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BASE-BASE PARACOMPACTNESS AND SUBSETS OF THE SORGENFREY LINE

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Abstract. A topological space X is called base-base paracompact (John E. Porter) if it has an open base $\mathcal{B}$ such that every base $\mathcal{B}' \subseteq \mathcal{B}$ has a locally finite subcover $\mathcal{C} \subseteq \mathcal{B}'$. It is not known if every paracompact space is base-base paracompact. We study subspaces of the Sorgenfrey line (e.g. the irrationals, a Bernstein set) as a possible counterexample.

Keywords: base-base paracompact space, coarse base, Sorgenfrey irrationals, totally imperfect set

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1. Introduction

The irrationals as a topological subspace of the reals have a *coarse* base [4], i.e. an open base that has no locally finite subcover. Also, the base of all bounded, open, convex sets in a reflexive, infinite-dimensional Banach space is coarse [4], [5]. A space is *totally paracompact* [8] if every open base has a locally finite subcover. Equivalently, if no base is coarse. In totally paracompact metric spaces small and large inductive dimensions coincide [8]. The irrationals with the usual metric topology are not totally paracompact [1], [4], [11], [12]. Non-metrizable, paracompact spaces that are not totally paracompact are the Sorgenfrey line and the Michael line [2], [14], [26]. A similar property that holds for all metrizabe spaces was defined by John E. Porter:

Definition 1.1 [21]. A space X is *base-base paracompact* if it has an open *base* $\mathcal{B}$ such that every *base* $\mathcal{B}'$ contained in $\mathcal{B}$ has a locally finite subcover $\mathcal{C}$. Equivalently, if there exists an open base $\mathcal{B}$ for X such that $\mathcal{B}$ contains no coarse base.

Base-base paracompact spaces are paracompact since every subcover is a refinement. Although base properties are stronger than covering properties, no example is known of a paracompact space that is not base-base paracompact [21], also [19], [20]. John E. Porter proved that base-base paracompact spaces are D-spaces [21] (i.e., for every open neighborhood assignment $\{U_x\colon x \in X\}$ there is a closed discrete $D \subseteq X$ such that $\bigcup\{U_x\colon x \in D\} = X$). Thus, it would be enough to find a paracompact space that is not a D-space, but this is an old problem of Eric van Douwen [7], [10].

Base-base paracompact spaces include base-cover paracompact and base-family paracompact spaces studied by the author in [16], [17], [18]. The latter two classes are distinct from each other and from paracompact spaces. Only the F_σ subspaces of the Sorgenfrey line are base-cover paracompact. Only countable subspaces of the Sorgenfrey line are base-family paracompact. Consistently, there are subspaces of the Sorgenfrey line which are not base-cover paracompact (i.e. not F_σ), yet that are base-base paracompact. Such is any Lusin set or, under MA, any uncountable set of cardinality less than continuum. These sets are Hurewicz [9], [13], and hence totally paracompact [6], see also [2], [12], [14], [15], [25]. G. Gruenhage gave a direct proof for Lusin subspaces that the base of all half-open intervals contains no coarse base.

A. Lelek [11] gave a necessary condition for a subset of a complete metric space to be totally paracompact. He constructed a coarse base $\mathcal{B}'$ such that if $\mathcal{C} \subseteq \mathcal{B}'$ is a point-finite family, then the complement of $\bigcup\mathcal{C}$ contains a Cantor set (that is, a homeomorphic copy of the Cantor middle-third set). In the context of base-base paracompactness, a similar construction would naturally be subject to the requirement that the elements of $\mathcal{B}'$ come from a base $\mathcal{B}$ that is given in advance. For some subspaces of the Sorgenfrey line, we show that such a construction works for *common* bases $\mathcal{B}$ defined below. It remains open if all bases for such subspaces are common.

2. Common bases for the Sorgenfrey line

Recall that the *Sorgenfrey line* S is the set of all reals having all half-open intervals $[a, b)$ as a base for its topology. If X is a subspace of S, then a *base* $\mathcal{B}$ *for* X *in* S is a family $\mathcal{B}$ of open subsets of S such that if $U \subseteq S$ is open and $x \in X \cap U$ then $x \in B \subseteq U$ for some $B \in \mathcal{B}$. We denote the set of integers $\{0, 1, \ldots\}$ by ω.

