N}})^2 \times \mathbb{Q}^+$ there exists $u \in \mathbb{N}^{\mathbb{N}}$ such that $A = \mathcal{U}_u$. For every $p \in \mathbb{Q}^+$ we set

$$U_p = \Big\{(a,x,y) \in (\mathbb{N}^{\mathbb{N}})^3 :\ \exists q_1,\dots,q_n \in \mathbb{Q}^+\ (\sum_{i\leq n} q_i < p),$$
$$\exists z_0,z_1,\dots,z_n \in \mathbb{N}^{\mathbb{N}}, z_0 = x, z_n = y,\ (\forall i \leq n\ (a,z_{i-1},z_i,q_i) \in \mathcal{U})\Big\}.$$

It is easy to check that $U_p$ is analytic. We define a pseudometric $\rho$ on $(\mathbb{N}^{\mathbb{N}})^2$ as follows. For $(a,x),(a',y) \in (\mathbb{N}^{\mathbb{N}})^2$ we set $\rho((a,x),(a,x)) = 0$ and $\rho((a,x),(a',y)) = \infty$ if $a \neq a'$. Otherwise we set

$$\rho((a,x),(a,y)) = \inf\big\{p \in \mathbb{Q}^+ :\ (a,x,y) \in U_p \text{ and } (a,y,x) \in U_p\big\}.$$

It is easy to check that $\rho$ is a pseudometric. It is also clear that for every $r > 0$ the set $\{((z,x),(z',y)) \in (\mathbb{N}^{\mathbb{N}})^4 :\ \rho((z,x),(z',y)) < r\}$ is analytic since this set is equal to

$$\bigcup_{p<r, p\in\mathbb{Q}^+} \big\{((z,x),(z',y)) \in (\mathbb{N}^{\mathbb{N}})^4 :\ z = z', (z,x,y) \in U_p, (z,y,x) \in U_p\big\}.$$

Now let $\sigma$ be an arbitrary analytic pseudometric on some standard Borel space, which we may assume without loss of generality is equal to $\mathbb{N}^{\mathbb{N}}$. For each $p \in \mathbb{Q}^+$ let $A_p$ be the analytic set $\{(x,y) \in (\mathbb{N}^{\mathbb{N}})^2 :\ \sigma(x,y) < p\}$. The set $\mathcal{A} = \bigcup_{p\in\mathbb{Q}^+} A_p \times \{p\}$ is analytic. Therefore there exists $u \in \mathbb{N}^{\mathbb{N}}$ such that $\mathcal{U}_u = \mathcal{A}$. Now, we easily verify that the mapping $(\mathbb{N}^{\mathbb{N}},\sigma) \ni y \mapsto (u,y) \in ((\mathbb{N}^{\mathbb{N}})^2,\rho)$ is an isometry and so it is the desired reduction. □

2.3. **Continuous orbit equivalence relations.** One of the main supplies of Borel and analytic equivalence relations comes from actions of Polish groups. Let $G$ be a Polish group and $X$ a Polish or standard Borel space.

Suppose that $G$ acts on $X$ in a continuous or Borel way. The corresponding *orbit equivalence relation* $E_G$ on $X$ is defined as $xE_Gy$ if and only if $\exists g \in G$ $(gx = y)$, where $x, y \in X$. The most important and most studied are the countable Borel equivalence relations. Nevertheless, there is now a rather developed theory also for actions of general (so typically not locally compact) Polish groups. See e.g. [29] for a reference. In particular, we highlight the fact that there is a universal orbit equivalence relation (see [23, Theorem 5.1.9], which can be, by the result of Gao and Kechris [24], realized as the canonical action of the isometry group of the Urysohn universal metric space $\mathbb{U}$ on the Effros-Borel space $F(\mathbb{U})$ (refer to [52] or [23] for information about the Urysohn space). Since then, several natural analytic equivalence relations have been proved to be bi-reducible with the universal orbit equivalence relations, including those that are not per se defined as orbit equivalences; see e.g. [50] and [53]. It is one of the main open problems whether the equivalence relations given by pseudometrics considered in this paper are bi-reducible with the universal orbit equivalence relations. We discuss more about this problem in the last section.

Here we try to demonstrate on examples that also non-discrete pseudometrics can be defined using actions of Polish groups. The definition is less natural though as it needs, besides the action of a group, some analytic metric on the given standard Borel space.

**Definition 38.** Let $G$ be a Polish group and let $X$ be a standard Borel $G$-space equipped with an analytic metric $d$ (or, more generally, an analytic pseudometric $d$), on which $G$ acts by isometries. We define an analytic pseudometric $\rho_{G,d}$ induced by the action of $G$ on $X$ as follows. For any $x, y \in X$ we set

$$\rho_{G,d}(x, y) = \inf\{d(gx, hy) : \; g, h \in G\}.$$

We call such pseudometrics *orbit pseudometrics.*

Clearly, $\rho_{G,d}$ is an analytic pseudometric. Indeed, fix any $r > 0$. Then $\{(x, y) \in X^2 : \; \rho_{G,d}(x, y) < r\} = \{(x, y) \in X^2 : \; \exists g, h \; (g, h, x, y) \in D_r\}$, where $D_r = \{(g, h, x, y) : \; d(gx, hy) < r\}$ is analytic as $d$ is analytic and the action is Borel.

*Remark* 39. Even when $G$ does not act on $X$ by isometries, it is possible to define the orbit pseudometric by setting

$$\widetilde{\rho_{G,d}}(x, y) = \inf\Big\{\sum_{i=1}^{n} \rho_{G,d}(z_i, z_{i+1}) : \; z_1 = x, z_{n+1} = y\Big\},$$

where $\rho_{G,d}$ is defined as above.

2.3.1. *CTR orbit pseudometrics.* Note that in the full generality provided by the previous definition, every analytic pseudometric is an orbit pseudometric for trivial reasons. If $\rho$ is any analytic pseudometric on a standard Borel space, then it is an orbit pseudometric of the trivial action of any Polish group (e.g. the trivial group). This leads us to impose some natural restrictions that make the definition more interesting.

**Definition 40.** Let $G$, $X$ and $d$ be as in Definition 38. If we additionally require that $d$ is a complete metric and refines some compatible topology on

$X$ (meaning that this topology defines the same Borel structure on $X$ and makes the action of $G$ continuous), then we say the orbit pseudometric $\rho_{G,d}$ is *CTR* (given by a complete topology-refining metric).

Our natural examples of orbit pseudometrics are CTR. The main result of this section, Theorem 42, works for CTR orbit pseudometrics.

**Examples.**

**1.** Let $G$ be a Polish group and $X$ a Borel $G$-space. Let $d$ be the trivial metric on $X$, i.e. $d(x,y) = 0$ if $x = y$, and $d(x,y) = 1$ otherwise. Then $\rho_{G,d}$ is an analytic pseudometric and $E_{\rho_{G,d}}$ is the standard orbit equivalence relation induced by the action of $G$. It was proved by Becker and Kechris in [5, Theorem 5.2.1] that there exists a Polish topology on $X$ inducing the same Borel structure so that the action of $G$ becomes continuous. This shows that $\rho_{G,d}$ is CTR.

**2.** Consider $X = \mathcal{M}^1_{1/2}$, $G = S_\infty$, $(\pi \cdot f)(m,n) = f(\pi^{-1}(m), \pi^{-1}(n))$ and $d(f,g) = \sup_{m,n} |g(m,n) - f(m,n)|$ for $\pi \in S_\infty$ and $f, g \in \mathcal{M}^1_{1/2}$. Then

$$\rho_{G,d}(f,g) = \inf\{\varepsilon : f \simeq_\varepsilon g\},$$

and $\rho_{G,d}$ is clearly a CTR orbit pseudometric. By Lemma 11 and Lemma 12, we have $\rho_{GH}(f,g) = (1/2)\rho_{G,d}(f,g)$ if one of the sides is less than 1/4. Therefore the Gromov-Hausdorff distance when restricted to $\mathcal{M}^1_{1/2}$ is Borel-uniformly continuous bi-reducible with a CTR orbit pseudometric. It follows from this observation and further results of this paper that most of the analytic pseudometrics investigated in this paper are Borel-uniformly continuous bi-reducible with a CTR orbit pseudometric, see Theorem A.

**3.** Fix some Polish metric space $M$. Let $X = F(M)$ be the Effros-Borel space of all closed subsets of $M$. Let $G$ be $\mathrm{Iso}(M)$, the isometry group of $M$ with the pointwise convergence topology. The canonical action of $G$ on $M$ extends to a Borel action of $G$ on $X$, so $X$ is naturally a Borel $G$-space. Let $d$ be the Hausdorff metric on $X$. Clearly $d$ is Borel and the action of $G$ is by isometries with respect to $d$. We note that the resulting distance $\rho_{G,d}$ appears in [18] where it is called *generalized $G$-Hausdorff distance*.

To see that $\rho_{G,d}$ is CTR, consider the *Wijsman topology* on $X$, which is the initial topology with respect to the maps $F \mapsto d_M(x,F)$, where $x \in M$. This topology is Polish (see [6, Theorem 4.3]) and it is compatible with the Effros-Borel structure on $X$ (see [5, Proposition 2.6.2]). Clearly the action of $G$ on $X$ with this topology is continuous. We leave to the reader to verify the easy fact that the Hausdorff metric $d$, which is also complete, refines this topology.

**4.** Fix some separable Banach space $E$ and let $X$ be the standard Borel space of all closed unit balls of closed linear subspaces of $E$, which we identify with the set of all linear subspaces of $E$. That is a Borel subset of $F(B_E)$. Let $G$ be $\mathrm{LIso}(E)$, the linear isometry group of $E$. The canonical action of $G$ on $E$ extends to a Borel action of $G$ on $X$. We define a Borel metric $d$ on $X$ so that $d(U,V)$, for $U, V \in X$, is the Hausdorff distance between $U$ and $V$. The action is again clearly by isometries and $\rho_{G,d}$ is an orbit pseudometric on the space $X$.

Let us check that $\rho_{G,d}$ is CTR. Consider the Wijsman topology restricted to the subspace of all closed unit balls of closed linear subspaces of $E$. This is a subset of the Polish space $F(E)$ with the Wijsman topology, therefore it is separable. We claim that it is closed. Let $F$ be a non-empty closed subset of $E$ that is not a unit ball of any closed linear subspace of $E$. It means that either

- $0 \notin F$;
- or $F \not\subseteq B_E$;
- or for some $x, y \in F$, $\frac{x+y}{2} \notin F$;
- or for some $x \in F$ and $r \in [-1/\|x\|, 1/\|x\|]$, $rx \notin F$.

It is easy to check though that each of these conditions is an open condition, i.e. defines an open neighborhood of $F$ of elements satisfying the same property. Let us show it for the third condition, the rest is left for the reader. Suppose that for some $x, y \in F$, $\frac{x+y}{2} \notin F$. Since $F$ is closed, we have $\delta = \mathrm{dist}_{\|\cdot\|}(\frac{x+y}{2}, F) > 0$. Let $O$ be an open neighborhood of $F$ defined as $\{C \in F(E) :\ \mathrm{dist}_{\|\cdot\|}(x, C) < \delta/2, \mathrm{dist}_{\|\cdot\|}(y, C) < \delta/2, \mathrm{dist}_{\|\cdot\|}(\frac{x+y}{2}, C) > 3\delta/4\}$. Pick any $C \in O$. There must exist $x', y' \in C$ such that $\|x - x'\| < \delta/2$, $\|y - y'\| < \delta/2$. However, we have $\frac{x'+y'}{2} \notin C$. Otherwise, since $\|\frac{x+y}{2} - \frac{x'+y'}{2}\| \leq \delta/2$, we would get $\mathrm{dist}_{\|\cdot\|}(\frac{x+y}{2}, C) \leq \delta/2$, which is a contradiction. This shows that the space of unit balls of Banach subspaces of $E$ with the Wijsman topology is a Polish space. Since the Borel structure of $X$ is the restriction of the Effros-Borel structure on $F(E)$ to $X$ and the Wijsman topology on $F(E)$ is compatible with it, we get that the restriction of the Wijsman topology on $X$ is compatible with the Borel structure of $X$. The action of $G$ on $X$ is continuous, which is again easily verified. The verification that $d$ refines the Wijsman topology is done as in the previous example.

Finally, to check that $d$ restricted to $X$ is complete, it suffices to check that $X$ is closed with respect to $d$ as $d$ is complete on $F(E)$. However, $X$ is closed already with respect to the coarser Wijsman topology.

**5.** Kadison and Kastler define in [33] a metric on the space of all *concrete* $C^*$-algebras, i.e. sub-$C^*$-algebras of $B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. For $A, C$, subalgebras of $B(\mathcal{H})$, their Kadison-Kastler distance $d_{KK}$ is again nothing but the Hausdorff distance $\rho_H^{B(\mathcal{H})}(B_A, B_C)$. Let $\mathcal{H}$ be now a fixed separable infinite-dimensional Hilbert space. We want to define a standard Borel space of all separable $C^*$-subalgebras of $B(\mathcal{H})$. Denote by $B_1(\mathcal{H})$ the closed unit ball in $B(\mathcal{H})$, that is, the set of all operators on $\mathcal{H}$ of norm bounded by 1. Consider the strong* operator topology ($SOT^*$) on $B_1(\mathcal{H})$. That is, a topology generated by the seminorms $B(\mathcal{H}) \ni T \mapsto (\|Tx\|^2 + \|T^*x\|^2)^{1/2}$, $x \in \mathcal{H}$; that is, a net $(T_\alpha)_\alpha$ of operators converges to an operator $T$ if and only if $(T_\alpha)_\alpha$, resp. $(T^*_\alpha)_\alpha$, converge to $T$, resp. $T^*$ in the strong operator topology. It is then easy to see that $(B_1(\mathcal{H}), SOT^*)$ is a Polish space and that all the standard operations such as addition, scalar multiplication, multiplication and involution are continuous with respect to this topology. See [12, Chapter I.3] for details.

Let $X$ be the Borel subset of $F(B_1(\mathcal{H}))$, with the Effros-Borel structure inherited from $B_1(\mathcal{H})$ with the $SOT^*$-topology, consisting of all closed unit balls of separable $C^*$-subalgebras of $B(\mathcal{H})$, which we identify with the set of all separable $C^*$-subalgebras of $B(\mathcal{H})$ (note that there are different codings of separable $C^*$-subalgebras of $B(\mathcal{H})$ in the literature, see e.g. [38]). We define a Borel metric $d$ on $X$ so that $d(A,B) = d_{KK}(\operatorname{span} A, \operatorname{span} B)$ for $A, B \in X$. Now consider the action of the Polish group $U(\mathcal{H})$, the group of all unitary operators of $\mathcal{H}$ equipped with the strong operator topology (equivalent with the strong* operator topology), on $X$ by conjugation. That is, for $A \in X$ and $\varphi \in U(\mathcal{H})$, we have

$$\varphi \cdot A = \{\varphi \circ T \circ \varphi^* :\ T \in A\}.$$

This action defines an orbit pseudometric $\rho_{U(\mathcal{H}),d}$ on $X$.

We now check that $\rho_{U(\mathcal{H}),d}$ is CTR. Fix some metric $p$ on $B_1(\mathcal{H})$ compatible with the $SOT^*$-topology. The Wijsman topology on $F(B_1(\mathcal{H}))$ induced by $p$ is again a Polish topology. We again show that $X$ is a closed subset of $F(B_1(\mathcal{H}))$ with respect to this topology. This is done similarly as in Example **4.**. Pick some closed subset $A \in F(B_1(\mathcal{H}))$. To verify that $A \in X$ we must check that

- $0 \in A$;
- if $x, y \in A$, then $\frac{x+y}{2} \in A$;
- if $x \in A$ and $c \in \mathbb{C}$ with $|c| \in (0, 1/\|x\|]$, then $cx \in A$;
- if $x, y \in A$, then $x \cdot y \in A$;
- if $x \in A$, then $x^* \in A$.

These are all closed conditions for the Wijsman topology, which is where we use that the operations are continuous with respect to the $SOT^*$ topology. The verifications are done similarly as in Example **4.**; let us show it for the last condition. Let $A \in F(B_1(\mathcal{H}))$ be such that for some $x \in A$, we have $x^* \notin A$. Then $\delta = p(x^*, A) > 0$. Since the $*$-operation is continuous in the $SOT^*$-topology, there exists $\gamma > 0$ such that if $p(x,z) < \gamma$, then $p(x^*, z^*) < \delta/2$, for $z \in B_1(\mathcal{H})$. Define an open neighborhood $O = \{B \in X : p(x, B) < \gamma, p(x^*, B) > \delta/2\}$ of $A$ in the Wijsman topology. Clearly, $A \in O$. If $B \in O$, then there is some $z \in B$ with $p(z,x) < \gamma$. By the assumption, $p(z^*, x^*) < \delta/2$. Since $B \in O$ and therefore $p(x^*, B) > \delta/2$, we get

$$p(z^*, B) \geq p(x^*, B) - p(x^*, z^*) > 0,$$

so $z^* \notin B$. It follows that the Wijsman topology on $X$ is compatible with the Borel structure of $X$. It is straightforward to check that the action of $G$ on $X$ is continuous. The completeness of $d$ will again follow as soon as we show that $d$ defines a topology which is finer than the Wijsman topology. We do it now. Pick some $x \in B_1(\mathcal{H})$, $A \in X$ and $\varepsilon$. We need to show that the set $O = \{B \in X :\ |p(x,B) - p(x,A)| < \varepsilon\}$ is open in the topology induced by $d$. We just show that there is $\delta > 0$ such that if $d(A,B) < \delta$, then $B \in O$. Here we shall without loss of generality assume that $p(y,z)$, for $y, z \in B_1(\mathcal{H})$, is equal to

$$\sum_{i=1}^{\infty} \frac{\|y(\xi_i) - z(\xi_i)\| + \|y^*(\xi_i) - z^*(\xi_i)\|}{2^{i+1}},$$

where $(\xi_i)_i$ is some countable dense subset of unit vectors of $\mathcal{H}$. Set $\delta = \varepsilon/2$, i.e. suppose that $d(A,B) < \varepsilon/2$. We claim that $B \in O$. Otherwise there either exists $z \in B$ such that $p(x,z) \leq p(x,A) - \varepsilon$, or for all $z \in B$ we have

$p(x,z) \geq p(x,A)+\varepsilon$. Suppose the former, the latter is treated analogously. Since $d(A,B)<\varepsilon/2$, there is $y \in A$ with $\|y-z\|<\varepsilon/2$. Therefore for all $i \in \mathbb{N}$, $\|y(\xi_i)-z(\xi_i)\|+\|y^*(\xi_i)-z^*(\xi_i)\|<\varepsilon$, so $p(y,z)<\varepsilon$. It follows that

$$p(x,A) \leq p(x,z)+p(z,y)<p(x,A),$$

which is a contradiction.

**6.** In the theory of graph limits (see [42] for a reference on this topic), a *graphon* is a symmetric measurable function $W:([0,1]^2,\lambda^2) \to [0,1]$, where $\lambda^2$ is the usual Lebesgue measure on $[0,1]^2$. Viewing each graphon $W$ as an element of $L^\infty([0,1]^2)$ we may equip the space $\mathcal{G}$ of all graphons with the weak*-topology coming from the identification of $L^\infty([0,1]^2)$ with $(L^1([0,1]^2))^*$, so that it becomes a compact Polish space, see [31, Theorem F.4].

Equip now the linear hull of $\mathcal{G}$ with the *cut norm* $\|\cdot\|_\square$ given by

$$\|W\|_\square = \sup_{S,T}\left|\int_{S\times T} W(x,y)\,\mathrm{d}x\,\mathrm{d}y\right|,$$

where the supremum is taken over all measurable sets $S,T \subseteq [0,1]$. Let now $G$ be the group of all measure preserving measurable bijections $\phi:[0,1]\to[0,1]$ equipped with the *weak topology*, i.e. subbasic neighborhoods of a transformation $\phi$, given by a measurable set $A\subseteq[0,1]$ and $\varepsilon>0$, are of the form $\{\psi:\ \lambda(\phi(A)\bigtriangleup\psi(A))<\varepsilon\}$. This is the strong (and also weak) topology, when the transformations are viewed as the corresponding unitary operators in $L^2([0,1])$. $G$ then becomes a Polish group, see [29, Lemma 2.11] and it acts naturally on the space $\mathcal{G}$ of graphons by

$$gW(x,y)=W(g^{-1}x,g^{-1}y),$$

which is obviously a continuous action. This action together with the cut norm $\|\cdot\|_\square$ gives a pseudometric on $\mathcal{G}$, called *cut distance* and denoted by $\delta_\square$ (see [42, Section 8.2.2]), defined as

$$\delta_\square(U,W)=\inf_{g,h\in G}\|gU-hW\|_\square.$$

One can check it is a Borel pseudometric. There is a connection between $\delta_\square$ and "graph limits", since the metric quotient of $(\mathcal{G},\delta_\square)$ corresponds to the space of graph limits, see [13, Section 3.4] for corresponding definitions and the result.

We claim that $\rho_{G,\|\cdot\|_\square}$ is a CTR orbit pseudometric. The cut norm $\|\cdot\|_\square$ refines the weak* topology (see [42, Lemma 8.22]) and we leave to the reader to verify that it is complete.

Let us note that we could work in a slightly more general setting and consider a graphon as a symmetric measurable function $W:(\Omega,\mathcal{A},\mu)^2\to[0,1]$, where $(\Omega,\mathcal{A},\mu)$ is a standard probability space, that is, a probability space defined on a standard Borel space. Even in this slightly more general setting we would end up with a Borel pseudometric $\delta_\square$.

2.3.2. *Non-reducibility of the equivalence $E_1$ into CTR orbit pseudometrics.*

**Definition 41.** The equivalence relation $E_1$ on $(2^\mathbb{N})^\mathbb{N}$ is defined by

$$xE_1y \quad\Leftrightarrow\quad \exists N\forall n\geq N: x(n)=y(n).$$

Recall that an equivalence relation $E$ is *hypersmooth* if it can be written as an increasing union $\bigcup_n E_n$ of smooth equivalence relations, i.e. equivalence relations Borel reducible to the identity relation. The relation $E_1$ is clearly hypersmooth and it plays a prominent role among hypersmooth equivalences as Kechris and Louveau prove in [39, Theorem 2.1] that every hypersmooth equivalence relation is either Borel reducible to $E_0$ on $2^{\mathbb{N}}$, where $xE_0y$ if and only if $\{n \in \mathbb{N} : x(n) \neq y(n)\}$ is finite, or it is Borel bi-reducible with $E_1$.

As they mention in [39, Section 4], it is a delicate question to decide for a given analytic equivalence relation $E$ whether $E_1 \leq_B E$. They show it does not happen when $E$ is Borel idealistic (see the end of this section), or when $E$ is an orbit equivalence relation. Here we extend their result for equivalences given by CTR orbit pseudometrics. Our result raises some problems that we discuss at the end of the section.

**Theorem 42.** *Let $\rho_{G,d}$ be a CTR orbit pseudometric on a standard Borel space $X$. Then $E_1$ is not Borel reducible to $E_{\rho_{G,d}}$.*

The proof of the theorem is inspired by the proof of [39, Theorem 4.2] as presented in [29, Section 8].

By the definition of a CTR orbit pseudometric, we shall without loss of generality assume that $X$ is a Polish space and the action of $G$ is continuous. We first need the following lemma.

By $\forall^*$ we mean "for all elements of a comeager set".

**Lemma 43** (based on [29, Lemma 3.17])**.** *Let $G$ and $H$ be Polish groups and $X$ and $Y$ be Polish $G$ and $H$-spaces. Let $Y$ be equipped with an analytic pseudometric $d$, on which $H$ acts by isometries. Suppose that $\theta : X \to Y$ is a Borel function such that*

$$\rho_{H,d}(\theta(x), \theta(g \cdot x)) = 0$$

*for all $x \in X$ and $g \in G$, i.e. $\theta$ is a Borel homomorphism from $E_G$ to $E_{\rho_{H,d}}$. Then, for every open neighborhood $W$ of the identity in $H$ and every $\varepsilon > 0$, there is a comeager set of $x \in X$ for which there is an open neighborhood $V$ of the identity in $G$ with*

$$\forall^* g \in V \exists h \in W : d(\theta(g \cdot x), h \cdot \theta(x)) < \varepsilon.$$

Let us fix a neighborhood $W$ and $\varepsilon > 0$. To prove the lemma, we need the following claim which is an analogue of a claim from the proof of [29, Lemma 3.17]. In fact, the argument demonstrating that the lemma follows from the claim is the same as in [29], and so we omit it.

**Claim 44.** *For all $x \in X$, there is a comeager set of $g_0 \in G$ for which there exists some open neighborhood $V$ of the identity in $G$ with*

$$\forall^* g_1 \in V \exists h \in W : d(\theta(g_1 g_0 \cdot x), h \cdot \theta(g_0 \cdot x)) < \varepsilon.$$

*Proof.* Fix $x \in X$ and choose a smaller open neighborhood $W'$ of $1_H$ with $(W')^{-1} = W'$ and $(W')^2 \subseteq W$. Taking a sequence $(h_i)_{i=1}^{\infty}$ in $H$ such that $\{W' \cdot h_i : i \in \mathbb{N}\}$ covers $H$, we obtain a cover of $G$ by the sets

$$C'_i = \{g \in G : \exists h \in W' \ (d(\theta(g \cdot x), hh_i \cdot \theta(x)) < \varepsilon/2)\}, \quad i \in \mathbb{N}.$$

Indeed, given $g \in G$, we have $\rho_{H,d}(\theta(g \cdot x), \theta(x)) = 0$, and so there is $h' \in H$ such that $d(\theta(g \cdot x), h' \cdot \theta(x)) < \varepsilon/2$. For some $i \in \mathbb{N}$ and $h \in W'$, we have $h' = hh_i$.

Since each $C'_i$ is analytic and therefore has the Baire property, there is an open set $O_i$ such that the symmetric difference of $C'_i$ and $O_i$ is meager. Set $C_i = C'_i \cap O_i$. Clearly, $C = \bigcup_{i\in\mathbb{N}} C_i$ is comeager. Take any $g_0 \in C$. There are $i \in \mathbb{N}$ with $g_0 \in C_i$. We put $V = O_i g_0^{-1}$. Then $\forall^* g_1 \in V$ we have $g_1 g_0 \in C_i$, so it suffices to check

$$g_0 \in C_i \,\&\, g_1 g_0 \in C_i \quad \Rightarrow \quad \exists h \in W : d(\theta(g_1 g_0 \cdot x), h \cdot \theta(g_0 \cdot x)) < \varepsilon$$

for $g_0, g_1 \in G$. There are $h', h'' \in W'$ such that $d(\theta(g_0 \cdot x), h'h_i \cdot \theta(x)) < \varepsilon/2$ and $d(\theta(g_1 g_0 \cdot x), h''h_i \cdot \theta(x)) < \varepsilon/2$. Since

$$\begin{aligned}
&d(\theta(g_1 g_0 \cdot x), h''(h')^{-1} \cdot \theta(g_0 \cdot x)) \\
&\quad \le d(\theta(g_1 g_0 \cdot x), h''h_i \cdot \theta(x)) + d(h''(h')^{-1} \cdot \theta(g_0 \cdot x), h''h_i \cdot \theta(x)) \\
&\quad = d(\theta(g_1 g_0 \cdot x), h''h_i \cdot \theta(x)) + d(\theta(g_0 \cdot x), h'h_i \cdot \theta(x)) < \varepsilon/2 + \varepsilon/2 = \varepsilon,
\end{aligned}$$

the choice $h = h''(h')^{-1}$ works. □

As mentioned above, the proof of Lemma 43 is finished as well.

