

ON THE LOCALIZATION OF THE SPECTRUM FOR
QUASI-SELFADJOINT EXTENSIONS OF A CARLEMAN OPERATOR

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Abstract. In the present work, using a formula describing all scalar spectral functions of a Carleman operator A of defect indices $(1, 1)$ in the Hilbert space $L^2(X, \mu)$ that we obtained in a previous paper, we derive certain results concerning the localization of the spectrum of quasi-selfadjoint extensions of the operator A .

Keywords: defect indices, integral operator, quasi-selfadjoint extension, spectral theory

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1. PRELIMINARIES

Let A be a closed symmetric operator with a dense domain $D(A)$ in a separable Hilbert space H endowed with an inner product (\cdot, \cdot) .

Let \mathfrak{M}_λ denote the range of the operator $(A - \lambda I)$, then its orthogonal complement in H

$$\mathfrak{N}_{\bar{\lambda}} = H \ominus \mathfrak{M}_\lambda$$

coincides with the eigenspace corresponding to the eigenvalue $\bar{\lambda}$ of the operator A^* . The sets $D(A)$, \mathfrak{N}_λ and $\mathfrak{N}_{\bar{\lambda}}$ ($\text{Im } \lambda \neq 0$) are linearly independent, hence according to von Neumann (see [1], [11]), the domain of the adjoint operator A^* admits the representation

$$(1.1) \quad D(A^*) = D(A) \oplus \mathfrak{N}_\lambda \oplus \mathfrak{N}_{\bar{\lambda}},$$

and

$$(1.2) \quad A^*f = Af_0 + \lambda\varphi_\lambda + \bar{\lambda}\varphi_{\bar{\lambda}}$$

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with $f_0 \in D(A)$, $\varphi_\lambda \in \mathfrak{N}_\lambda$ and $\varphi_{\bar{\lambda}} \in \mathfrak{N}_{\bar{\lambda}}$. The numbers $m = \dim \mathfrak{N}_\lambda$ and $n = \dim \mathfrak{N}_{\bar{\lambda}}$ do not change when λ belongs to the half-plane $\text{Im } \lambda > 0$. Then A is said to be of defect indices (m, n) . The formulas (1.1) and (1.2) show that A is selfadjoint iff it is of defect indices $(0, 0)$.

Further, let M and \widetilde{M} be two subspaces of H such that $M \subset \widetilde{M}$. The number n is called the dimension of \widetilde{M} modulo M (denoted $\dim_M \widetilde{M}$, i.e. $\dim \widetilde{M} = n \pmod{M}$) if there is n , and no more than n vectors f_1, f_2, \dots, f_n in \widetilde{M} such that

$$\alpha_1 f_1 + \alpha_2 f_2 + \dots + \alpha_n f_n \in M$$

implies that

$$\alpha_1 = \alpha_2 = \dots = \alpha_n = 0.$$

A quasi-selfadjoint extension of A of defect indices (m, m) ($m < \infty$) is an arbitrary linear operator B which satisfies the conditions

$$\begin{aligned} A &\subset B \subset A^*, \\ \dim D(B) &= m \pmod{D(A)} \end{aligned}$$

but is not a selfadjoint extension of the operator A .

For simplicity we restrict ourselves to the case of operators of defect indices $(1, 1)$. We shall assume that the operator A is simple (i.e. there exists no subspace invariant under A such that the restriction of A to this subspace is selfadjoint).

We recall that a number λ is called a regular point of the operator A if the operator $(A - \lambda I)^{-1}$ (I denotes the identity operator in H) exists, is bounded, and is defined in the whole space. The spectrum of the operator A is defined as the complement of the set of its regular points. In ([1], Appendix I, Section 5), it is proved that the spectrum of a quasi-selfadjoint extension B of a simple symmetric operator A of defect indices $(1, 1)$ consists of the spectral kernel (i.e., the complement of the set of all points of regular type) of A and the eigenvalues, and the set of the eigenvalues lies wholly either in the upper or in the lower half-plane.

