ON MAGIC JOINS OF GRAPHS

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Abstract. A graph is called magic (supermagic) if it admits a labeling of the edges by pairwise different (and consecutive) integers such that the sum of the labels of the edges incident with a vertex is independent of the particular vertex. In this paper we characterize magic joins of graphs and we establish some conditions for magic joins of graphs to be supermagic.

Keywords: magic graph, supermagic graph, join of graphs

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1. Introduction

We consider finite undirected graphs without loops, multiple edges and isolated vertices. If G is a graph, then $V(G)$ and $E(G)$ stand for the vertex set and the edge set of G, respectively. Cardinalities of these sets are called the *order* and the *size* of G.

Let a graph G and a mapping f from $E(G)$ into positive integers be given. The *index-mapping* of f is the mapping f^* from $V(G)$ into positive integers defined by

$$f^*(v) = \sum_{e \in E(G)} \eta(v,e) f(e) \quad \text{for every } v \in V(G),$$

where $\eta(v,e)$ is equal to 1 when e is an edge incident with a vertex v, and 0 otherwise. An injective mapping f from $E(G)$ to positive integers is called a *magic labeling* of G for an *index* λ if its index-mapping f^* satisfies

$$f^*(v) = \lambda \quad \text{for all } v \in V(G).$$

A magic labeling f of G is called a *supermagic labeling* of G if the set $\{f(e)\colon e \in E(G)\}$ consists of consecutive positive integers. We say that a graph G is *supermagic* (*magic*) whenever there exists a supermagic (magic) labeling of G.

The concept of magic graphs was introduced by Sedláček [10]. The regular magic graphs are characterized in [3]. Two different characterizations of all magic graphs are given in [9] and [8]. Supermagic graphs were introduced by Stewart [12]. There is by now a considerable number of papers published on supermagic graphs; we single out [6], [4], [7], [11], [1] as being more particularly relevant to the present paper, and refer the reader to [5] for comprehensive references.

Let $G \cup H$ denote the disjoint union of graphs G and H. The *join* $G \oplus H$ of the disjoint graphs G and H is the graph $G \cup H$ together with all edges joining vertices of $V(G)$ and vertices of $V(H)$. In this paper we will deal with magic and supermagic joins of graphs.

2. Magic graphs

In this section, we characterize magic joins of graphs. Since, except for the complete graph of order 2, no graph with less than 5 vertices is magic, we consider only joins of order at least 5. Moreover, if G and H are edgeless graphs then $G \oplus H$ is isomorphic to the complete bipartite graph $K_{m,n}$, where m and n are the orders of G and H. Since the graph $K_{m,n}$ is magic (and also supermagic) if and only if $m = n \neq 2$, we consider only joins $G \oplus H$ where $E(G) \cup E(H) \neq \emptyset$.

We say that a graph G is of *type* $\mathcal{A}$ if it has two edges e_1, e_2 such that $G - \{e_1, e_2\}$ is a balanced bipartite graph with a partition V_1, V_2, and the edge e_i joins two vertices of V_i. A graph G is of *type* $\mathcal{B}$ if it has two edges e_1, e_2 such that $G - \{e_1, e_2\}$ has a component C which is a balanced bipartite graph with partition V_1, V_2, and e_i joins a vertex of V_i with a vertex of $V(G) - V(C)$. For any non-empty subset S of $V(G)$, $\Gamma_G(S)$ denotes the set of vertices in G adjacent to a vertex in S.

In what follows we will use the following characterization of magic graphs.

Proposition 1 (Jeurissen [8]). *A connected non-bipartite graph G is magic if and only if G is neither of type $\mathcal{A}$ nor of type $\mathcal{B}$, and $|\Gamma_G(S)| > |S|$ for every independent non-empty subset S of $V(G)$.*

Magic graphs $G \oplus K_1$ were characterized in [11]. Now we can prove an extension.

Theorem 1. *Let G and H be graphs such that $|V(G)| \geqslant |V(H)| \geqslant 1$, $|V(G)| + |V(H)| \geqslant 5$ and $|E(G)| + |E(H)| \geqslant 1$. The graph $G \oplus H$ is magic if and only if the following conditions are satisfied:*

(i) $|\Gamma_G(S)| + |V(H)| > |S|$ *for every independent subset S of $V(G)$.*

(ii) *G does not contain an isolated edge when $|V(H)| = 1$.*

(iii) $|E(H)| \geqslant 1$ *and* $|E(G)| + |E(H)| \geqslant 3$ *when* $|V(H)| = |V(G)|$.

