COMPOSITION OPERATORS ON MUSIELAK-ORLICZ SPACES OF BOCHNER TYPE

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Abstract. The invertible, closed range, compact, Fredholm and isometric composition operators on Musielak-Orlicz spaces of Bochner type are characterized in the paper.

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1. Introduction and preliminaries

Let $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{N}$ denote the set of reals, non-negative reals and the set of natural numbers respectively. Let (G,Σ,μ) be a σ-finite measure space. Denote by $L^0 = L^0(G)$ the set of all μ-equivalence classes of complex-valued measurasble functions defined on G. A function $M\colon\ G\times\mathbb{R}\to[0,\infty)$ is said to be a Musielak-Orlicz function if $M(\cdot,u)$ is measurable for each $u\in\mathbb{R}$, $M(t,u)=0$ if and only if $u=0$ and $M(t,\cdot)$ is convex, even, not identically equal to zero and $M(t,u)/u\to 0$ as $u\to 0$ for μ-a.e. $t\in G$. Define on L^0 a convex modular ϱ_M by

$$\varrho_M(f)=\int_G M(t,f(t))\,\mathrm{d}\mu$$

for every $f\in L^0$. By the Musielak-Orlicz space L_M we mean

$$L_M=\{f\in L^0\colon\ \varrho_M(\lambda f)<\infty \text{ for some } \lambda>0\}.$$

Its subspace E_M is defined as

$$E_M = \{f \in L^0\colon\ \varrho_M(\lambda f) < \infty \text{ for any } \lambda > 0\}.$$

The space L_M equipped with the Luxemberg norm

$$\|f\|_M = \inf\{\lambda > 0\colon\ \varrho_M(f/\lambda) \leqslant 1\}$$

is a Banach space (see [14], [15]). For every Musielak-Orlicz function M we define the complementary function $M^*(t,v)$ as

$$M^*(t,v) = \sup_{u>0}\{u|v| - M(t,u)\colon\ v \geqslant 0 \text{ and } t \in G \text{ a.e.}\}.$$

It is easy to see that $M^*(t,v)$ is also a Musielak-Orlicz function. We say that a Musielak-Orlicz function M satisfies the Δ_2-conditions (write $M \in \Delta_2$) if there exists a constant $k > 2$ and a measurable non-negative function f such that $\varrho_M(f) < \infty$ and

$$M(t,2u) \leqslant kM(t,u)$$

for every $u \geqslant f(t)$ and for $t \in G$ a.e. For more details see ([1], [6], [12], [18]). Throughout this paper we assume that M satisfies the Δ_2-conditions.

We now define the types of spaces considered in this paper. For a Banach space $(X, \|\cdot\|_X)$, denote by $L^0(X)$ the family of strongly measurable functions $f\colon G \to X$, identifying functions which are equal μ-almost everywhere in G. Define a new modular $\tilde\varrho_M$ on $L^0(X)$ by

$$\tilde\varrho_M(f) = \int_G M(t, \|f(t)\|)\,\mathrm{d}\mu.$$

Let

$$L_M(G,X) = \{f \in L^0(X)\colon\ \|f(t)\| = \|f(t)\|_X \in L_M\}.$$

Then $L_M(G,X)$ becomes a Banach space with the norm

$$\|f\| = \big\|\|f(t)\|_X\big\|_M = \inf\{\lambda\colon\ \tilde\varrho_M(f/\lambda) \leqslant 1\}$$

and it is called a Musielak-Orlicz space of Bochner type, see [4].

If T is a non-singular measurable transformation, then the measure μT^{-1} is absolutely continuous with respect to the measure μ. Hence by the Radon-Nikodym derivative theorem there exists a positive measurable function f_0 such that $\mu(T^{-1}(E)) = \int_E f_0 \, \mathrm{d}\mu$ for every $E \in \Sigma$. The function f_0 is called the Radon-Nikodym derivative of the measure μT^{-1} with respect to the measure μ. It is denoted by $f_0 = \mathrm{d}\mu T^{-1}/\mathrm{d}\mu$.