*Proof of Theorem 42.* In order to get a contradiction, let us assume that $\theta : (2^{\mathbb{N}})^{\mathbb{N}} \to X$ is a Borel map with

$$xE_1 y \Leftrightarrow \rho_{G,d}(\theta(x), \theta(y)) = 0, \quad x, y \in (2^{\mathbb{N}})^{\mathbb{N}}.$$

Due to [36, Lemma 11.8.2], we may assume that $\theta$ is continuous. We notice first that, by Lemma 43, the subset of $(2^{\mathbb{N}})^{\mathbb{N}}$ given by

$$\begin{aligned}
C_n = \big\{x : &\forall W \text{ nbhd of } 1_G, \forall\varepsilon > 0, \exists V \text{ nbhd of } (0, 0, \dots), \forall^* a \in V, \exists g \in W \\
&d\big(\theta(x(1), \dots, x(n-1), x(n) +_2 a, x(n+1), \dots), g \cdot \theta(x)\big) < \varepsilon\big\}
\end{aligned}$$

is comeager for every $n \in \mathbb{N}$. More precisely, we apply the lemma on every $W$ from a countable neighborhood basis of $1_G$ and on every $\varepsilon$ of the form $1/k$, $G$ and $X$ from the lemma are $2^{\mathbb{N}}$ and $(2^{\mathbb{N}})^{\mathbb{N}}$ with the action $a \cdot x = (x(1), \dots, x(n-1), x(n) +_2 a, x(n+1), \dots)$, and $H$ and $Y$ from the lemma are the current $G$ and $X$.

Using the Kuratowski-Ulam theorem (see e.g. [29, Theorem 2.46]), we can pick $x \in (2^{\mathbb{N}})^{\mathbb{N}}$ such that

$$\begin{aligned}
&x = (x(1), x(2), x(3), \dots) \in C_1, \\
&\forall^* a_1 \in 2^{\mathbb{N}} : (a_1, x(2), x(3), \dots) \in C_2, \\
&\forall^* a_1 \in 2^{\mathbb{N}}\, \forall^* a_2 \in 2^{\mathbb{N}} : (a_1, a_2, x(3), \dots) \in C_3, \\
&\quad\vdots
\end{aligned}$$

Indeed, for every $n \in \mathbb{N}$ the set

$$D'_n = \big\{(x(n), x(n+1), \dots) : \ C_n^{(x(n),\dots)} \text{ is comeager}\big\}$$

is comeager, where $C_n^{(x(n),\dots)} = \{(x(1), \dots, x(n-1)) : \ (x(1), \dots, x(n), \dots) \in C_n\}$. Set

$$D_n = \big\{x \in (2^{\mathbb{N}})^{\mathbb{N}} : \ (x(1), \dots, x(n-1)) \in (2^{\mathbb{N}})^{n-1}, (x(n), x(n+1), \dots) \in D'_n\big\}.$$

Then each $D_n$ is comeager in $(2^{\mathbb{N}})^{\mathbb{N}}$ and we can take $x \in \bigcap_n D_n$.

Let $p$ be a compatible complete metric on $G$ and let $g_0 = 1_G$. We show that it is possible to find $y = (y(1), y(2), y(3), \dots) \in (2^{\mathbb{N}})^{\mathbb{N}}$ and $g_1, g_2, \dots \in G$ such that, denoting

$$x_n = (y(1), \dots, y(n), x(n+1), \dots), \quad n = 0, 1, \dots,$$

we have for every $n \in \mathbb{N}$ that

(i) $y(n) \neq x(n)$,
(ii) $p(g_n, g_{n-1}) < 1/2^n$,
(iii) $d(g_n \cdot \theta(x_n), g_{n-1} \cdot \theta(x_{n-1})) < 1/2^n$,
(iv) it holds that

$$\begin{aligned}
&x_n = (y(1), \dots, y(n), x(n+1), x(n+2), x(n+3), \dots) \in C_{n+1},\\
&\forall^* a_{n+1} \in 2^{\mathbb{N}} : (y(1), \dots, y(n), a_{n+1}, x(n+2), x(n+3), \dots) \in C_{n+2},\\
&\forall^* a_{n+1} \in 2^{\mathbb{N}}\, \forall^* a_{n+2} \in 2^{\mathbb{N}} : (y(1), \dots, y(n), a_{n+1}, a_{n+2}, x(n+3), \dots) \in C_{n+3},\\
&\quad\vdots
\end{aligned}$$

Let us note first that (iv) already holds for $n = 0$ due to the choice of $x_0 = x$. Assume that $n \in \mathbb{N}$ and that $y(1), \dots, y(n-1)$ and $g_1, \dots, g_{n-1}$ are already found. Let $W = \{g \in G : p(g_{n-1}g^{-1}, g_{n-1}) < 1/2^n\}$. Since $x_{n-1} \in C_n$, considering $\varepsilon = 1/2^n$ in the definition of $C_n$, we have

$$\begin{aligned}
&\exists V \text{ nbhd of } (0, 0, \dots), \forall^* a \in V, \exists g \in W :\\
&\qquad d(\theta(y(1), \dots, y(n-1), x(n) +_2 a, x(n+1), \dots), g \cdot \theta(x_{n-1})) < 1/2^n.
\end{aligned}$$

Let us take such open neighborhood $V$. We obtain from condition (iv) for $n-1$ that

$$\begin{aligned}
&\forall^* a_n \in x(n) +_2 V : (y(1), \dots, y(n-1), a_n, x(n+1), x(n+2), \dots) \in C_{n+1},\\
&\forall^* a_n \in x(n) +_2 V,\ \forall^* a_{n+1} \in 2^{\mathbb{N}} :\ (y(1), \dots, a_n, a_{n+1}, x(n+2), \dots) \in C_{n+2},\\
&\quad\vdots
\end{aligned}$$

Hence, we can choose $y(n) \in x(n) +_2 V$ such that $y(n) \neq x(n)$,

$$\exists g \in W :\ d(\theta(y(1), \dots, y(n-1), y(n), x(n+1), \dots), g \cdot \theta(x_{n-1})) < 1/2^n$$

and

$$\begin{aligned}
&(y(1), \dots, y(n-1), y(n), x(n+1), x(n+2), \dots) \in C_{n+1},\\
&\forall^* a_{n+1} \in 2^{\mathbb{N}} : (y(1), \dots, y(n-1), y(n), a_{n+1}, x(n+2), \dots) \in C_{n+2},\\
&\quad\vdots
\end{aligned}$$

Provided with $g \in W$ such that $d(\theta(x_n), g \cdot \theta(x_{n-1})) < 1/2^n$, we define $g_n = g_{n-1}g^{-1}$. Then $p(g_n, g_{n-1}) < 1/2^n$ (due to the choice of $W$) and $d(g_n \cdot \theta(x_n), g_{n-1} \cdot \theta(x_{n-1})) = d(\theta(x_n), g \cdot \theta(x_{n-1})) < 1/2^n$. Therefore, our choice of $y(n)$ and $g_n$ works, as conditions (i)–(iv) hold for $n$.

So, we have seen that there are appropriate $y = (y(1), y(2), y(3), \dots)$ and $g_1, g_2, \dots$ indeed. It is clear from (i) that $(x, y) \notin E_1$. To obtain the promised contradiction, we provide a series of simple arguments resulting to $(x, y) \in E_1$.

Notice that $x_n \to y$. By (ii), the sequence $g_1, g_2, \dots$ has a limit $g$ in $G$. Considering the continuity of $\theta$, we obtain $g_n \cdot \theta(x_n) \to g \cdot \theta(y)$ in the original topology of $X$. By (iii), the sequence $g_n \cdot \theta(x_n)$ is Cauchy in $(X, d)$. Since $d$ is a complete and the topology refining metric, this sequence has a limit in $(X, d)$ which is nothing but the same point $g \cdot \theta(y)$. Using (iii) again, we arrive at

$$d(g \cdot \theta(y), g_n \cdot \theta(x_n)) < 1/2^{n+1} + 1/2^{n+2} + \cdots = 1/2^n,$$

and so

$$\rho_{G,d}(\theta(y), \theta(x_n)) < 1/2^n,$$

for every $n \in \mathbb{N}$. Since $xE_1x_n$, we have $\rho_{G,d}(\theta(x), \theta(x_n)) = 0$. Thus,

$$\rho_{G,d}(\theta(x), \theta(y)) \leq \rho_{G,d}(\theta(x), \theta(x_n)) + \rho_{G,d}(\theta(y), \theta(x_n)) < 0 + 1/2^n$$

for every $n \in \mathbb{N}$. For this reason, $\rho_{G,d}(\theta(x), \theta(y)) = 0$ and, consequently, $xE_1y$. □

**Corollary 45.** *The relation $E_1$ is not Borel reducible to $E_\rho$, where $\rho$ is any of the pseudometrics from Theorem A (1)or $\rho$ is any of the CTR pseudometrics mentioned in the examples above.*

*Proof.* Theorem 42 can be directly applied to the CTR orbit pseudometrics from the examples above. The pseudometric from Example **2.** is Borel-uniformly continuous bi-reducible with $\rho_{GH}$ restricted on $\mathcal{M}^1_{1/2}$. The other distances from the statement are reducible to it by Theorem A. □

The corollary has an important consequence that ought to be investigated further. It has been conjectured (see e.g. the introduction in [36], or [30, Conjecture 1] and [29, Question 10.9] where it was stated for Borel equivalence relations) that the equivalence relation $E_1$ is the least equivalence which is not Borel reducible to an orbit equivalence relation. This combined with Corollary 45 suggests two different scenarios:

- If the conjecture is true (for the analytic equivalence relations), then the equivalence relations given by pseudometrics from Theorem A (1) are reducible to an orbit equivalence relation. That would actually imply that they are bi-reducible with the universal orbit equivalence relation, see the discussion in Section 6.
- If one feels that the equivalence relations given by pseudometrics from Theorem A (1) should be different from orbit equivalence relations, then he is led to the reconsideration of the conjecture.

**Conjecture 46.** *The equivalence $E_1$ is* not *the least analytic equivalence relation which is not Borel reducible to an orbit equivalence.*

We note that the conjecture that $E_1$ is the least equivalence which is not Borel reducible to an orbit equivalence relation has been verified affirmatively by Solecki for the special class of equivalence relations $E_\mathcal{I}$ on $2^\mathbb{N}$, where $\mathcal{I}$ is an ideal of subsets of $\mathbb{N}$. We then have $xE_\mathcal{I}y$ if and only if $\{n \in \mathbb{N} : x(n) \neq y(n)\} \in \mathcal{I}$. Note that one can view each such ideal $\mathcal{I}$ as a subgroup of $2^\mathbb{N}$. Call $\mathcal{I}$ *polishable* if there exists a Polish group topology on $\mathcal{I}$ producing the same Borel structure as the standard topology on $\mathcal{I}$ inherited from $2^\mathbb{N}$. The following follows from [51, Theorem 2.1] and [36, Corollary 11.8.3].

**Theorem 47** (Solecki)**.** *Let $\mathcal{I}$ be an analytic ideal on $\mathbb{N}$. Then $E_1 \leq_B E_{\mathcal{I}}$ if and only if $\mathcal{I}$ is not polishable.*

It is mentioned in [23, Chapter 8] that a plausible conjecture is that orbit equivalence relations coincide with the idealistic equivalence relations, which is motivated by the fact that Kechris and Louveau prove in [39] that $E_1$ is not Borel reducible to any Borel idealistic equivalence relation. Note that Kechris and Louveau in [39] pose the problem whether Borel idealistic equivalence relations coincide with Borel equivelence relations $E$ such that $E_1 \not\leq_B E$.

Recall that an equivalence relation $E$ on a standard Borel space $X$ is *idealistic* if for every equivalence class $C \subseteq X$ of $E$ there is a $\sigma$-ideal $\mathcal{I}_C$ of subsets of $C$ such that

- $C \notin \mathcal{I}_C$;
- for every Borel set $A \subseteq X^2$ the set $\{x \in X : \{y \in [x]_E : (x,y) \in A\} \in \mathcal{I}_{[x]_E}\}$ is Borel.

In view of the conjecture, the following is natural to be investigated.

**Question 48.** *Are the equivalences $E_\rho$, where $\rho$ is a CTR orbit pseudometric, idealistic?*

## 3. Reductions

In this section we prove the reducibility results of the paper. The section is divided into three parts, one dealing with reductions between pseudometrics on the spaces of metric spaces, the remaining two dealing with reductions where pseudometrics on the space of Banach spaces are involved.

### 3.1. Reductions between pseudometrics on spaces of metric spaces.

**Theorem 49.** *For every positive real numbers $p$, $q$, there is an injective Borel-uniformly continuous reduction from $\rho_{GH}$ on $\mathcal{M}_p$ to $\rho_{GH}$ on $\mathcal{M}^q$.*

*Moreover, the reduction is not only Borel-uniformly continuous, but also Borel-Lipschitz on small distances.*

*Proof.* First, note that for every positive real numbers $p$, $q$ there is an injective Borel-uniformly continuous (and even Borel-Lipschitz) reduction from $\rho_{GH}$ on $\mathcal{M}_p$ to $\rho_{GH}$ on $\mathcal{M}_5$ (the reduction is $\mathcal{M}_p \ni d \mapsto \frac{5d}{p} \in \mathcal{M}_5$) and from $\rho_{GH}$ on $\mathcal{M}^3$ to $\rho_{GH}$ on $\mathcal{M}^q$ (the reduction is $\mathcal{M}^3 \ni d \mapsto \frac{qd}{3} \in \mathcal{M}^q$). Hence, it suffices to show that there is an injective Borel-uniformly continuous reduction from $\rho_{GH}$ on $\mathcal{M}_5$ to $\rho_{GH}$ on $\mathcal{M}^3$.

The strategy of the proof is the following. First, we will describe a construction which to each $(M, d_M) \in \mathcal{M}_5$ assigns $(\tilde{M}, d_{\tilde{M}}) \in \mathcal{M}^3$. We will show that for every $M, N \in \mathcal{M}_5$ we have

$$\begin{aligned} \rho_{GH}(M,N) < 1 &\implies \rho_{GH}(\tilde{M},\tilde{N}) \leq \rho_{GH}(M,N), \\ \rho_{GH}(\tilde{M},\tilde{N}) < \tfrac{1}{6} &\implies \rho_{GH}(M,N) \leq 5\rho_{GH}(\tilde{M},\tilde{N}). \end{aligned}$$

Finally, we will show that it is possible to make such an assignment in a Borel way, that is, find an injective Borel mapping $\mathcal{M}_5 \ni d \mapsto \tilde{d} \in \mathcal{M}^3$ such that for every $M = (\mathbb{N}, d)$, $\tilde{M}$ is isometric to $(\mathbb{N}, \tilde{d})$.

*First step: Construction of $\tilde{M}$*
Consider $(M, d_M) \in \mathcal{M}_5$ where $M = \{m_n : n \in \mathbb{N}\}$. For any two distinct $i, j \in \mathbb{N}$ we set $I^M_{i,j} = \{k \in \mathbb{Z} : \ |k| < d_M(m_i, m_j)/2\}$. Note that by our assumption on the minimal distance in $M$ we have that $|I^M_{i,j}| \geq 5$. We set

$$\tilde{M} = M \cup \big\{p^M_{i,j,k} : \ i < j \in \mathbb{N}, k \in I^M_{i,j}\big\}.$$

In the sequel, when the metric space in question is clear, we shall denote the points $p^M_{i,j,k}$ just by $p_{i,j,k}$, and $I^M_{i,j}$ just by $I_{i,j}$. We define a partial distance $d'$ as follows. Fix $i < j \in \mathbb{N}$. We set $d'(m_i, m_j) = d_M(m_i, m_j)$. For any $k \in I_{i,j}$ we set $d'(m_i, p_{i,j,k}) = d_M(m_i, m_j)/2 + k$, $d'(m_j, p_{i,j,k}) = d_M(m_i, m_j)/2 - k$. Finally, for $k, k' \in I_{i,j}$ we set $d'(p_{i,j,k}, p_{i,j,k'}) = |k' - k|$. The function $d'$ is then extended to the whole $\tilde{M}$ as the greatest extension of $d'$ (graph metric), which we denote as $\hat{d}_{\tilde{M}}$, and finally we take its minimum with the constant 3, that is for $x, y \in \tilde{M}$ we set

$$d_{\tilde{M}}(x,y) = \begin{cases} \min\{d'(x,y), 3\} & \text{if } (x,y) \in \mathrm{dom}(d'),\\ \min\{d'(x,m_i) + d'(m_i,y), 3\} & \text{if there are } i, j, j', k, k' \text{ with}\\ & j \neq j', x = p_{\min\{i,j\},\max\{i,j\},k} \text{ and}\\ & y = p_{\min\{i,j'\},\max\{i,j'\},k'},\\ 3 & \text{otherwise.} \end{cases}$$

It is easy to verify that $(\tilde{M}, d_{\tilde{M}}) \in \mathcal{M}^3$.

*Second step: for any $\varepsilon \in (0,1)$ and any $M, N \in \mathcal{M}_5$, we have*

$$\rho_{GH}(M,N) < \varepsilon \Rightarrow \rho_{GH}(\tilde{M}, \tilde{N}) \leq \varepsilon.$$

Fix any $\varepsilon \in (0,1)$ and $M, N \in \mathcal{M}_5$ with $\rho_{GH}(M,N) < \varepsilon$. By Lemma 12, there exists a permutation $\pi : \mathbb{N} \to \mathbb{N}$ such that for any $i \neq j \in \mathbb{N}$, $|d_M(m_i, m_j) - d_N(n_{\pi(i)}, n_{\pi(j)})| < 2\varepsilon$. Note that, since $\varepsilon < 1$, we have either $I^M_{i,j} = I^N_{\pi(i),\pi(j)}$ or there exists $k \in \mathbb{N}$ with either $\{-k, k\} = I^M_{i,j} \setminus I^N_{\pi(i),\pi(j)}$ or $\{-k, k\} = I^N_{\pi(i),\pi(j)} \setminus I^M_{i,j}$. Define a correspondence $\mathcal{R}$ on $\tilde{M} \times \tilde{N}$ as a union of $\mathcal{R}_1$, $\mathcal{R}_2$ and $\mathcal{R}_3$, where

$$\begin{aligned} \mathcal{R}_1 :=& \big\{(m_i, n_{\pi(i)}) : \ i \in \mathbb{N}\big\},\\ \mathcal{R}_2 :=& \bigcup_{\{i<j:\ \pi(i)<\pi(j)\}} \Big( \big\{(p^M_{i,j,k}, p^N_{\pi(i),\pi(j),k}) : \ k \in I^M_{i,j} \cap I^N_{\pi(i),\pi(j)}\big\} \cup \\ & \big\{(p^M_{i,j,-k}, n_{\pi(i)}), (p^M_{i,j,k}, n_{\pi(j)}) : \ k \in I^M_{i,j} \setminus I^N_{\pi(i),\pi(j)}, k > 0\big\} \cup \\ & \big\{(m_i, p^N_{\pi(i),\pi(j),-k}), (m_j, p^N_{\pi(i),\pi(j),k}) : \ k \in I^N_{\pi(i),\pi(j)} \setminus I^M_{i,j}, k > 0\big\} \Big), \end{aligned}$$

and $\mathcal{R}_3$ is defined in a similar way as $\mathcal{R}_2$: we make the union over $\{i < j : \pi(j) < \pi(i)\}$ and replace $(p^M_{i,j,k}, p^N_{\pi(i),\pi(j),k})$, $(m_i, p^N_{\pi(i),\pi(j),-k})$, $(m_j, p^N_{\pi(i),\pi(j),k})$ by $(p^M_{i,j,k}, p^N_{\pi(j),\pi(i),-k})$, $(m_i, p^N_{\pi(j),\pi(i),k})$, $(m_j, p^N_{\pi(j),\pi(i),-k})$, respectively.

It is straightforward to check that the correspondence $\mathcal{R}$ witnesses that $\rho_{GH}(\tilde{M}, \tilde{N}) \leq \varepsilon$.

*Third step: for any $\varepsilon \in (0, 1/6]$ and any $M, N \in \mathcal{M}_5$, we have*

$$\rho_{GH}(\tilde{M}, \tilde{N}) < \varepsilon \Rightarrow \rho_{GH}(M,N) \leq 5\varepsilon.$$

Let $\varepsilon \leq 1/6$. Suppose that for some $M, N \in \mathcal{M}_5$ we have $\rho_{GH}(\tilde{M}, \tilde{N}) < \varepsilon$. By Fact 9 there is a correspondence $\mathcal{R} \subseteq \tilde{M} \times \tilde{N}$ such that for every $x, x' \in \tilde{M}$ and $y, y' \in \tilde{N}$ such that $(x, y) \in \mathcal{R}$ and $(x', y') \in \mathcal{R}$ we have $|d_{\tilde{M}}(x, x') - d_{\tilde{N}}(y, y')| < 2\varepsilon$. Let us consider the relations $\pi$ and $\tau$ on $\mathbb{N}$ given by

$$\begin{aligned}\pi :=& \big\{(i,j) \in \mathbb{N} \times \mathbb{N} : \ \exists y \in \tilde{N} : \ (m_i, y) \in \mathcal{R} \ \& \ d_{\tilde{N}}(y, n_j) < 3\varepsilon\big\},\\ \tau :=& \big\{(j,i) \in \mathbb{N} \times \mathbb{N} : \ \exists x \in \tilde{M} : \ (x, n_j) \in \mathcal{R} \ \& \ d_{\tilde{M}}(m_i, x) < 3\varepsilon\big\}.\end{aligned}$$

We shall prove that $\pi$ is a bijection with inverse $\tau$ and that this bijection, due to Lemma 11, witnesses $\rho_{GH}(M, N) \leq 5\varepsilon$.

First, we shall prove that $\mathrm{dom}(\pi) = \mathbb{N}$. Fix $i \in \mathbb{N}$. If there is $n_j \in N$ with $(m_i, n_j) \in \mathcal{R}$, we have $(i, j) \in \pi$. Otherwise, pick $p^N_{h,j,k}$ with $(m_i, p^N_{h,j,k}) \in \mathcal{R}$. First, assume we have $k \leq 0$. For $l > i$ pick $a_l = p^M_{i,l,k(l)}$ with $d_{\tilde{M}}(m_i, a_l) \in [2\varepsilon, 1 + 2\varepsilon)$ and find $b_l \in \tilde{N}$ with $(a_l, b_l) \in \mathcal{R}$. For $l, l' > i$, $l \neq l'$, we have $d_{\tilde{M}}(a_l, a_{l'}) \geq 4\varepsilon$; hence, $d_{\tilde{N}}(b_l, b_{l'}) \geq 2\varepsilon > 0$ and $b_l \neq b_{l'}$. Moreover, $d_{\tilde{M}}(m_i, a_l) \in [2\varepsilon, 1{+}2\varepsilon)$ implies that $d_{\tilde{N}}(p^N_{h,j,k}, b_l) \in (0, 1{+}4\varepsilon)$, and so by the definition of $d_{\tilde{N}}$, using the fact that obviously $d_{\tilde{N}}(n_j, p^N_{h,j,k}) \geq \frac{5}{2} \geq 1 + 4\varepsilon$, we have that

$$\begin{aligned}b_l \in \{n_h\} \cup \big\{p^N_{h,j,k'} : \ |k' - k| = 1\big\} \cup \big\{p^N_{j',h,k'} : \ j' < h, k' \in I_{j',h}\big\} \cup \\ \cup \big\{p^N_{h,j',k'} : \ h < j', k' \in I_{h,j'}\big\}.\end{aligned}$$

Since the first three sets are finite and we have infinitely many $b_l$'s, there are $l, l' > i$ with $b_l = p_{h,j',k'}$ and $b_{l'} = p_{h,j'',k''}$ for some $h < j' \neq j''$ and some $k', k''$. Then

$$\begin{aligned}2d_{\tilde{N}}(p^N_{h,j,k}, n_h) &= d_{\tilde{N}}(p^N_{h,j,k}, b_l) + d_{\tilde{N}}(p^N_{h,j,k}, b_{l'}) - d_{\tilde{N}}(b_l, b_{l'}) < \\ &< d_{\tilde{M}}(m_i, a_l) + d_{\tilde{M}}(m_i, a_{l'}) - d_{\tilde{M}}(a_l, a_{l'}) + 6\varepsilon = 6\varepsilon.\end{aligned}$$

Therefore, we have $d_{\tilde{N}}(p^N_{h,j,k}, n_h) < 3\varepsilon$ and $(i, h) \in \pi$. Similarly, for $k > 0$ we get $(i, j) \in \pi$. Therefore, $i \in \mathrm{dom}(\pi)$ and since $i \in \mathbb{N}$ was arbitrary, $\mathrm{dom}(\pi) = \mathbb{N}$.

Analogously, $\mathrm{dom}(\tau) = \mathbb{N}$. Fix some $(i, j) \in \pi$ and $(j, k) \in \tau$. There are $y \in \tilde{N}$ and $x \in \tilde{M}$ with $\{(m_i, y), (x, n_j)\} \subseteq \mathcal{R}$ and $\max\{d_{\tilde{N}}(y, n_j), d_{\tilde{M}}(m_k, x)\} < 3\varepsilon$. Hence,

$$d_{\tilde{M}}(m_i, m_k) \leq d_{\tilde{M}}(m_i, x) + d_{\tilde{M}}(x, m_k) < (d_{\tilde{N}}(y, n_j) + 2\varepsilon) + 3\varepsilon < 8\varepsilon$$

which implies $d_{\tilde{M}}(m_i, m_k) < 3$ and $i = k$. Therefore, $\tau \circ \pi = \mathrm{id}$. Analogously, $\pi \circ \tau = \mathrm{id}$. Therefore, $\pi$ is a bijection with $\pi^{-1} = \tau$.

Let us recall that $\hat{d}_{\tilde{N}}$ is our notation for the greatest extension of $d'_{\tilde{N}}$, that is, $\hat{d}_{\tilde{N}} \supset d_N$ is a metric with $d_{\tilde{N}} = \min\{\hat{d}_{\tilde{N}}, 3\}$. In order to see that $\pi$ witnesses $d_M \simeq_{10\varepsilon} d_N$, we shall use the following claim.

**Claim 50.** *Let us have $(m_i, y) \in \mathcal{R}$ and $(m_{i'}, y') \in \mathcal{R}$ for some $i \neq i'$ and $y, y' \in \tilde{N}$. Then*

$$\hat{d}_{\tilde{N}}(y, y') \leq d_M(m_i, m_{i'}) + 4\varepsilon.$$

*Proof.* We may suppose that $i < i'$. Pick integers $u, v$ with $d_{\tilde{M}}(m_i, p_{i,i',u}) \in [\frac{3}{2}, \frac{5}{2})$ and $d_{\tilde{M}}(m_{i'}, p_{i,i',v}) \in [\frac{3}{2}, \frac{5}{2})$. Moreover, for every $u \leq k \leq v$, pick some

$y_k \in \tilde{N}$ such that $(p_{i,i',k}, y_k) \in \mathcal{R}$. We check first that $d_{\tilde{N}}(y_k, y_{k+1}) \leq 1$ for $u \leq k \leq v-1$.