P r o o f. Assume that the graph $G \oplus H$ is not magic. Since $|E(G)| + |E(H)| \geqslant 1$, $G \oplus H$ is a non-bipartite graph. According to Proposition 1, we consider the following cases.

A. Suppose that $G \oplus H$ is of type $\mathcal{A}$. As $G \oplus H$ is a balanced bipartite graph with two added edges it has at least 6 vertices and the corresponding partition is $V(G)$, $V(H)$. Added edges join two vertices in $V(G)$ and two vertices in $V(H)$. Therefore, $|V(H)| = |V(G)|$ and $|E(H)| = |E(G)| = 1$, i.e., condition (iii) is not satisfied.

B. Suppose that $G \oplus H$ is of type $\mathcal{B}$. The edge-connectivity of $G \oplus H$ is at most two in this case. Thus, $|V(H)| \leqslant 2$. If $|V(H)| = 2$ then any 2-edge cut of $G \oplus H$ consists of edges incident to an isolated vertex v of G. The graph $G \oplus H - v$ is bipartite only if both G and H are edgeless graphs, contrary to $|E(G)| + |E(H)| \geqslant 1$. If $|V(H)| = 1$ then any edge cut of $G \oplus H$ contains an edge incident to u, where $V(H) = \{u\}$. As u is adjacent to all other vertices, it is not a vertex of the bipartite component C of $G \oplus H - \{e_1, e_2\}$. Therefore, $V(C) \subset V(G)$, $|V(C)| = 2$ and the unique edge of C is an isolated edge of G, contrary to (ii).

C. There exists an independent set S of vertices of $G \oplus H$ such that $|\Gamma_{G\oplus H}(S)| \leqslant |S|$. The set S is independent, thus either $S \subseteq V(H)$ or $S \subseteq V(G)$. If $S \subseteq V(H)$, then $V(G) \subseteq \Gamma_{G\oplus H}(S)$. Therefore, $S = V(H)$ and $|V(H)| = |V(G)|$, i.e., condition (iii) is not satisfied. If $S \subseteq V(G)$, then $\Gamma_{G\oplus H}(S) = V(H) \cup \Gamma_G(S)$ and condition (i) is not satisfied.

The converse implication is obvious. □

A *complete k-partite graph* is a graph whose vertices can be partitioned into $k \geqslant 2$ disjoint classes $V_1, \ldots, V_k$ such that two vertices are adjacent if and only if they belong to distinct classes. If $|V_i| = n_i$ for all $i = 1, \ldots, k$, then the complete k-partite graph is denoted by $K_{n_1,\ldots,n_k}$. If $n_i = n$ for all $i = 1, \ldots, k$, then the complete k-partite graph is regular of degree $(k-1)n$ and is denoted by $K_{k[n]}$. Similarly, if $n_i = n$ for all $i = 1, \ldots, k$ and $n_{k+1} = p$ then the complete $(k+1)$-partite graph is denoted by $K_{p,k[n]}$.

Clearly, $K_{n_1,\ldots,n_k} = \overline{K}_{n_k} \oplus K_{n_1,\ldots,n_{k-1}}$, where $\overline{K}_{n_k}$ is an edgeless graph of order n_k. According to Theorem 1 we have immediately

Corollary 1. *Let $k \geqslant 3$ and $1 \leqslant n_1 \leqslant \ldots \leqslant n_k$ be integers. The complete k-partite graph $K_{n_1,\ldots,n_k}$ is magic if and only if*

$$n_1 + \ldots + n_{k-1} > n_k \qquad \textit{and} \qquad n_1 + \ldots + n_k \geqslant 5.$$

The problem of characterizing supermagic joins of general graphs seems to be difficult. In the next sections we present only some necessary and some sufficient conditions.

3. Necessary conditions

Any supermagic graph is magic. Thus the conditions of Theorem 1 are necessary for $G \oplus H$ to be supermagic. In this section we establish some other necessary conditions.

Theorem 2. *Let G be a graph with n vertices and m edges. Let g be the greatest common divisor of integers n and $2m$, and let $\nu = n/g$, $\varepsilon = 2m/g$. If G admits a supermagic labeling onto $\{a, a+1, \ldots, a+m-1\}$ for an index λ, then g is a divisor of $(m-1)m$ and there exists an integer t such that*

$$\lambda = \varepsilon\tau \qquad \text{and} \qquad a = \frac{1-m}{2} + \nu\tau,$$

where

$$\frac{m+1}{2\nu} \leqslant \tau = \begin{cases} t & \text{for odd } m, \\ t + \frac{1}{2} & \text{for even } m. \end{cases}$$

