MEAN-VALUE THEOREM FOR VECTOR-VALUED FUNCTIONS

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Abstract. For a differentiable function $\mathbf{f}\colon I \to \mathbb{R}^k$, where I is a real interval and $k \in \mathbb{N}$, a counterpart of the Lagrange mean-value theorem is presented. Necessary and sufficient conditions for the existence of a mean $M\colon I^2 \to I$ such that

$$\mathbf{f}(x) - \mathbf{f}(y) = (x-y)\mathbf{f}'(M(x,y)), \quad x, y \in I,$$

are given.

Similar considerations for a theorem accompanying the Lagrange mean-value theorem are presented.

Keywords: Lagrange mean-value theorem, mean, Darboux property of derivative, vector-valued function

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1. Introduction

Let $I \subset \mathbb{R}$ be an interval. Recall that a function $M\colon I^2 \to \mathbb{R}$ is said to be a *mean* in I if, for all $x, y \in I$,

$$\min(x,y) \leqslant M(x,y) \leqslant \max(x,y).$$

A mean M in I is called *strict* if these inequalities are sharp whenever $x \neq y$, and *symmetric* if $M(x,y) = M(y,x)$ for all $x, y \in I$.

If M is a mean in I then, obviously, $M(J^2) = J$ for any subinterval $J \subset I$; in particular M is *reflexive*, i.e.

$$M(x,x) = x, \quad x \in I.$$

Note also that if a function $M\colon I^2 \to \mathbb{R}$ is reflexive and (strictly) increasing with respect to each variable, then it is a (strict) mean I. In the sequel such a mean is called (*strictly*) *increasing.*

According to the Lagrange mean value theorem, *if $f\colon I \to \mathbb{R}$ is a differentiable function, then there exists a strict mean $M\colon I^2 \to I$ such that*

$$f(x) - f(y) = f'(M(x,y))(x-y), \quad x, y \in I.$$

Moreover, if f' is one-to-one, then $M = M_f$ is unique, continuous, symmetric, strictly increasing, and

$$M_f(x,y) = (f')^{-1}\Big(\frac{f(x)-f(y)}{x-y}\Big), \quad x, y \in I,\ x \neq y.$$

This result can be extended to functions $f\colon I \to \mathbb{R}^k$ as follows.

Theorem 1. *Let $k \in \mathbb{N}$. If $\mathbf{f}\colon I \to \mathbb{R}^k$, $\mathbf{f} = (f_1, \ldots, f_k)$ is differentiable in an interval I, then there exists a vector of strict means $\mathbf{M} = (M_1, \ldots, M_k)\colon I \times I \to I^k$ such that*

$$\mathbf{f}(x) - \mathbf{f}(y) = (x-y)(f_1'(M_1(x,y)), \ldots, f_k'(M_k(x,y))), \quad x, y \in I. \tag{1.1}$$

Moreover, if $f_1', \ldots, f_k'$ are one-to-one, then $\mathbf{M} = \mathbf{M_f}$ is unique, continuous, symmetric, the mean $M_i = M_{f_i}$ is strictly increasing and

$$M_{f_i}(x,y) = (f_i')^{-1}\Big(\frac{f_i(x)-f_i(y)}{x-y}\Big), \quad x, y \in I,\ x \neq y;\ i = 1, \ldots, k.$$

P r o o f. It is enough to apply the Lagrange mean-value theorem to each of the coordinate functions. □

This leads to a natural question when there is a unique mean M such that $M_1 = \ldots = M_k = M$; in particular, when formula 1.1 can be written in the form

$$\mathbf{f}(x) - \mathbf{f}(y) = (x-y)\mathbf{f}'(M(x,y)), \quad x, y \in I?$$

The answer is given by the following

Theorem 2. *Let $k \in \mathbb{N}$, $k \geqslant 2$, be fixed. Suppose that $\mathbf{f}\colon I \to \mathbb{R}^k$, $\mathbf{f} = (f_1, \ldots, f_k)$ is differentiable in an interval I and such that $f_1', \ldots, f_k'$ are one-to-one. Then the following conditions are equivalent:*