In order to get a contradiction, let us assume that $d_{\tilde{N}}(y_k, y_{k+1}) > 1$ for some $u \leq k \leq v-1$. Note that $\hat{d}_{\tilde{N}}(y_k, y_{k+1}) \in (1, 1+2\varepsilon)$, since $d_{\tilde{N}}(y_k, y_{k+1}) < d_{\tilde{M}}(p_{i,i',k}, p_{i,i',k+1}) + 2\varepsilon = 1 + 2\varepsilon$, and thus $\hat{d}_{\tilde{N}}(y_k, y_{k+1}) = d_{\tilde{N}}(y_k, y_{k+1})$ in particular. For this reason, there is $j$ such that $\hat{d}_{\tilde{N}}(y_k, n_j) + \hat{d}_{\tilde{N}}(n_j, y_{k+1}) = \hat{d}_{\tilde{N}}(y_k, y_{k+1})$. There is $l \in \{k, k+1\}$ such that $\hat{d}_{\tilde{N}}(y_l, n_j) < \frac{1}{2}(1+2\varepsilon) = \frac{1}{2} + \varepsilon$. Consequently, $d_{\tilde{M}}(p_{i,i',l}, m_{\tau(j)}) < \frac{1}{2} + \varepsilon + 2\varepsilon + 3\varepsilon \leq \frac{3}{2}$. On the other hand, since $u \leq l \leq v$, we have $d_{\tilde{M}}(p_{i,i',l}, m_p) \geq \frac{3}{2}$ for any $p$. This completes the verification of $d_{\tilde{N}}(y_k, y_{k+1}) \leq 1$.

Finally, using $d_{\tilde{N}}(y, y_u) < d_{\tilde{M}}(m_i, p_{i,i',u}) + 2\varepsilon < \frac{5}{2} + 2\varepsilon < 3$ and $d_{\tilde{N}}(y_v, y') < d_{\tilde{M}}(p_{i,i',v}, m_{i'}) + 2\varepsilon < \frac{5}{2} + 2\varepsilon < 3$, we get

$$\begin{aligned}
\hat{d}_{\tilde{N}}(y, y') &\leq \hat{d}_{\tilde{N}}(y, y_u) + \sum_{k=u}^{v-1} \hat{d}_{\tilde{N}}(y_k, y_{k+1}) + \hat{d}_{\tilde{N}}(y_v, y') \\
&= d_{\tilde{N}}(y, y_u) + \sum_{k=u}^{v-1} d_{\tilde{N}}(y_k, y_{k+1}) + d_{\tilde{N}}(y_v, y') \\
&\leq d_{\tilde{M}}(m_i, p_{i,i',u}) + 2\varepsilon + \sum_{k=u}^{v-1} 1 + d_{\tilde{M}}(p_{i,i',v}, m_{i'}) + 2\varepsilon \\
&= \hat{d}_{\tilde{M}}(m_i, p_{i,i',u}) + \sum_{k=u}^{v-1} \hat{d}_{\tilde{M}}(p_{i,i',k}, p_{i,i',k+1}) + \hat{d}_{\tilde{M}}(p_{i,i',v}, m_{i'}) + 4\varepsilon \\
&= d_M(m_i, m_{i'}) + 4\varepsilon,
\end{aligned}$$

which provides the desired inequality. □

Now, for every $i, i' \in \mathbb{N}$, $i \neq i'$, consider $y$ and $y'$ from $\tilde{N}$ witnessing that $(i, \pi(i)) \in \pi$ and $(i', \pi(i')) \in \pi$. We have

$$\begin{aligned}
d_N(n_{\pi(i)}, n_{\pi(i')}) = \hat{d}_{\tilde{N}}(n_{\pi(i)}, n_{\pi(i')}) &\leq \hat{d}_{\tilde{N}}(n_{\pi(i)}, y) + \hat{d}_{\tilde{N}}(y, y') + \hat{d}_{\tilde{N}}(y', n_{\pi(i')}) \\
&\leq 3\varepsilon + d_M(m_i, m_{i'}) + 4\varepsilon + 3\varepsilon.
\end{aligned}$$

Analogously, we get $d_M(m_i, m_{i'}) \leq d_N(n_{\pi(i)}, n_{\pi(i')}) + 10\varepsilon$; hence, $\pi$ witnesses $d_M \simeq_{10\varepsilon} d_N$ and by Lemma 11, $\rho_{GH}(M, N) \leq 5\varepsilon$.

*Fourth step: there is an injective Borel mapping $\mathcal{M}_5 \ni d \mapsto \tilde{d} \in \mathcal{M}^3$ such that for every $M = (\mathbb{N}, d)$, $\tilde{M}$ is isometric to $(\mathbb{N}, \tilde{d})$.*

Split $\mathbb{N}$ into two disjoint infinite subsets $A$ and $B$ enumerated as $A = \{a_n : n \in \mathbb{N}\}$ and $B = \{b_n : n \in \mathbb{N}\}$. Moreover, let $\{(c_n, d_n) : n \in \mathbb{N}\}$ be the enumeration of the set $\{(n, m) \in \mathbb{N}^2 : n < m\}$ given by

$$\underbrace{(1,2)}_{(c_1,d_1)}, \quad \underbrace{(1,3)}_{(c_2,d_2)}, \underbrace{(2,3)}_{(c_3,d_3)}, \quad \ldots, \quad (1,n), (2,n), \ldots, (n-1,n), \quad \ldots$$

Take arbitrary $M = (\mathbb{N}, d) \in \mathcal{M}_5$ enumerated as above as $\{m_i : i \in \mathbb{N}\}$, where $m_i = i$ for every $i \in \mathbb{N}$. For further use we put $I^d_{i,j} := I^M_{i,j}$ and $p^d_{i,j,k} := p^M_{i,j,k}$ for $i < j$ and $k \in I^M_{i,j}$. Let us inductively construct bijection $\pi_d : \mathbb{N} \to \tilde{M}$. We put $\pi_d(a_n) := m_n$ for every $n \in \mathbb{N}$. Now consider

$(c_1, d_1)$. In $\tilde{M}$ there are finitely many, in fact $|I^d_{c_1,d_1}|$-many, points $p_{c_1,d_1,k}$, $k \in I_{c_1,d_1}$. We enumerate them in an increasing order as $\pi_d(b_1), \ldots, \pi_d(b_{N_1})$, where $N_1 = |I_{c_1,d_1}|$. Then we enumerate the points $p_{c_2,d_2,k}$, $k \in I_{c_2,d_2}$, as $\pi_d(b_{N_1+1}), \ldots, \pi_d(b_{N_1+N_2})$, where $N_2 = |I_{c_2,d_2}|$, and so on. Finally, we define $\tilde{d} \in \mathcal{M}^3$ as $\tilde{d}(n,k) := d_{\tilde{M}}(\pi_d(n), \pi_d(k))$, $(n,k) \in \mathbb{N} \times \mathbb{N}$.

Now, it is not difficult to see that the map $\mathcal{M}_5 \ni d \mapsto \tilde{d} \in \mathcal{M}^3$ is Borel and injective. □

**Theorem 51.** *There is an injective Borel-Lipschitz on small distances reduction from $\rho_{GH}$ on $\mathcal{M}$ to $\rho_{GH}$ on $\mathcal{M}_p$, where $p > 0$ is arbitrary. Moreover, the reduction $F : \mathcal{M} \to \mathcal{M}_p$ is not only Borel but even continuous and for every $q > 0$ and $d \in \mathcal{M}^q$ we have $F(d) \in \mathcal{M}_p^{q+p}$. In particular, for $q > 0$ we have*

$$(\rho_{GH} \restriction \mathcal{M}) \leq_{B,u} (\rho_{GH} \restriction \mathcal{M}_1) \quad \text{and} \quad (\rho_{GH} \restriction \mathcal{M}^q) \leq_{B,u} (\rho_{GH} \restriction \mathcal{M}_p^{q+p}).$$

*Proof.* Fix $p > 0$. To each metric $d \in \mathcal{M}$ on $\mathbb{N}$ we associate a metric $\tilde{d}$ on $\mathbb{N}^2$. For any two distinct points $(m,n), (m',n') \in \mathbb{N}^2$ we set

$$\tilde{d}((m,n),(m',n')) = d(m,m') + p.$$

We *claim* that $\rho_{GH}(d,e) \geq \rho_{GH}(\tilde{d},\tilde{e})$, and $\rho_{GH}(d,e) \leq \rho_{GH}(\tilde{d},\tilde{e})$ whenever $\rho_{GH}(\tilde{d},\tilde{e}) < p/2$.

Indeed, fix $\varepsilon > 0$ and $d, e \in \mathcal{M}$. Suppose that $\rho_{GH}(d,e) < \varepsilon$. By Fact 9, there is a correspondence $\mathcal{R} \subseteq \mathbb{N}^2$ such that for $m, m', n, n' \in \mathbb{N}$ we have $|d(m,m') - e(n,n')| < 2\varepsilon$ whenever $m\mathcal{R}n$ and $m'\mathcal{R}n'$. It is straightforward to see that there exists a permutation $\pi$ of $\mathbb{N}^2$ such that, for every $(m,n), (m',n') \in \mathbb{N}^2$, we have $m\mathcal{R}m'$ whenever $\pi(m,n) = (m',n')$. It follows that for every $(m,n) \neq (m',n') \in \mathbb{N}^2$ we have

$$\begin{aligned}\big|\tilde{d}((m,n),(m',n')) - &\tilde{e}(\pi(m,n),\pi(m',n'))\big| = \\ &= \big|d(m,m') - e(\pi_1(m,n),\pi_1(m',n'))\big| < 2\varepsilon,\end{aligned}$$

where $\pi_1(m,n)$ is the first coordinate of $\pi(m,n)$. Therefore $\tilde{d} \simeq_{2\varepsilon} \tilde{e}$, so by Lemma 11 we have $\rho_{GH}(\tilde{d},\tilde{e}) \leq \varepsilon$. On the other hand, if $\rho_{GH}(\tilde{d},\tilde{e}) < \varepsilon < p/2$ then, by Lemma 12, there is a permutation $\pi$ of $\mathbb{N}^2$ witnessing $\tilde{d} \simeq_{2\varepsilon} \tilde{e}$ and then applying Fact 9 to the correspondence

$$\mathcal{R} := \big\{(i,j) \in \mathbb{N}^2 : \exists k, l \in \mathbb{N}\ \pi(i,k) = (j,l)\big\},$$

we easily obtain $\rho_{GH}(d,e) \leq \varepsilon$.

Fix some bijection $\varphi : \mathbb{N} \to \mathbb{N}^2$ and define $F(d) \in \mathcal{M}_p$ by $F(d)(i,j) := \tilde{d}(\varphi(i),\varphi(j))$, $(i,j) \in \mathbb{N}^2$. Then obviously $(\mathbb{N}, F(d))$ is isometric to $(\mathbb{N}^2, \tilde{d})$ and the mapping $F : \mathcal{M} \to \mathcal{M}_p$ is the reduction. Moreover, it is easy to see that $F$ is one-to-one and continuous. □

**Corollary 52.** *Fix real numbers $0 < p < q$. Then there is an injective Borel-uniformly continuous reduction from $\rho_{GH}$ on $\mathcal{M}$ to $\rho_{GH}$ on $\mathcal{M}_p^q$.*

*Moreover, the reduction is not only Borel-uniformly continuous, but also Borel-Lipschitz on small distances.*

*Proof.* By Theorems 51 and 49 we have

$$(\rho_{GH} \restriction \mathcal{M}) \leq_{B,u} (\rho_{GH} \restriction \mathcal{M}_1) \leq_{B,u} (\rho_{GH} \restriction \mathcal{M}^{q-p}) \leq_{B,u} (\rho_{GH} \restriction \mathcal{M}_p^q).$$

□

**Theorem 53.** *Fix real numbers $p < q$. Then the identity map on $\mathcal{M}_p^q$ is a Borel-uniformly continuous reduction from $\rho_{GH}$ to $\rho_L$ and from $\rho_L$ to $\rho_{GH}$.*

*Moreover, the identity is not only Borel-uniformly continuous, but also Borel-Lipschitz on small distances.*

*Proof.* Take some $d, e \in \mathcal{M}_p^q$ and suppose that $\rho_{GH}(d,e) < \varepsilon < p/2$. By Lemma 12 there is a permutation $\pi \in S_\infty$ witnessing that $d \simeq_{2\varepsilon} e$. The permutation $\pi$ also defines a bi-Lipschitz map between $(\mathbb{N}, d)$ and $(\mathbb{N}, e)$. Let us compute the Lipschitz constant of $\pi$. We have

$$\mathrm{Lip}(\pi) = \sup_{m \neq n \in \mathbb{N}} \frac{e(\pi(m), \pi(n))}{d(m,n)} \leq \sup_{m \neq n \in \mathbb{N}} \frac{d(m,n) + 2\varepsilon}{d(m,n)} \leq 1 + \frac{2\varepsilon}{p}.$$

The same argument shows that $\mathrm{Lip}(\pi^{-1}) \leq 1 + \frac{2\varepsilon}{p}$ and so we have $\rho_L(d,e) \leq \log(1 + \frac{2\varepsilon}{p}) \leq \frac{2\varepsilon}{p}$.

Conversely, suppose that $\rho_L(d,e) < \varepsilon < 1$. Then there is a bi-Lipschitz map $\pi : (\mathbb{N}, d) \to (\mathbb{N}, e)$ such that $\max\{\mathrm{Lip}(\pi), \mathrm{Lip}(\pi^{-1})\} \leq 1 + \delta$, where $1 + \delta < \exp(\varepsilon)$. So for any $m, n \in \mathbb{N}$ we have

$$\begin{aligned} |d(m,n) - e(\pi(m), \pi(n))| &\leq \max\big\{|(1+\delta)e(\pi(m),\pi(n)) - e(\pi(m),\pi(n))|, \\ &\quad |(1+\delta)d(m,n) - d(m,n)|\big\} = \max\big\{\delta e(\pi(m),\pi(n)), \delta d(m,n)\big\} \leq \delta q. \end{aligned}$$

Thus, we have $d \simeq_{\delta q} e$ and, by Lemma 11, $\rho_{GH}(d,e) \leq \frac{\delta q}{2} < \frac{q(\exp(\varepsilon)-1)}{2} \leq \frac{q(\exp(1)-1)}{2}\varepsilon$. □

Next, we show that Lipschitz and Gromov-Hausdorff distances are Borel-uniformly continuous bi-reducible.

**Theorem 54.** *There is an injective Borel-uniformly continuous reduction from $\rho_L$ on $\mathcal{M}$ to $\rho_{GH}$ on $\mathcal{M}$.*

*Moreover, the reduction is not only Borel-uniformly continuous, but also Borel-Lipschitz on small distances.*

*Proof.* For every $d \in \mathcal{M}$, we define a metric $\tilde{d}$ on $(\mathbb{N} \times \mathbb{Z}) \cup \{\clubsuit\}$ by

$$\tilde{d}\big((i,k),(j,l)\big) = |10k - 10l| + \min\big\{1, 2^{\min\{k,l\}} d(i,j)\big\},$$
$$\tilde{d}\big((i,k), \clubsuit\big) = |10k + 4| + 1.$$

We leave it to the reader to verify the elementary fact that $\tilde{d}$ is a metric.

Since $d \mapsto \tilde{d}$ is an injective continuous mapping from $\mathcal{M}$ into $\mathbb{R}^{((\mathbb{N}\times\mathbb{Z})\cup\{\clubsuit\})^2}$, it is easy to show that there is an injective continuous mapping $f : \mathcal{M} \to \mathcal{M}$ such that $M_{f(d)}$ is isometric to the completion of $((\mathbb{N} \times \mathbb{Z}) \cup \{\clubsuit\}, \tilde{d})$. Hence, to prove the theorem, it is sufficient to show that

$$\rho_{GH}(\tilde{d}, \tilde{e}) \leq (\exp \rho_L(d,e)) - 1$$

and

$$\rho_{GH}(\tilde{d}, \tilde{e}) < 1/4 \quad \Rightarrow \quad \rho_L(d,e) \leq \log(1 + 24\rho_{GH}(\tilde{d}, \tilde{e}))$$

for every $d, e \in \mathcal{M}$.

Assume that $(\exp \rho_L(d,e)) - 1 < \varepsilon$ and pick some $L : M_d \to M_e$ with $\mathrm{Lip}\, L < 1 + \varepsilon$ and $\mathrm{Lip}\, L^{-1} < 1 + \varepsilon$. We define a correspondence

$$\mathcal{R} = \{(\clubsuit, \clubsuit)\} \cup \big\{\big((i,k),(j,k)\big) : d(i, L^{-1}(j)) < 2^{-k-1}\varepsilon, e(L(i), j) < 2^{-k-1}\varepsilon\big\}.$$

Our aim is to show that $|\tilde{d}(a,a') - \tilde{e}(b,b')| < 2\varepsilon$ whenever $a\mathcal{R}b$ and $a'\mathcal{R}b'$. In the case that $a = a' = \clubsuit$ (and so $b = b' = \clubsuit$), this is trivial. Assume that $a \neq \clubsuit = a'$ (and so $b \neq \clubsuit = b'$), and denote $a = (i,k), b = (j,k)$. We want to show that $||10k+4|+1-|10k+4|-1| < 2\varepsilon$, which is obvious. We obtain the same conclusion in the case $a = \clubsuit \neq a'$. So, assume that $a \neq \clubsuit \neq a'$ (and so $b \neq \clubsuit \neq b'$), and denote $a = (i,k), b = (j,k), a' = (i',k'), b' = (j',k')$. We want to show that $||10k - 10k'| + \min\{1, 2^{\min\{k,k'\}} d(i,i')\} - |10k - 10k'| - \min\{1, 2^{\min\{k,k'\}} e(j,j')\}| < 2\varepsilon$, which can be slightly simplified to

$$\big|\min\big\{1, 2^{\min\{k,k'\}} d(i,i')\big\} - \min\big\{1, 2^{\min\{k,k'\}} e(j,j')\big\}\big| < 2\varepsilon.$$

Due to the symmetry, it is sufficient to show that the number under the absolute value is less than $2\varepsilon$. If $1 \leq 2^{\min\{k,k'\}} e(j,j')$, then we just write $\min\{1, 2^{\min\{k,k'\}} d(i,i')\} - 1 \leq 0 < 2\varepsilon$. In the opposite case that $1 > 2^{\min\{k,k'\}} e(j,j')$, we write

$$\begin{aligned} d(i,i') &\leq d(L^{-1}(j), L^{-1}(j')) + d(i, L^{-1}(j)) + d(i', L^{-1}(j')) \\ &< (1+\varepsilon)e(j,j') + 2^{-k-1}\varepsilon + 2^{-k'-1}\varepsilon \\ &\leq (1+\varepsilon)e(j,j') + 2^{-\min\{k,k'\}-1}\varepsilon + 2^{-\min\{k,k'\}-1}\varepsilon \end{aligned}$$

and

$$\begin{aligned} &\min\big\{1, 2^{\min\{k,k'\}} d(i,i')\big\} - \min\big\{1, 2^{\min\{k,k'\}} e(j,j')\big\} \\ &\quad \leq 2^{\min\{k,k'\}} d(i,i') - 2^{\min\{k,k'\}} e(j,j') \\ &\quad < 2^{\min\{k,k'\}} (1+\varepsilon) e(j,j') + 2^{-1}\varepsilon + 2^{-1}\varepsilon - 2^{\min\{k,k'\}} e(j,j') \\ &\quad = 2^{\min\{k,k'\}} \varepsilon \cdot e(j,j') + \varepsilon < \varepsilon + \varepsilon = 2\varepsilon. \end{aligned}$$

By Fact 9, this completes the verification of $\rho_{GH}(\tilde{d}, \tilde{e}) \leq \varepsilon$.

Now, assume that $\rho_{GH}(\tilde{d}, \tilde{e}) < \varepsilon < 1/4$ for some $d, e \in \mathcal{M}$. This is witnessed by a correspondence $\mathcal{R} \subseteq [(\mathbb{N} \times \mathbb{Z}) \cup \{\clubsuit\}]^2$ provided by Fact 9. We verify first that $\clubsuit\mathcal{R}\clubsuit$, showing that there is no $m$ with $m\mathcal{R}\clubsuit$ but $\clubsuit$. If $m\mathcal{R}\clubsuit$, then there are points which have distance to $m$ in $(5 - 2\varepsilon, 5 + 2\varepsilon)$ and in $(7 - 2\varepsilon, 7 + 2\varepsilon)$. If two points have distance in $(3, 7)$, then these points belong to $(\mathbb{N} \times \{0\}) \cup \{\clubsuit\}$. As $(5 - 2\varepsilon, 5 + 2\varepsilon) \subseteq (3, 7)$, we obtain $m \in (\mathbb{N} \times \{0\}) \cup \{\clubsuit\}$. The case $m \in \mathbb{N} \times \{0\}$ is also excluded, since distances of these points to other points belong to the set $[0,1] \cup \{5\} \cup [10, \infty)$, which is disjoint from $(7 - 2\varepsilon, 7 + 2\varepsilon)$.

It follows that

$$(i,k)\mathcal{R}(j,l) \quad \Rightarrow \quad k = l. \tag{3}$$

Indeed, we have $2 > 2\varepsilon > |\tilde{d}((i,k),\clubsuit) - \tilde{e}((j,l),\clubsuit)| = ||10k+4| - |10l+4||$, and this is possible only if $k = l$.

By (3), the relations

$$\mathcal{R}_k = \big\{(i,j) \in \mathbb{N}^2 : (i,k)\mathcal{R}(j,k)\big\}, \quad k \in \mathbb{Z},$$

are correspondences. Let us show that

$$i\mathcal{R}_k j,\ i'\mathcal{R}_k j' \ \&\ d(i,i') \leq 2^{-k-1} \quad \Rightarrow \quad e(j,j') \leq d(i,i') + 2^{1-k}\varepsilon. \tag{4}$$

Using $\tilde{e}((j,k),(j',k)) < \tilde{d}((i,k),(i',k)) + 2\varepsilon$, we obtain

$$\min\big\{1, 2^k e(j,j')\big\} < \min\big\{1, 2^k d(i,i')\big\} + 2\varepsilon \leq 2^k d(i,i') + 2\varepsilon.$$

In particular, using $d(i,i') \leq 2^{-k-1}$, the minimum on the left hand side is less than $1/2 + 2\varepsilon$, hence less than 1 and equal to $2^k e(j,j')$. Thus, (4) follows.

Further, we show that

$$i\mathcal{R}_k j \ \&\ i\mathcal{R}_{k'} j' \quad \Rightarrow \quad e(j,j') \leq 2^{1-\min\{k,k'\}}\varepsilon. \tag{5}$$

Using $|\tilde{e}((j,k),(j',k')) - \tilde{d}((i,k),(i,k'))| < 2\varepsilon$, we obtain $||10k - 10k'| + \min\{1, 2^{\min\{k,k'\}}e(j,j')\} - |10k - 10k'|| < 2\varepsilon$. That is,

$$\min\{1, 2^{\min\{k,k'\}}e(j,j')\} < 2\varepsilon,$$

which gives (5).

Let us consider the decreasing sequence of correspondences given by

$$\mathcal{R}^*_s = \bigcup_{k=s}^{\infty} \mathcal{R}_k = \big\{(i,j) \in \mathbb{N}^2 : (\exists k \geq s)\big((i,k)\mathcal{R}(j,k)\big)\big\}, \quad s = 1, 2, \ldots,$$

and let us observe that

$$e\text{-}\operatorname{diam}(i\mathcal{R}^*_s) \leq 2^{1-s}\varepsilon, \quad d\text{-}\operatorname{diam}(\mathcal{R}^*_s j) \leq 2^{1-s}\varepsilon. \tag{6}$$

The first inequality follows from (5), the second one holds due to the symmetry.

We claim moreover that

$$i\mathcal{R}^*_s j,\ i'\mathcal{R}^*_s j' \ \&\ d(i,i') \geq 2^{-s-2} \quad \Rightarrow \quad e(j,j') \leq (1+24\varepsilon)d(i,i'). \tag{7}$$

Let $k \in \mathbb{Z}$ be such that $2^{-k-2} \leq d(i,i') < 2^{-k-1}$. We have $k \leq s$, and so $i\mathcal{R}^*_k j, i'\mathcal{R}^*_k j'$ in particular. Let us pick $n, n' \in \mathbb{N}$ such that $i\mathcal{R}_k n, i'\mathcal{R}_k n'$. We obtain from (4) that $e(n,n') \leq d(i,i') + 2^{1-k}\varepsilon$. Also, we obtain from (6) that $e(j,n) \leq 2^{1-k}\varepsilon$ and $e(j',n') \leq 2^{1-k}\varepsilon$. By the triangle inequality,

$$e(j,j') \leq d(i,i') + 2^{1-k}\varepsilon + 2^{1-k}\varepsilon + 2^{1-k}\varepsilon = d(i,i') + 24 \cdot 2^{-k-2}\varepsilon \leq (1+24\varepsilon)d(i,i'),$$

which gives (7). Let us note that, due to the symmetry, we have also

$$i\mathcal{R}^*_s j,\ i'\mathcal{R}^*_s j' \ \&\ e(j,j') \geq 2^{-s-2} \quad \Rightarrow \quad d(i,i') \leq (1+24\varepsilon)e(j,j'). \tag{8}$$

Finally, applying Lemma 20 and using (6), (7) and (8), we obtain $\rho_L(d,e) \leq \log(1+24\varepsilon)$. □

**Theorem 55.** *The pseudometrics $\rho_{GH}$ on $\mathcal{M}$ and $\rho_L$ on $\mathcal{M}$ are Borel-uniformly continuous bi-reducible.*

*Proof.* By Corollary 52 and Theorem 53 we get

$$\rho_{GH} \ \leq_{B,u} \ (\rho_{GH} \restriction \mathcal{M}^4_2) \ \leq_{B,u} \ (\rho_L \restriction \mathcal{M}^4_2) \ \leq_{B,u} \ \rho_L.$$

For the other direction, we use Theorem 54. □

Finally, using an analogous proof, we obtain that "the coarse Lipschitz distance" on metric spaces is reducible to the Gromov-Hausdorff distance. We will see later it is actually bi-reducible with it (Theorem 63).

**Theorem 56.** *There is an injective Borel-uniformly continuous reduction from $\rho_{HL}$ on $\mathcal{M}$ to $\rho_{GH}$ on $\mathcal{M}$.*

*Proof.* Denote by $\mathbb{N}^-$ the set $\{k \in \mathbb{Z} : \ k \leq 0\}$. For every $d \in \mathcal{M}$, we define a metric $\tilde{d}$ on $(\mathbb{N} \times \mathbb{N}^-) \cup \{\clubsuit\}$ by

$$\tilde{d}\big((i,k),(j,l)\big) = |10k - 10l| + \min\big\{1, 2^{\min\{k,l\}} d(i,j)\big\},$$

$$\tilde{d}\big((i,k),\clubsuit\big) = |10k + 4| + 1.$$

Note that this is the same construction which we used already in the proof of Theorem 54 with the exception that the underlying set is $(\mathbb{N} \times \mathbb{N}^-) \cup \{\clubsuit\}$ and in the proof of Theorem 54 it is $(\mathbb{N} \times \mathbb{Z}) \cup \{\clubsuit\}$. Hence, to prove the theorem, it is sufficient to show that for every $\varepsilon > 0$ there are $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$\rho_{GH}(\tilde{d}, \tilde{e}) < \delta_1 \Rightarrow \rho_{HL}(d,e) \leq \varepsilon$$

and

$$\rho_{HL}(d,e) < \delta_2 \Rightarrow \rho_{GH}(\tilde{d}, \tilde{e}) \leq \varepsilon$$

for every $d, e \in \mathcal{M}$.

By Lemma 29, there exists $\delta' > 0$ such that

$$\begin{aligned} d \text{ and } e \text{ are } HL(\delta')\text{-close} \quad &\Rightarrow \quad \rho_{HL}(d,e) < \varepsilon, \\ \rho_{HL}(d,e) < \delta' \quad &\Rightarrow \quad d \text{ and } e \text{ are } HL(\varepsilon)\text{-close}. \end{aligned}$$

We claim that it suffices to put $\delta_1 = \min\{\frac{1}{5}, \frac{\delta'}{24}\}$ and $\delta_2 = \delta'$.