Proof. Suppose that $f\colon\ E(G) \to \{a, a+1, \ldots, a+m-1\}$ is a supermagic labeling of G for an index λ. Then we have

$$\begin{aligned} n\lambda &= \sum_{v\in V(G)} \sum_{e\in E(G)} \eta(v,e) f(e) = 2 \sum_{e\in E(G)} f(e) \\ &= 2[a + \ldots + (a+m-1)] = (2a+m-1)m. \end{aligned}$$

Thus, $n\lambda - 2ma = (m-1)m$, i.e., the pair (λ, a) is a solution of the linear Diophantine equation

$$nx - 2my = (m-1)m. \tag{1}$$

It is well known that this equation has a solution if and only if $g = \gcd(n, 2m)$ divides $(m-1)m$. Moreover, if (x_0, y_0) is one solution then all other solutions are given by $(x_0 + 2mg^{-1}t, y_0 + ng^{-1}t)$ for any integer t.

If m is odd then $(1-m)/2$ is an integer and $n0 - 2m(1-m)/2 = (m-1)m$. Therefore, $(0, (1-m)/2)$ is a solution of (1) and so there exists an integer t such that

$$\lambda = 0 + \frac{2m}{g}t = \varepsilon\tau \quad \text{and} \quad a = \frac{1-m}{2} + \frac{n}{g}t = \frac{1-m}{2} + \nu\tau.$$

If m is even then $g = \gcd(n, m)$, because $g = 2\gcd(n, m)$ does not divide $(m-1)m$. Thus, ε is even and ν is odd. Consequently, $\varepsilon/2$ and $(\nu + 1 - m)/2$ are integers. As

$$n\frac{\varepsilon}{2} - 2m\frac{\nu+1-m}{2} = n\frac{2m}{2g} - m\frac{n}{g} + (m-1)m = (m-1)m,$$

$(\varepsilon/2, (\nu+1-m)/2)$ is a solution of the equation (1) and so there exists an integer t such that

$$\lambda = \frac{\varepsilon}{2} + \frac{2m}{g}t = \varepsilon\tau \quad \text{and} \quad a = \frac{\nu+1-m}{2} + \frac{n}{g}t = \frac{1-m}{2} + \nu\tau.$$

Finally, a is a positive integer and so $(1-m)/2 + \nu\tau = a \geqslant 1$, which implies $\tau \geqslant (m+1)/2\nu$. □

Corollary 2. *Let α, β be positive integers and let p, q be odd positive integers.*

(i) *If $\alpha > \beta$ then there exists no supermagic graph with $2^\alpha p$ vertices and $2^\beta q$ edges.*

(ii) *If $\alpha = \beta$ then there exists no supermagic graph with $2^\alpha p$ vertices and $2^\beta q$ edges having a component of odd order.*

P r o o f. Let G be a graph of order $n = 2^\alpha p$ and size $m = 2^\beta q$ and let f be a supermagic labeling of G onto $\{a, a+1, \ldots, a+m-1\}$ for an index λ. Suppose that g and ε are as in Theorem 2.

If $\alpha > \beta$ then $g = 2^{\beta+1}\gcd(p,q)$. Clearly, in this case g does not divide $(m-1)m = (2^\beta q - 1)2^\beta q$, contrary to Theorem 2.

Assume that G has a component C of odd order. If $\alpha = \beta$ then $g = 2^\beta \gcd(p,q)$ and $\varepsilon = 2q/\gcd(p,q)$. According to Theorem 2, $\lambda = \varepsilon(t + \frac{1}{2}) = (2t+1)q/\gcd(p,q)$. Therefore, λ is odd. Hence $\lambda|V(C)|$ is odd, too. On the other hand,

$$|V(C)|\lambda = \sum_{v\in V(C)} \sum_{e\in E(G)} \eta(v,e)f(e) = 2\sum_{e\in E(C)} f(e),$$

a contradiction. □

Note that a special case of the previous result concerning regular graphs is proved in [6]. An assertion equivalent to (i) is proved in [4].

Let S be a non-empty set of vertices of a graph G. The average degree of S is denoted by $d(S)$, i.e., $d(S) = |S|^{-1} \sum_{v\in S} \deg(v)$. Evidently, $d(G) = d(V(G)) = 2|E(G)|/|V(G)|$.

Lemma 1. *Let G be a graph as in Theorem 2. Any independent set S of G satisfies*

$$2\nu|d(S) - d(G)|\tau \leqslant d(S)(m - |S|d(S)).$$

P r o o f. The set of edges incident to a vertex of S is denoted by E_S. Evidently, $|E_S| = |S|d(S)$. Suppose that $f\colon E(G) \to \{a, a+1, \ldots, a+m-1\}$ is a supermagic labeling of G for an index λ. If $d(S) = d(G)$ then the desired inequality is satisfied. Thus, we consider the following two cases.