We first discuss the intuition behind the definition that follows. Suppose that $\mathcal{B}'$ is a base, and we are to pick open sets from it, one at a time, to form a cover $\mathcal{C} \subseteq \mathcal{B}'$. If we pick sets that are too big too often, then these sets may overlap too much, and as a result we may end up with a cover $\mathcal{C}$ that is not locally finite, or even not point-finite. If we pick sets that are too small then there will be too many gaps that are not covered, and at the end $\mathcal{C}$ may not be a cover. Think of the usual construction of the Cantor set, as being an attempt to use the middle-third open intervals to form a cover of the unit interval, but at the end the Cantor set is exactly the part that was not covered.

Suppose that we want to make it difficult for $\mathcal{C}$ to be a point-finite cover, then what we want is for $\mathcal{B}'$ to have only sets that are either too big or too small. Suppose we are to construct $\mathcal{B}'$ first, with $\mathcal{B}' \subseteq \mathcal{B}$, where $\mathcal{B}$ is given. To do this, we need $\mathcal{B}$ to have enough sets of suitable sizes. The following definition works.

Definition 2.1. Assume that X is dense in S. Call a base $\mathcal{B}$ for X in S a *common base* if there are an interval T and sets A_n, $n \in \omega$, such that $T \cap X = \bigcup_{n\in\omega} A_n$ and for each interval $I \subseteq T$ and each n there are $\varepsilon > 0$ and an interval $J \subseteq I$ such that for each $x \in J \cap A_n$ there is $B \in \mathcal{B}$ (depending on I, n, ε, J and x) with $[x, x+\varepsilon) \subseteq B \subseteq [x, \infty) \cap I$.

Recall that a non-empty set of reals is *perfect* if it is closed and has no isolated points. A set of reals is *totally imperfect* if it does not contain any perfect set.

Theorem 2.2. *Let X be a dense subspace of the Sorgenfrey line S such that $S \setminus X$ is dense and totally imperfect. Then every common base $\mathcal{B}$ for X contains a coarse base $\mathcal{B}'$. Thus, if X is base-base paracompact, then only a base that is not common could possibly witness this.*

Proof. Note that in the above definition we may replace "for each $x \in J \cap A_n$" by "for each $x \in J \cap \Big(\bigcup_{m\leqslant n} A_m\Big)$". We may also assume that the length $\lambda(J) < \varepsilon$ and therefore the right endpoint of J belongs to $[x, x+\varepsilon)$, and to B.

Let $\Sigma = \{s\colon s$ is a finite sequence of non-negative integers$\}$. If $s = \langle k, l, \ldots, i\rangle$ and $j \in \omega$ let $s^\frown\langle j\rangle$ denote the sequence $\langle k, l, \ldots, i, j\rangle$. I_s and J_s will always denote left-closed, right-open intervals, with the left endpoint of J_s in $S \setminus X$, and its right endpoint in X. Start with any $I_\emptyset \subseteq T$ (where $\emptyset$ is the empty sequence). This defines I_s for all s with $|s| = 0$, where $|s|$ is the length of s. Recursively, assume $n \geqslant 0$ and I_s were defined whenever $|s| \leqslant n$. We will define I_s for $|s| = n+1$.

For each s with $|s| = n$ fix $\varepsilon_s > 0$ and $J_s \subseteq I_s$ with $\lambda(J_s) < \varepsilon_s$ such that for every $x \in J_s \cap \Big(\bigcup_{m\leqslant n} A_m\Big)$ we can fix $B_s(x) \in \mathcal{B}$ with $[x, x+\varepsilon_s) \subseteq B_s(x) \subseteq [x, \infty) \cap I_s$. Using a sequence of points decreasing to the left endpoint of J_s, represent J_s minus its left endpoint as the disjoint union of countably many left-closed, right-open intervals $I_{s^\frown\langle l\rangle}$, $l \in \omega$, where the left endpoint of $I_{s^\frown\langle l\rangle}$ is the right endpoint of $I_{s^\frown\langle l+1\rangle}$, i.e. $I_{s^\frown\langle l+1\rangle}$ is "the next and to the left of" $I_{s^\frown\langle l\rangle}$. We also require that $\lambda(I_s) < |s|^{-1}$ for each $s \in \Sigma$, and that the right endpoint of J_s, and therefore the right endpoints of all $I_{s^\frown\langle l\rangle}$, $l \in \omega$, are bounded away a distance at least ε_s from the right endpoint of I_s. It is easily seen by induction on $|s|$ that $I_s \cap I_{s'} = \emptyset$ if $|s| = |s'|$ and $s \neq s'$.

If σ is an infinite sequence of non-negative integers let $\sigma|n$ denote the sequence of the first n many members of σ. There is a unique point $p_\sigma \in \bigcap_{n\in\omega} I_{\sigma|n}$. Such a p_σ may or may not be in X. If a point $x \in X$ happens to be p_σ for some σ we say that x is of type one. Then $\{x\} = \bigcap_{n\in\omega} J_{\sigma|n}$. Pick $n(x)$ with $x \in A_{n(x)}$. Then the family $\mathcal{B}_1(x) = \{B_{\sigma|n}(x)\colon n \geqslant n(x)\}$ is a local base at x.