Assume that $\rho_{GH}(\tilde{d}, \tilde{e}) < \delta_1$. This is witnessed by a correspondence $\mathcal{R} \subseteq [(\mathbb{N} \times \mathbb{N}^-) \cup \{\clubsuit\}]^2$. Then, using verbatim the same arguments as in the proof of Theorem 54, the relation $\mathcal{R}_0 = \{(i,j) \in \mathbb{N}^2 : \ (i,0)\mathcal{R}(j,0)\}$ is a correspondence and whenever $i\mathcal{R}_0 j$ and $i'\mathcal{R}_0 j'$, we have

$$\begin{aligned} d(i,i') \leq \tfrac{1}{2} \quad &\Rightarrow \quad e(j,j') \leq d(i,i') + 2\delta_1, \\ d(i,i') \geq \tfrac{1}{4} \quad &\Rightarrow \quad e(j,j') \leq (1 + 24\delta_1) d(i,i'); \end{aligned}$$

and similarly for the symmetric situation when the roles of $d$ and $e$ are changed. In particular, $\mathcal{R}_0$ witnesses the fact that $d$ and $e$ are $HL(24\delta_1)$-close and since $24\delta_1 \leq \delta'$, we have $\rho_{HL}(d,e) < \varepsilon$.

Assume that $\rho_{HL}(d,e) < \delta_2$. Then $d$ and $e$ are $HL(\varepsilon)$-close, which is witnessed by a correspondence $\mathcal{R}' \subseteq \mathbb{N}^2$. Similarly as in the proof of Theorem 54, we define a correspondence

$$\mathcal{R} = \{(\clubsuit, \clubsuit)\} \cup \big\{((i,k),(j,k)) : \ i\mathcal{R}'j, \ k \leq 0\big\}.$$

Our aim is to show that $|\tilde{d}(a,a') - \tilde{e}(b,b')| < 2\varepsilon$ whenever $a\mathcal{R}b$ and $a'\mathcal{R}b'$. Using verbatim the same arguments as in the proof of Theorem 54, it is sufficient to show that for $a = (i,k)$, $b = (j,k)$, $a' = (i',k')$, $b' = (j',k')$ with $1 > 2^{\min\{k,k'\}} e(j,j')$, $i\mathcal{R}'j$, $i'\mathcal{R}'j'$ we have

$$2^{\min\{k,k'\}} d(i,i') - 2^{\min\{k,k'\}} e(j,j') < 2\varepsilon. \tag{9}$$

Fix $a, a', b, b'$ as above. If $e(j,j') \leq 1$, using that $\mathcal{R}'$ witnesses $d$ and $e$ are $HL(\varepsilon)$-close, we get $2^{\min\{k,k'\}}(d(i,i') - e(j,j')) < 2\varepsilon$. On the other hand, if $e(j,j') \geq 1$, we get $2^{\min\{k,k'\}}(d(i,i') - e(j,j')) \leq 2^{\min\{k,k'\}}((1+\varepsilon)e(j,j') - e(j,j')) = 2^{\min\{k,k'\}} \varepsilon e(j,j') < \varepsilon$. Hence, (9) holds and so the correspondence $\mathcal{R}$ witnesses that $\rho_{GH}(\tilde{d}, \tilde{e}) \leq \varepsilon$. $\square$

3.2. **Reductions from pseudometrics on $\mathcal{B}$ to pseudometrics on $\mathcal{M}$.** We start with a reduction from the Banach-Mazur distance to the Lipschitz distance. An essential ingredient is Lemma 22.

**Theorem 57.** *There is a Borel-uniformly continuous reduction from $\rho_{BM}$ to $\rho_L$ on $\mathcal{M}_p^q$, where $0 < p < q$.*

*Moreover, the reduction is not only Borel-uniformly continuous, but also Borel-Lipschitz on small distances.*

*Proof.* Without loss of generality, we assume that $p = 2$ and $q = 15$. The structure of the proof is the following. First, we describe a construction which to each $\nu \in \mathcal{B}$ assigns a metric space $M_\nu$. Next, we show that for $\nu, \lambda \in \mathcal{B}$ we have $\rho_L(M_\nu, M_\lambda) \leq \rho_{BM}(\nu, \lambda)$ and

$$\rho_L(M_\nu, M_\lambda) < \log(\tfrac{4}{3}) \implies \rho_{BM}(\nu, \lambda) \leq 2\rho_L(M_\nu, M_\lambda).$$

Finally, we show it is possible to make such an assignment in a Borel way.

Fix some countable sequence $(c_i)_{i\in\mathbb{N}}$ of positive real numbers such that for every positive real number $r > 0$ there exists $i \geq 7$ such that $c_i \cdot r \in (2, 9/4)$. Also, let $\pi : \mathbb{Q} \to \mathbb{N}\setminus\{1\}$ be some bijection and let $\preceq$ be some linear order on $V$. To each $\nu \in \mathcal{B}$ we assign a countable metric space $M_\nu$ with the following underlying set:

$$V \;\cup\; \left\{p_{a,b}^{m,k} :\; a \preceq b \in V, m \geq 7, k \leq m\right\} \cup$$
$$\cup \left\{f_{a,q}^{m} :\; a \in V, q \in \mathbb{Q}, m \leq \pi(q)\right\} \;\cup\; \left\{x_{a,b}^{i} :\; a \preceq, b \in V, i \leq 3\right\}.$$

The metric $d_\nu$ on $M_\nu$ is defined as follows.

- For every $a \neq b \in V$ we set $d_\nu(a, b) = 15$.
- For every $a \preceq b \in V$, $m \geq 7$ we define the number $K_{a,b}^m$ to be $\max\{2, \min\{3, c_m \cdot \nu(a-b)\}\}$. Then we set $d_\nu(a, p_{a,b}^{m,1}) = d_\nu(p_{a,b}^{m,1}, p_{a,b}^{m,2}) = \ldots = d_\nu(p_{a,b}^{m,m}, b) = K_{a,b}^m$.
- For every $a \in V$ and $q \in \mathbb{Q}$ we set $d_\nu(a, f_{a,q}^1) = 7$, $d_\nu(f_{a,q}^1, f_{a,q}^2) = \ldots = d_\nu(f_{a,q}^{\pi(q)}, qa) = 10$.
- For every $a \preceq b \in V$ we set $d_\nu(a, x_{a,b}^1) = d_\nu(b, x_{a,b}^2) = d_\nu(x_{a,b}^1, x_{a,b}^3) = d_\nu(x_{a,b}^2, x_{a,b}^3) = d_\nu(x_{a,b}^3, a + b) = 5$.
- On the rest of $M_\nu^2$, we take the greatest extension of $d_\nu$ defined above with 15 as the upper bound, which is nothing but the graph metric (bounded by 15).

We shall call the pairs of elements from $M_\nu$, for which the distance was defined directly before taking the extension, *edges*. In order to simplify some notation, whenever we write $p_{b,a}^{m,k}$, where $a \preceq b$, we mean the element $p_{a,b}^{m,k}$. Also by $p_{a,b}^{m,0}$ we mean the element $a$, and by $p_{a,b}^{m,m+1}$ we mean the element $b$. We shall call the pairs $p_{a,b}^{m,k}, p_{a,b}^{m,k+1}$ *neighbors.*

Consider two norms $\nu, \lambda \in \mathcal{B}$. Denote the elements of $M_\lambda$ by $V \cup \{q_{a,b}^{m,k} : a \preceq b \in V, m \geq 7, k \leq m\} \cup \{g_{a,q}^m : a \in V, q \in \mathbb{Q}, m \leq \pi(q)\} \cup \{y_{a,b}^i : a \preceq b \in V, i \leq 3\}$ and the numbers $\max\{2, \min\{3, c_m \cdot \lambda(a - b)\}\}$ by $L_{a,b}^m$.

We claim that $\rho_L(M_\nu, M_\lambda) \leq \rho_{BM}(\nu, \lambda)$. If $\rho_{BM}(\nu, \lambda) < \varepsilon$, by Lemma 22, there exists a surjective $\mathbb{Q}$-linear isomorphism $T : (V, \nu) \to (V, \lambda)$ with

$\|T\|\|T^{-1}\| < \exp(\varepsilon)$. Fix $\varepsilon' > 0$. We may assume that $\min\{\|T\|, \|T^{-1}\|\} \geq 1-\varepsilon'$. We use $T$ to define a bi-Lipschitz bijection $T' : M_\nu \to M_\lambda$. For every $a \in V$ we set $T'(a) = T(a)$ and for all elements of the form $p^{m,k}_{a,b}$, $f^m_{a,q}$, and $x^i_{a,b}$, with appropriate indices, whenever $T(a) \preceq T(b)$, we set $T'(p^{m,k}_{a,b}) = q^{m,k}_{T(a),T(b)}$, $T'(f^m_{a,q}) = g^m_{T(a),q}$, and $T'(x^i_{a,b}) = y^i_{T(a),T(b)}$; similarly, if $T(b) \prec T(a)$, we set $T'(p^{m,k}_{a,b}) = q^{m+1-k,k}_{T(b),T(a)}$, $T'(f^m_{a,q}) = g^m_{T(a),q}$, $T'(x^1_{a,b}) = y^2_{T(b),T(a)}$, $T'(x^2_{a,b}) = y^1_{T(b),T(a)}$ and $T'(x^3_{a,b}) = y^3_{T(b),T(a)}$. Let us compute the Lipschitz constants of $T'$. If $a = b$, then obviously $\frac{L^m_{T(a),T(b)}}{K^m_{a,b}} = 1$. Otherwise, we have

$$\begin{aligned}\frac{L^m_{T(a),T(b)}}{K^m_{a,b}} &= \frac{\max\{2, \min\{3, c_m \cdot \lambda(T(a)-T(b))\}\}}{\max\{2, \min\{3, c_m \cdot \nu(a-b)\}\}} \\ &\leq \max\left\{1, \frac{\lambda(T(a)-T(b))}{\nu(a-b)}\right\} \leq \max\{1, \|T\|\} \leq \|T\| + \varepsilon',\end{aligned}$$

where in the first inequality we used the easy fact that for $x, y > 0$ we have $\frac{\max\{2,\min\{3,x\}\}}{\max\{2,\min\{3,y\}\}} \leq \max\{1, \frac{x}{y}\}$. It follows that $\mathrm{Lip}(T') \leq \|T\| + \varepsilon'$. Indeed, it follows from the definition of $T'$ that it maps edges onto edges. Moreover, for every edge $(x, y) \in M^2_\nu$ we have $d_\lambda(T'(x), T'(y)) \leq (\|T\| + \varepsilon')d_\nu(x, y)$, so the same inequality extends to the graph metrics – the extensions of $d_\nu$ and $d_\lambda$ on the whole $M_\nu$ and $M_\lambda$ respectively. We obtain in particular that $\mathrm{Lip}(T) \leq (\|T\|+\varepsilon')\|T^{-1}\|/(1-\varepsilon')$. Since an analogous inequality holds for $\mathrm{Lip}((T')^{-1})$ and $\varepsilon' > 0$ was arbitrary, we have $\rho_L(M_\nu, M_\lambda) \leq \log(\|T\|\|T^{-1}\|) < \varepsilon$. Thus, we conclude that $\rho_L(M_\nu, M_\lambda) \leq \rho_{BM}(\nu, \lambda)$.

Conversely, assume that $\exp(\rho_L(M_\nu, M_\lambda)) < 4/3$, that is, there exists a bijection $T : M_\nu \to M_\lambda$ with $\mathrm{Lip}(T) < 4/3$ and $\mathrm{Lip}(T^{-1}) < 4/3$. We will show that $\rho_{BM}(\nu, \lambda) \leq 2\rho_L(M_\nu, M_\lambda)$.

First we claim that $T$ maps $V \subseteq M_\nu$ bijectively onto $V \subseteq M_\lambda$. Indeed, the points $a \in V \subseteq M_\nu$ are characterized as those points $x$ of $M_\nu$ for which there exist infinitely many points $y \in M_\nu$ with $\nu(x - y) \leq 3$. On the other hand, the points from $M_\nu \setminus V$ are characterized as those points $x$ of $M_\nu$ for which there are at most two points distinct from $x$ of distance less than 4 from $x$. Since $\mathrm{Lip}(T) < 4/3$, we get $T(V) \subseteq V$ and similarly we have $T^{-1}(V) \subseteq V$, which proves the claim. We denote by $S$ the induced bijection between $(V, \nu)$ and $(V, \lambda)$.

We claim that $S$ is $\mathbb{Q}$-linear. Let us check that it is homogeneous for all rationals, that is, $S(qa) = qS(a)$ for all $a \in V$ and $q \in \mathbb{Q}$, which in particular gives that $S(0) = 0$. For each $a \in V \subseteq M_\nu$ and $q \in \mathbb{Q}$ there is a path of points $a, f^1_{a,q}, \ldots, f^{\pi(q)}_{a,q}, qa$. The map $T$ must send this path to some path $T(a), g^1_{T(a),q'}, \ldots, g^{\pi(q')}_{T(a),q'}, q'T(a)$. However, $q'$ is determined by the length of the path which must be the same as the length of the former path. Therefore $q' = q$ and $S(qa) = T(qa) = qT(a) = qS(a)$. Next, we show that for $a \neq b \in V$ we have $S(a + b) = S(a) + S(b)$. There is a "triangle of paths" formed by the points $a, b, x^1_{a,b}, x^2_{a,b}, x^3_{a,b}, a + b$. $T$ must preserve this triangle, so it maps it to a triangle formed by the points

$T(a), T(b), y^1_{T(a),T(b)}, y^2_{T(a),T(b)}, y^3_{T(a),T(b)}, T(a)+T(b)$. That shows that $S(a+b) = T(a+b) = T(a)+T(b) = S(a)+S(b)$.

It remains to compute the Lipschitz constant of $S$, resp. $S^{-1}$, as a map from $(V,\nu)$ to $(V,\lambda)$. In order to do it, we claim that for every $a \preceq b$, $a \neq b$, $m \geq 7$ and $k \leq m$ we have $T(p^{m,k}_{a,b}) = q^{m,k}_{S(a),S(b)}$ if $S(a) \preceq S(b)$, and $T(p^{m,k}_{a,b}) = q^{m,m+1-k}_{S(b),S(a)}$ if $S(b) \preceq S(a)$. We only treat the former case, the other is treated analogously. First observe that $T(p^{m,1}_{a,b}) = q^{m',k'}_{S(a),b'}$, for some $m'$ and $b'$, and $k' = 1$ or $k' = m'$. Indeed, $p^{m,1}_{a,b}$ is a neighbor of $a$, so $d_\nu(a, p^{m,1}_{a,b}) \leq 3$. Therefore $d_\lambda(S(a), T(p^{m,1}_{a,b})) < 4$, so $S(a)$ and $T(p^{m,1}_{a,b})$ are also neighbors. Analogously, we show that for every $0 \leq k \leq m$ we have that $T(p^{m,k}_{a,b})$ and $T(p^{m,k+1}_{a,b})$ are neighbors, which implies that $T$ indeed maps the 'path' $a, p^{m,1}_{a,b}, p^{m,2}_{a,b}, \ldots, p^{m,m}_{a,b}, b$ onto the path $S(a), q^{m,1}_{S(a),S(b)}, q^{m,2}_{S(a),S(b)}, \ldots, q^{m,m}_{S(a),S(b)}, S(b)$.

We are now ready to compute the Lipschitz constants. We do it for $S$. Pick some $a \preceq b$, $a \neq b$. We want to compute $\frac{\lambda(S(a)-S(b))}{\nu(a-b)}$. We consider only the case when $S(a) \preceq S(b)$, the other case is analogous. By the choice of $(c_i)_{i\in\mathbb{N}}$, there exists $m \geq 7$ such that $c_m \cdot \nu(a-b) \in (2, 9/4)$. It follows that $d_\nu(a, p^{m,1}_{a,b}) \in (2, 9/4)$, so we have

$$d_\lambda(S(a), q^{m,1}_{S(a),S(b)}) = d_\lambda(T(a), T(p^{m,1}_{a,b})) \leq \mathrm{Lip}(T) d_\nu(a, p^{m,1}_{a,b}) < 3,$$

which implies that

$$\lambda(S(a)-S(b)) \leq \frac{d_\lambda(S(a), q^{m,1}_{S(a),S(b)})}{c_m} \leq \frac{\mathrm{Lip}(T) d_\nu(a, p^{m,1}_{a,b})}{c_m} = \mathrm{Lip}(T)\nu(a-b).$$

That shows that $\|S\| \leq \mathrm{Lip}(T)$. Analogously, we get $\|S^{-1}\| \leq \mathrm{Lip}(T^{-1})$; hence, we have $\rho_{BM}(\nu,\lambda) \leq 2\log\max\{\mathrm{Lip}(T), \mathrm{Lip}(T^{-1})\}$. Considering all bi-Lipschitz maps $T$ with $\mathrm{Lip}(T) < 4/3$ and $\mathrm{Lip}(T^{-1}) < 4/3$, we obtain $\rho_{BM}(\nu,\lambda) \leq 2\rho_L(M_\nu, M_\lambda)$ whenever $\rho_L(M_\nu, M_\lambda) < \log(4/3)$.

Finally, to verify that the map $\mathcal{B} \ni \nu \to (M_\nu, d_\nu)$ is Borel, let us denote by $N$ the underlying set of $M_\nu$ (which is the same for every $\nu \in \mathcal{B}$). Now it suffices to fix some bijection $\phi : \mathbb{N} \to N$ and check that the distances in $M_\nu$ depend on distances of $\nu$ in a continuous (when considering $\nu$ as a member of $\mathbb{R}^V$) way. $\square$

A consequence of the last theorem and Theorem 53 is that the Banach-Mazur distance is Borel-uniformly continuous reducible to the Gromov-Hausdorff distance. We will see later it is actually bi-reducible with it.

**Corollary 58.** *We have $\rho_{BM} \leq_{B,u} \rho_{GH}$.*

Next we show that 'the coarse Lipschitz distance' on Banach spaces is reducible to the Gromov-Hausdorff distance. Again, we will see later it is actually bi-reducible with it (Theorem 63).

The reduction is obtainable already from Theorem 56. However, the proof which follows is in this concrete case more natural and gives a slightly better result, that is, the reduction is even Borel-Lipschitz on small distances.

**Theorem 59.** *There is an injective Borel-uniformly continuous reduction from $\rho_{HL}$, equivalently $\rho_N$, on $\mathcal{B}$ to $\rho_{GH}$ on $\mathcal{M}$.*

*Moreover, the reduction is not only Borel-uniformly continuous, but also Borel-Lipschitz on small distances.*

*Proof.* To every separable Banach space $(X, \|\cdot\|_X)$ we associate a metric space $(X, d_X)$ whose underlying set is unchanged, and for every $x, y \in X$ we set $d_X(x, y) = \min\{\|x - y\|_X, 1\}$. We claim that the map $(X, \|\cdot\|_X) \to (X, d_X)$ is the desired reduction.

Fix some separable Banach spaces $X$ and $Y$. Suppose first that $\rho_{HL}(X, Y) = \rho_N(X, Y) < K$, for some $K > 0$, where the first equality follows from Proposition 26. So there exist nets $\mathcal{N}_X \subseteq X$ and $\mathcal{N}_Y \subseteq Y$ and a bi-Lipschitz map $T : \mathcal{N}_X \to \mathcal{N}_Y$ with $\log\max\{\mathrm{Lip}(T), \mathrm{Lip}(T^{-1})\} < K$. Pick any $\varepsilon > 0$. By rescaling the nets $\mathcal{N}_X$ and $\mathcal{N}_Y$ if necessary we may assume (as in the proof of Proposition 26) the nets are an $(a, \varepsilon)$-net, resp. an $(a', \varepsilon)$-net, for some $a, a' > 0$. Since $(\mathcal{N}_X, d_X)$ and $(\mathcal{N}_Y, d_Y)$ belong to $\mathcal{M}^1_{\min(a,a')}$ we get from Theorem 53 that $\rho_{GH}((\mathcal{N}_X, d_X), (\mathcal{N}_Y, d_Y)) \leq (\exp(K) - 1)/2$. Since $\rho_{GH}((X, d_X), (\mathcal{N}_X, d_X)) \leq \varepsilon$, $\rho_{GH}((Y, d_Y), (\mathcal{N}_Y, d_Y)) \leq \varepsilon$, and since $\varepsilon$ was arbitrary, we get that $\rho_{GH}((X, d_X), (Y, d_Y)) \leq \frac{\exp(K)-1}{2} \leq \frac{\exp(1)-1}{2} K$ whenever $K < 1$.

Conversely, suppose that $\rho_{GH}((X, d_X), (Y, d_Y)) < K$, where $K < 1/4$. By Lemma 13 there exists a bijection $\phi : X \to Y$ witnessing the Gromov-Hausdorff distance, i.e. for every $x, y \in X$ we have $|d_X(x, y) - d_Y(\phi(x), \phi(y))| < 2K$ (note that although Lemma 13 was stated only for countable dense subsets of perfect metric spaces, by transfinite recursion it can be proved also for the completions). We aim to show that $\phi$ is large scale bi-Lipschitz for $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$. Pick any $x, y \in X$ with $\|x - y\|_X \geq 1$. Find points $x_0 = x, x_1, x_2, \ldots, x_{n-1}, x_n = y$ such that $\sum_{i=0}^{n-1} \|x_i - x_{i+1}\| = \|x - y\|$, $n \leq 3\|x - y\|_X$ and for every $i < n$ we have $\|x_i - x_{i+1}\| \leq 1/2$. Notice that for every $i < n$ we have $\|x_i - x_{i+1}\|_X = d_X(x_i, x_{i+1}) \leq 1/2$. So $d_Y(\phi(x_i), \phi(x_{i+1})) \leq d_X(x_i, x_{i+1}) + 2K < 1$, therefore $\|\phi(x_i) - \phi(x_{i+1})\|_Y = d_Y(\phi(x_i), \phi(x_{i+1})) \leq \|x_i - x_{i+1}\|_X + 2K$.

Now we compute

$$\begin{aligned}\|\phi(x) - \phi(y)\|_Y &\leq \sum_{i=0}^{n-1} \|\phi(x_i) - \phi(x_{i+1})\|_Y \leq \|x - y\|_X + 2K(3\|x - y\|_X) \\ &= (1 + 6K)\|x - y\|_X.\end{aligned}$$

Along with the analogous computations for $\phi^{-1}$ we get that $\max\{\mathrm{Lip}_1(\phi), \mathrm{Lip}_1(\phi^{-1})\} \leq 1 + 6K$, where

$$\mathrm{Lip}_1(\phi) = \sup_{\substack{x,y \in X \\ \|x-y\|_X \geq 1}} \frac{\|\phi(x) - \phi(y)\|_Y}{\|x - y\|_X}.$$

Now it suffices to choose some maximal 2-separated set $\mathcal{N}_X$ in $(X, \|\cdot\|_X)$, which is a net in $X$. Its image $\phi[\mathcal{N}_X]$, denoted by $\mathcal{N}_Y$, is a net in $Y$. Indeed, we claim that for each $x \neq y \in \mathcal{N}_X$ we have $\|\phi(x) - \phi(y)\|_Y > 1$. Otherwise, there is $z \in Y$ such that $\|\phi(x) - z\|_Y \leq 1/2$ and $\|z - \phi(y)\|_Y \leq 1/2$. This implies that $\|x - y\|_X \leq \|x - \phi^{-1}(z)\|_X + \|\phi^{-1}(z) - y\|_X < \|\phi(x) - z\|_Y + \|z - \phi(y)\|_Y + 4K \leq 2$, a contradiction. Finally, we claim that for every

$y \in Y$ we can find $y' \in \mathcal{N}_Y$ with $\|y - y'\|_Y < 4$. Pick any $y \in Y$. Then there exist $x_1, x_2, x_3 \in X$ and $x' \in \mathcal{N}_X$ such that $\max\{\|\phi^{-1}(y) - x_1\|_X, \|x_1 - x_2\|_X, \|x_2 - x_3\|_X, \|x_3 - x'\|_X\} \leq 1/2$. Set $y' = \phi(x')$. We get that $\|y - y'\|_Y \leq \|y - \phi(x_1)\|_Y + \|\phi(x_1) - \phi(x_2)\|_Y + \|\phi(x_2) - \phi(x_3)\|_Y + \|\phi(x_3) - y'\|_Y \leq 2 + 8K < 4$. So we have verified that $\mathcal{N}_Y$ is a net. It is bi-Lipschitz with $\mathcal{N}_X$ as witnessed by $\phi$. So we get the estimate $\rho_{HL}((X, \|\cdot\|_X), (Y, \|\cdot\|_Y)) = \rho_N((X, \|\cdot\|_X), (Y, \|\cdot\|_Y)) \leq \log(1 + 6K) \leq 6K$.

Finally, we observe that the map $(X, \|\cdot\|_X) \to (X, d_X)$ can be viewed as a Borel function from $\mathcal{B}$ to $\mathcal{M}$. Recall that elements of $\mathcal{B}$ are norms on a countable infinite-dimensional $\mathbb{Q}$-vector space denoted by $V$. By fixing a bijection $f : V \to \mathbb{N}$ we associate to each $\|\cdot\| \in \mathcal{B}$ a metric $d \in \mathcal{M}$ such that for every $n, m \in \mathbb{N}$ we have $d(n, m) = \min\{1, \|f^{-1}(n) - f^{-1}(m)\|\}$. This is clearly Borel. □

*Remark* 60. Observe that the only geometric property of Banach spaces that we used in the proof, besides that Banach spaces are cones so that $\rho_{HL}$ and $\rho_N$ agree on them (see Remark 27), was that Banach spaces are geodetic metric spaces. Clearly, it is sufficient that they are length spaces, i.e. between every two points $x, y$ there is a path of length $d(x, y) + \varepsilon$, where $\varepsilon > 0$ is arbitrary. Therefore it follows from the proof of Theorem 59 that there is a reduction from $\rho_{HL}$ (or $\rho_N$) on cones that are length spaces to $\rho_{GH}$ on metric spaces.

Finally, we present the proof of the reduction that involves the Kadets distance.

**Theorem 61.** *There is an injective Borel-uniformly continuous reduction from $\rho_K$ on $\mathcal{B}$ to $\rho_{GH}$ on $\mathcal{M}$.*

*Moreover, the reduction is not only Borel-uniformly continuous, but also Borel-Lipschitz on small distances.*

We need the following lemma first.

**Lemma 62.** *Let $X$ and $Y$ be two separable Banach spaces and fix countable dense subsets $(x_i)_i$ and $(y_i)_i$ of the spheres $S_X$ and $S_Y$ respectively. Then $\rho_K(X, Y) < \varepsilon$, for some $\varepsilon > 0$, implies that there exists a bijection $\pi \in S_\infty$ such that for every finite $F \subseteq \mathbb{N}$ and every $(\delta_i)_{i \in F} \subseteq \{-1, 1\}$ we have*

$$\left| \Big\| \sum_{i \in F} \delta_i x_i \Big\|_X - \Big\| \sum_{i \in F} \delta_i y_{\pi(i)} \Big\|_Y \right| < 2|F|\varepsilon.$$

*Proof.* We may suppose that $X$ and $Y$ are subspaces of a Banach space $Z$ and that we have $\rho_H^Z(B_X, B_Y) < \varepsilon$. First we claim that for every $x \in S_X$ there exists $y \in S_Y$ such that $\|x - y\| < 2\varepsilon$. Analogously, for every $y \in S_Y$ there exists such $x \in S_X$. Indeed, by definition for every $x \in S_X$ there exists $y' \in B_Y$ with $\|x - y'\| < \varepsilon$. So we can take $y = y'/\|y'\|$ and we have $\|y - y'\| < \varepsilon$, so we are done by the triangle inequality. Now since $S_X$ and $S_Y$ are perfect metric spaces, by a back-and-forth argument (see e.g. the proof of Lemma 13), we get a bijection $\pi \in S_\infty$ such that for every $i \in \mathbb{N}$ we have $\|x_i - y_{\pi(i)}\| < 2\varepsilon$. We claim that $\pi$ is as desired.

