Turing instability and Turing patterns

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Outline

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- Well mixed chemical reactor
- Chemical reactor with free diffusion
- Turing instability and Turing patterns

Introduction

THE CHEMICAL BASIS OF MORPHOGENESIS

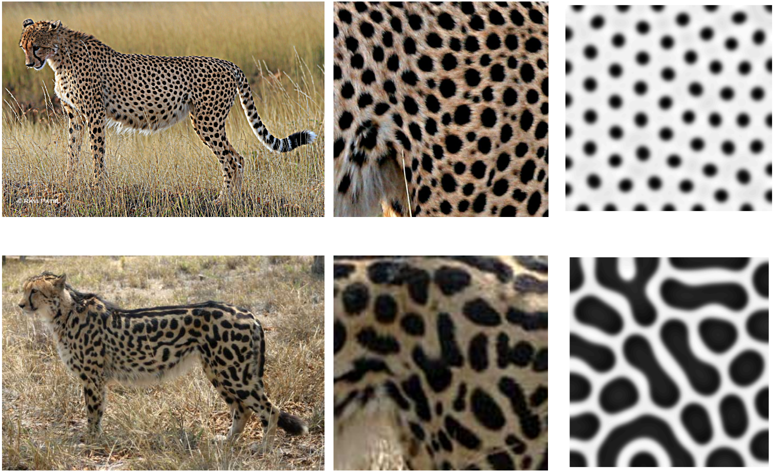
By A. M. TURING, F.R.S. *University of Manchester*

(*Received* 9 *November* 1951—*Revised* 15 *March* 1952)

It is suggested that a system of chemical substances, called morphogens, reacting together and diffusing through a tissue, is adequate to account for the main phenomena of morphogenesis. Such a system, although it may originally be quite homogeneous, may later develop a pattern or structure due to an instability of the homogeneous equilibrium, which is triggered off by random disturbances. Such reaction-diffusion systems are considered in some detail in the case of an isolated ring of cells, a mathematically convenient, though biologically unusual system. The investigation is chiefly concerned with the onset of instability. It is found that there are six essentially different forms which this may take. In the most interesting form stationary waves appear on the ring. It is suggested that this might account, for instance, for the tentacle patterns

Over 6 000 citations in WoS.

Turing instability – pattern formation



Chemical kinetics in well mixed reactor

Law of mass action: The rate of a chemical reaction is directly proportional to the product of concentrations of reactants.

Schnakenberg system [J. Schnakenberg 1979]

$$2U + V \xrightarrow{k_1} 3U \qquad \emptyset \overset{k_2,k_3}{\rightleftharpoons} U \qquad \emptyset \xrightarrow{k_4} V$$

Well mixed reactor

Concentrations $u = u(t)$, $v = v(t)$ of U, V satisfy

$$\frac{du}{dt} = k_1 u^2 v + k_2 - k_3 u$$
$$\frac{dv}{dt} = -k_1 u^2 v + k_4$$

Initial condition

$$u(0) = u_0 \quad \text{and} \quad v(0) = v_0$$

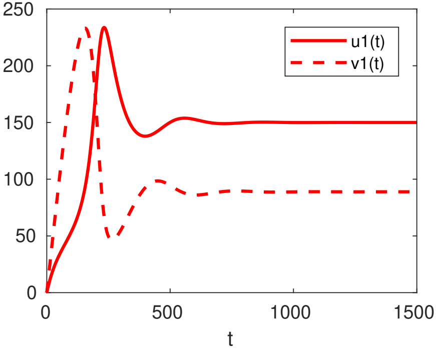
Example

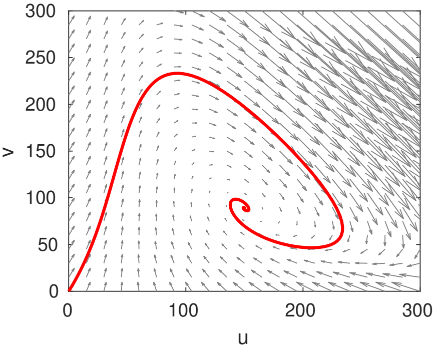
Schnakenberg system in a well mixed reactor