Call an $x \in X$ of type two if x is not of type one. If $x \in X \setminus I_\emptyset$ then clearly x is of type two: Then let $\mathcal{B}_2(x) = \{B \in \mathcal{B}\colon x \in B \subseteq S \setminus I_\emptyset\}$. If $x \in X \cap I_\emptyset$ and x is of type two then there is an $s \in \Sigma$ such that $x \in I_s$ but $x \notin I_{s^\frown\langle l\rangle}$ for any $l \in \omega$, or equivalently $x \notin J_s$. Let $\mathcal{B}_2(x) = \{B \in \mathcal{B}\colon x \in B \subseteq I_s \setminus J_s\}$. If x is of type two then $\mathcal{B}_2(x)$ is a local base at x. Note that if $p \in X$ is of type one and $x \in X$ is of type two then no member of $\mathcal{B}_2(x)$ contains p.

Let $\mathcal{B}_1 = \bigcup\{\mathcal{B}_1(x)\colon x \text{ is of type one}\}$ and $\mathcal{B}_2 = \bigcup\{\mathcal{B}_2(x)\colon x \text{ is of type two}\}$. Then the family $\mathcal{B}' = \mathcal{B}_1 \cup \mathcal{B}_2$ is a base for X in S contained in $\mathcal{B}$.

Suppose that $\mathcal{C} \subseteq \mathcal{B}'$ and $\mathcal{C}$ is point-finite at each $x \in X$. Then for each $s \in \Sigma$ there could be at most finitely many $x \in J_s \cap \Big(\bigcup_{n\leqslant|s|} A_n\Big)$ for which $B_s(x) \in \mathcal{C}$, since all such $B_s(x)$ contain the right endpoint of J_s which is in X. Let x_s be the minimal such x (and if there are no such x let $x_s = \infty$). Then x_s is strictly larger than the left endpoint of J_s since the latter is not in X. Hence $I_{s^\frown\langle l\rangle}$ is to the left of x_s for infinitely many l. Recursively we may construct the smallest set $\Sigma' \subset \Sigma$ and simultaneously pick distinct k_s and l_s for each $s \in \Sigma'$ such that: (a) $\emptyset \in \Sigma'$, and (b) $I_{s^\frown\langle k_s\rangle}$ and $I_{s^\frown\langle l_s\rangle}$ both are to the left of x_s, and therefore they do not intersect $B_s(x)$, if $B_s(x) \in \mathcal{C}$ for some x. Note that they also do not intersect any $B_{s'}(x')$ with $|s'| = |s|$ and $s' \neq s$.

Hence the set $P = \bigcap_{n\in\omega} \big(\bigcup\{I_s\colon s \in \Sigma', |s| = n\}\big)$ is a Cantor set (in the usual topology of the reals) that does not intersect any element of $\mathcal{C} \cap \mathcal{B}_1$. Since $S \setminus X$ is totally imperfect there is $p \in X \cap P$. Then p is not covered by $\mathcal{C} \cap \mathcal{B}_1$. Since p is of type one, p is not covered by $\mathcal{C} \cap \mathcal{B}_2$ either. Therefore $\mathcal{C}$ does not cover X. $\square$

3. Examples and problems

If X is as in Theorem 2.2 we do not know if every base for X in S is common. If so, then X would be paracompact but not base-base paracompact. The base of all half-open intervals is common (which by Theorem 2.2 implies that e.g. the irrationals as a subspace of S are not totally paracompact, relating to Problem 3.1 of [2]). Given any common base $\mathcal{B}$, for simplicity consisting of half-open intervals, and any partition $\{E_m\colon m \in \omega\}$ of X, we obtain another common base $\tilde{\mathcal{B}}$ by removing from $\mathcal{B}$ all

$[x, x+t)$ with $x \in E_m$ and $t > 1/m$. The sets A_n from Definition 2.1 that work for $\mathcal{B}$ need not work for $\tilde{\mathcal{B}}$, but the sets $A_n \cap E_m$, $n, m \in \omega$, would.