Take any finite subset $F \subseteq \mathbb{N}$ and $(\delta_i)_{i\in F} \subseteq \{-1, 1\}$. Then we have

$$\left|\Big\|\sum_{i\in F}\delta_i x_i\Big\| - \Big\|\sum_{i\in F}\delta_i y_{\pi(i)}\Big\|\right| \leq \Big\|\sum_{i\in F}\delta_i(x_i - y_{\pi(i)})\Big\| \leq \sum_{i\in F}\|x_i - y_{\pi(i)}\| < \\ < 2|F|\varepsilon,$$

and we are done. □

*Proof of Theorem 61.* The structure of the proof is the following. First, we describe a construction which to each separable Banach space $X$ assigns a metric space $M_X$. Next, we show that for every two separable Banach spaces $X$ and $Y$ we have $\rho_{GH}(M_X, M_Y) \leq 2\rho_K(X, Y)$ and

$$\rho_{GH}(X, Y) < 1 \implies \rho_K(X, Y) \leq 17\rho_{GH}(M_X, M_Y).$$

Finally, we show it is possible to make such an assignment in a Borel way.

Let $X$ be a separable Banach space. Fix a countable dense subset $D_X = \{x_i : i \in \mathbb{N}\} \subseteq S_X$ of the unit sphere of $X$ that is symmetric, that is, for every $x \in D_X$ we also have $-x \in D_X$. Set $M_X = D_X \cup \{p_{F,k} : F \in [\mathbb{N}]^{<\omega} \setminus \{\emptyset\}, k \in F\}$. We define a metric $d_X$ on $M_X$ as follows:

$$d_X(x_i, x_j) = \|x_i - x_j\|_X,$$
$$d_X(x_i, p_{F,k}) = 10 + \|x_i - x_k\|_X,$$
$$d_X(p_{F,i}, p_{F,j}) = 15 + \frac{\|\sum_{k\in F} x_k\|_X}{|F|}, \quad i \neq j \in F,$$
$$d_X(p_{F,i}, p_{G,j}) = 20, \quad F \neq G.$$

Fix separable Banach spaces $X$ and $Y$. The space $M_Y = D_Y \cup \{q_{F,k} : F \in [\mathbb{N}]^{<\omega} \setminus \{\emptyset\}, k \in F\}$, where $D_Y = \{y_i : i \in \mathbb{N}\} \subseteq S_Y$ is symmetric countable dense, is constructed analogously as $M_X$.

We claim that for every $\varepsilon > 0$ with $\rho_K(X, Y) < \varepsilon$ we have $\rho_{GH}(M_X, M_Y) \leq 2\varepsilon$. Indeed, fix $\varepsilon > 0$ with $\rho_K(X, Y) < \varepsilon$ and use Lemma 62 applied to countable dense sequences $(x_i)_i$ and $(y_i)_i$ of the spheres $S_X$ and $S_Y$ respectively. The bijection $\pi$ from Lemma 62 induces a bijection $\phi : M_X \to M_Y$ defined as follows:

$$\phi(x_i) = y_{\pi(i)},$$
$$\phi(p_{F,j}) = q_{\pi[F],\pi(j)}.$$

We claim that for every $x, y \in M_X$ we have $|d_X(x, y) - d_Y(\phi(x), \phi(y))| < 4\varepsilon$, i.e. $M_X \simeq_{4\varepsilon} M_Y$. We consider several cases:

**Case 1.** $(x, y) = (x_i, x_j)$ for some $i, j \in \mathbb{N}$: then we have

$$|d_X(x_i, x_j) - d_Y(y_{\pi(i)}, y_{\pi(j)})| = |\|x_i - x_j\|_X - \|y_{\pi(i)} - y_{\pi(j)}\|_Y| < 4\varepsilon.$$

**Case 2.** $x = x_i$, $y = p_{F,k}$ for some $i \in \mathbb{N}$, $F \subseteq \mathbb{N}$, $k \in F$: then we have

$$|d_X(x_i, p_{F,k}) - d_Y(y_{\pi(i)}, q_{\pi[F],\pi(k)})| = |\|x_i - x_k\|_X - \|y_{\pi(i)} - y_{\pi(k)}\|_Y| < 4\varepsilon.$$

**Case 3.** $x = p_{F,j}$, $y = p_{F,k}$ for some $F \subseteq \mathbb{N}$, $j \neq k \in F$: then we have

$$|d_X(p_{F,j}, p_{F,k}) - d_Y(q_{\pi[F],\pi(j)}, q_{\pi[F],\pi(k)})| = \frac{|\|\sum_{i\in F} x_i\|_X - \|\sum_{i\in F} y_{\pi(i)}\|_Y|}{|F|} \\ < \frac{2|F|\varepsilon}{|F|} = 2\varepsilon.$$

**Case 4.** $x = p_{F,i}$, $y = p_{G,j}$, for $F \neq G \subseteq \mathbb{N}$, $i \in F$, $j \in G$: then we have

$$|d_X(p_{F,i}, p_{G,j}) - d_Y(q_{\pi[F],\pi(i)}, q_{\pi[G],\pi(j)})| = 20 - 20 = 0.$$

Hence, $d_X \simeq_{4\varepsilon} d_Y$ and, by Lemma 11, we get $\rho_{GH}(M_X, M_Y) \leq 2\varepsilon$ which proves the claim.

Conversely, suppose now that $\rho_{GH}(M_X, M_Y) < \varepsilon$, where $\varepsilon \in (0,1)$. By Fact 9 there exists a correspondence $\mathcal{R} \subseteq M_X \times M_Y$ such that for every $x, y \in M_X$ and $x', y' \in M_Y$, if $x\mathcal{R}x'$ and $y\mathcal{R}y'$, then $|d_X(x,y) - d_Y(x',y')| < 2\varepsilon$. Pick some $i \neq j \in \mathbb{N}$, a finite subset $F \subseteq \mathbb{N}$, $k \neq k' \in F$. Set $u_1 = x_i$, $u_2 = x_j$, $u_3 = p_{F,k}$, $u_4 = p_{F,k'}$. We find elements $v_1, \ldots, v_4 \in M_Y$ such that $u_i \mathcal{R} v_i$ for $i \leq 4$. We get the following observations:

- Since $d_X(u_1, u_2) \in [0,2]$ we get that $d_Y(v_1, v_2) \in [0,4]$, so we deduce that for every $n \in \mathbb{N}$ and every $y \in M_Y$ such that $x_n \mathcal{R} y$ we have $y = y_m$ for some $m \in \mathbb{N}$. Conversely, for every $n \in \mathbb{N}$ and every $x \in M_X$ such that $x \mathcal{R} y_n$ we have $x = x_m$ for some $m \in \mathbb{N}$.
- Since $d_X(u_3, u_4) \in [15,16]$ we get that $d_Y(v_3, v_4) \in [13,18]$. So we deduce that for every finite subsets $G, G' \subseteq \mathbb{N}$ and $l \in G$, $l' \in G'$, and every $y, y' \in M_Y$ such that $p_{G,l} \mathcal{R} y$ and $p_{G',l'} \mathcal{R} y'$ there are finite subsets $H, H' \subseteq \mathbb{N}$ and $h \in H$, $h' \in H'$ such that $y = p_{H,h}$, $y' = p_{H',h'}$ and $G = G'$ if and only if $H = H'$.

To summarize, $\mathcal{R}$ induces a bijection $\phi$ between $M_X \setminus (x_i)_i$ and $M_Y \setminus (y_i)_i$. Moreover, for every finite $F \subseteq \mathbb{N}$ there is a unique finite set, which we shall denote by $\varphi(F)$, such that $\phi$ is a bijection between $\{p_{F,i} : \ i \in F\}$ and $\{q_{\varphi(F),j} : \ j \in \varphi(F)\}$. For every $i \in F$, by $\varphi_F(i)$ we shall the denote the element $i' \in \varphi(F)$ such that $q_{\varphi(F),i'} = \phi(p_{F,i})$.

On the other hand, $\mathcal{R}$, when restricted on $D_X \times D_Y$, is a correspondence between $D_X$ and $D_Y$ witnessing that $\rho_{GH}(D_X, D_Y) \leq \varepsilon$. Since $d_X \restriction D_X$ and $d_Y \restriction D_Y$ are perfect metric spaces, by a back-and-forth argument (see e.g. the proof of Lemma 13) we construct a bijection $\phi' \subseteq \mathcal{R}$ between $D_X$ and $D_Y$ such that $|d_X(x_i, x_j) - d_Y(\phi'(x_i), \phi'(x_j))| < 2\varepsilon$. Taking the union of the bijections $\phi$ and $\phi'$ we get a bijection, which we shall still denote by $\phi$, between $M_X$ and $M_Y$ such that for every $x, y \in M_X$, $|d_X(x,y) - d_Y(\phi(x), \phi(y))| < 2\varepsilon$.

Pick now an arbitrary finite $F \subseteq \mathbb{N}$. We want to estimate the expression

$$\left| \Big\| \sum_{k \in F} x_k \Big\|_X - \Big\| \sum_{k \in F} \phi(x_k) \Big\|_Y \right|.$$

We may suppose $F$ contains at least two elements. Take any $i \neq i' \in F$ and set $G = \varphi(F)$ and $j = \varphi_F(i)$, $j' = \varphi_F(i')$. Since we have $d_X(p_{F,i}, p_{F,i'}) = 15 + \| \sum_{k \in F} x_k \|_X / |F|$ and $d_Y(q_{G,j}, q_{G,j'}) = 15 + \| \sum_{k \in G} y_k \|_Y / |G|$, and moreover $|d_X(p_{F,i}, p_{F,i'}) - d_Y(q_{G,j}, q_{G,j'})| < 2\varepsilon$ we get

$$\left| \Big\| \sum_{k \in F} x_k \Big\|_X - \Big\| \sum_{k \in G} y_k \Big\|_Y \right| < 2\varepsilon |F|.$$

So we try to estimate

$$\left|\Big\|\sum_{k\in G} y_k\Big\|_Y - \Big\|\sum_{k\in F}\phi(x_k)\Big\|_Y\right|.$$

Pick any $k \in F$ and let $k' = \varphi_F(k) \in G$. We have $d_X(x_k, p_{F,k}) = 10$ and $d_Y(\phi(x_k), q_{G,k'}) = 10 + \|\phi(x_k) - y_{k'}\|_Y$. Therefore, since

$$\big|d_X(x_k, p_{F,k}) - d_Y(\phi(x_k), q_{G,k'})\big| = |d_X(x_k, p_{F,k}) - d_Y(\phi(x_k), \phi(p_{F,k})| < 2\varepsilon,$$

we get that $\|\phi(x_k) - y_{k'}\|_Y < 2\varepsilon$. This implies that

$$\left|\Big\|\sum_{k\in G} y_k\Big\|_Y - \Big\|\sum_{k\in F}\phi(x_k)\Big\|_Y\right| < 2\varepsilon|F|,$$

which in turn implies that

$$\left|\Big\|\sum_{k\in F} x_k\Big\|_X - \Big\|\sum_{k\in F}\phi(x_k)\Big\|_Y\right| < 4\varepsilon|F|.$$

Note that the last inequality in particular implies that $\phi$ is almost symmetric. Pick any $x \in D_X$. Since also $-x \in D_X$, the previous inequality implies

$$\Big|\|x - x\|_X - \|\phi(x) + \phi(-x)\|_Y\Big| = \| - \phi(x) - \phi(-x)\|_Y < 8\varepsilon. \tag{10}$$

We set $E = \{e \in X : \ e = qx, q \in \mathbb{Q}^+ \cup \{0\}, x \in D_X\}$ and $F = \{f \in Y : f = qy, q \in \mathbb{Q}^+ \cup \{0\}, y \in D_Y\}$. Clearly $E$ and $F$ are $\mathbb{Q}$-homogeneous dense subsets of $X$ and $Y$ respectively. We define a correspondence $\mathcal{R}_0 \subseteq E \times F$, in fact a bijection, such that $e\mathcal{R}_0 f$ if and only if there are $x \in D_X$ and $q \in \mathbb{Q}^+$ such that $e = qx$, $f = q\phi(x)$. So for every pair $(e, f)$ such that $e\mathcal{R}_0 f$ we have $\|e\|_X = \|f\|_Y$. We now claim that $\mathcal{R}_0$ is such that

$$\left|\Big\|\sum_{i\le n} u_i\Big\|_X - \Big\|\sum_{i\le n} u'_i\Big\|_Y\right| \le 8\varepsilon\Big(\sum_{i\le n}\|u_i\|_X\Big) \tag{11}$$

for all $(u_i)_i \subseteq E$ and $(u'_i)_i \subseteq F$, where for all $i \le n$ we have $u_i\mathcal{R}_0 u'_i$.

Fix such a sequence $(u_i)_{i\le n} \subseteq E$. The corresponding sequence $(u'_i)_i$ is then determined uniquely. First we claim that without loss of generality we may suppose that $(u_i)_{i\le n} \subseteq D_X$. Indeed, by the homogeneity of the inequality above, we may assume that each $u_i$ is a positive integer multiple of some $x \in D_X$. Since we allow repetitions in the sequence $(u_i)_{i\le n}$, each element of the form $kx$, where $k \in \mathbb{N}$ and $x \in D_X$ can be replaced by $k$-many repetitions of the element $x$.

Next we show how we may approximate the sequence $(u_i)_i$, in which we allow repetitions, by a sequence $(a_i)_i \subseteq D_X$ in which we do not allow repetitions. For each $i \le n$, choose some $a_i \in D_X$ such that $\|a_i - u_i\|_X < \varepsilon$. Let $(a'_i)_i \subseteq D_Y$ be the elements such that for all $i \le n$ we have $a_i\mathcal{R}_0 a'_i$. Since by the assumption we have $|\|u_i - a_i\|_X - \|u'_i - a'_i\|_Y| < 2\varepsilon$, we get $\|u'_i - a'_i\|_Y < 3\varepsilon$. Notice that for such sequences we get, by the computations above,

$$\left|\Big\|\sum_{i\le n} a_i\Big\|_X - \Big\|\sum_{i\le n} a'_i\Big\|_Y\right| \le 4\varepsilon n.$$

This, together with the inequalities $\|u_i - a_i\|_X < \varepsilon$ and $\|u'_i - a'_i\|_Y < 3\varepsilon$ implies that

$$\left| \Big\| \sum_{i \leq n} u_i \Big\|_X - \Big\| \sum_{i \leq n} u'_i \Big\|_Y \right| \leq 8\varepsilon n,$$

which proves the inequality (11).

Before we are in the position to apply Lemma 16 we need to guarantee that $\mathcal{R}_0$ is $\mathbb{Q}$-homogeneous. Note that so far it is only closed under multiplication by positive rationals. This will be fixed in the last step.

Set $\bar{\mathcal{R}} = \mathcal{R}_0 \cup -\mathcal{R}_0$, where $-\mathcal{R}_0 = \{(x,y) : \ (-x)\mathcal{R}_0(-y)\}$. Now $\bar{\mathcal{R}}$ is clearly $\mathbb{Q}$-homogeneous. Pick now an arbitrary sequence $(u_i)_{i \leq n} \subseteq E$ and the sequence $(u'_i)_i \subseteq F$ such that for all $i \leq n$ we have $u_i \bar{\mathcal{R}} u'_i$. For each $i \leq n$, pick $u''_i \in F$ such that $u_i \mathcal{R}_0 u''_i$. Either $u''_i = u'_i$, or by (10) we get $\|u''_i - u'_i\|_Y \leq 8\varepsilon \|u_i\|_X$. From these inequalities and from (11), which gives us

$$\left| \Big\| \sum_{i \leq n} u_i \Big\|_X - \Big\| \sum_{i \leq n} u''_i \Big\|_Y \right| \leq 8\varepsilon \Big( \sum_{i \leq n} \|u_i\|_X \Big),$$

we get the estimate

$$\left| \Big\| \sum_{i \leq n} u_i \Big\|_X - \Big\| \sum_{i \leq n} u'_i \Big\|_Y \right| \leq 16\varepsilon \Big( \sum_{i \leq n} \|u_i\|_X \Big).$$

The application of Lemma 16 then gives us that $\rho_K(X,Y) < 17\varepsilon$.

It remains to see that it is possible to find an injective and Borel map $f : \mathcal{B} \to \mathcal{M}$ such that $f(X)$ is isometric to $(M_X, d_X)$ for every $X \in \mathcal{B}$. Each Banach space $X$ is coded as a norm $\|\cdot\|_X \in \mathcal{B}$ which is defined on a countable infinite-dimensional vector space over $\mathbb{Q}$ denoted by $V$. First we need to select in a Borel way a countable dense symmetric subset of $S_X$. Pick $D \subseteq V \setminus \{0\}$ such that $D$ contains exactly one element of $\{tv : \ t > 0\}$ for every $v \in V \setminus \{0\}$. Fix some bijection $g : D \to \mathbb{N}$ and define a metric $d'_X$ on $\mathbb{N}$ as follows: for $n, m \in \mathbb{N}$ we set

$$d'_X(n,m) = \left\| \frac{g^{-1}(n)}{\|g^{-1}(n)\|_X} - \frac{g^{-1}(m)}{\|g^{-1}(m)\|_X} \right\|_X.$$

This corresponds to selecting a countable dense symmetric subset of $S_X$ with the metric inherited from $\|\cdot\|_X$. Clearly, the assignment $\|\cdot\|_X \to d'_X$ is injective and Borel. Then we only add to $\mathbb{N}$ a fixed countable set $\{p_{F,k} : \ F \in [\mathbb{N}]^{<\omega} \setminus \{\emptyset\}, k \in F\}$ and define the metric $d_X$ on the union of these two countable sets using the norm $\|\cdot\|_X$. Finally, we reenumerate this countable set so that $d_X$ is defined on $\mathbb{N}$, and so belongs to $\mathcal{M}$. That is clearly one-to-one and Borel. $\square$

### 3.3. **Reductions from pseudometrics on $\mathcal{M}$ to pseudometrics on $\mathcal{B}$.**

This subsection is devoted to the proof of the following result.

**Theorem 63.** *There is an injective Borel-uniformly continuous reduction from $\rho_{GH}$ on $\mathcal{M}^1_{1/2}$ to each of the distances $\rho_K, \rho_{BM}, \rho_L, \rho_U, \rho_N, \rho^{\mathcal{B}}_{GH}$ on $\mathcal{B}$.*

*Moreover, for the distances $\rho_K$ and $\rho_{BM}$, the reduction is not only Borel-uniformly continuous, but also Borel-Lipschitz.*

The definition of our reduction is based on a simple geometric idea of renorming of the Hilbert space $\ell_2$. However, the proof that the idea works is technical and splits into many steps.

Let us denote by $e_n$ the sequence in $\ell_2$ that has 1 at the $n$-th place and 0 elsewhere. Let us moreover denote

$$e_{n,m} = \frac{1}{\sqrt{2}}(e_n + e_m), \quad \{n, m\} \in [\mathbb{N}]^2.$$

The following fact can be verified by a simple computation.

**Lemma 64.** *For $\{n, m\}, \{n', m'\} \in [\mathbb{N}]^2$, we have*

$$\begin{aligned} \|e_{n,m} - e_{n',m'}\|_{\ell_2} &= 1 && \text{if } |\{n, m\} \cap \{n', m'\}| = 1, \\ \|e_{n,m} + e_{n',m'}\|_{\ell_2} &= \sqrt{3} && \text{if } |\{n, m\} \cap \{n', m'\}| = 1, \\ \|e_{n,m} \pm e_{n',m'}\|_{\ell_2} &= \sqrt{2} && \text{if } \{n, m\}, \{n', m'\} \text{ are disjoint.} \end{aligned}$$

*Hence, the set of all vectors $\pm e_{n,m}$ is 1-separated.*

Let us fix numbers $\alpha$ and $\delta$ such that

$$1 < \alpha < \alpha + \delta \leq \frac{200}{199}.$$

For every $f : [\mathbb{N}]^2 \to [0, 1]$, we define an equivalent norm $\|\cdot\|_f$ on $\ell_2$ by

$$\|x\|_f = \sup\Big(\{\|x\|_{\ell_2}\} \cup \Big\{\frac{1}{\sqrt{2}} \cdot \big(\alpha + \delta \cdot f(n, m)\big) \cdot |x_n + x_m| : n \neq m\Big\}\Big)$$

for $x = (x_n)_{n=1}^{\infty} \in \ell_2$. This is an equivalent norm indeed, as $\frac{1}{\sqrt{2}} \cdot |x_n + x_m| = |\langle x, e_{n,m}\rangle| \leq \|x\|_{\ell_2}$, and consequently

$$\|x\|_{\ell_2} \leq \|x\|_f \leq \frac{200}{199}\|x\|_{\ell_2}, \quad x \in \ell_2.$$

Let us define

$$P_{n,m} = \Big\{x \in \ell_2 : \|x\|_{\ell_2} \leq \frac{1}{\sqrt{2}} \cdot \big(\alpha + \delta \cdot f(n, m)\big) \cdot (x_n + x_m)\Big\}.$$

It follows from the following lemma that any non-zero $x \in \ell_2$ belongs to at most one set $\pm P_{n,m}$.

**Lemma 65.** *Let us denote $h = \alpha + \delta \cdot f(n, m)$. If $\|x\|_{\ell_2} = 1$, then*

$$x \in P_{n,m} \quad \Leftrightarrow \quad \|x - e_{n,m}\|_{\ell_2} \leq \sqrt{\frac{2(h-1)}{h}}.$$

*In particular,*

$$x \in P_{n,m} \quad \Rightarrow \quad \|x - e_{n,m}\|_{\ell_2} \leq \frac{1}{10}.$$

*Proof.* We compute

$$\begin{aligned} \|x - e_{n,m}\|_{\ell_2} \leq \sqrt{\frac{2(h-1)}{h}} &\Leftrightarrow \|x\|_{\ell_2}^2 - 2\langle x, e_{n,m}\rangle + \|e_{n,m}\|_{\ell_2}^2 \leq \frac{2(h-1)}{h} \\ &\Leftrightarrow 2\langle x, e_{n,m}\rangle \geq \frac{2}{h} \\ &\Leftrightarrow \frac{1}{\sqrt{2}} \cdot h \cdot (x_n + x_m) \geq 1 \\ &\Leftrightarrow x \in P_{n,m}. \end{aligned}$$

Finally, since $h \leq \alpha + \delta \leq \frac{200}{199}$, we have $\sqrt{\frac{2(h-1)}{h}} \leq \frac{1}{10}$. $\square$

Our proof of Theorem 63 is based on the following technical lemma.

**Lemma 66.** *Let $f, g : [\mathbb{N}]^2 \to [0,1]$. If $\rho_K((\ell_2, \|\cdot\|_f), (\ell_2, \|\cdot\|_g)) < \eta$ for some $\eta$ satisfying $0 < \eta < \frac{1}{100}, \eta < \frac{1}{10} \cdot \frac{\sqrt{\alpha^2-1}}{\alpha}$ and $\eta \leq \frac{1}{2}(1 - \frac{1}{\alpha})$, then*

$$\exists \pi \in S_\infty \forall \{n,m\} \in [\mathbb{N}]^2 : \left|g(\pi(n), \pi(m)) - f(n,m)\right| < \frac{3}{\delta} \cdot \eta.$$

Due to the assumption of the lemma, we can pick a Banach space $Z$ and isometries $I : (\ell_2, \|\cdot\|_f) \to Z$ and $J : (\ell_2, \|\cdot\|_g) \to Z$ such that

$$\rho_H^Z\big(I(B_{(\ell_2, \|\cdot\|_f)}), J(B_{(\ell_2, \|\cdot\|_g)})\big) < \eta.$$

We need to prove the following claim first.

**Claim 67.** *For every $\{n,m\} \in [\mathbb{N}]^2$, there are $\{k,l\} \in [\mathbb{N}]^2$ and $s \in \{-1,1\}$ such that*

$$\left\|Ie_{n,m} - s \cdot Je_{k,l}\right\|_Z < \frac{1}{7}$$

*and, moreover,*

$$f(n,m) - g(k,l) < \frac{3}{\delta} \cdot \eta.$$

*Proof.* We prove the claim in eight steps. Let us fix $\{n,m\} \in [\mathbb{N}]^2$ and keep the notation $h = \alpha + \delta \cdot f(n,m)$ throughout the proof. Analogously as above, we define

$$Q_{k,l} = \Big\{x \in \ell_2 : \|x\|_{\ell_2} \leq \frac{1}{\sqrt{2}} \cdot \big(\alpha + \delta \cdot g(k,l)\big) \cdot (x_k + x_l)\Big\}.$$

1. step: We show that there is $x$ orthogonal to $e_{n,m}$ such that $\|x\|_{\ell_2} = 1$ and $\|Ix - \pm Je_{k,l}\|_Z \geq \frac{1}{2}$ for all $k \neq l$. This is an easy consequence of the fact that the vectors $\pm Je_{k,l}$ are 1-separated (which follows from Lemma 64 and from $\|\cdot\|_g \geq \|\cdot\|_{\ell_2}$). Indeed, let $E \subseteq \ell_2$ be a two-dimensional subspace orthogonal to $e_{n,m}$. Let us pick $x \in S_E$. If $\|Ix - \pm Je_{k,l}\|_Z \geq \frac{1}{2}$ for all $k,l$, we are done. In the opposite case, there is a point $w = Je_{k,l}$ or $w = -Je_{k,l}$ for which $\|Ix - w\|_Z < \frac{1}{2}$. Since $I(S_E)$ is a closed curve in $Z$ with diameter at least 2, we can find $x' \in S_E$ such that $\|Ix' - w\|_Z = \frac{1}{2}$. Then $x'$ works, as the distance of $Ix'$ to other vectors is at least $1 - \frac{1}{2}$ by the triangle inequality.

2. step: We denote

$$p_+ = \frac{1}{h}e_{n,m} + \frac{\sqrt{h^2-1}}{h}x, \quad p_- = \frac{1}{h}e_{n,m} - \frac{\sqrt{h^2-1}}{h}x.$$

It is easy to see that

$$\|p_+\|_f = \|p_-\|_f = 1.$$

Let us choose $q_+$ and $q_-$ with $\|q_+\|_g \leq 1, \|q_-\|_g \leq 1$, satisfying

$$\|Ip_+ - Jq_+\|_Z < \eta, \quad \|Ip_- - Jq_-\|_Z < \eta.$$

3. step: We show that

$$\frac{\|q_+ + q_-\|_g}{2} > 1 - \eta.$$

Since $\|(Ip_+ + Ip_-)/2 - (Jq_+ + Jq_-)/2\|_Z < \eta$, we have

$$\frac{\|q_+ + q_-\|_g}{2} = \Big\|\frac{1}{2}(Jq_+ + Jq_-)\Big\|_Z > \Big\|\frac{1}{2}(Ip_+ + Ip_-)\Big\|_Z - \eta = \Big\|\frac{1}{h}e_{n,m}\Big\|_f - \eta.$$

As $\|e_{n,m}\|_f = h$, the desired inequality follows.