A. Suppose $d(S) > d(G)$. Then we have

$$|S|\lambda = \sum_{v\in S} \sum_{e\in E(G)} \eta(v,e)f(e) = \sum_{e\in E_S} f(e)$$

$$\geqslant a + \ldots + (a + |E_S| - 1) = |E_S|\Big(a + \frac{|E_S|-1}{2}\Big).$$

Using Theorem 2 we get

$$|S|\varepsilon\tau \geqslant |E_S|\Big(\frac{1-m}{2} + \nu\tau + \frac{|E_S|-1}{2}\Big).$$

As $d(G) = 2m/n = \varepsilon/\nu$ and $|E_S| = |S|d(S)$, we have

$$2\nu(d(G) - d(S))\tau \geqslant d(S)(|S|d(S) - m),$$

which implies the desired inequality.

B. Suppose $d(S) < d(G)$. Then we have

$$|S|\lambda = \sum_{v\in S} \sum_{e\in E(G)} \eta(v,e)f(e) = \sum_{e\in E_S} f(e)$$

$$\leqslant (a + m - 1) + \ldots + (a + m - |E_S|) = |E_S|\Big(a + m - \frac{|E_S|+1}{2}\Big).$$

Using Theorem 2 we get

$$|S|\varepsilon\tau \leqslant |E_S|\Big(\frac{1-m}{2} + \nu\tau + m - \frac{|E_S|+1}{2}\Big).$$

As $d(G) = 2m/n = \varepsilon/\nu$ and $|E_S| = |S|d(S)$, we have

$$2\nu(d(G) - d(S))\tau \leqslant d(S)(m - |S|d(S)),$$

which implies the desired inequality. □

Theorem 3. *Let G be a supermagic graph of order n and size m. Any independent set S of G satisfies*

$$(m+1)\Big|d(S) - \frac{2m}{n}\Big| \leqslant d(S)(m - |S|d(S)).$$

P r o o f. Let G be a graph as in Theorem 2 and let S be any independent set of G. We can assume that $d(S) \neq d(G)$, because the desired inequality is trivial otherwise. According to Theorem 2, $\tau \geqslant (m+1)/2\nu$. By Lemma 1,

$$\tau \leqslant \frac{d(S)(m - |S|d(S))}{2\nu|d(S) - d(G)|} \quad \text{and so} \quad \frac{m+1}{2\nu} \leqslant \frac{d(S)(m - |S|d(S))}{2\nu|d(S) - d(G)|}.$$

As $d(G) = 2m/n$, we get the desired inequality after routine manipulation. □

For joins we have

Corollary 3. *Let G be a graph of order n and average degree d. If $\overline{K}_p \oplus G$ is a supermagic graph then*

$$n \leqslant \frac{1}{4p}(d^2 + 4pd + 2p^2 - 2 + \omega),$$

where $\omega = \sqrt{d^4 + 8pd^3 + 20p^2d^2 + 16p^3d + 4p^4 - 4d^2 + 8p^2 + 4}$.

P r o o f. The join $\overline{K}_p \oplus G$ has $n + p$ vertices and $nd/2 + np$ edges. The set of vertices of $\overline{K}_p$ is an independent set of $\overline{K}_p \oplus G$ and each of its p vertices has degree n. According to Theorem 3 we have

$$\Bigl(n\frac{d}{2} + np + 1\Bigr)\Bigl|n - \frac{nd + 2np}{n + p}\Bigr| \leqslant n\Bigl(n\frac{d}{2} + np - pn\Bigr).$$

After routine manipulation we get

$$(nd + 2np + 2)|n - p - d| \leqslant nd(n + p).$$

If $n \leqslant p + d$, then the inequality is satisfied because it is easy to see that $\omega \geqslant 2p^2 - d^2 + 2$.

If $n > p + d$, then $(nd + 2np + 2)(n - p - d) \leqslant nd(n + p)$. This implies

$$2pn^2 - (d^2 + 4pd + 2p^2 - 2)n - 2(p + d) \leqslant 0,$$

and the desired bound follows. □

Note that a special case of the previous result concerning $p = 1$ is proved in [11].