Associated with each σ-finite subalgebra $\Sigma_0 \subset \Sigma$ there exists an operator $E = E^{\Sigma_0}$, which is called the conditional expectation operator, on the set of all non-negative measurable functions f or for each $f \in L^0(G, \Sigma, \mu)$, and is uniquely determined by the following conditions:

(1) $E(f)$ is Σ_0-measurable, and

(2) if A is any Σ_0-measurable set for which $\int_A f \, \mathrm{d}\mu$ exists, we have $\int_A f \, \mathrm{d}\mu = \int_A E(f) \, \mathrm{d}\mu$.

The transfromation E has the following properties:

- ▷ $E(f \cdot g \circ T) = E(f) \cdot (g \circ T)$;
- ▷ if $f \geqslant g$ almost everywhere, then $E(f) \geqslant E(g)$ almost everywhere;
- ▷ $E(1) = 1$;
- ▷ $E(f)$ has the form $E(f) = g \circ T$ for exactly one σ-measurable function g. In particular, $g = E(f) \circ T^{-1}$ is a well defined measurable function.
- ▷ $|E(fg)|^2 \leqslant (E|f|^2)(E|g|^2)$. This is the Cauchy-Schwartz inequality for conditional expectation.
- ▷ For $f > 0$ almost everywhere, $E(f) > 0$ almost everywhere.
- ▷ If φ is a convex function, then $\varphi(E(f)) \leqslant E(\varphi(f))\mu$-almost everywhere. For deeper study of properties of E see [11].

Let $T\colon\ G \to G$ be a non-singular measurable transformation. Then we can define a composition transformation

$$C_T\colon\ L_M(G, X) \to L_M(G, X)$$

by

$$(C_T f)(t) = f(T(t)), \quad \forall\, t \in G.$$

If C_T is continuous, we call it a composition operator induced by T. In the early 1930's the composition operators were used to study problems in mathematical physics and especially classical mechanics, see Koopman [5]. In those days these operators were known as substitution operators. The systematic study of composition operators has relatively a very short history. It was started by Nordgren in 1968 in his paper [17]. After this, the study of composition operators has been extended in several directions by several mathematicians. For more details on these operators we refer to ([7], [13], [16], [19], [20]). In particular, for the study of composition operators on Orlicz and Orlicz-Lorentz spaces one can refer to ([2], [3], [8], [9], [10]) and references therein.

2. Composition operators

In this section we characterize invertibility, closed range, Fredholmness and compactness of composition operators on Musielak-Orlicz spaces of Bochner type.

Theorem 2.1. *Let $T\colon G \to G$ be a measurable transformation. Then $C_T\colon L_M(G,X) \to L_M(G,X)$ is bounded if and only if there exists $k > 0$ such that*

$$E[M(I\circ T^{-1}(t), x)]f_0(t) \leqslant M(t, kx)$$

for every $x \in X$ and for μ-almost all $t \in G$.

Proof. Let $f \in L_M(G,X)$. Then

$$\begin{aligned}\int_G M\Big(t, \frac{\|(f\circ T)(t)\|}{k\|f\|}\Big)\,\mathrm{d}\mu &= \int_G E\Big[M\Big(I\circ T^{-1}(t), \frac{\|f(t)\|}{k\|f\|}\Big)\Big]f_0(t)\,\mathrm{d}\mu\\ &\leqslant \int_G M\Big(t, \frac{\|f(t)\|}{\|f\|}\Big)\,\mathrm{d}\mu \leqslant 1.\end{aligned}$$

Therefore $\|C_T f\| \leqslant k\|f\|$ for all $f \in L_M(G,X)$. Hence C_T is bounded.

Conversely, suppose that the condition is not fulfilled. Then for every positive integer k there exists $x_k \in X$ and a measurable subset E_k such that

$$E[M(I\circ T^{-1}(t), x_k)]f_0(t) > M(t, kx_k)$$

for almost every $t \in E_k$. Choose a measurable subset F_k of E_k such that $\chi_{F_k} \in L_M(G,X)$. Let $f_k = x_k\chi_{F_k}$. Then

$$\begin{aligned}\int_G M\Big(t, \frac{k\|f_k(t)\|}{\|C_T f_k\|}\Big)\,\mathrm{d}\mu &= \int_{F_k} M\Big(t, \frac{\|kx_k\|}{\|C_T f_k\|}\Big)\,\mathrm{d}\mu\\ &\leqslant \int_G E\Big[M\Big(I\circ T^{-1}(t), \frac{\|x_k\chi_{F_k}(t)\|}{\|C_T f_k\|}\Big)\Big]f_0(t)\,\mathrm{d}\mu\\ &= \int_G M\Big(t, \frac{\|(C_T f_k)(t)\|}{\|C_T f_k\|}\Big)\,\mathrm{d}\mu \leqslant 1.\end{aligned}$$

This shows that $\|C_T f_k\| \geqslant k\|f_k\|$, which contradicts the boundedness of C_T. Hence the condition of the theorem is fulfilled.