4. step: We show that

$$\|q_+ - q_-\|_g = \|q_+ - q_-\|_{\ell_2},$$

as $q_+ - q_-$ does not belong to any $\pm Q_{k,l}$. Let us denote

$$u = \frac{h}{2\sqrt{h^2-1}} \cdot (q_+ - q_-) \quad \text{and} \quad z = \frac{1}{\|u\|_{\ell_2}} \cdot u.$$

Then

$$\begin{aligned}\|Ju - Ix\|_Z &= \frac{h}{2\sqrt{h^2-1}} \cdot \big\|(Jq_+ - Jq_-) - (Ip_+ - Ip_-)\big\|_Z \\ &< \frac{h}{2\sqrt{h^2-1}} \cdot 2\eta = \frac{h}{\sqrt{h^2-1}} \cdot \eta \le \frac{\alpha}{\sqrt{\alpha^2-1}} \cdot \eta < \frac{1}{10}\end{aligned}$$

by an assumption of Lemma 66. By the choice of $x$ (1. step), we obtain for all $k \neq l$ that

$$\|u - \pm e_{k,l}\|_g = \|Ju - \pm Je_{k,l}\|_Z > \|Ix - \pm Je_{k,l}\|_Z - \frac{1}{10} \ge \frac{1}{2} - \frac{1}{10} = \frac{2}{5}.$$

Also, it is easy to check that

$$\big|\|u\|_{\ell_2} - 1\big| < \frac{1}{199} + \frac{1}{10}.$$

Indeed, we have $\|u\|_{\ell_2} \le \|u\|_g < \|x\|_f + \frac{1}{10} \le \frac{200}{199} \cdot \|x\|_{\ell_2} + \frac{1}{10} = 1 + \frac{1}{199} + \frac{1}{10}$ and $\|u\|_{\ell_2} \ge \frac{199}{200} \cdot \|u\|_g > \frac{199}{200} \cdot (\|x\|_f - \frac{1}{10}) \ge \frac{199}{200} \cdot (\|x\|_{\ell_2} - \frac{1}{10}) = \frac{199}{200} \cdot (1 - \frac{1}{10}) > 1 - (\frac{1}{199} + \frac{1}{10})$.

Since $z = u - (\|u\|_{\ell_2} - 1)z$ and $\|z\|_{\ell_2} = 1$, we obtain for all $k \neq l$ that

$$\|z - \pm e_{k,l}\|_{\ell_2} \ge \|u - \pm e_{k,l}\|_{\ell_2} - \big|\|u\|_{\ell_2} - 1\big|\|z\|_{\ell_2} > \frac{199}{200} \cdot \frac{2}{5} - \Big(\frac{1}{199} + \frac{1}{10}\Big) > \frac{1}{10}.$$

By Lemma 65, $z$ does not belong to $\pm Q_{k,l}$. The same holds for $q_+ - q_-$, as it is a multiple of $z$.

5. step: We show that

$$\frac{\|q_+ + q_-\|_{\ell_2}}{2} < \frac{1}{h} + \eta.$$

By the parallelogram law,

$$\|q_+ + q_-\|_{\ell_2}^2 + \|q_+ - q_-\|_{\ell_2}^2 = 2\|q_+\|_{\ell_2}^2 + 2\|q_-\|_{\ell_2}^2 \le 2\|q_+\|_g^2 + 2\|q_-\|_g^2 \le 4.$$

Using the conclusion of the previous step,

$$\|q_+ - q_-\|_{\ell_2} = \|q_+ - q_-\|_g > \|p_+ - p_-\|_f - 2\eta \ge \|p_+ - p_-\|_{\ell_2} - 2\eta = 2 \cdot \frac{\sqrt{h^2-1}}{h} - 2\eta,$$

$$\|q_+ - q_-\|_{\ell_2}^2 > 4 \cdot \frac{h^2-1}{h^2} - 8 \cdot \frac{\sqrt{h^2-1}}{h} \cdot \eta + 4\eta^2 > 4 \cdot \frac{h^2-1}{h^2} - 8 \cdot \frac{1}{h} \cdot \eta$$

and

$$\|q_+ + q_-\|_{\ell_2}^2 < 4 - 4 \cdot \frac{h^2-1}{h^2} + 8 \cdot \frac{1}{h} \cdot \eta < \frac{4}{h^2} + 8 \cdot \frac{1}{h} \cdot \eta + 4\eta^2 = 4 \cdot \Big(\frac{1}{h} + \eta\Big)^2.$$

The desired inequality follows.

6. step: We realize that $q_+ + q_-$ belongs to some $Q_{k,l}$ or $-Q_{k,l}$. In the opposite case, we obtain $\|q_+ + q_-\|_g = \|q_+ + q_-\|_{\ell_2}$ and

$$1 - \eta < \frac{\|q_+ + q_-\|_g}{2} = \frac{\|q_+ + q_-\|_{\ell_2}}{2} < \frac{1}{h} + \eta \leq \frac{1}{\alpha} + \eta,$$

which is disabled by an assumption of Lemma 66.

7. step: Let $q_+ + q_-$ belong to $s \cdot Q_{k,l}$, where $s \in \{-1, 1\}$. We show that

$$\big\|Ie_{n,m} - s \cdot Je_{k,l}\big\|_Z < \frac{1}{7}$$

for such $k, l$ and $s$. If we denote

$$a = \frac{1}{\|q_+ + q_-\|_{\ell_2}} \cdot (q_+ + q_-),$$

then $a$ belongs to $s \cdot Q_{k,l}$ as well. Lemma 65 provides

$$\|a - s \cdot e_{k,l}\|_{\ell_2} \leq \frac{1}{10},$$

and so

$$\|Ja - s \cdot Je_{k,l}\|_Z = \|a - s \cdot e_{k,l}\|_g \leq \frac{200}{199} \cdot \frac{1}{10}.$$

Since $\|(Ip_+ + Ip_-)/2 - (Jq_+ + Jq_-)/2\|_Z < \eta < \frac{1}{100}$, we obtain

$$\Big\|\frac{1}{h} Ie_{n,m} - \frac{\|q_+ + q_-\|_{\ell_2}}{2} Ja\Big\|_Z < \frac{1}{100}$$

and

$$\|Ie_{n,m} - Ja\|_Z < \frac{1}{100} + \Big(1 - \frac{1}{h}\Big)\|Ie_{n,m}\|_Z + \Big(1 - \frac{\|q_+ + q_-\|_{\ell_2}}{2}\Big)\|Ja\|_Z.$$

Since $\|e_{n,m}\|_f = h$, $\|a\|_{\ell_2} = 1$ and

$$\frac{\|q_+ + q_-\|_{\ell_2}}{2} \geq \frac{199}{200} \cdot \frac{\|q_+ + q_-\|_g}{2} > \frac{199}{200} \cdot (1 - \eta) > \frac{199}{200} \cdot \frac{99}{100},$$

we obtain

$$\|Ie_{n,m} - Ja\|_Z < \frac{1}{100} + (h - 1) + \Big(1 - \frac{199}{200} \cdot \frac{99}{100}\Big) \cdot \frac{200}{199}$$

and

$$\begin{aligned} \|Ie_{n,m} - s \cdot Je_{k,l}\|_Z &\leq \|Ie_{n,m} - Ja\|_Z + \|Ja - s \cdot Je_{k,l}\|_Z \\ &< \frac{1}{100} + \frac{1}{199} + \Big(1 - \frac{199}{200} \cdot \frac{99}{100}\Big) \cdot \frac{200}{199} + \frac{200}{199} \cdot \frac{1}{10} \\ &< \frac{1}{7}. \end{aligned}$$

8. step: Let $q_+ + q_-$ belong to $Q_{k,l}$ or $-Q_{k,l}$. We show that

$$f(n, m) - g(k, l) < \frac{3}{\delta} \cdot \eta$$

for such $k$ and $l$. If we denote $h' = \alpha + \delta \cdot g(k, l)$, then the elements of $\pm Q_{k,l}$ fulfill

$$\|w\|_g = \frac{1}{\sqrt{2}} \cdot h' \cdot |w_k + w_l| = h' \cdot |\langle w, e_{k,l}\rangle| \leq h' \cdot \|w\|_{\ell_2}, \quad w \in \pm Q_{k,l}.$$

We obtain

$$1 - \eta < \frac{\|q_+ + q_-\|_g}{2} \le h' \cdot \frac{\|q_+ + q_-\|_{\ell_2}}{2} < h' \cdot \Big(\frac{1}{h} + \eta\Big),$$

and so

$$h \cdot (1 - \eta) < h' \cdot (1 + h\eta).$$

It follows that

$$\delta \cdot \big(f(n,m) - g(k,l)\big) = h - h' < h\eta \cdot (1 + h') \le \frac{200}{199} \cdot \Big(1 + \frac{200}{199}\Big) \cdot \eta < 3\eta,$$

which provides the desired inequality. □

*Proof of Lemma 66.* For every $\{n,m\} \in [\mathbb{N}]^2$, Claim 67 provides $\Sigma(n,m) \in [\mathbb{N}]^2$ and $s(n,m) \in \{-1,1\}$ such that

$$\big\|Ie_{n,m} - s(n,m) \cdot Je_{\Sigma(n,m)}\big\|_Z < \frac{1}{7}$$

and, moreover,

$$f(n,m) - g(\Sigma(n,m)) < \frac{3}{\delta} \cdot \eta.$$

Let us make a series of observations concerning $\Sigma(n,m)$ and $s(n,m)$.

(a) If $\{n,m\}$ and $\{n',m'\}$ have exactly one common element, then the same holds for $\Sigma(n,m)$ and $\Sigma(n',m')$. Indeed, using Lemma 64, we can compute

$$\begin{aligned}
&\big\|s(n,m) \cdot e_{\Sigma(n,m)} - s(n',m') \cdot e_{\Sigma(n',m')}\big\|_{\ell_2} \\
&\qquad \le \big\|s(n,m) \cdot e_{\Sigma(n,m)} - s(n',m') \cdot e_{\Sigma(n',m')}\big\|_g \\
&\qquad < \|e_{n,m} - e_{n',m'}\|_f + \frac{1}{7} + \frac{1}{7} \le \frac{200}{199} \cdot 1 + \frac{1}{7} + \frac{1}{7} < \sqrt{2}, \\
&\big\|s(n,m) \cdot e_{\Sigma(n,m)} - s(n',m') \cdot e_{\Sigma(n',m')}\big\|_{\ell_2} \\
&\qquad \ge \frac{199}{200} \cdot \big\|s(n,m) \cdot e_{\Sigma(n,m)} - s(n',m') \cdot e_{\Sigma(n',m')}\big\|_g \\
&\qquad > \frac{199}{200} \cdot \Big(\|e_{n,m} - e_{n',m'}\|_f - \frac{1}{7} - \frac{1}{7}\Big) \ge \frac{199}{200} \cdot \Big(1 - \frac{1}{7} - \frac{1}{7}\Big) > 0,
\end{aligned}$$

and it is sufficient to apply Lemma 64 again (in fact, we obtain also $s(n,m) = s(n',m')$).

(b) If $\{n,m\}$ and $\{n',m'\}$ are disjoint, then the same holds for $\Sigma(n,m)$ and $\Sigma(n',m')$. This can be shown by the same method as above. This time, we have $\|e_{n,m} - e_{n',m'}\|_{\ell_2} = \sqrt{2}$ and

$$\big\|s(n,m) \cdot e_{\Sigma(n,m)} - s(n',m') \cdot e_{\Sigma(n',m')}\big\|_{\ell_2} < \frac{200}{199} \cdot \sqrt{2} + \frac{1}{7} + \frac{1}{7} < \sqrt{3},$$

$$\big\|s(n,m) \cdot e_{\Sigma(n,m)} - s(n',m') \cdot e_{\Sigma(n',m')}\big\|_{\ell_2} > \frac{199}{200} \cdot \Big(\sqrt{2} - \frac{1}{7} - \frac{1}{7}\Big) > 1.$$

(c) For every $n \in \mathbb{N}$, there is $\pi(n) \in \mathbb{N}$ such that $\pi(n) \in \Sigma(n,m)$ for all $m \neq n$. Assume the opposite and pick distinct $p, q$ different from $n$. By (a), we can denote the elements of $\Sigma(n,p)$ and $\Sigma(n,q)$ by $a, b, c$ in the way that

$$\Sigma(n,p) = \{a,b\}, \quad \Sigma(n,q) = \{a,c\}.$$

By our assumption, there is $m \neq n$ such that $a$ does not belong to $\Sigma(n,m)$. Then the only possibility for $\Sigma(n,m)$ allowed by (a) is

$$\Sigma(n,m) = \{b,c\}.$$

Pick some $r$ different from $n,m,p,q$. Then there is no possibility for $\Sigma(n,r)$ allowed by (a). Indeed, no set has exactly one common element with all sets $\{a,b\}, \{a,c\}$ and $\{b,c\}$.

(d) The function $\pi$ is injective. Indeed, assume that $n \neq m$ and pick distinct $p,q$ different from $n$ and $m$. Then $\pi(n)$ and $\pi(m)$ belong to the sets $\Sigma(n,p)$ and $\Sigma(m,q)$ that are disjoint by (b).

(e) As $\pi$ is injective, we have $\Sigma(n,m) = \{\pi(n), \pi(m)\}$ for all $\{n,m\}$, and we can write

$$\big\| Ie_{n,m} - s(n,m) \cdot Je_{\pi(n),\pi(m)} \big\|_Z < \frac{1}{7}$$

and

$$f(n,m) - g(\pi(n),\pi(m)) < \frac{3}{\delta} \cdot \eta.$$

(f) Due to the symmetry, there is an injective function $\xi : \mathbb{N} \to \mathbb{N}$ with the property that

$$\big\| Je_{k,l} - s'(k,l) \cdot Ie_{\xi(k),\xi(l)} \big\|_Z < \frac{1}{7}$$

and

$$g(k,l) - f(\xi(k),\xi(l)) < \frac{3}{\delta} \cdot \eta$$

for all $\{k,l\}$ and for a suitable $s'(k,l) \in \{-1,1\}$.

(g) We have $\pi(\xi(k)) = k$ for every $k$ and, consequently, $\pi$ is surjective. Let $k \in \mathbb{N}$ be given. For every $l \neq k$, we obtain

$$\begin{aligned}
&\big\| e_{k,l} - s'(k,l)s(\xi(k),\xi(l)) \cdot e_{\pi(\xi(k)),\pi(\xi(l))} \big\|_g \\
&\qquad \le \big\| Je_{k,l} - s'(k,l) \cdot Ie_{\xi(k),\xi(l)} \big\|_Z \\
&\qquad\quad + |s'(k,l)| \cdot \big\| Ie_{\xi(k),\xi(l)} - s(\xi(k),\xi(l)) \cdot Je_{\pi(\xi(k)),\pi(\xi(l))} \big\|_Z \\
&\qquad < \frac{1}{7} + \frac{1}{7}.
\end{aligned}$$

Due to Lemma 64, this is possible only if $\{k,l\} = \{\pi(\xi(k)), \pi(\xi(l))\}$ and $s'(k,l)s(\xi(k),\xi(l)) = 1$. If we pick distinct $l_1$ and $l_2$ different from $k$, then $k \in \{\pi(\xi(k)), \pi(\xi(l_1))\} \cap \{\pi(\xi(k)), \pi(\xi(l_2))\} = \{\pi(\xi(k))\}$.

(h) We check that $\pi$ works. We already know that $\pi \in S_\infty$ and $\pi^{-1} = \xi$. Thus, we obtain

$$g(\pi(n),\pi(m)) - f(n,m) = g(\pi(n),\pi(m)) - f(\xi(\pi(n)),\xi(\pi(m))) < \frac{3}{\delta} \cdot \eta$$

for every $\{n,m\} \in [\mathbb{N}]^2$. Finally, combining this with an above inequality,

$$\big| g(\pi(n),\pi(m)) - f(n,m) \big| < \frac{3}{\delta} \cdot \eta,$$

which completes the proof of the lemma. $\square$

*Proof of Theorem 63.* During the proof, we make no difference between a metric $f \in \mathcal{M}^1_{1/2}$ and the corresponding function $f : [\mathbb{N}]^2 \to [1/2,1]$. For $f \in \mathcal{M}^1_{1/2}$, we can thus consider the norm $\|\cdot\|_f$ defined above. It is clear that there is an injective Borel mapping from $\mathcal{M}^1_{1/2}$ into $\mathcal{B}$ such that the image of $f$ is isometric to $(\ell_2, \|\cdot\|_f)$ (it is sufficient to restrict the norm $\|\cdot\|_f$ to $V$).

To prove the first part of the theorem, we show a series of inequalities that illustrates that the Gromov-Hausdorff distance of $M_f$ and $M_g$ and all the involved distances between $(\ell_2, \|\cdot\|_f)$ and $(\ell_2, \|\cdot\|_g)$ are uniformly equivalent.

(1) We show that

$$\rho_{BM}\big((\ell_2, \|\cdot\|_f), (\ell_2, \|\cdot\|_g)\big) \le C\rho_{GH}(M_f, M_g)$$

for every $f, g \in \mathcal{M}^1_{1/2}$. If $\rho_{GH}(M_f, M_g) \ge \frac{1}{4}$, then

$$\rho_{BM}\big((\ell_2, \|\cdot\|_f), (\ell_2, \|\cdot\|_g)\big) \le 2\log\left(\frac{200}{199}\right) \le 2\log\left(\frac{200}{199}\right) \cdot 4\rho_{GH}(M_f, M_g).$$

Assuming $\rho_{GH}(M_f, M_g) < \frac{1}{4}$, we pick $r$ with $\rho_{GH}(M_f, M_g) < r < \frac{1}{4}$. Since $f, g \in \mathcal{M}_p$ and $\rho_{GH}(M_f, M_g) < p/2$ for $p = 1/2$, Lemma 12 provides $\pi \in S_\infty$ such that

$$\big|g(\pi(n), \pi(m)) - f(n, m)\big| \le 2r, \quad \{n, m\} \in [\mathbb{N}]^2.$$

Let us consider the isometry $T : \ell_2 \to \ell_2$ which maps $e_n$ to $e_{\pi(n)}$. For $x \in \ell_2$, we have

$$\begin{aligned}\|Tx\|_g &= \sup\Big(\{\|Tx\|_{\ell_2}\} \cup \Big\{\frac{1}{\sqrt{2}} \cdot \big(\alpha + \delta \cdot g(k, l)\big) \cdot |(Tx)_k + (Tx)_l| : k \ne l\Big\}\Big)\\ &= \sup\Big(\{\|x\|_{\ell_2}\} \cup \Big\{\frac{1}{\sqrt{2}} \cdot \big(\alpha + \delta \cdot g(\pi(n), \pi(m))\big) \cdot |x_n + x_m| : n \ne m\Big\}\Big),\end{aligned}$$

and so

$$\begin{aligned}|\|Tx\|_g - \|x\|_f| &\le \sup\Big\{\frac{1}{\sqrt{2}} \cdot \delta \cdot \big|g(\pi(n), \pi(m)) - f(n, m)\big| \cdot |x_n + x_m| : n \ne m\Big\}\\ &\le \delta \cdot 2r \cdot \|x\|_{\ell_2}.\end{aligned}$$

It follows that $\|Tx\|_g \le (1 + 2\delta r)\|x\|_f$ and $\|x\|_f \le (1 + 2\delta r)\|Tx\|_g$. We obtain $\rho_{BM}((\ell_2, \|\cdot\|_f), (\ell_2, \|\cdot\|_g)) \le 2\log(1 + 2\delta r) \le 2 \cdot 2\delta r$. As $r$ could be chosen arbitrarily close to $\rho_{GH}(M_f, M_g)$, we arrive at

$$\rho_{BM}\big((\ell_2, \|\cdot\|_f), (\ell_2, \|\cdot\|_g)\big) \le 4\delta\rho_{GH}(M_f, M_g).$$

Therefore, the choice $C = \max\{8\log(\frac{200}{199}), 4\delta\}$ works.

(2) It is easy to check that

$$\begin{aligned}2\rho_N\big((\ell_2, \|\cdot\|_f), (\ell_2, \|\cdot\|_g)\big) &\le \rho_U\big((\ell_2, \|\cdot\|_f), (\ell_2, \|\cdot\|_g)\big)\\ &\le 2\rho_L\big((\ell_2, \|\cdot\|_f), (\ell_2, \|\cdot\|_g)\big) \le \rho_{BM}\big((\ell_2, \|\cdot\|_f), (\ell_2, \|\cdot\|_g)\big)\end{aligned}$$

for every $f, g \in \mathcal{M}^1_{1/2}$.

(3) By [20, Proposition 2.1], we have

$$\rho^{\mathcal{B}}_{GH}\big((\ell_2, \|\cdot\|_f), (\ell_2, \|\cdot\|_g)\big) \le e^{2\rho_N((\ell_2, \|\cdot\|_f), (\ell_2, \|\cdot\|_g))} - 1$$

for every $f, g \in \mathcal{M}^1_{1/2}$.

(4) There is a function $\varphi : (0, 1] \to (0, 1]$ with $\lim_{\varepsilon\to 0}\varphi(\varepsilon) = 0$ such that

$$\rho_K\big((\ell_2, \|\cdot\|_f), (\ell_2, \|\cdot\|_g)\big) \le C\varphi\Big(\rho^{\mathcal{B}}_{GH}\big((\ell_2, \|\cdot\|_f), (\ell_2, \|\cdot\|_g)\big)\Big)$$

for every $f, g \in \mathcal{M}^1_{1/2}$. Considering any $0 < r < 1$, a function provided by [35, Theorem 3.6] (denoted $f$ there) works. Indeed, if we adopt some notation from [35], then [35, Theorem 3.7] provides

$$\begin{aligned}\rho_K\big((\ell_2, \|\cdot\|_f), (\ell_2, \|\cdot\|_g)\big) &\leq C(r) \cdot \kappa_0\big((\ell_2, \|\cdot\|_f)\big) \cdot d_r\big((\ell_2, \|\cdot\|_f), (\ell_2, \|\cdot\|_g)\big)\\ &\leq C(r) \cdot \frac{200}{199} \cdot \kappa_0(\ell_2) \cdot \varphi\Big(\rho^{\mathcal{B}}_{GH}\big((\ell_2, \|\cdot\|_f), (\ell_2, \|\cdot\|_g)\big)\Big).\end{aligned}$$

(5) We show that

$$\rho_{GH}(M_f, M_g) \leq C\rho_K\big((\ell_2, \|\cdot\|_f), (\ell_2, \|\cdot\|_g)\big)$$

for every $f, g \in \mathcal{M}^1_{1/2}$. Let us denote

$$d = \rho_K\big((\ell_2, \|\cdot\|_f), (\ell_2, \|\cdot\|_g)\big), \quad \eta_{max} = \min\Big\{\frac{1}{100}, \frac{1}{10} \cdot \frac{\sqrt{\alpha^2 - 1}}{\alpha}, \frac{1}{2}\Big(1 - \frac{1}{\alpha}\Big)\Big\}.$$

If $d \geq \eta_{max}$, then

$$\rho_{GH}(M_f, M_g) \leq 1 \leq \frac{1}{\eta_{max}} \cdot d.$$

Assuming $d < \eta_{max}$, we pick $d < \eta < \eta_{max}$. Then Lemma 66 can be applied, and we obtain $f \simeq_{2\varepsilon} g$ for $\varepsilon = \frac{1}{2} \cdot \frac{3}{\delta} \cdot \eta$. By Lemma 11, we get $\rho_{GH}(M_f, M_g) \leq \frac{1}{2} \cdot \frac{3}{\delta} \cdot \eta$. As $\eta$ can be chosen arbitrarily close to $d$, we arrive at

$$\rho_{GH}(M_f, M_g) \leq \frac{1}{2} \cdot \frac{3}{\delta} \cdot d.$$

It follows that the choice $C = \max\{\frac{1}{\eta_{max}}, \frac{3}{2\delta}\}$ works.

Finally, concerning the moreover part of the theorem, it remains to notice that

$$\rho_K\big((\ell_2, \|\cdot\|_f), (\ell_2, \|\cdot\|_g)\big) \leq \rho_{BM}\big((\ell_2, \|\cdot\|_f), (\ell_2, \|\cdot\|_g)\big)$$

by [47, Proposition 6.2] (or [20, Proposition 2.1]), and it is sufficient to use the inequalities proven in (1) and (5). □

## 4. Boreless of equivalence classes

Infinitary logic is an extension of the standard first order logic to the logic where one allows the formulas to have infinitely many quantifiers or infinite conjunctions/disjunctions. The number of conjunctions, resp. quantifiers is typically bounded by infinite cardinal numbers $\lambda$, resp. $\kappa$, and the symbol $L_{\lambda,\kappa}$ is then used to denote such logic. We refer an interested reader to [43] for more information. The most relevant for us is the $L_{\omega_1,\omega}$ logic, where one allows countably many conjunctions and disjunctions. The expressive power of such logic is obviously stronger than that of the standard logic, hence it allows to 'see' properties not recognized by the standard logic. Indeed, it is an application of the Scott analysis, a technique from the infinitary model theory (see e.g. [43] or [23] for a reference on this subject), to approximate the analytic equivalence relation of isomorphism $\approx$ on a Polish space $\mathcal{C}$ of countable structures of certain type by an $\omega_1$-chain of Borel equivalence relations $(\approx_\alpha)_{\alpha<\omega_1}$. For each structure $M \in \mathcal{C}$ Scott analysis defines a certain countable ordinal $\alpha_M$, called the Scott rank of $M$ such that for any $N \in \mathcal{C}$, $M \approx N$ if and only if $M \approx_{\alpha_M+\omega} N$.

The infinitary logic was generalized to the setting of infinitary continuous logic by Ben Yaacov and Iovino in [10], where the authors apply it in the Banach space theory. Furthermore, a generalization of the classical Scott analysis for the infinitary continuous logic was provided in [9], where it was shown that analogously to the approximation of the isomorphism relation for countable structures, the Gromov-Hausdorff and Kadets analytic pseudometrics on metric, resp. Banach spaces can be approximated by an $\omega_1$-chain of Borel pseudometrics. That is, if we consider e.g. the Kadets distance $\rho_K$, then there exists a chain $\rho_1 \leq \rho_2 \leq \ldots \leq \rho_\omega \leq \rho_{\omega+1} \leq \ldots$ of Borel pseudometrics on $\mathcal{B}$ such that for every Banach space $X \in \mathcal{B}$ there is a countable ordinal $\alpha_X$ such that for every $Y \in \mathcal{B}$, $\rho_K(X, Y) = 0$ if and only if $\rho_{\alpha_X}(X, Y) = 0$. The consequence of this is the following result.

**Theorem 68** (see Corollary 8.3 and Theorem 8.9 in [9])**.** *The equivalence classes of $E_{\rho_{GH}}$ and $E_{\rho_K}$ are Borel.*

The aim of this section is to develop a complementary approach to that one from [9] using games, which also provides alternative means how to prove Theorem 68. For the reader who is familiar with techniques of model theory we mention that the games we use here can be viewed as generalizations of the Ehrenfeucht-Fraïssé game to the metric setting. Recall that if $\mathcal{C}$ is some class of countable structures, then for $M, N \in \mathcal{C}$ we have $M \cong N$ if and only if Player II has a winning strategy in an appropriate version of Ehrenfeucht-Fraïssé game for $M$ and $N$ of length $\omega$. Analogously, for each countable ordinal $\alpha$ one can define a variant of the Ehrenfeucht-Fraïssé game played with $M$ and $N$ with parameter $\alpha$ such that Player II has a winning strategy there if and only if $M \equiv_\alpha N$, where $\equiv_\alpha$ is a certain Borel approximation of $\cong$.

Let $\mathcal{X}$ be a standard Borel space and $X$ a countable set. A typical example of $\mathcal{X}$ is a class of separable metric structures (e.g. Polish metric spaces or separable Banach spaces) that can be described as completion of some countable metric structures with a countable set $X$ as an underlying set (more precisely, we assume in this case that $\mathcal{X}$ can be identified with some Borel subset of $\mathbb{R}^{X \times X}$).

Denote by $\mathcal{C}$ the Polish space of all correspondences $\mathcal{R} \subseteq X \times X$, which is a $G_\delta$ subset of the Polish space $\mathcal{P}(X \times X)$. Let $f : \mathcal{C} \times \mathcal{X} \times \mathcal{X} \to [0, \infty]$ be a Borel function.