(i) *there is a unique mean M in I such that*

$$\mathbf{f}(x) - \mathbf{f}(y) = (x-y)\mathbf{f}'(M(x,y)), \quad x, y \in I;$$

(ii) *there are $a_i, b_i, c_i \in \mathbb{R}$, $a_i \neq 0$, for $i = 1, \ldots, k$, and a function $f\colon I \to \mathbb{R}$ such that*

$$f_i(x) = a_i f(x) + b_i x + c_i, \quad x \in I.$$

Proof. Since $f_1', \ldots, f_k'$ are one-to-one, each of these derivatives is strictly monotonic and continuous. To show this assume, to the contrary, that f_i' is not strictly monotonic. Then there are $x, y, z \in I$, $x < y < z$ such that either

$$f_i'(x) < f_i'(y) \quad \text{and} \quad f_i'(y) > f_i'(z)$$

or

$$f_i'(x) > f_i'(y) \quad \text{and} \quad f_i'(y) < f_i'(z).$$

By the Darboux property of the derivative, in each of these two cases we would find $u \in [x,y]$ and $v \in [y,z]$ such that $f_i'(u) = f_i'(v)$. This is a contradiction, as f_i' is one-to-one. Now, the monotonicity of f_i' and the Darboux property imply the continuity of f_i'.

In particular, we have shown that each of the functions f_i is either strictly convex or strictly concave in the interval I.

Assume that condition (i) is satisfied. Then, by Theorem 1, $M = M_{f_i}$ for $i = 1, \ldots, k$, whence

$$(f_i')^{-1}\Big(\frac{f_i(x) - f_i(y)}{x-y}\Big) = (f_1')^{-1}\Big(\frac{f_1(x) - f_1(y)}{x-y}\Big), \quad x, y \in I,\ x \neq y,$$

for all $i = 2, \ldots, k$. Setting

$$f := f_1, \quad \varphi_i := f_i' \circ (f_1')^{-1},$$

we hence get

$$\frac{f_i(x) - f_i(y)}{x-y} = \varphi_i\Big(\frac{f(x) - f(y)}{x-y}\Big), \quad x, y \in I,\ x \neq y;\ i \in \{2, \ldots, k\}.$$

Since $f = f_1$ is strictly convex or strictly concave, it follows that (cf. [2], Theorem 1) there exist $a_i, b_i, c_i \in \mathbb{R}$ such that

$$f_i(x) = a_i f(x) + b_i x + c_i, \quad x \in I;\ i = 2, \ldots, k.$$

The strict convexity or strict concavity of f_i implies that $a_i \neq 0$ for $i = 2, \ldots, k$, which completes the proof of the implication (i) $\Longrightarrow$ (ii).

Since the converse implication is easy to verify, the proof is complete. □

R e m a r k. At the beginning of the proof of Theorem 2 we have observed that all the derivatives $f_1', \ldots, f_k'$ are continuous and strictly monotonic. Therefore, in the proof of the implication (i) $\Longrightarrow$ (ii) one could apply the following

Lemma 1. *Let $f, g\colon I \to \mathbb{R}$ be differentiable and such that f' and g' are continuous and strictly monotonic. Then $M_g = M_f$ if, and only if, there are $a, b, c \in \mathbb{R}$, $a \neq 0$, such that*

$$h(x) = af(x) + bx + c, \quad x \in I.$$

This lemma is a consequence of a result due to Berrone and Moro (cf. Corollary 7 in [1]).

2. The counterparts of Theorems 1 and 2

In [3] the following counterpart of the Lagrange mean-value theorem has been proved. *If a real function f defined on an interval $I \subset \mathbb{R}$ is differentiable, and f' is one-to-one, then there exists a unique mean function $M\colon f'(I) \times f'(I) \to f'(I)$ such that*

$$\frac{f(x) - f(y)}{x - y} = M(f'(x), f'(y)), \quad x, y \in I,\ x \neq y.$$

Obviously, this result can also be extended to functions $f\colon I \to \mathbb{R}^k$. We have the following

Theorem 3. *Let $k \in \mathbb{N}$. If $\mathbf{f}\colon I \to \mathbb{R}^k$, $\mathbf{f} = (f_1, \ldots, f_k)$ is differentiable in an interval I and $f_1', \ldots, f_k'$ are one-to-one, then there exists a unique vector of means $\mathbf{M} = (M_1, \ldots, M_k)$, $M_i\colon f_i'(I) \times f_i'(I) \to f_i'(I)$, $i = 1, \ldots, k$, such that*

$$\mathbf{f}(x) - \mathbf{f}(y) = (x - y)(M_1(f_1'(x), f_1'(y)), \ldots, M_k(f_k'(x), f_k'(y))), \quad x, y \in I.$$

Moreover, $\mathbf{M} = \mathbf{M_f}$ is continuous for each $i = 1, \ldots, k$, the mean $M_i = M_{f_i}$ is symmetric, strictly increasing, and

$$M_i(u, v) = \frac{f_i((f_i')^{-1}(u)) - f_i((f_i')^{-1}(v))}{(f_i')^{-1}(u) - (f_i')^{-1}(v)}, \quad u, v \in f_i'(I),\ u \neq v.$$