Corollary 4. *Let G be a graph of order n and size m. If $\overline{K}_p \oplus G$ is a supermagic graph then*

$$p \leqslant \frac{1}{2n}\bigl(n^2 - 2m - 1 + \sqrt{n^4 + 4n^2m - 4m^2 + 2n^2 - 4m + 1}\bigr).$$

P r o o f. The join $\overline{K}_p \oplus G$ has $n + p$ vertices and $m + np$ edges. The set of vertices of $\overline{K}_p$ is an independent set of $\overline{K}_p \oplus G$ and each of its p vertices has degree n. According to Theorem 3 we have

$$(m + np + 1)\Bigl|n - \frac{2m + 2np}{n + p}\Bigr| \leqslant n(m + np - pn).$$

After routine manipulation we obtain

$$(m + np + 1)|n^2 - np - 2m| \leqslant nm(n + p).$$

If $n^2 \geqslant np+2m$, then $p \leqslant n-2m/n$. However, the desired upper bound is greater than $n-2m/n = ((n^2-2m-1)+(n^2-2m+1))/2n$ because $n^4+4n^2m-4m^2+2n^2-4m+1 = (n^2-2m+1)^2+8m(n^2-m)$ and $n^2 > m$.

If $n^2 < np+2m$, then $(m+np+1)(np+2m-n^2) \leqslant nm(n+p)$ and consequently

$$n^2p^2-(n^2-2m-1)np-(2n^2m+n^2-2m^2-2m) \leqslant 0.$$

This inequality immediately implies the desired bound. □

Since $K_{p,k[n]} = \overline{K}_p \oplus K_{k[n]}$ and $K_{k[n]}$ is a graph with kn vertices and $k(k-1)n^2/2$ edges, we immediately get

Corollary 5. *Let p, n and $k \geqslant 2$ be positive integers. If $K_{p,k[n]}$ is a supermagic graph then*

$$p \leqslant \frac{1}{2kn}\big(kn^2-1+\sqrt{k^2(2k^2-1)n^4+2kn^2+1}\big).$$

4. Sufficient conditions

Clearly, if f is a supermagic labeling of a regular graph G, then $f+\kappa$, for every integer $\kappa > -\min\{f(e)\colon e \in E(G)\}$, is a supermagic labeling of G, too. Therefore, a regular graph G is supermagic if and only if it admits a supermagic labeling $f\colon E(G) \to \{1,2,\ldots,|E(G)|\}$. This can be generalized as follows.

Theorem 4. *A graph G is supermagic if and only if there are a bijection $g\colon E(G) \to \{1,2,\ldots,|E(G)|\}$ and a non-negative integer κ such that*

$$g^*(u)-g^*(v) = \kappa(\deg(v)-\deg(u)) \quad \textit{for all } u,v \in V(G).$$

Proof. Suppose that $f\colon E(G) \to \{a, a+1, \ldots, a+|E(G)|-1\}$ is a supermagic labeling of a graph G for an index λ. Consider the mapping g defined by

$$g(e) = f(e)-a+1 \quad \text{for all } e \in E(G).$$

Evidently, g is a bijection from $E(G)$ onto $\{1,2,\ldots,|E(G)|\}$. Moreover, $g^*(w) = \lambda-(a-1)\deg(w)$ for any vertex $w \in V(G)$. Therefore

$$g^*(u)-g^*(v) = (a-1)(\deg(v)-\deg(u))$$

for all $u,v \in V(G)$.

Assume that $g\colon E(G) \to \{1, 2, \ldots, |E(G)|\}$ is a bijection satisfying $g^*(u) - g^*(v) = \kappa(\deg(v) - \deg(u))$ for all $u, v \in V(G)$. Consider the mapping f given by

$$f(e) = g(e) + \kappa \quad \text{for all } e \in E(G).$$

Clearly, f is a bijection from $E(G)$ onto $\{\kappa + 1, \kappa + 2, \ldots, \kappa + |E(G)|\}$. Moreover, $f^*(w) = g^*(w) + \kappa \deg(w)$ for any vertex $w \in V(G)$. This implies

$$f^*(u) - f^*(v) = (g^*(u) - g^*(v)) + \kappa(\deg(u) - \deg(v)) = 0$$

for all $u, v \in V(G)$, which means that f is a supermagic labeling of G. □

Let $\mathfrak{S}(n; d, r)$ denote the family of all d-regular graphs of order n which admit a supermagic labeling f onto $\{1, 2, \ldots, nd/2\}$ such that the set of edges $\{e\colon f(e) \leqslant nr/2\}$ induces an r-factor of a graph. Clearly, $G \in \mathfrak{S}(n; d, r)$ is a supermagic graph (and $d \geqslant 3$ when $n > 2$). On the other hand, any supermagic d-regular graph of order n belongs to $\mathfrak{S}(n; d, 0)$ and $\mathfrak{S}(n; d, d)$. Moreover, it is easy to see that $G \in \mathfrak{S}(n; d, r)$ if and only if $G \in \mathfrak{S}(n; d, d - r)$.