Suppose that the function $\rho : \mathcal{X} \times \mathcal{X} \to [0, \infty]$, defined as

$$\rho(d, p) = \inf_{\mathcal{R}} f(\mathcal{R}, d, p)$$

is a pseudometric on $\mathcal{X}$. For every two finite sets $E, F \subseteq X$ and $\mathcal{R} \subseteq X \times X$ let $\mathcal{R}^{E,F} = \{(x, y) : x\mathcal{R}y, x \in E, y \in F\}$. By $\mathcal{C}^{E,F}$ we shall denote the finite set $\{\mathcal{R}^{E,F} : \mathcal{R} \in \mathcal{C}\}$. Suppose that for every pair of finite sets $E, F \subseteq X$ there are functions $f^{E,F} : \mathcal{C}^{E,F} \times \mathcal{X} \times \mathcal{X} \to [0, \infty]$ satisfying the following five conditions.

(1) *Monotonicity with respect to inclusion*, that is,

$$f^{E,F}(\mathcal{R}^{E,F}, d, p) \leq f^{E',F'}((\mathcal{R}')^{E',F'}, d, p)$$

whenever finite sets $E, E', F, F'$ are such that $E \subseteq E'$ and $F \subseteq F'$, and $\mathcal{R} \subseteq \mathcal{R}'$.

(2) *Continuity in upward unions*, that is

$$f(\mathcal{R},d,p)=\sup_{E,F} f^{E,F}(\mathcal{R}^{E,F},d,p)=\lim_{E,F} f^{E,F}(\mathcal{R}^{E,F},d,p),$$

where the limit is taken over pairs of finite sets that increase in inclusion and eventually cover $X$.

(3) *Borelness*, that is, for every $E,F$ finite subsets of $X$, every $\mathcal{R}\in\mathcal{C}^{E,F}$ and every $d\in\mathcal{X}$, the mapping $\mathcal{X}\ni e\mapsto f^{E,F}(\mathcal{R},d,e)$ is Borel.

(4) *Symmetry*, that is $f^{E,F}(\mathcal{R}^{E,F},d,p)=f^{F,E}((\mathcal{R}^{E,F})^{-1},p,d)$ for all finite subsets $E,F\subseteq X$, every $\mathcal{R}\in\mathcal{C}$ and $d,p\in\mathcal{X}$.

(5) *Transitivity*, that is,

$$f^{E,G}((\mathcal{R}')^{F,G}\circ\mathcal{R}^{E,F},d,e)\le f^{E,F}(\mathcal{R}^{E,F},d,p)+f^{F,G}((\mathcal{R}')^{F,G},p,e)$$

for all finite subsets $E,F,G\subseteq X$, $\mathcal{R},\mathcal{R}'\in\mathcal{C}$ and $d,p,e\in\mathcal{X}$.

**Examples**

**1. Gromov-Hausdorff distance.** For a correspondence $\mathcal{R}\subseteq\mathbb{N}\times\mathbb{N}$ and two metrics $d,p\in\mathcal{M}$ we set

$$f(\mathcal{R},d,p)=\sup\big\{|d(n,m)-p(n',m')|/2:\ n\mathcal{R}n',m\mathcal{R}m'\big\}.$$

For finite sets $E,F\subseteq\mathbb{N}$, the function $f^{E,F}$ is defined analogously. Using Fact 9 it is easy to check that the functions $f,f^{E,F}$ have the desired properties and $\rho_{GH}$ is defined using them.

**2. Kadets distance.** For a correspondence $\mathcal{R}\subseteq V\times V$ and two norms $\|\cdot\|_X,\|\cdot\|_Y\in\mathcal{B}$, we define

$$f(\mathcal{R},\|\cdot\|_X,\|\cdot\|_Y)=\sup\Bigg\{\frac{1}{n}\bigg|\Big\|\sum_{i\le n}x_i\Big\|_X-\Big\|\sum_{i\le n}y_i\Big\|_Y\bigg|:\ \forall i\le n:\ \big(x_i\mathcal{R}y_i\text{ or }x_i(-\mathcal{R})y_i\big),\ \|x_i\|_X\le 1,\ \|y_i\|_Y\le 1\Bigg\}$$

if $\|x\|_X\le 1\Leftrightarrow\|y\|_Y\le 1$ for all $x,y$ with $x\mathcal{R}y$; otherwise we set

$$f(\mathcal{R},\|\cdot\|_X,\|\cdot\|_Y)=\infty.$$

Functions $f^{E,F}$ are defined analogously. Let us show that $\rho=\rho_K$.

Assume that $\rho_K(\|\cdot\|_X,\|\cdot\|_Y)<\varepsilon$, and let $\delta>0$ and a correspondence $\mathcal{R}\subseteq V\times V$ be provided by Lemma 16. We define

$$\mathcal{R}'=\big\{(x,y):x\mathcal{R}y,\|x\|_X\le 1,\ \|y\|_Y\le 1\big\}\cup\big\{(x,y):\|x\|_X>1,\ \|y\|_Y>1\big\}.$$

This is still a correspondence, and it is easy to check that $f(\mathcal{R}',\|\cdot\|_X,\|\cdot\|_Y)\le\varepsilon-\delta$. It follows that $\rho(\|\cdot\|_X,\|\cdot\|_Y)<\varepsilon$.

Assume that $\rho(\|\cdot\|_X,\|\cdot\|_Y)<\varepsilon$, and let $\mathcal{R}\subseteq V\times V$ be a correspondence with $f(\mathcal{R},\|\cdot\|_X,\|\cdot\|_Y)\le\varepsilon-2\delta$ for some $\delta>0$. Let us consider

$$\mathcal{R}'=\big\{(qx,qy):q\in\mathbb{Q},x\mathcal{R}y,\|x\|_X\le 1,\|y\|_Y\le 1,\max\{\|x\|_X,\|y\|_Y\}\ge\tfrac{\varepsilon-2\delta}{\varepsilon-\delta}\big\}.$$

If $\kappa>0$ is arbitrarily small, then every $u\in V$ admits some $v\in V$ such that $u\mathcal{R}'v$ and $\|v\|_Y\le(1+\kappa)\|u\|_X$. Indeed, assuming without loss of generality that $u\neq 0$ and choosing $s\in\mathbb{Q}_+$ with $\max\{\frac{1}{1+\kappa},\frac{\varepsilon-2\delta}{\varepsilon-\delta}\}\le\|su\|_X\le 1$, we can find $v \in V$ such that $(su)\mathcal{R}(sv)$. As $f(\mathcal{R}, \|\cdot\|_X, \|\cdot\|_Y) < \infty$, we have $\|sv\|_Y \leq 1$, and so $u\mathcal{R}'v$. Moreover, $\|v\|_Y \leq 1/s \leq (1+\kappa)\|u\|_X$.

Analogously, every $v \in V$ admits some $u \in V$ such that $u\mathcal{R}'v$ and $\|u\|_X \leq (1+\kappa)\|v\|_Y$. This is a relaxed version of an assumption of Lemma 16, nevertheless, it is not difficult to check that the proof of the lemma still works. For this reason, to prove that $\rho_K(\|\cdot\|_X, \|\cdot\|_Y) < \varepsilon$, it remains to check that

$$\Big|\Big\|\sum_{i\leq n} u_i\Big\|_X - \Big\|\sum_{i\leq n} v_i\Big\|_Y\Big| \leq (\varepsilon-\delta)\Big(\sum_{i\leq n}\max\{\|u_i\|_X, \|v_i\|_Y\}\Big)$$

for all $(u_i)_i \subseteq V$ and $(v_i)_i \subseteq V$, where for all $i$, $u_i\mathcal{R}'v_i$.

So, let $u_i$ and $v_i$ with $u_i\mathcal{R}'v_i$ be given for $1 \leq i \leq n$. There are $q_i, x_i, y_i$ such that

$$(u_i, v_i) = (q_i x_i, q_i y_i), q_i \in \mathbb{Q}, x_i\mathcal{R}y_i, \|x_i\|_X \leq 1, \|y_i\|_Y \leq 1, \max\{\|x_i\|_X, \|y_i\|_Y\} \geq \tfrac{\varepsilon-2\delta}{\varepsilon-\delta}$$

for $1 \leq i \leq n$. Notice first that in the definition of $f(\mathcal{R}, \|\cdot\|_X, \|\cdot\|_Y)$, we can repeat each $(x_i, y_i)$ arbitrarily many times, and we can also switch its sign. Hence, for any $k_i \in \mathbb{Z}, 1 \leq i \leq n$, we have

$$\Big|\Big\|\sum_{i\leq n} k_i x_i\Big\|_X - \Big\|\sum_{i\leq n} k_i y_i\Big\|_Y\Big| \leq (\varepsilon-2\delta)\sum_{i\leq n}|k_i|.$$

Due to the homogeneity of this inequality, integers $k_i$ can be replaced by rationals. It follows that

$$\begin{aligned}\Big|\Big\|\sum_{i\leq n} u_i\Big\|_X - \Big\|\sum_{i\leq n} v_i\Big\|_Y\Big| &\leq (\varepsilon-2\delta)\sum_{i\leq n}|q_i| \\ &\leq (\varepsilon-2\delta)\sum_{i\leq n}|q_i|\cdot\tfrac{\varepsilon-\delta}{\varepsilon-2\delta}\cdot\max\{\|x_i\|_X, \|y_i\|_Y\} \\ &= (\varepsilon-\delta)\sum_{i\leq n}\max\{\|u_i\|_X, \|v_i\|_Y\}.\end{aligned}$$

**3. Banach-Mazur distance.** For a correspondence $\mathcal{R} \subseteq V \times V$ and two norms $\|\cdot\|_X, \|\cdot\|_Y \in \mathcal{B}$ we define

$$f(\mathcal{R}, \|\cdot\|_X, \|\cdot\|_Y) = \infty$$

if $\mathcal{R}$ does not extend to a graph of a surjective linear isomorphism $T : X \to Y$; otherwise we set

$$f(\mathcal{R}, \|\cdot\|_X, \|\cdot\|_Y) = \log\|T\| + \log\|T^{-1}\|,$$

if $\mathcal{R}$ extends to a graph of such an operator $T$. Using Lemma 22, we check that indeed $\rho_{BM}(\|\cdot\|_X, \|\cdot\|_Y) = \inf\{f(\mathcal{R}, \|\cdot\|_X, \|\cdot\|_Y) : \mathcal{R}$ is a correspondence on $V \times V\}$. We define $f^{E,F}(\mathcal{R}^{E,F}, \|\cdot\|_X, \|\cdot\|_Y)$ to be the number $\log\|T\| + \log\|T^{-1}\|$ where $T : \operatorname{span}\operatorname{Dom}(\mathcal{R}^{E,F}) \to \operatorname{span}\operatorname{Rng}(\mathcal{R}^{E,F})$ is the unique linear operator whose graph extends the relation $\mathcal{R}^{E,F}$ provided it exists. Otherwise, we set $f^{E,F}(\mathcal{R}^{E,F}, \|\cdot\|_X, \|\cdot\|_Y) = \infty$. It is easy to check that functions $f^{E,F}$ have the desired properties.

*Remark* 69. Another example considered here may involve $X = \mathbb{N}$ and *a logic* action of $S_\infty$. That is, a canonical action of $S_\infty$ on a space of the form $\prod_{i\in I} 2^{\mathbb{N}^{n_i}}$. The interpretation is as follows. Let $L = \{R_i : i \in I\}$

be a countable relational language, where each $R_i$ is a relational symbol of arity $n_i \in \mathbb{N}$. Each element $x \in X_L = \prod_{i\in I} 2^{\mathbb{N}^{n_i}}$ corresponds to a relational structure $A_x$ with domain $\mathbb{N}$ such that the tuple $k_1, \ldots, k_{n_i} \in \mathbb{N}$ satisfies the relation $R_i^{A_x}$ if and only if $x(i)(k_1, \ldots, k_{n_i}) = 1$. We refer the reader to [23, Chapter 3.6] for details. Note there that logic actions form the canonical Borel $S_\infty$-spaces ([23, Theorem 3.6.1]).

Suppose moreover that the language $L$ is finite and that $X_L$ is a *topometric space* as defined by Ben Yaacov in [7]. That is, $X_L$ is equipped with a metric $d$ which refines the compact Polish topology of $X_L$ and which is lower semi-continuous with respect to this Polish topology. Suppose that the action of $S_\infty$ is by isometries. One may then define

$$f(\mathcal{R}, x, y) = \begin{cases} d(\pi \cdot x, y), & \mathcal{R} = \pi \in S_\infty, \\ \infty, & \mathcal{R} \notin S_\infty. \end{cases}$$

Then the corresponding function $\rho$ is nothing but the orbit pseudometric $\rho_{S_\infty, d}$, which is moreover CTR provided that $d$ is complete. The continuity of the action and especially the lower semi-continuity of $d$ allow to define $f^{E,F}(\mathcal{R}^{E,F}, x, y)$ naturally as

$$\inf_{\mathcal{R}' \in S_{E,F}} \inf_{x' \in E_x, y' \in F_y} f(\mathcal{R}', x', y'),$$

where $S_{E,F}$ is the open set $\{\mathcal{R}' \in \mathcal{C} : (\mathcal{R}')^{E,F} = \mathcal{R}^{E,F}\}$; $E_x$, resp. $F_y$ is the neighborhood of $x$, resp. of $y$ determined by $E$, $F$ respectively. These last two neighborhoods are open by our assumption that $L$ is finite. We do not know if these conditions suffice to prove that the functions $f^{E,F}$ are Borel, or further restrictions on $d$ are necessary.

Fix now $d, p \in \mathcal{X}$, two finite tuples $\bar{x}$ and $\bar{y}$ from $X$ of the same length (the first is supposed to be from $(X, d)$, the second from $(X, p)$), and some $\varepsilon > 0$. Let $\mathcal{G}(d\bar{x}, p\bar{y}, \varepsilon)$ denote a game of two players I and II. At the $i$-th step Player I chooses either a point $m_i \in X$ that is supposed to extend the tuple $\bar{x}m_1 \ldots m_{i-1}$, or a point $n_i \in X$, that is supposed to extend the tuple $\bar{y}n_1 \ldots n_{i-1}$. In the former case, Player II responds by playing an element $n_i \in X$ extending the tuple $\bar{y}n_1 \ldots n_{i-1}$, or in the latter case a point $m_i \in X$ extending the tuple $\bar{x}m_1 \ldots m_{i-1}$. The game has countably many steps in which the players produce infinite sequences $(m_i)_i$ and $(n_i)_i$ which define a relation $\mathcal{R}$ where $x_j \mathcal{R} y_j$, for $j \leq |\bar{x}|$, and $m_i \mathcal{R} n_i$, for $i \in \mathbb{N}$. Note that this is not in general a bijection as the players are allowed to repeat the elements. As $\mathcal{R} \subseteq X \times X$, one may see $\mathcal{R}$ as a correspondence between certain subsets $X_1, X_2 \subseteq X$. By $\mathcal{R}_i$, for $i \in \mathbb{N}$, we define the subset of $X \times X$ given by the tuples $\bar{x}m_1 \ldots m_i$ and $\bar{y}n_1 \ldots n_i$. Analogously, we use the notation $f^i$ for the function $f^{\bar{x}m_1\ldots m_i, \bar{y}n_1\ldots n_i}$. By $f^0$ we mean the function $f^{\bar{x},\bar{y}}$ and by $\mathcal{R}_0$ the subset of $X \times X$ given by the tuples $\bar{x}$ and $\bar{y}$.

At the end Player II wins if and only if $f^i(\mathcal{R}_i, d, p) < \varepsilon$, for all $i \in \mathbb{N}$.

If the tuples $\bar{x}$, $\bar{y}$ are empty, we denote the game just by $\mathcal{G}(d, p, \varepsilon)$.

*Remark* 70. It may well happen that $d = p$. Since we still need to distinguish between these two copies during the game (as the role of $d$ and $p$ is clearly not symmetric), we say that Player I in her first move either chooses a point $m_1 \in X$ which extends $\bar{x}$, or chooses a point $n_1 \in X$ which extends $\bar{y}$.

After $i-1$-many steps when the players have produced tuples $\bar{x}m_1 \dots m_{i-1}$, $\bar{y}n_1 \dots n_{i-1}$, Player I either chooses $m_i \in X$ extending $\bar{x}m_1 \dots m_{i-1}$, or chooses a point $n_i \in X$ extending $\bar{y}n_1 \dots n_{i-1}$.

**Lemma 71.** *For every $d, p \in \mathcal{X}$, we have $\rho(d,p) = 0$ if and only if for every $\varepsilon > 0$ Player II has a winning strategy in $\mathcal{G}(d,p,\varepsilon)$.*

*Proof.* Fix $d$ and $p$ from $\mathcal{X}$. Suppose that $\rho(d,p) = 0$. Fix some $\varepsilon > 0$. By the assumption there exists a correspondence $\mathcal{R} \subseteq X \times X$ such that $f(\mathcal{R}, d, p) < \varepsilon$. Now Player II can use $\mathcal{R}$ as his strategy. That is, if Player I plays some $m_i \in X$, then Player II responds by playing some $n_i \in X$ such that $m_i \mathcal{R} n_i$; or vice versa. It is clear that this is a winning strategy.

Conversely, suppose that Player II has a winning strategy in $\mathcal{G}(d,p,\varepsilon)$ for every $\varepsilon > 0$. Fix some $\varepsilon > 0$. Player I can play so that $\bigcup_i \{m_i\} = \bigcup_i \{n_i\} = X$. At the end the players produce a correspondence $\mathcal{R} \subseteq X \times X$ and by our assumptions we get

$$f(\mathcal{R}, d, p) = \sup_i f^i(\mathcal{R}_i, d, p) \leq \varepsilon,$$

thus $\rho(d,p) \leq f(\mathcal{R}, d, p) \leq \varepsilon$. $\square$

Let $\alpha < \omega_1$ be now a countable ordinal, $\varepsilon > 0$ and $\bar{x}$ and $\bar{y}$ tuples of the same length from $X$. By $\mathcal{G}(d\bar{x}, p\bar{y}, \varepsilon, \alpha)$ we denote a game which is similar in its rules to $\mathcal{G}(d\bar{x}, p\bar{y}, \varepsilon)$, however in the first step Player I moreover chooses an ordinal $\alpha_1 < \alpha$. In the $i$-th step, Player I chooses moreover an ordinal $\alpha_i < \alpha_{i-1} < \dots < \alpha_1 < \alpha$. The length of each play is finite, where the last step is when Player I chooses 0 as an ordinal. Analogously as above, Player II wins if $f^k(\mathcal{R}_k, d, p) < \varepsilon$, where $k \in \mathbb{N}$ is such that $\alpha_k = 0$, and $\mathcal{R}_k$ is the correspondence produced by the players at the end of the game. If $\alpha = 0$, then the game is decided in the very beginning and we set that Player II wins if $f^0(\mathcal{R}_0, d, p) < \varepsilon$.

Let $\varepsilon > 0$, $\alpha$ be a countable ordinal, and $\bar{x}$ and $\bar{y}$ be tuples of the same length. For every $(X, d) \in \mathcal{X}$, denote by $E(d, \bar{x}, \bar{y}, \varepsilon, \alpha)$ the set of all $p \in \mathcal{X}$ such that Player II has a winning strategy in the game $\mathcal{G}(d\bar{x}, p\bar{y}, \varepsilon, \alpha)$. Again, if the tuples are empty, we write just $E(d, \varepsilon, \alpha)$ instead of $E(d, \emptyset, \emptyset, \varepsilon, \alpha)$.

**Lemma 72.** *$E(d, \bar{x}, \bar{y}, \varepsilon, \alpha)$ is Borel.*

*Proof.* We shall prove it by induction on $\alpha$. Suppose that $\alpha = 0$. Then the game is decided from the beginning and Player II wins if $f^0(\mathcal{R}_0, d, p) < \varepsilon$, where $\mathcal{R}_0$ is the correspondence given by the tuples $\bar{x}$ and $\bar{y}$. That is by definition a Borel condition, so $E(d, \bar{x}, \bar{y}, \varepsilon, 0)$ is Borel.

Now suppose that $\alpha > 0$ and we have checked that $E(d, \bar{u}, \bar{v}, \varepsilon, \beta)$ is Borel for all tuples $\bar{u}$ and $\bar{v}$, and $\beta < \alpha$.

If $\alpha$ is limit, then $E(d, \bar{x}, \bar{y}, \varepsilon, \alpha)$ is just $\bigcap_{\beta<\alpha} E(d, \bar{x}, \bar{y}, \varepsilon, \beta)$ which is Borel by assumption. So suppose that $\alpha = \beta + 1$ for some $\beta$. Then by definition

$$\begin{aligned} E(d, \bar{x}, \bar{y}, \varepsilon, \alpha) = &\left( \bigcap_{m \in X} \bigcup_{n \in X} E(d, \bar{x}m, \bar{y}n, \varepsilon, \beta) \right) \cap \\ &\cap \left( \bigcap_{n \in X} \bigcup_{m \in X} E(d, \bar{x}m, \bar{y}n, \varepsilon, \beta) \right) \end{aligned}$$

which is Borel. □

**Lemma 73.** *Let $d, p, e \in \mathcal{X}$ and let $\bar{x}$, $\bar{y}$, and $\bar{z}$ be tuples from $X$ of the same length. Let $\alpha$ be a countable ordinal, $\varepsilon > 0$ and $\varepsilon' > 0$.*

*(1) If Player II has a winning strategy in $\mathcal{G}(d\bar{x}, p\bar{y}, \varepsilon, \alpha)$ and in $\mathcal{G}(p\bar{y}, e\bar{z}, \varepsilon')$ then he also has a winning strategy in $\mathcal{G}(d\bar{x}, e\bar{z}, \varepsilon + \varepsilon', \alpha)$.*

*(2) If Player II has a winning strategy in $\mathcal{G}(d\bar{x}, e\bar{z}, \varepsilon, \alpha)$ and in $\mathcal{G}(p\bar{y}, e\bar{z}, \varepsilon')$ then he also has a winning strategy in $\mathcal{G}(d\bar{x}, p\bar{y}, \varepsilon + \varepsilon', \alpha)$.*

*Proof.* We only prove (1); (2) is proved analogously.

We shall prove it by induction on $\alpha$. It is clear that the statement holds for $\alpha = 0$. Now suppose that $\alpha > 0$ and the statement of the lemma holds for every $\beta < \alpha$. In order to shorten the description of the proof, let us denote by $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{G}_3$ the games $\mathcal{G}(d\bar{x}, p\bar{y}, \varepsilon, \alpha)$, $\mathcal{G}(p\bar{y}, e\bar{z}, \varepsilon')$ and $\mathcal{G}(d\bar{x}, e\bar{z}, \varepsilon + \varepsilon', \alpha)$, respectively.

Suppose that Player I in $\mathcal{G}_3$ plays some ordinal $\alpha_1 < \alpha$ and a point $m_1 \in X$ extending $\bar{x}$. Then we play the same move in game $\mathcal{G}_1$, and let Player II use his winning strategy in this game and pick a point $n_1 \in X$ such that Player II has a winning strategy in $\mathcal{G}(d\bar{x}m_1, p\bar{y}n_1, \varepsilon, \alpha_1)$.

Further, let Player I play $n_1$ in the game $\mathcal{G}_2$, and let Player II use his winning strategy in this game and pick a point $w_1 \in X$ such that Player II has a winning strategy in $\mathcal{G}(p\bar{y}n_1, e\bar{z}w_1, \varepsilon')$. By the inductive assumption, Player II has a winning strategy in $\mathcal{G}(d\bar{x}m_1, e\bar{z}w_1, \varepsilon + \varepsilon', \alpha_1)$; hence we let Player II play $w_1$ as his response in the game $\mathcal{G}_3$. If Player I plays in her first move a point extending $\bar{z}$, then we proceed analogously. □

**Definition 74.** Fix $d \in \mathcal{X}$. For every $\varepsilon > 0$ and two tuples $\bar{x}$ and $\bar{y}$ from $X$ of the same length let $\alpha(\bar{x}, \bar{y}, \varepsilon)$ be the least ordinal $\alpha$ such that Player II does <u>not</u> have a winning strategy in the game $\mathcal{G}(d\bar{x}, d\bar{y}, \varepsilon, \alpha)$. If Player II has a winning strategy in $\mathcal{G}(d\bar{x}, d\bar{y}, \varepsilon, \alpha)$ for every $\alpha < \omega_1$, then we set $\alpha(\bar{x}, \bar{y}, \varepsilon) = -1$.

We define a *Scott rank* of $d$, $\alpha_d$ in symbols, to be
$\sup\{\alpha(\bar{x}, \bar{y}, \varepsilon) : (\bar{x}, \bar{y}) \text{ appropriate elements of the same length}, \varepsilon > 0\}$.

**Lemma 75.** *Let $\varepsilon > 0$ and $\bar{x}$, $\bar{y}$ be tuples of the same length from $X$.*

*If Player II has a winning strategy in $\mathcal{G}(d\bar{x}, d\bar{y}, \varepsilon, \alpha_d)$ then he also has a winning strategy in $\mathcal{G}(d\bar{x}, d\bar{y}, \varepsilon, \alpha_d + 1)$.*

*Proof.* If it were not true, then we would have $\alpha(\bar{x}, \bar{y}, \varepsilon) = \alpha_d + 1 > \alpha_d \geq \alpha(\bar{x}, \bar{y}, \varepsilon)$, which is a contradiction. □

Our aim is now to define a set of those $p \in \mathcal{X}$ such that $\rho(d, p) = 0$, for a fixed element $d \in \mathcal{X}$. In the following, for a subset $A \subseteq \mathcal{X}$ we denote by $A^c$ the complement $\mathcal{X} \setminus A$. Also, we agree that $X^0$ denotes an empty sequence. We set

$$I_d = \bigcap_{\varepsilon \in \mathbb{Q}^+} \Bigg( E(d, \varepsilon, \alpha_d) \cap \bigcap_{\substack{n \in \mathbb{N} \cup \{0\} \\ \bar{x}, \bar{y} \in X^n}} \Bigg( E^c(d, \bar{x}, \bar{y}, \varepsilon, \alpha_d) \cup \bigcap_{\varepsilon' \in \mathbb{Q}^+} E(d, \bar{x}, \bar{y}, \varepsilon + \varepsilon', \alpha_d + 1) \Bigg) \Bigg).$$

It follows from Lemma 72 that $I_d$ is a Borel set. To translate the definition above into words, it says that $p \in I_d$ if and only if for every $\varepsilon > 0$ we have that Player II has a winning strategy in the game $\mathcal{G}(d, p, \varepsilon, \alpha_d)$ and, for all tuples of the same length (possibly empty tuples) $\bar{x}$, $\bar{y}$, if Player II has a winning strategy in the game $\mathcal{G}(d\bar{x}, p\bar{y}, \varepsilon, \alpha_d)$ then for every $\varepsilon' > 0$ he has a winning strategy in the game $\mathcal{G}(d\bar{x}, p\bar{y}, \varepsilon + \varepsilon', \alpha_d + 1)$.

**Theorem 76.** *Let $d \in \mathcal{X}$ be arbitrary. For every $p \in \mathcal{X}$ we have that $\rho(d, p) = 0$ if and only if $p \in I_d$.*

*Proof.* Fix some $d \in \mathcal{X}$ and also pick some $p \in \mathcal{X}$. By Lemma 71 it suffices to check that $p \in I_d$ if and only if Player II has a winning strategy in $\mathcal{G}(d, p, \varepsilon)$ for every $\varepsilon > 0$.