Theorem 2.2. *Let $T\colon G \to G$ be a measurable transformation. Then $C_T\colon L_M(G,X) \to L_M(G,X)$ has closed range if and only if*

$$E[M(I\circ T^{-1}(t),x)]f_0(t) \geqslant M(t,\delta x)$$

for μ-almost all $t \in G\setminus T(G)$ and $\delta > 0$.

Proof. Suppose that the condition of the theorem is fulfilled. Let $f \in \overline{\operatorname{ran} C_T}$. Then there exists a sequence $\{g_n\}$ in ran C_T such that $g_n \to f$. Write $g_n = C_T f_n$. Then $C_T f_n \to f$. It follows that $\{C_T f_n\}$ is a Cauchy sequence. Then there exists a positive integer n_0 such that $\|C_T f_n - C_T f_m\| < \varepsilon$, for all $m, n \geqslant n_0$. Hence

$$\begin{aligned}\int_G M\Bigl(t, \frac{\delta\|f_n(t)-f_m(t)\|}{\|g_n-g_m\|}\Bigr)\,\mathrm{d}\mu &\leqslant \int_G E\Bigl[M\Bigl(I\circ T^{-1}(t), \frac{\|f_n(t)-f_m(t)\|}{\|g_n-g_m\|}\Bigr)\Bigr] f_0(t)\,\mathrm{d}\mu\\ &= \int_G \Bigl(t, \frac{\|f_n(T(t))-f_m(T(t))\|}{\|g_n-g_m\|}\Bigr)\,\mathrm{d}\mu\\ &= \int_G M\Bigl(t, \frac{\|g_n(t)-g_m(t)\|}{\|g_n-g_m\|}\Bigr)\,\mathrm{d}\mu \leqslant 1.\end{aligned}$$

This prove that

$$\delta\|f_n - f_m\| \leqslant \|g_n - g_m\|, \ \forall\, m,n \geqslant n_0.$$

Hence $\{f_n\}$ is a Cauchy sequence in $L_M(G,X)$. In view of completeness there exists $g \in L_M(G,X)$ such that $f_n \to g$. Thus $C_T f_n \to C_T g$, that is $g_n \to C_T f$ so that $f = C_T g \in \operatorname{ran} C_T$. This proves that $\operatorname{ran} C_T$ is closed.

Conversely, suppose C_T has closed range. If the condition of the theorem is not satisfied, then for every positive integer k there exist a measurable subset E_k and $x_k \in X$ such that

$$E[M(I\circ T^{-1}(t),x_k)]f_0(t) < M(t, x_k/k)$$

for μ-almost all $t \in E_k$. Choose a measurable subset F_k of E_k such that $\chi_{F_k} \in L_M(G,X)$ and $f_k = k\chi_{F_k}$. Now

$$\begin{aligned}\int_G M\Bigl(t, \frac{k\|(C_T f_k)(t)\|}{\|f_k\|}\Bigr)\,\mathrm{d}\mu &\leqslant \int_{F_k} E\Bigl[M\Bigl(I\circ T^{-1}(t), \frac{\|kx_k\|}{\|f_k\|}\Bigr)\Bigr] f_0(t)\,\mathrm{d}\mu\\ &= \int_G M\Bigl(t, \frac{\|f_k(t)\|}{\|f_k\|}\Bigr)\,\mathrm{d}\mu \leqslant 1.\end{aligned}$$

This proves that

$$\|C_T f_k\| \leqslant \frac{1}{k}\|f_k\|$$

so that C_T is not bounded away from zero. Hence the condition of the theorem must be satisfied.