Now we present a construction of supermagic irregular graphs.

Theorem 5. *Let p, n, d and r be integers such that $d \geqslant r \geqslant 0$, $n > d \geqslant 3$ and $n > p \geqslant 2$. Put*

$$\varpi = \frac{1}{2}((p + d) + n((d^2/2 - rn - 1) + p(p + 2d - r - n)))$$

and suppose that the following conditions are satisfied:
(i) $n \equiv p \pmod 2$;
(ii) $n - p - d \neq 0$;
(iii) $\varpi(n - p - d) \geqslant 0$;
(iv) $\varpi \equiv 0 \pmod{|n - p - d|}$.
If G is any regular graph belonging to $\mathfrak{S}(n; d, r)$ then the join $\overline{K}_p \oplus G$ is a supermagic graph.

Proof. Let $H = (\overline{K}_p \oplus G) - E(G)$. Evidently, H is a spanning subgraph of $\overline{K}_p \oplus G$ isomorphic to $K_{p,n}$. As $n > p \geqslant 2$ and $n \equiv p \pmod 2$, the complete bipartite graph $K_{p,n}$ is degree-magic (see [1]). This means there is a bijection $h\colon E(H) \to \{1, 2, \ldots, pn\}$ such that $h^*(w) = \frac{1}{2}(1 + pn)\deg_H(w)$ for every vertex $w \in V(H)$.

Since G belongs to $\mathfrak{S}(n; d, r)$ there is a supermagic labeling f of G onto $\{1, 2, \ldots, nd/2\}$ such that the set of edges $\{e \in E(G)\colon f(e) \leqslant nr/2\}$ induces an r-factor of G. Clearly, $f^*(v) = (1 + nd/2)d/2$ for every vertex $v \in V(G)$.

Consider a mapping g from $E(\overline{K}_p \oplus G)$ into positive integers given by

$$g(e) = \begin{cases} f(e) & \text{if } e \in E(G) \text{ and } f(e) \leqslant nr/2, \\ h(e) + nr/2 & \text{if } e \in E(H), \\ f(e) + np & \text{if } e \in E(G) \text{ and } f(e) > nr/2. \end{cases}$$

Evidently, g is a bijection from $E(\overline{K}_p \oplus G)$ onto $\{1, 2, \ldots, np + nd/2\}$. For any vertex $v \in V(G)$ we have

$$\begin{aligned} g^*(v) &= f^*(v) + (d-r)np + h^*(v) + \frac{1}{2}pnr \\ &= \frac{1}{2}(1 + nd/2)d + (d-r)np + \frac{1}{2}(1+np)p + \frac{1}{2}pnr \\ &= \frac{1}{2}\Big((p+d) + n\Big(\frac{1}{2}d^2 + p^2 + 2pd - pr\Big)\Big). \end{aligned}$$

Similarly, for every vertex $u \in V(\overline{K}_p)$ we have

$$\begin{aligned} g^*(u) = h^*(u) + \frac{1}{2}nnr &= \frac{1}{2}(1+np)n + \frac{1}{2}n^2 r \\ &= \frac{1}{2}(1 + np + nr)n. \end{aligned}$$

Thus, $\varpi = g^*(v) - g^*(u)$ for any $v \in V(G)$ and $u \in V(\overline{K}_p)$. According to (iii) and (iv), there is a non-negative integer κ such that $\varpi = \kappa(n - p - d)$. Therefore,

$$g^*(v) - g^*(u) = \kappa(n-p-d) = \kappa(\deg(u) - \deg(v))$$

for any $v \in V(G)$ and $u \in V(\overline{K}_p)$. So $g^*(v) - g^*(u) = \kappa(\deg(u) - \deg(v))$ for any u, v of $\overline{K}_p \oplus G$. By Theorem 4, $\overline{K}_p \oplus G$ is a supermagic graph. □

Corollary 6. *Let G be a supermagic d-regular graph of order $n \geqslant 6$, where $3 \leqslant d \equiv 1 \pmod 2$ and $n \equiv 2 \pmod 4$. For $p \in \{n-d+1, n-d-1\}$, $p \geqslant 2$, the join $\overline{K}_p \oplus G$ is a supermagic graph.*

P r o o f. Suppose that $p = n - d + 1$. A graph G belongs to $\mathfrak{S}(n; d, d)$ and conditions (i), (ii), (iv) of Theorem 5 are satisfied in this case. Moreover,

$$\begin{aligned} \varpi &= \frac{1}{2}\Big(n + 1 + n\Big(\frac{1}{2}d^2 + n - dn - d\Big)\Big) = \frac{1}{2}\Big((1-d)n(n+1) + 1 + \frac{1}{2}nd^2\Big) \\ &< \frac{1}{2}\Big((1-d)n(n+1) + \frac{1}{2}n(n+1)d\Big) = \frac{1}{2}\Big(1 - d + \frac{1}{2}d\Big)n(n+1) < 0. \end{aligned}$$

Thus, (iii) is also satisfied.