E x a m p l e 3.1. Let $S \setminus X$ be dense and totally imperfect. Since $|X| = 2^\omega = \mathfrak{c}$ we may list $X = \{x_\alpha \colon \alpha < \mathfrak{c}\}$. As in the proof that under MA there is a scale [24], we may find a family of monotone increasing functions $\{f_\alpha \colon \alpha < \mathfrak{c}\}$ such that if $\beta < \alpha$ then $f_\alpha(n)$ goes to ∞ faster, as $n \to \infty$, than $f_\beta(n)$ does (e.g. $\lim\limits_{n\to\infty} f_\alpha(n)/f_\beta(n+k) = \infty$ for all k). Then $1/f_\alpha(n)$ goes to 0 faster than $1/f_\beta(n)$ does. If $\mathcal{B}_{x_\alpha} = \{[x_\alpha, x_\alpha + 1/f_\alpha(n)) \colon n \in \omega\}$ then $\mathcal{B} = \bigcup\limits_{\alpha<\mathfrak{c}} \mathcal{B}_{x_\alpha}$ is a base for X in S.

We do not know if the base $\mathcal{B}$ in the above example is common or not. But the mere variety of "essentially different" local bases at different points (as in $\mathcal{B}$ above) is not enough to produce a base that is not common, as the next example shows.

E x a m p l e 3.2. Consider $S \cap [0,1]$ instead of S. Let X be the set of all dyadic irrationals in $[0,1]$, i.e. all sums $x = \sum\limits_{k=1}^{\infty} a_k/2^k$ where $a_k \in \{0,1\}$ and both $a_k = 0$ and $a_k = 1$ occur infinitely often. Define a local base $\mathcal{B}_x$ at x as $\mathcal{B}_x = \{[x, x+2^{-k}) \colon a_k = 1\}$. Clearly $\mathcal{B}_x = \{[x, x+1/f_x(n)) \colon n \in \omega\}$ for a unique monotone increasing f_x. Then given any $g \colon \omega \to \omega$ there is $x \in X$ such that $f_x(n) \geqslant g(n)$ for all n. Nevertheless, we now show that the base for X obtained in this way is common. Let $T = [0,1)$ and $A_n = X$ for all n. Given any interval $I \subseteq T$, fix a finite sequence u of 0's and 1's such that $[u] \subseteq I$ where $[u]$ is the set of all $x \in [0,1]$ whose dyadic representation $\langle a_1, a_2, \ldots\rangle$ starts with u. Let $J = [u^\frown\langle 0\rangle^\frown\langle 1\rangle]$. Then for each x in $J \cap X$ there is an element $B(x)$ in $\mathcal{B}_x$ corresponding to the 1 at the end of $u^\frown\langle 0\rangle^\frown\langle 1\rangle$. All these $B(x)$ have the same length $\varepsilon = 2^{-|u^\frown\langle 0\rangle^\frown\langle 1\rangle|}$ and are contained in I.

Peter de Caux [3] proved that every finite power of S is a hereditarily D-space. He used a special base, described below, easily seen to be common, too.

E x a m p l e 3.3 [3]. The base $\mathcal{B}$ consists of all $[x, t_i)$ where $i \in \omega$ and t_i is the smallest integer multiple of 2^{-i} larger than x.

Q u e s t i o n 3.4. Let X be as in Theorem 2.2. (a) Is there a base for X that is not common? (b) Is X an example of a space that is not base-base paracompact?

Since Lusin subsets of S are base-base paracompact, one may inquire about other "small" subsets of S, including some that exist in ZFC, which leads to the following question:

Q u e s t i o n 3.5. Is every Marczewski null subspace of the Sorgenfrey line base-base paracompact? (A set M of real numbers is *Marczewski null* if for each perfect set P there is a perfect set Q contained in $P \setminus M$.)

Recall that $w(X)$ is the *weight* of X, i.e. the minimal possible cardinality of a base. X is *base-paracompact* [21], [22] if it has an open base $\mathcal{B}$ with $|\mathcal{B}| = w(X)$, such that every open cover has a locally finite refinement with elements of $\mathcal{B}$. (A base with only the latter property is called *fine* in [4], [12].) Base-base paracompact spaces are base-paracompact [21], but these two properties seem quite different for the following reason. Suppose $\mathcal{B}_1 \subseteq \mathcal{B} \subseteq \mathcal{B}_2$ are bases for a space X. If $\mathcal{B}$ witnesses base-base paracompactness, then so does $\mathcal{B}_1$, but $\mathcal{B}_2$ need not. The opposite holds for base-paracompactness: If $\mathcal{B}$ witnesses base-paracompactness, then so does $\mathcal{B}_2$ as long as $|\mathcal{B}_2| = w(X)$, but $\mathcal{B}_1$ need not. It is not known if paracompact spaces are base-paracompact [21], [22], see also [19], [20]. But, it is known that Lindelöf spaces, in particular every subspace of S, are base-paracompact [21], [22], see also [23].

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