We first show the left-to-right implication. So we fix some $\varepsilon > 0$. Since $p \in I_d$, by definition $p \in E(d, \varepsilon/2, \alpha_d)$. Then it follows, also from the definition of $I_d$, that in fact $p \in E(d, 3\varepsilon/4, \alpha_d + 1)$. We start playing the game $\mathcal{G}(d, p, \varepsilon)$, which in the sequel will be denoted just by $\mathcal{G}$, with players I and II, however on the side we will also play several auxiliary games. We let Player I play her turn in $\mathcal{G}$, which is, say, an element $m_1$. We consider an auxiliary game $\mathcal{G}(d, p, 3\varepsilon/4, \alpha_d + 1)$ and we force the first player there to copy her move from $\mathcal{G}$ and moreover play the ordinal $\alpha_d$. Since the second player has a winning strategy in $\mathcal{G}(d, p, 3\varepsilon/4, \alpha_d + 1)$, we let him use the strategy and then use his response, say a point $n_1$, as a move of Player II in the main game $\mathcal{G}$. Now by definition Player II has a winning strategy in the game $\mathcal{G}(dm_1, pn_1, 3\varepsilon/4, \alpha_d)$. However, by the definition of $I_d$ this immediately implies that he has a winning strategy also in $\mathcal{G}(dm_1, pn_1, 7\varepsilon/8, \alpha_d + 1)$. We again let Player I play her second turn in the main game $\mathcal{G}$, which is, say, a point $n_2$. We consider an auxiliary game $\mathcal{G}(dm_1, pn_1, 7\varepsilon/8, \alpha_d + 1)$ and we force the first player there to copy her move from $\mathcal{G}$, i.e. playing $n_2$, and moreover play the ordinal $\alpha_d$. Since the second player has a winning strategy in $\mathcal{G}(dm_1, pn_1, 7\varepsilon/8, \alpha_d + 1)$, we let him use his strategy, which is, say, a point $m_2$, and then we use this response in the main game $\mathcal{G}$. Now by definition Player II has a winning strategy in the game $\mathcal{G}(dm_1m_2, pn_1n_2, 7\varepsilon/8, \alpha_d)$. However, again by the definition of $I_d$ this implies that he actually has a winning strategy also in $\mathcal{G}(dm_1m_2, pn_1n_2, 15\varepsilon/16, \alpha_d + 1)$.

It is now clear that by using the winning strategies from these auxiliary games we get some strategy for Player II in the main game $\mathcal{G}$ which is winning.

We now prove the reverse implication. So we suppose that Player II has a winning strategy in the game $\mathcal{G}(d, p, \varepsilon)$ for every $\varepsilon > 0$. We show that $p \in I_d$. We need to show that for every $\varepsilon > 0$ and $\varepsilon' > 0$ we have that Player II has a winning strategy in the game $\mathcal{G}(d, p, \varepsilon, \alpha_d)$ and that for all tuples of elements of the same length $\bar{x}$, $\bar{y}$, if he has a winning strategy in $\mathcal{G}(d\bar{x}, p\bar{y}, \varepsilon, \alpha_d)$ then he has also a winning strategy in $\mathcal{G}(d\bar{x}, p\bar{y}, \varepsilon + \varepsilon', \alpha_d + 1)$.

The former is clear. Indeed, if Player II has a winning strategy in $\mathcal{G}(d, p, \varepsilon)$, which is our assumption, then he also has a winning strategy in $\mathcal{G}(d, p, \varepsilon, \alpha_d)$. So we must show the latter. Fix some tuples $\bar{x}, \bar{y}$ of the same length. We need to show that if Player II has a winning strategy in $\mathcal{G}(d\bar{x}, p\bar{y}, \varepsilon, \alpha_d)$, then for every $\varepsilon' > 0$ he has a winning strategy also in $\mathcal{G}(d\bar{x}, p\bar{y}, \varepsilon + \varepsilon', \alpha_d + 1)$.

Since $\rho(d,p)=0$, Player II has a winning strategy in the game $\mathcal{G}(p,d,\varepsilon'/2)$, so there exists a tuple $\bar{z}$ of elements of the same length as $\bar{y}$ such that Player II has a winning strategy in the game $\mathcal{G}(p\bar{y},d\bar{z},\varepsilon'/2)$.

Since, by Lemma 73, Player II has a winning strategy in $\mathcal{G}(d\bar{x},d\bar{z},\varepsilon+\varepsilon'/2,\alpha_d)$, by Lemma 75 we get that Player II has a winning strategy in $\mathcal{G}(d\bar{x},d\bar{z},\varepsilon+\varepsilon'/2,\alpha_d+1)$. By Lemma 73 again, Player II has a winning strategy in the game $\mathcal{G}(d\bar{x},p\bar{y},\varepsilon+\varepsilon',\alpha_d+1)$. □

The following corollary immediately follows from Theorem 76 and the fact that $I_d$ is a Borel subset of $\mathcal{X}$.

**Corollary 77.** *For every $d\in\mathcal{X}$ the set $\{p\in\mathcal{X}:\ \rho(d,p)=0\}$ is Borel.*

## 5. Distances are not Borel

Although the pseudometrics considered in this paper are analytic, it follows from the results of [9] and from the results of Sections 3 and 4 that the equivalence classes of all these pseudometrics, except the uniform distance for which we do not know the answer, are Borel. It is then of interest to investigate whether for any pseudometric $\rho$ from our list and an element $A$ from the domain of $\rho$ we have that the sets $\{B:\ \rho(A,B)\leq r\}$ are Borel, where $r>0$. This question was raised for the Gromov-Hausdorff and Kadets distances in [9, Question 8.4]. The goal of this section is to provide a negative answer and to show that whenever $\rho$ is a pseudometric to which the Kadets distance is reducible, then there are $\rho$-balls which are not Borel.

Let us introduce some notation and definitions first. By $\mathcal{P}(\mathbb{N})$ we denote the set of all subsets of $\mathbb{N}$ endowed with the coarsest topology for which $\{A\in\mathcal{P}(\mathbb{N}):n\in A\}$ is clopen for every $n$. Obviously, $\mathcal{P}(\mathbb{N})$ is nothing else than a copy of the Cantor space $2^{\mathbb{N}}$. Further, by $K(\mathcal{P}(\mathbb{N}))$ we mean the hyperspace of all compact subsets of $\mathcal{P}(\mathbb{N})$ endowed by the Vietoris topology.

For $E\subseteq\mathbb{N}$ and $x\in c_{00}$, we denote by $Ex$ the element of $c_{00}$ given by $Ex(n)=x(n)$ for $n\in E$ and $Ex(n)=0$ for $n\notin E$.

If $X$ and $Y$ are Banach spaces, then by $X\oplus_1 Y$ we mean the direct sum $X\oplus Y$ with the norm $\|(x,y)\|=\|x\|+\|y\|$. If $X_1,X_2,\dots$ is a sequence of Banach spaces, then its $\ell_1$-sum $(\bigoplus X_n)_{\ell_1}$ is defined as the space of all sequences $x=(x_1,x_2,\dots),x_k\in X_k$, such that $\|x\|:=\sum_{k=1}^\infty\|x_k\|<\infty$.

We say that a sequence $G_1,G_2,\dots$ of finite-dimensional Banach spaces is *dense* if for any finite-dimensional Banach space $G$ and any $\varepsilon>0$, there is $n\in\mathbb{N}$ such that $\dim G_n=\dim G$ and $\rho_{BM}(G_n,G)<\varepsilon$.

In the context of Banach spaces, by a basis we mean a Schauder basis. By a *basic sequence* we mean a basis of its closed linear span. A basis $\{x_i\}_{i=1}^\infty$ of a Banach space $X$ is said to be *shrinking* if

$$X^*=\overline{\operatorname{span}}\{x_1^*,x_2^*,\dots\},$$

where $x_1^*,x_2^*,\dots$ is the dual basic sequence $x_n^*:\sum_{i=1}^\infty a_ix_i\mapsto a_n$.

We say that a sequence $x_1,x_2,\dots$ of non-zero vectors in a Banach space $X$ is *$c$-equivalent to the standard basis of $\ell_1$* if $\|\sum_{k=1}^n\alpha_kx_k\|\geq\frac{1}{c}\sum_{k=1}^n\|\alpha_kx_k\|$ for all $n\in\mathbb{N}$ and $\alpha_1,\dots,\alpha_n\in\mathbb{R}$.

**Proposition 78.** *Let us consider the space*

$$X = \Big(\bigoplus G_n\Big)_{\ell_1},$$

*where $G_1, G_2, \dots$ is a dense sequence of finite-dimensional spaces. Then, for every $\varepsilon > 0$, there exists a Borel mapping $\mathfrak{S} : K(\mathcal{P}(\mathbb{N})) \to \mathcal{B}$ such that*

(a) *if $\mathcal{A} \in K(\mathcal{P}(\mathbb{N}))$ contains an infinite set, then $\rho_{BM}(\mathfrak{S}(\mathcal{A}), X) \le \varepsilon$, and thus $\rho_K(\mathfrak{S}(\mathcal{A}), X) \le \varepsilon$,*

(b) *if $\mathcal{A} \in K(\mathcal{P}(\mathbb{N}))$ consists of finite sets only, then $\mathfrak{S}(\mathcal{A})$ contains a normalized 1-separated shrinking basic sequence, and thus $\rho_K(\mathfrak{S}(\mathcal{A}), X) \ge 1/8$.*

*Proof.* To find an appropriate mapping $\mathfrak{S}$, we apply a construction provided in [4, §1(a)] and a simple idea from [40, Remark 3.10(vii)]. Let us recall first that, for $\mathcal{A} \in K(\mathcal{P}(\mathbb{N}))$ and $0 < \theta < 1$, a Tsirelson type space $T[\mathcal{A}, \theta]$ is defined as the completion of $c_{00}$ under the implicitly defined norm

$$\|x\|_{\mathcal{A},\theta} = \max\Big\{\|x\|_\infty, \theta \sup \sum_{k=1}^{n} \|E_k x\|_{\mathcal{A},\theta}\Big\},$$

where the "sup" is taken over all finite families $\{E_1, \dots, E_n\}$ of finite subsets of $\mathbb{N}$ such that

$$\exists A \in \mathcal{A}\; \exists m_1, \dots, m_n \in A : \; m_1 \le E_1 < m_2 \le E_2 < \dots < m_n \le E_n.$$

Given $\varepsilon > 0$, we put $\theta = e^{-\varepsilon}$ and

$$X_{\mathcal{A}} = T[\mathcal{A}_1, \theta] \oplus_1 X, \quad \mathcal{A} \in K(\mathcal{P}(\mathbb{N})),$$

where $\mathcal{A}_1 = \{A \cup \{1\} : A \in \mathcal{A}\}$. We check that $X_{\mathcal{A}}$ satisfies the requirements (a) and (b) on $\mathfrak{S}(\mathcal{A})$.

(a) We observe that there is a sequence of finite-dimensional spaces whose $\ell_1$-sum has the Banach-Mazur distance to $T[\mathcal{A}_1, \theta]$ at most $\varepsilon$. Since $\mathcal{A}$ contains an infinite set, we can put $m_1 = 1$ and find numbers $1 < m_2 < m_3 < \dots$ such that $\{m_1, m_2, m_3, \dots\} \in \mathcal{A}_1$. Considering $E_k = \{m_k, \dots, m_{k+1} - 1\}$ for every $k \in \mathbb{N}$, we obtain

$$e^{-\varepsilon} \sum_{k=1}^{\infty} \|E_k x\|_{\mathcal{A}_1,\theta} \le \|x\|_{\mathcal{A}_1,\theta} \le \sum_{k=1}^{\infty} \|E_k x\|_{\mathcal{A}_1,\theta}, \quad x \in c_{00},$$

(the first inequality holds due to the definition of $\|\cdot\|_{\mathcal{A}_1,\theta}$ and the choice $\theta = e^{-\varepsilon}$, the second one is just the triangle inequality). So, the sequence $\operatorname{span}\{e_n : n \in E_k\}, k = 1, 2, \dots$, works.

It follows that the same holds for $X_{\mathcal{A}}$ and that an appropriate sequence of finite-dimensional spaces can be chosen to be dense. Indeed, we can collect all spaces $\operatorname{span}\{e_n : n \in E_k\}$ with all $G_n$'s. Thus, to show that $\rho_{BM}(X, X_{\mathcal{A}}) \le \varepsilon$, it is sufficient to realize that $\rho_{BM}(X, Y) = 0$ for

$$Y = \Big(\bigoplus H_n\Big)_{\ell_1},$$

where $H_1, H_2, \dots$ is another dense sequence of finite-dimensional spaces.

Given $\delta > 0$, we obtain by a back-and-forth argument that there are a bijection $\pi : \mathbb{N} \to \mathbb{N}$ and linear surjective mappings $L_n : G_n \to H_{\pi(n)}$ such that $\|L_n\| \leq 1$ and $\|L_n^{-1}\| \leq e^\delta$. Then the operator

$$L : X \to Y, \quad \sum_{n=1}^{\infty} x_n \mapsto \sum_{n=1}^{\infty} L_n x_n, \quad (x_n \in G_n),$$

satisfies $\|L\| \leq 1$ and its inverse

$$L^{-1} : Y \to X, \quad \sum_{n=1}^{\infty} y_n \mapsto \sum_{n=1}^{\infty} L_n^{-1} y_n, \quad (y_n \in H_{\pi(n)}),$$

satisfies $\|L^{-1}\| \leq e^\delta$. Hence, $\rho_{BM}(X, Y) \leq \delta$. As $\delta > 0$ could be arbitrary, we arrive at $\rho_{BM}(X, Y) = 0$.

Finally, using [47, Proposition 6.2] (or [20, Proposition 2.1]), we arrive at $\rho_K(X_\mathcal{A}, X) \leq \rho_{BM}(X_\mathcal{A}, X) \leq \varepsilon$.

(b) Since $\mathcal{A}$ consists of finite sets, the canonical basis $e_n = \mathbf{1}_{\{n\}}$ of $c_{00}$ is a shrinking basis of $T[\mathcal{A}_1, \theta]$ (to show this, it is possible to adapt the part (a) of the proof of [4, Proposition 1.1] if we consider $\theta_k = \theta$ and $\mathcal{M}_k = \mathcal{A}_1$ for every $k$). As $T[\mathcal{A}_1, \theta] \subseteq X_\mathcal{A}$, we obtain that $e_1, e_2, \dots$ is a shrinking basic sequence as desired. In order to get a contradiction, let us assume that $\rho_K(X, Y) < 1/8$ where $Y = X_\mathcal{A}$.

Let us choose some $\eta$ with $\rho_K(X, Y) < \eta < 1/8$. Let $\iota_X$ and $\iota_Y$ be linear isometric embeddings of $X$ and $Y$ into a Banach space $Z$ such that $\rho_H^Z(\iota_X(B_X), \iota_Y(B_Y)) < \eta$. Let $x'_1, x'_2, \dots$ be points in $B_X$ such that

$$\|\iota_X(x'_n) - \iota_Y(e_n)\|_Z < \eta, \quad n \in \mathbb{N}.$$

It is straightforward to check that the sequence $x_n = x'_n / \|x'_n\|_X$ fulfills

$$\|\iota_X(x_n) - \iota_Y(e_n)\|_Z < 2\eta, \quad n \in \mathbb{N}.$$

This sequence is $(1 - 4\eta)$-separated, as $\|x_n - x_m\|_X > \|e_n - e_m\|_{\mathcal{A}_1, \theta} - 4\eta \geq 1 - 4\eta$ for $n \neq m$.

We employ the fact that the space $X$ has the 1-strong Schur property (by [26, p. 57], a space is said to have the 1-*strong Schur property* if, for any $\delta \in (0, 2]$, any $c > 2/\delta$ and any normalized $\delta$-separated sequence, there is a subsequence which is $c$-equivalent to the standard basis of $\ell_1$). This fact follows for instance from [34, Proposition 4.1] and the observation that the proof of [34, Theorem 1.3] works for $X$.

We obtain that $x_1, x_2, \dots$ has a subsequence $x_{n_k}$ which is 4-equivalent to the standard basis of $\ell_1$. It follows that

$$\Big\| \sum_{k=1}^{l} \lambda_k e_{n_k} \Big\|_{\mathcal{A}_1, \theta} \geq \Big\| \sum_{k=1}^{l} \lambda_k x_{n_k} \Big\|_X - \sum_{k=1}^{l} |\lambda_k| \cdot 2\eta \geq \Big(\frac{1}{4} - 2\eta\Big) \sum_{k=1}^{l} |\lambda_k|$$

for every $l \in \mathbb{N}$ and $\lambda_1, \dots, \lambda_l \in \mathbb{R}$. Therefore, $e_1, e_2, \dots$ has a subsequence equivalent to the standard basis of $\ell_1$. This is a contradiction, as $e_1, e_2, \dots$ is a shrinking basic sequence at the same time.

So, (a) and (b) are proven for $X_\mathcal{A}$, and it remains to show that there is a Borel mapping $\mathfrak{S} : K(\mathcal{P}(\mathbb{N})) \to \mathcal{B}$ such that $\mathfrak{S}(\mathcal{A})$ is isometric to $X_\mathcal{A}$ for every $\mathcal{A} \in K(\mathcal{P}(\mathbb{N}))$. Let $x_1, x_2, \dots$ be a sequence of linearly independent vectors in $X$ whose linear span is dense in $X$. For every $\mathcal{A} \in K(\mathcal{P}(\mathbb{N}))$, we define the norm on $V$ given by

$$\left\|(q_j)_{j=1}^{\infty}\right\| = \Big\|\sum_{k=1}^{\infty} q_{2k-1}e_k\Big\|_{\mathcal{A}_1,\theta} + \Big\|\sum_{k=1}^{\infty} q_{2k}x_k\Big\|_X.$$

In this way, the coded space is isometric to $X_{\mathcal{A}}$.

Thus, we need just to check that the defined mapping is Borel, i.e., that the function

$$\mathcal{A} \in K(\mathcal{P}(\mathbb{N})) \quad \mapsto \quad \Big\|\sum_{k=1}^{\infty} q_{2k-1}e_k\Big\|_{\mathcal{A}_1,\theta} + \Big\|\sum_{k=1}^{\infty} q_{2k}x_k\Big\|_X$$

is Borel for a fixed $(q_j)_{j=1}^{\infty} \in V$. If we pick $l \in \mathbb{N}$ such that $q_j = 0$ for every $j > l$, then it is not difficult to show that the value of the function depends only on $\{A \cap \{1, \ldots, l\} : A \in \mathcal{A}\}$ (an analogous statement in the dual setting was discussed in [40], see [40, Fact 3.4]). For this reason, $K(\mathcal{P}(\mathbb{N}))$ can be decomposed into finitely many clopen sets on which the function is constant. □

It is quite easy now to prove that the distances studied in the present work are not Borel. We use the following classical result that can be found e.g. in [37, (27.4)].

**Theorem 79** (Hurewicz)**.** *The set*

$$\mathfrak{H} = \Big\{\mathcal{A} \in K(\mathcal{P}(\mathbb{N})) : \mathcal{A} \text{ contains an infinite set}\Big\}$$

*is a complete analytic subset of $K(\mathcal{P}(\mathbb{N}))$. In particular, it is not Borel.*

**Theorem 80.** *Let $\rho$ be an analytic pseudometric on a standard Borel space $P$. If $\rho_K$ on $\mathcal{B}$ is Borel-uniformly continuous reducible to $\rho$ on $P$, then the pseudometric $\rho$ is not Borel. In fact, there is $x \in P$ such that the function $\rho(x, \cdot)$ is not Borel.*

*Proof.* Let $f : \mathcal{B} \to P$ be a Borel uniform embedding. We can find $\eta > 0$ and $\varepsilon > 0$ such that

$$\rho(f(Y), f(Z)) < \eta \Rightarrow \rho_K(Y, Z) < \frac{1}{8}$$

and

$$\rho_K(Y, Z) \leq \varepsilon \Rightarrow \rho(f(Y), f(Z)) < \eta$$

for all $Y, Z \in \mathcal{B}$. Let $X$ be as in Proposition 78 and let $\mathfrak{S}$ be a mapping provided for $\varepsilon$. We claim that the set

$$\Phi = \{p \in P : \rho(p, f(X)) < \eta\}$$

is not Borel, and thus that the function $\rho(f(X), \cdot)$ is not Borel.

Due to Theorem 79, it is sufficient to realize that $(f \circ \mathfrak{S})^{-1}(\Phi) = \mathfrak{H}$, i.e.,

$$\mathcal{A} \in \mathfrak{H} \quad \Leftrightarrow \quad f(\mathfrak{S}(\mathcal{A})) \in \Phi.$$

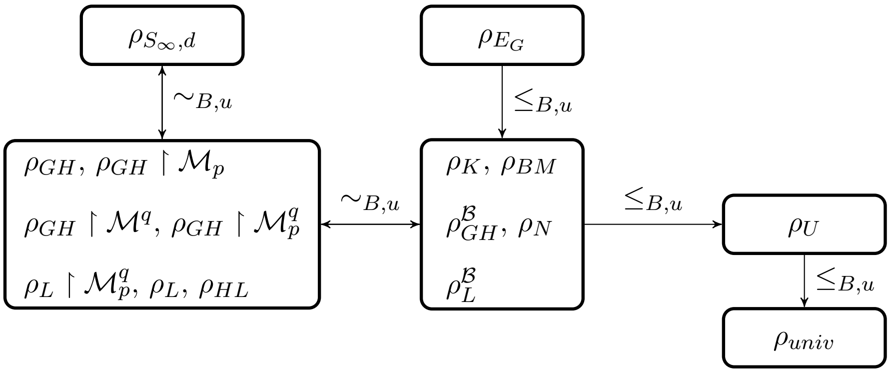
If $\mathcal{A} \in K(\mathcal{P}(\mathbb{N}))$ contains an infinite set, then $\rho_K(\mathfrak{S}(\mathcal{A}), X) \leq \varepsilon$, and so $\rho(f(\mathfrak{S}(\mathcal{A})), f(X)) < \eta$. If $\mathcal{A} \in K(\mathcal{P}(\mathbb{N}))$ consists of finite sets only, then $\rho_K(\mathfrak{S}(\mathcal{A}), X) \geq \frac{1}{8}$, and so $\rho(f(\mathfrak{S}(\mathcal{A})), f(X)) \geq \eta$. □

The following corollary then immediately follows from the results of Section 3. In particular, it answers in negative Question 8.4 from [9].

**Corollary 81.** *Let $\rho$ be any pseudometric from the following list: $\rho_{GH}$, $\rho_{GH} \upharpoonright \mathcal{M}_p$, $\rho_{GH} \upharpoonright \mathcal{M}_p^q$, $\rho_{GH}^{\mathcal{B}}$, $\rho_K$, $\rho_L$, $\rho_N$, $\rho_U$, $\rho_{BM}$. Then there exists $A$ from the domain of $\rho$ so that the function $\rho(A, \cdot)$ is not Borel.*

## 6. Concluding remarks and open problems

The following diagram summarizes the reducibility results we have proved in this paper (and includes also the reducibility $\rho_{E_G} \leq_{B,u} \rho_{GH}^{\mathcal{B}}$ discussed below). The reducibilities which are not explicitly mentioned in the diagram are not known to us. By $\rho_{E_G}$ we denote the pseudometric induced by the universal orbit equivalence relation $E_G$; by $\rho_{S_\infty,d}$ we denote the CTR orbit pseudometric given in Section 2.3, Example **2.**; by $\rho_{univ}$ we mean the universal analytic pseudometric which exists by Theorem 37; all the remaining pseudometrics are explained in Section 2.




We believe there is enough space for investigating other reductions. The interested reader can find many more distances for which their exact place in the reducibility diagram is not known. This includes the uniform distance $\rho_U$, or distances that we mentioned in Section 2 but left untouched, such as the completely bounded Banach-Mazur distance or e.g. the orbit version of the Kadison-Kastler distance.

In our paper, we have focused mainly on positive results, i.e. showing the reducibility between pseudometrics. Any rich theory should however contain also the 'negative results'. Within the standard theory of definable equivalence relations, it is often the negative results, results demonstrating that some equivalence relations are not reducible to some other ones, that form the most interesting and challenging part of the theory. Hjorth's theory of turbulence is one of the main examples of the latter, see [29]. For some pseudometrics it is clear that they do not reduce to each other for trivial reasons, e.g. the Gromov-Hausdorff distance to the cut distance on graphons defined in the section on orbit pseudometrics, as one is analytic non-Borel, while the other is Borel. Some more interesting non-reducibility results would be welcome.

**Problem 82.** *Find some 'natural pseudometric from functional analysis or metric geometry' that is not bi-reducible with the Gromov-Hausdorff distance.*

Note that, by our results, it would be sufficient to find such natural pseudometric which is not Borel-uniformly continuous reducible to a CTR orbit pseudometric.

It follows from the result of Zielinski in [53], that the homeomorphism relation on compact metrizable spaces is bi-reducible with the universal orbit equivalence relation, and from the results of Amir ([2]) and of Dutrieux and Kalton ([20]) that the equivalences $E_{\rho_{GH}}$ and $E_{\rho_{BM}}$ are above the universal orbit equivalence in the sense of Borel reducibility. More thorough discussion about this fact is in [9, Remark 8.5]. By our results from Section 3 this is also true for $E_\rho$, where $\rho$ is any pseudometric from the set $\{\rho_{GH}, \rho_{GH} \upharpoonright \mathcal{M}_p, \rho_{GH} \upharpoonright \mathcal{M}^q, \rho_{GH} \upharpoonright \mathcal{M}_p^q, \rho_{GH}^{\mathcal{B}}, \rho_K, \rho_{HL}, \rho_N, \rho_L, \rho_L \upharpoonright \mathcal{M}_p^q, \rho_U, \rho_{BM}\}$. By our further results, all these equivalences $E_\rho$ (except $E_{\rho_U}$ for which we do not know the answer) have Borel classes. This suggests the main open question, already stated in [9] for $\rho_{GH}$ and $\rho_K$.

**Question 83.** *Are the equivalence relations $E_\rho$, where $\rho$ is from the list above, Borel reducible to an orbit equivalence relation?*

If the answer is negative, these relations would form an interesting rather unexplored class of analytic equivalence relations with Borel classes. Moreover, it would shed light on some long standing conjectures about the Borel equivalence relation $E_1$, as this equivalence is not Borel reducible to $E_\rho$, where $\rho$ is as above. We refer the reader to Section 2.3, where we discuss these issues related to $E_1$.

Note however that even if the answer were positive, our particular reducibility results would still be of interest, as we work with a quantitative notion of reducibility (the Borel-uniformly continuous reducibility) that is stronger than the standard Borel reducibility.

Clemens, Gao and Kechris prove in [17] (see also [24]) that the isometry relation on the Effros-Borel space $F(\mathbb{U})$, where $\mathbb{U}$ is the Urysohn universal metric space, is Borel bi-reducible with the orbit equivalence relation induced by the canonical action of $\mathrm{Iso}(\mathbb{U})$ on $F(\mathbb{U})$. One may ask if there is a continuous version of this result.

**Question 84.** *Let $d$ be the Hausdorff distance on $F(\mathbb{U})$, $\mathrm{Iso}(\mathbb{U}) \curvearrowright F(\mathbb{U})$ the canonical action and $\rho$ the corresponding orbit pseudometric (see Section 2 for a definition). Is $\rho_{GH}$ Borel-uniformly continuous bi-reducible with $\rho$? Or does at least one of the inequalities $\rho \leq_{B,u} \rho_{GH}$, $\rho_{GH} \leq_{B,u} \rho$ hold?*

We have proved that there exists a universal analytic pseudometric $\rho$ with respect to Borel-uniformly continuous reducibility (even Borel-isometric reducibility) in Theorem 37. The corresponding equivalence relation is clearly the complete analytic equivalence relation. Several natural equivalence relations have been shown to be bireducible with the complete analytic one, e.g. the bi-Lipschitz homeomorphism of Polish metric spaces, or linear isomorphism of separable Banach spaces, see [22].

**Problem 85.** *Is there a natural pseudometric that is bi-reducible with the universal analytic pseudometric $\rho$?*

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