$k_1 = 10^{-6}, \quad k_2 = 1, \quad k_3 = 0.02, \quad k_4 = 2$

Initial condition 1: $u_0 = 0$

$v_0 = 0$





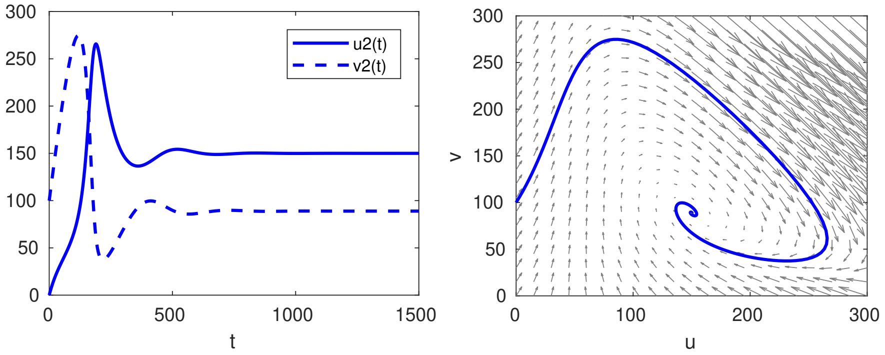
Example

Schnakenberg system in a well mixed reactor

$k_1 = 10^{-6}, \quad k_2 = 1, \quad k_3 = 0.02, \quad k_4 = 2$

Initial condition 2: $u_0 = 0$

$v_0 = 100$

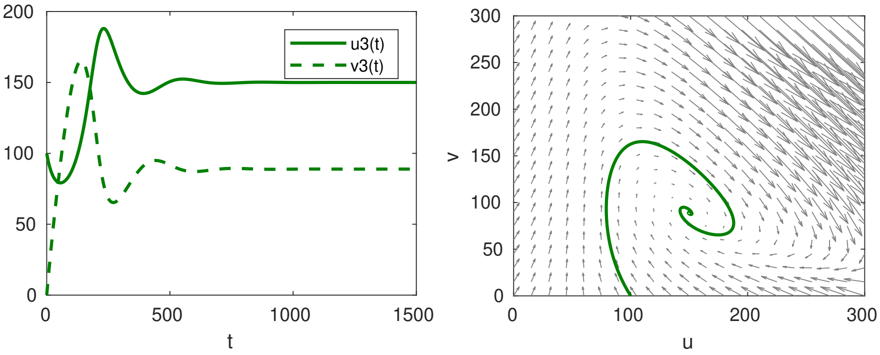


Example

Schnakenberg system in a well mixed reactor

$k_1 = 10^{-6}, \quad k_2 = 1, \quad k_3 = 0.02, \quad k_4 = 2$

Initial condition 3: $u_0 = 100$
$v_0 = 0$

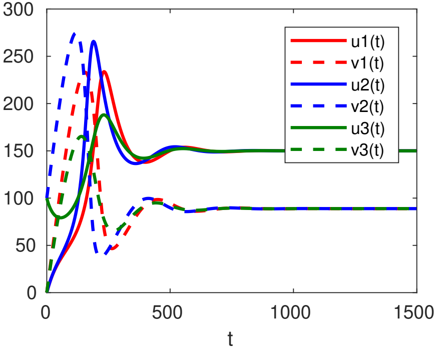


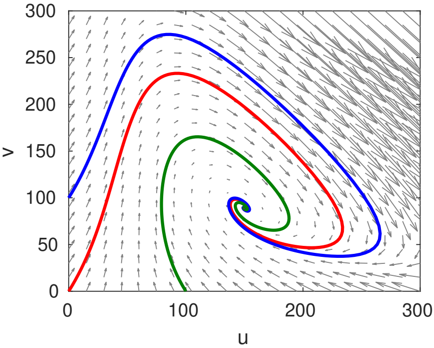
Example

Schnakenberg system in a well mixed reactor

$k_1 = 10^{-6}, \quad k_2 = 1, \quad k_3 = 0.02, \quad k_4 = 2$

Initial condition 3: $u_0 = 100$
$v_0 = 0$





Stability

Well mixed reactor

$$\frac{du}{dt} = k_1 u^2 v + k_2 - k_3 u$$
$$\frac{dv}{dt} = -k_1 u^2 v + k_4$$

Stationary state

$$u_s = \frac{k_4 + k_2}{k_3}, \quad v_s = \frac{k_4}{k_1 u_s^2}$$

Example

$$u_s = 150, \quad v_s = 88.889$$

Chemical kinetics in a reactor with free diffusion

Concentrations $u = u(t,x)$, $v = v(t,x)$ of U, V satisfy

$$\frac{\partial u}{\partial t} = d_1 \frac{\partial^2 u}{\partial x^2} + k_1 u^2 v + k_2 - k_3 u \quad \text{in } \Omega = (0, L)$$
$$\frac{\partial v}{\partial t} = d_2 \frac{\partial^2 v}{\partial x^2} - k_1 u^2 v + k_4 \quad \text{in } \Omega$$