Assume that $p = n - d - 1 \geqslant 2$. A graph G belongs to $\mathfrak{S}(n; d, 0)$ and conditions (i), (ii), (iv) of Theorem 5 are satisfied. Moreover,

$$\begin{aligned}\varpi &= \frac{1}{2}\Big(n - 1 + n\Big(nd - n - \frac{1}{2}d^2\Big)\Big) > \frac{n}{2}\Big(nd - n - \frac{1}{2}d^2\Big)\\ &= \frac{n}{4}(n^2 - 2n - n^2 + 2nd - d^2) = \frac{n}{4}(n(n-2) - (n-d)^2) > 0,\end{aligned}$$

because $n > d \geqslant 3$. Therefore, (iii) is also satisfied. □

Corollary 7. *Let G be a supermagic d-regular graph of order $n > 6$, where $4 \leqslant d \equiv 0 \pmod 2$ and $n \equiv 1 \pmod 2$. For $p \in \{n - d + 2, n - d - 2\}$, $p > 2$, the join $\overline{K}_p \oplus G$ is a supermagic graph.*

Proof. Suppose that $p = n - d + 2$. A graph G belongs to $\mathfrak{S}(n; d, d)$ and conditions (i) and (ii) of Theorem 5 are satisfied in this case. We have

$$\begin{aligned}\varpi &= \frac{1}{2}\Big(n + 2 + n\Big(\frac{1}{2}d^2 + 2n - dn - 2d + 3\Big)\Big)\\ &= \frac{1}{2}\Big((2-d)(n+1)^2 + \frac{1}{2}d(2 + nd)\Big)\\ &< \frac{1}{2}\Big((2-d)(n+1)^2 + \frac{1}{2}(n+1)^2 d\Big) = \frac{1}{2}\Big(2 - d + \frac{1}{2}d\Big)(n+1)^2 \leqslant 0.\end{aligned}$$

This implies that (iii) is satisfied. Moreover,

$$\varpi = \frac{2-d}{2}(n+1)^2 + \frac{d}{2}\frac{2 + nd}{2}.$$

As d is even and n is odd, $(2-d)/2$, $d/2$, $(2+nd)/2$ are integers and either $d/2$ or $(2+nd)/2$ is an even integer. Thus, ϖ is an even integer, and (iv) is also satisfied in this case.

Assume that $p = n - d - 2 > 2$. A graph G belongs to $\mathfrak{S}(n; d, 0)$ and conditions (i) and (ii) of Theorem 5 are satisfied. We have

$$\begin{aligned}\varpi &= \frac{1}{2}\Big(n - 2 + n\Big(nd - 2n - \frac{1}{2}d^2 + 3\Big)\Big) = \frac{1}{2}\Big(nd\Big(n - \frac{1}{2}d\Big) - 2(n-1)^2\Big)\\ &> \frac{n}{2}\Big(nd - 2n - \frac{1}{2}d^2\Big) > \frac{n^2}{2}\Big(d - 2 - \frac{1}{2}d\Big) \geqslant 0,\end{aligned}$$

because $d \geqslant 4$. Therefore, (iii) is satisfied. Moreover,

$$\varpi = n\frac{d}{2}\Big(n - \frac{d}{2}\Big) - (n-1)^2.$$

As d is even and n is odd, $d/2$ and $n - d/2$ are integers and one of them is even. Thus, ϖ is an even integer, and (iv) is also satisfied. □

In [6], supermagic regular complete multipartite graphs $K_{k[n]}$ are characterized. According to the characterization, Corollary 6 and Corollary 7, we immediately get

Corollary 8. *Let p, n, $k \geqslant 2$ be positive integers such that one of the following conditions is satisfied:*
(i) $p = 2$, $6 \leqslant k \equiv 2 \pmod 4$, $n = 1$;
(ii) $p = n - 1$, $k \equiv 2 \pmod 4$, $3 \leqslant n \equiv 1 \pmod 2$;
(iii) $p = n + 1$, $k \equiv 2 \pmod 4$, $3 \leqslant n \equiv 1 \pmod 2$;
(iv) $p = 3$, $7 \leqslant k \equiv 1 \pmod 2$, $n = 1$;
(v) $p = 5$, $k \equiv 1 \pmod 2$, $n = 3$;
(vi) $p = n - 2$, $k \equiv 1 \pmod 2$, $5 \leqslant n \equiv 1 \pmod 2$;
(vii) $p = n + 2$, $k \equiv 1 \pmod 2$, $5 \leqslant n \equiv 1 \pmod 2$.
Then the complete multipartite graph $K_{p,k[n]}$ is supermagic.