Theorem 2.3. *Suppose $C_T \in B(L_M(G,X))$. Then C_T is invertible if and only if*

(i) *T is invertible a.e.;*

(ii) *there exists $\delta > 0$ such that $M(T(t),x) \leqslant M(t,\delta x)$ a.e.*

P r o o f. Suppose that C_T is invertible. We show that T is invertible. If T is not surjective a.e., then choose a measurable subset $E \subset G \setminus T(G)$ such that $\chi_E \in L_M(G,X)$. Then $C_T\chi_E = 0$ which indicates that C_T has a non-trivial kernel. Hence T is surjective. If C_T is onto, then C_T has closed range. Therefore the condition (ii) is satisfied as $T(G) = G$. We next show that $T^{-1}(\Sigma) = \Sigma$. Clearly $T^{-1}(\Sigma) \subset \Sigma$. For the reverse inclusion, let $E \in \Sigma$. Since C_T is onto, there exists $g \in L_M(G,X)$ such that $C_T g = \chi_E$, and it follows that there exists a measurable subset F such that $g = \chi_F$. Hence $C_T\chi_F = \chi_E$ or $T^{-1}(F) = E$ a.e. Then $E \in T^{-1}(\Sigma)$. Therefore $T^{-1}(\Sigma) = \Sigma$ which proves that T is invertible.

Conversely, suppose that the conditions of the theorem are satisfied. Let T^{-1} be the inverse of T. The condition (ii) implies that $C_{T^{-1}}$ is a bounded operator as

$$C_T C_{T^{-1}} = C_{T^{-1}} C_T = I.$$

Hence C_T is invertible.

Theorem 2.4. *Let (G,Σ,μ) be a non-atomic measure space. Then no composition operator C_T on $L_M(G,X)$ is compact.*

P r o o f. Let for some $\varepsilon > 0$, the set

$$E_\varepsilon = \{t \in G\colon\ E[M(I \circ T^{-1}(t),x)]f_0(t) \geqslant M(t,\varepsilon x)\}$$

be of positive measure. Since μ is non-atomic, we can find measurable subsets $E_{n+1} \subset E_n \subset E \subset E_\varepsilon$ such that $\mu(E_\varepsilon) < \infty$ and $\mu(E_{n+1}) = \frac{1}{2}\mu(E_n)$. Let $e_n(t) = \|\chi_{E_n}(t)\|/\|\chi_{E_n}\|$. Then $\|e_n\| = 1$. Therefore the sequence $\{e_n\}$ is a bounded sequence. Consider

$$\begin{aligned}\int_G M\Big(t, \frac{\|\varepsilon e_n(t)\|}{\|C_T e_n\|}\Big)\,\mathrm{d}\mu &\leqslant \int_{E_n} M\Big(t, \frac{\varepsilon}{\|\chi_{E_n}\|\,\|C_T e_n\|}\Big)\,\mathrm{d}\mu \\ &\leqslant \int_{E_n} E\Big[M\Big(I \circ T^{-1}(t), \frac{1}{\|\chi_{E_n}\|\,\|C_\varphi e_n\|}\Big)\Big] f_0(t)\,\mathrm{d}\mu \\ &= \int_G M\Big(t, \frac{\|(C_T e_n)(t)\|}{\|C_T e_n\|}\Big)\,\mathrm{d}\mu \leqslant 1.\end{aligned}$$

Hence $\|C_T e_n\| \geqslant \varepsilon$. This proves that C_T cannot be compact. Hence $\mu(E_\varepsilon) = 0$, i.e.

$$E[M(I \circ T^{-1}(t),x)]f_0(t) < M(t,\varepsilon x)$$

for every μ-almost $t \in T$ and for all $x \in X$. Then

$$\int_G M\Big(t, \frac{\|(C_T\chi_E)(t)\|}{\varepsilon\|\chi_E\|}\Big)\,\mathrm{d}\mu = \int_G E\Big[M\Big(I \circ T^{-1}(t), \frac{\|\chi_E(t)\|}{\varepsilon\|\chi_E\|}\Big)\Big] f_0(t)\,\mathrm{d}\mu$$
$$< \int_G M\Big(t, \frac{\|\chi_E(t)\|}{\|\chi_E\|}\Big)\,\mathrm{d}\mu \leqslant 1$$

and therefore $\|C_T\chi_E\| \leqslant \varepsilon\|\chi_E\|$. Since ε is arbitrary, we have $\|C_T\chi_E\| = 0$. In other words $C_T\chi_E = 0$. Since simple functions are dense in $L_M(G,X)$ it follows that $C_T = 0$, which is again a contradiction. Hence no composition operator C_T on $L_M(G,X)$ is compact.