Boundary conditions:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} = 0 \quad \text{at points 0 and } L$$

Initial condition:

$$u(0,x) = u_0(x) \quad \text{and} \quad v(0,x) = v_0(x) \quad \text{for all } x \in \Omega$$

Stationary state:

$$u_s = \frac{k_4 + k_2}{k_3}, \quad v_s = \frac{k_4}{k_1 u_s^2}$$

Examples

$\Omega = (0, 1)$

$k_1 = 10^{-6}, \quad k_2 = 1, \quad k_3 = 0.02, \quad k_4 = 2$

Example 1. $d_1 = d_2 = 1$ [Video "Schnak_II_diff=1.avi"]

Example 2. $d_1 = d_2 = 10^{-5}$ [Video "Schnak_II_diff=1e-5.avi"]

Example 3. $d_1 = 10^{-5}$, $d_2 = 10^{-3}$
Initial condition is a centered bump [Video "Schnak_II_C.avi"]

Example 4. $d_1 = 10^{-5}$, $d_2 = 10^{-3}$
Initial condition is an asymmetrically placed bump
[Video "Schnak_II_D.avi"]

Definitions

Turing instability occurs if stationary solution (u_s, v_s) is

(a) stable with respect to spatially homogeneous disturbances

(b) unstable with respect to spatial disturbances

Turing pattern is a spatially nonhomogeneous stationary solution.

Turing instability

Reaction-diffusion system:

$$\frac{\partial u}{\partial t} = d_1 \frac{\partial^2 u}{\partial x^2} \overbrace{+k_1 u^2 v + k_2 - k_3 u}^{f(u,v)} \quad \text{in } \Omega = (0, L)$$
$$\frac{\partial v}{\partial t} = d_2 \frac{\partial^2 v}{\partial x^2} \underbrace{-k_1 u^2 v + k_4}_{g(u,v)} \quad \text{in } \Omega$$

Linearisation matrix:

$$A = \begin{bmatrix} f_u & f_v \\ g_u & g_v \end{bmatrix} (u_s, v_s)$$

Turing conditions:

(T1) $\operatorname{tr} A < 0$

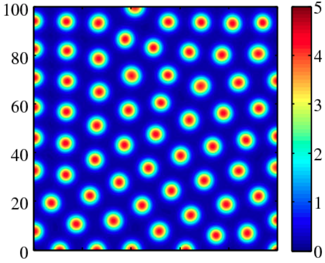
(T2) $\det A > 0$

(T3) $d_2 f_u - d_1 g_v > 0$

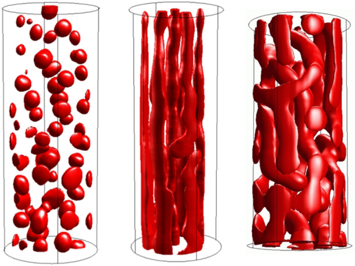
(T4) $4 d_1 d_2 \det A < (d_2 f_u + d_1 g_v)^2$

Further development

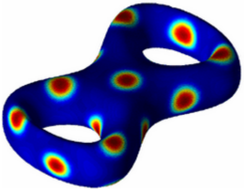
Patterns in 2D



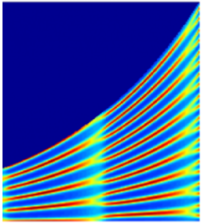
Patterns in 3D



Patterns on manifolds



Patterns on growing domains



Our research

Nonsmooth kinetics

$$\frac{\partial u}{\partial t} = d_1 \Delta u + f(u, v)$$
$$\frac{\partial v}{\partial t} = d_2 \Delta v + g(u, v) + \tau v^-$$

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Unilateral regulation breaks regularity of Turing patterns

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We consider a reaction-diffusion system undergoing Turing instability and augment it by an additional unilateral source term. We investigate its influence on the Turing instability and on the character of resulting patterns. The nonsmooth positively homogeneous unilateral term τv^- has favorable properties, but the standard linear stability analysis cannot be performed. We illustrate the importance of the nonsmoothness by a numerical case study, which shows that the Turing instability can considerably change if we replace this term by its arbitrarily precise smooth approximation. However, the nonsmooth unilateral term and all its approximations yield qualitatively similar patterns although not necessarily developing from small disturbances of the spatially homogeneous steady state. Further, we show that the unilateral source breaks the approximate symmetry and regularity of the classical patterns and yields asymmetric and irregular patterns. Moreover, a given system with a unilateral source produces spatial patterns even for diffusion parameters with ratios closer to 1 than the same system without any unilateral term.

Conclusion

Turing instability and Turing patterns

are mutually independent phenomena.

Thank you for your attention

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