We conclude this paper with a sufficient condition for $K_{p,2[n]}$ to be supermagic, but first we prove the following auxiliary result.

Lemma 2. *Let n be an odd positive integer. The complete bipartite graph $K_{n,n}$ belongs to $\mathfrak{S}(2n; n, r)$ for every integer r, $0 \leqslant r \leqslant n$.*

P r o o f. In the proof we use Latin squares. A Latin square of order n is a square matrix of order n such that every row and column is a permutation of integers $\{1, 2, \ldots, n\}$. Two Latin squares $A = (a_{i,j})$ and $B = (b_{i,j})$ of order n are called orthogonal, if all n^2 ordered pairs $[a_{i,j}, b_{i,j}]$, $i, j \in \{1, 2, \ldots, n\}$, are different. In [2] it is proved that two orthogonal Latin squares of order n exist if and only if $n \notin \{2, 6\}$. Therefore, there exist two orthogonal Latin squares A and B of order n in this case.

Denote the vertices of $K_{n,n}$ by $u_1, u_2, \ldots u_n, v_1, v_2, \ldots, v_n$ in such a way that $\{u_1, u_2, \ldots, u_n\}$ and $\{v_1, v_2, \ldots, v_n\}$ are its maximal independent sets. Consider a mapping $f\colon\ E(K_{n,n}) \to \{1, 2, \ldots, n^2\}$ given by

$$f(u_i v_j) = (a_{i,j} - 1)n + b_{i,j}.$$

Since A and B are Latin squares of order n, $1 \leqslant f(u_i v_j) \leqslant n^2$ for every edge $u_i v_j$ of $K_{n,n}$. Moreover, $f(u_i v_j) = f(u_s v_t)$ if and only if $[a_{i,j}, b_{i,j}] = [a_{s,t}, b_{s,t}]$ and so f is injective, because A, B are orthogonal. As $|E(K_{n,n})| = n^2$, f is a bijection. For any vertex u_i we have

$$\begin{aligned} f^*(u_i) &= \sum_{j=1}^{n} f(u_i v_j) = \sum_{j=1}^{n} (a_{i,j} - 1)n + \sum_{j=1}^{n} b_{i,j} \\ &= (0 + 1 + \ldots + n - 1)n + (1 + 2 + \ldots + n) = \frac{n}{2}(n^2 + 1). \end{aligned}$$

Similarly, $f^*(v_j) = \frac{1}{2}n(n^2+1)$ for any vertex v_j. Therefore, f is a supermagic labeling of $K_{n,n}$.

Let H_r be a subgraph of $K_{n,n}$ induced by $\{e \in E(K_{n,n})\colon f(e) \leqslant nr\}$. Then $\Gamma_{H_r}(u_i) = \{v_j\colon a_{i,j} \leqslant r\}$ and $\deg_{H_r}(u_i) = |\Gamma_{H_r}(u_i)| = r$. Similarly, $\deg_{H_r}(v_j) = |\{u_i\colon a_{i,j} \leqslant r\}| = r$. The subgraph H_r is r-regular and so $K_{n,n}$ belongs to $\mathfrak{G}(2n; n, r)$. □

Theorem 6. *Let $s \geqslant t$ be positive integers. Then the complete tripartite graph $K_{2t,2[2s+1]}$ is supermagic.*

P r o o f. The graph $K_{2t,2[2s+1]}$ is isomorphic to $\overline{K}_{2t} \oplus K_{2[2s+1]}$. According to Lemma 2, $K_{2[2s+1]}$ belongs to $\mathfrak{G}(4s+2; 2s+1, t)$ and conditions (i) and (ii) of Theorem 5 are satisfied in this case. Moreover, we have

$$\begin{aligned}\varpi &= \frac{1}{2}\Big(2t + 2s + 1 + 2(2s+1)\Big(\frac{1}{2}(2s+1)^2 - 2t(2s+1) - 1 + 2t^2\Big)\Big)\\ &= (2s - 2t + 1)((2s+1)(s-t) + s).\end{aligned}$$

Now it is easy to see that $\varpi > 0$ and conditions (iii) and (iv) are satisfied. By theorem 5, $K_{2t,2[2s+1]}$ is a supermagic graph. □

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