Corollary 2.5. *If T is non-atomic, then no non-zero composition operator is compact.*

Theorem 2.6. *Let $C_T \in B(L_M(G,X))$. Then C_T is Fredholm if and only if C_T is invertible.*

Proof. Suppose C_T is Fredholm. Then C_T has closed range. Therefore, there exists $\varepsilon > 0$ such that

$$E[M(I \circ T^{-1}(t), x)] f_0(t) \geqslant M(t, \varepsilon x) \tag{1}$$

for μ-almost all $t \in T(G)$ and for all $x \in X$. If $T(G) \neq G$ a.e., then there exists $E \in \Sigma$ such that $E \subset G \setminus T(G)$. Therefore $C_T\chi_E = 0$ a.e. Hence $\ker C_T$ is infinite dimensional because for every subset $F \subset E$, we have $C_T\chi_E = 0$. This is a contradiction as $\ker C_T$ is assumed to be finite dimensional. Hence $T(G) = G$ a.e., i.e. T is surjective. Next, if T is injective, then $T^{-1}(\Sigma) \neq \Sigma$, so that the range C_T is not dense. Hence by the Hahn Banach theorem there exists a bounded linear functional $g^* \in L_M^*(G,X)$ such that $g^*(\operatorname{ran} C_T) = 0$. Let $E = \operatorname{supp} g^*$. Partition E into a sequence of disjoint measurable subsets E_n such that $E = \bigcup_{n=1}^{\infty} E_n$. Let $g_n^* = g^*\chi_{E_n}$. Then again $(g^*\chi_{E_n})(\operatorname{ran} C_T) = 0$. But $\ker C_T^* = (\overline{\operatorname{ran} C_T})^{\perp}$. This proves that $\ker C_T^*$ is infinite dimensional, which is again a contradiction. Therefore $\overline{\operatorname{ran} C_T} = L_M(G,X)$. We can conclude that C_T is bounded away from zero and therefore C_T is invertible.

Theorem 2.7. *Suppose $M(t,x) = M_1(t)M_2(x)$. Then C_T is an isometry if and only if*

$$E[M_1(T^{-1}(t))] f_0(t) = M_1(t).$$

P r o o f. Suppose that the condition of the theorem is fulfilled. Then for $f \neq 0$ in $L_M(G,X)$,

$$\begin{aligned}\int_G M\Big(t, \frac{\|f(T(t))\|}{\|f\|}\Big)\,\mathrm{d}\mu &= \int_G M_1(t) M_2\Big(\frac{\|f(T(t))\|}{\|f\|}\Big)\,\mathrm{d}\mu \\ &= \int_G E\Big[M_1(I \circ T^{-1}(t)) M_2\Big(\frac{\|f(t)\|}{\|f\|}\Big)\Big] f_0(t)\,\mathrm{d}\mu \\ &= \int M\Big(t, \frac{\|f(t)\|}{\|f\|}\Big)\,\mathrm{d}\mu \leqslant 1.\end{aligned}$$

Therefore $\|C_T f\| \leqslant \|f\|$. In the same way we can easily prove $\|f\| \leqslant \|C_T f\|$. Hence $\|C_T f\| = \|f\|$, i.e. C_T is an isometry.

Conversely, suppose C_T is an isometry. Let $F \in \Sigma$ be such that $\chi_F \in L_M(G,X)$. Then

$$\|C_T \chi_F\| = \|\chi_F\|$$

implies that

$$\frac{1}{M_2^{-1}\big[1/\int_{T^{-1}(F)} M_1(t)\,\mathrm{d}\mu\big]} = \frac{1}{M_2^{-1}\big[1/\int_F M_1(t)\,\mathrm{d}\mu\big]},$$

which further implies that

$$\int_{T^{-1}(F)} M_1(t)\,\mathrm{d}\mu = \int_F M_1(t)\,\mathrm{d}\mu$$

or

$$\int_F E\big[M_1(T^{-1}(t))\big] f_0(t)\,\mathrm{d}\mu = \int_F M_1(t)\,\mathrm{d}\mu.$$

This is true for every F such that $\chi_F \in L_M(G,X)$. Hence we can conclude that

$$E\big[M_1(T^{-1}(t))\big] f_0(t) = M_1(t)$$

for μ-almost all $t \in G$.

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