To answer the question when the means M_i, $i = 1, \dots, k$, are equal, we need

Lemma 2. *Let $I \subset \mathbb{R}$ be an interval and let $F, g, h\colon I \to \mathbb{R}$. Suppose that h and, for any $y \in I$, the function*

$$(I \setminus \{y\}) \ni x \mapsto \frac{g(x) - g(y)}{x - y}$$

are one-to-one. If

$$\frac{F(x) - F(y)}{h(x) - h(y)} = \frac{g(x) - g(y)}{x - y}, \quad x, y \in I,\ x \neq y, \tag{2.1}$$

then there are $a, b, c \in \mathbb{R}$, $a \neq 0$ such that

$$h(x) = ax + b, \quad F(x) = ag(x) + c, \quad x \in I.$$

Proof. Without any loss of generality we can assume that $0 \in I$ and that $g(0) = h(0) = 0$. From 2.1 we have

$$F(x) - F(y) = \frac{g(x) - g(y)}{x - y}[h(x) - h(y)], \quad x, y \in I,\ x \neq y.$$

Since $F(x) - F(y) = [F(x) - F(z)] + [F(z) - F(y)]$, we get

$$\frac{g(x) - g(y)}{x - y}[h(x) - h(y)] = \frac{g(x) - g(z)}{x - z}[h(x) - h(z)] + \frac{g(z) - g(y)}{z - y}[h(z) - h(y)]$$

for all $x, y, z \in I$, $x \neq y \neq z \neq x$, whence, after simple calculations,

$$g(x)K(x, y, z) = L(x, y, z), \quad x, y, z \in I,\ x \neq y \neq z \neq x,$$

where

$$K(x, y, z) := h(x)(y - z) + h(z)(x - y) + [h(y)z - h(z)y]$$

and

$$\begin{aligned} L(x, y, z) := {} & xh(x)[g(y) - g(z)] + h(x)[g(z)y - g(y)z] + x[g(z)h(z) - g(y)h(y)] \\ & + \frac{g(z) - g(y)}{z - y}[h(z) - h(y)](x - y)(x - z) + [g(y)h(y)z - g(z)h(z)y]. \end{aligned}$$

Setting in this equality $x = 0$ we obtain

$$\frac{g(z) - g(y)}{z - y}[h(z) - h(y)]yz + [g(y)h(y)z - g(z)h(z)y] = 0, \quad y, z \in I,\ y \neq z,$$

whence, after simple calculations,

$$\Big(\frac{g(y)}{y}-\frac{g(z)}{z}\Big)\Big(\frac{h(y)}{y}-\frac{h(z)}{z}\Big)=0,\quad y,z\in I\setminus\{0\},\ y\neq z.$$

This equality and the injectivity assumption of the function

$$I\setminus\{0\}\ni\frac{g(x)}{x}\to\mathbb{R}$$

imply that there is $a\in\mathbb{R}$ such that $h(x)/x=a$ for all $x\in I\setminus\{0\}$. As $h(0)=0$, we get $h(x)=ax$ for all $x\in I$. Since the remaining results are obvious, the proof is complete. □

Using the idea of the proof of this lemma we prove

R e m a r k 1. Let $I\subset\mathbb{R}$ be an interval. Suppose that the functions $F,g,h\colon\ I\to\mathbb{R}$ are one-to-one. Then

$$\frac{F(x)-F(y)}{x-y}=g(x)+h(y),\quad x,y\in I,\ x\neq y, \tag{2.2}$$

if, and only if, there are $a,b,c,d\in\mathbb{R}$ such that

$$F(x)=ax^2+2bx+d,\quad g(x)=ax+b-c,\quad h(x)=ax+b+c,\qquad x\in I.$$

P r o o f. Suppose that the functions $F,g,h\colon\ I\to\mathbb{R}$ satisfy equation 2.2. Interchanging x and y in 2.2 we conclude that $g(x)+h(y)=g(y)+h(x)$, i.e.

$$h(x)-g(x)=h(y)-g(y),\quad x\in I,$$

whence, for some $c\in\mathbb{R}$,

$$h(x)=g(x)+2c,\quad x\in I. \tag{2.3}$$

Hence, setting

$$G(x):=g(x)+c,\quad x\in I, \tag{2.4}$$

we can write equation 2.2 in the form

$$\frac{F(x)-F(y)}{x-y}=G(x)+G(y),\quad x,y\in I,\ x\neq y,$$

or, equivalently,

$$F(x) - F(y) = [G(x) + G(y)](x - y), \quad x, y \in I,\ x \neq y.$$

Since $F(x) - F(y) = [F(x) - F(z)] + [F(z) - F(y)]$, we get

$$[G(x) + G(y)](x - y) = [G(x) + G(z)](x - z) + [G(z) + G(y)](z - y)$$

for all $x, y, z \in I$, $x \neq y \neq z \neq x$. Taking here $z := (1-t)x + ty$, after a simplification we obtain

$$G((1-t)x + ty) = tG(x) + (1-t)G(y), \quad x, y \in I,\ x \neq y,\ t \in (0,1),$$

that is, G is an affine function. Consequently, there are $a, b \in \mathbb{R}$, such that $G(x) = ax + b$ for all $x \in I$. From 2.4 and 2.3 we get

$$g(x) = ax + b - c, \quad h(x) = ax + b + c, \quad x \in I.$$

Substituting these functions into 2.2 we get

$$\frac{F(x) - F(y)}{x - y} = a(x + y) + 2b, \quad x, y \in I,\ x \neq y,$$

whence

$$F(x) - ax^2 - 2bx = F(y) - ay^2 - 2by, \quad x, y \in I,\ x \neq y.$$

It follows that, for some $d \in \mathbb{R}$,

$$F(x) = ax^2 + 2bx + d, \quad x \in I.$$

Since the converse implication is obvious, the proof is complete. □

Theorem 4. *Let $k \in \mathbb{N}$, $k \geqslant 2$, be fixed. Suppose that $\mathbf{f}\colon I \to \mathbb{R}^k$, $\mathbf{f} = (f_1, \ldots, f_k)$ is differentiable in an interval I and $f_1', \ldots, f_k'$ are one-to-one. Then the following conditions are equivalent:*

(i) *there is a unique mean M such that*

$$\mathbf{f}(x) - \mathbf{f}(y) = (x - y)(M(f_1'(x), f_1'(y)), \ldots, M(f_k'(x), f_k'(y))), \quad x, y \in I;$$

(ii) *there are $c_1, \ldots, c_k \in \mathbb{R}$, and a differentiable function $g\colon I \to \mathbb{R}$ with one-to-one derivative such that*

$$f_i(x) = g(x) + c_i, \quad x \in I,\ i = 1, \ldots, k,$$

and

$$M(u,v)=\frac{g((g')^{-1}(u))-g((g')^{-1}(v))}{(g')^{-1}(u)-(g')^{-1}(v)},\quad u,v\in g'(I),\ u\neq v.$$

Proof. Assume (i). Then

$$\frac{f_i(x)-f_i(y)}{x-y}=M(f_i'(x),f_i'(y)),\quad x,y\in I,\ x\neq y,\ i=1,\dots,k,$$

whence, for each $i=1,\dots,k$,

$$M(u,v)=\frac{f_i((f_i')^{-1}(u))-f_i((f_i')^{-1}(v))}{(f_i')^{-1}(u)-(f_i')^{-1}(v)},\quad u,v\in f_i'(I),\ u\neq v.$$

Taking $g:=f_1$ we get, for each $i=1,\dots,k$,

$$\frac{f_i((f_i')^{-1}(u))-f_i((f_i')^{-1}(v))}{(f_i')^{-1}(u)-(f_i')^{-1}(v)}=\frac{g((g')^{-1}(u))-g((g')^{-1}(v))}{(g')^{-1}(u)-(g')^{-1}(v)},\quad u,v\in f_i'(I),\ u\neq v.$$

Let us fix arbitrary $i\in\{2,3,\dots,k\}$ and put

$$h_i:=(f_i')^{-1}\circ g',\qquad F_i:=f_i\circ(f_i')^{-1}\circ g'.$$

Hence, taking arbitrary $x,y\in I$, $x\neq y$, and setting $u:=g'(x)$, $v:=g'(y)$ in the above equality, we obtain

$$\frac{F_i(x)-F_i(y)}{h_i(x)-h_i(y)}=\frac{g(x)-g(y)}{x-y},\quad x,y\in I,\ x\neq y.$$

By Lemma 2, there are $a_i,b_i,c_i\in\mathbb{R}$, $a_i\neq 0$ such that

$$h_i(x)=a_ix+b_i,\quad F_i(x)=a_ig(x)+c_i,\quad x\in I. \tag{2.5}$$

By the definition of h_i, we get

$$f_i'(a_ix+b_i)=g'(x),\quad x\in I,\ i=1,\dots,k.$$

Since the domains of all functions f_i are the same, it follows that $a_i=1$, $b_i=0$, and $f_i'=g'$ for each $i=2,\dots,k$. Now from the latter of formulas 2.5, we obtain $f_i=g+c_i$ for each $i=2,\dots,k$, which completes the proof of the implication (i) $\Longrightarrow$ (ii). Since the converse implication is obvious, the proof is complete. □

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