2. CARLEMAN OPERATORS OF SECOND CLASS

One can find necessary information about Carleman operators, for example, in [7], [13], [18], [19], [20]. Let X be an arbitrary set, μ a σ -finite measure on X (μ is defined on a σ -algebra of subsets of X , we do not indicate this σ -algebra), $L^2(X, \mu)$ the Hilbert space of square integrable functions with respect to μ . For short, instead of writing ‘ μ -measurable’, ‘ μ -almost everywhere’ and ‘ $d\mu(x)$ ’ we write ‘measurable’, ‘a.e.’ and ‘ dx ’.

A linear operator $A: D(A) \longrightarrow L^2(X, \mu)$, where the domain $D(A)$ is a dense linear manifold in $L^2(X, \mu)$, is said to be integral if there exists a measurable function K on $X \times X$, a kernel, such that, for every $f \in D(A)$,

$$(2.1) \quad Af(x) = \int_X K(x, y)f(y) \, dy \quad \text{a.e.}$$

A kernel K on $X \times X$ is a Carleman kernel if $K(x, y) \in L^2(X, \mu)$ for almost every fixed x , that is to say

$$\int_X |K(x, y)|^2 \, dy < \infty \quad \text{a.e.}$$

The integral operator A defined by (2.1) is called a Carleman operator if K is a Carleman kernel. Since the closure of a Carleman operator always exists and is itself a Carleman operator [20], we can suppose also that A is closed.

Now we consider the Carleman integral operators (2.1) of second class that were introduced in [7], [3] generated by symmetric kernels of the form

$$K(x, y) = \sum_{p=0}^{\infty} a_p \psi_p(x) \overline{\psi_p(y)},$$

where the overbar denotes complex conjugation. Here $\{\psi_p(x)\}_{p=0}^{\infty}$ is an orthonormal sequence in $L^2(X, \mu)$ such that

$$\sum_{p=0}^{\infty} |\psi_p(x)|^2 < \infty \quad \text{a.e.},$$

and $\{a_p\}_{p=0}^{\infty}$ is a real number sequence verifying

$$\sum_{p=0}^{\infty} a_p^2 |\psi_p(x)|^2 < \infty \quad \text{a.e.}$$

We call $\{\psi_p(x)\}_{p=0}^{\infty}$ a Carleman sequence (we refer for instance to [20], Section 6.2).

Moreover, we assume that there exists a number sequence $\{\gamma_p\}_{p=0}^{\infty} \neq 0$ such that

$$(2.2) \quad \sum_{p=0}^{\infty} \gamma_p \psi_p(x) = 0 \quad \text{a.e.}$$

and

$$(2.3) \quad \sum_{p=0}^{\infty} \left| \frac{\gamma_p}{a_p - \lambda} \right|^2 < \infty.$$

Under the conditions (2.2) and (2.3), the symmetric operator $A = (A^*)^*$ is of defect indices (1, 1) (see [3]) with

$$A^* f(x) = \sum_{p=0}^{\infty} a_p(f, \psi_p) \psi_p(x),$$

$$D(A^*) = \left\{ f \in L^2(X, \mu) : \sum_{p=0}^{\infty} a_p(f, \psi_p) \psi_p(x) \in L^2(X, \mu) \right\}.$$

Moreover, in [4], we saw that

$$\begin{cases} \varphi_\lambda(x) = \sum_{p=0}^{\infty} \frac{\gamma_p}{a_p - \lambda} \psi_p(x) \in \mathfrak{N}_{\bar{\lambda}}, & \lambda \in \mathbb{C}, \lambda \neq a_k, k = 1, 2, \dots, \\ \varphi_{a_k}(x) = \psi_k(x) \end{cases}$$

with $\mathfrak{N}_{\bar{\lambda}}$ the defect space of A .

We denote by \mathfrak{L}_ψ the sub-space of $L^2(X, \mu)$ generated by the sequence $\{\psi_p(x)\}_{p=0}^{\infty}$. It is clear that the orthogonal complement $\mathfrak{L}_\psi^\perp = L^2(X, \mu) \ominus \mathfrak{L}_\psi$ is contained in $D(A)$ and cancels the operator A . As \mathfrak{L}_ψ is reduced by A (see [1]), we consider A on \mathfrak{L}_ψ . Then we have (see [4]) for all $f \in \mathfrak{L}_\psi$ and for almost all $x \in X$:

$$(2.4) \quad f(x) = \int_{-\infty}^{+\infty} \frac{(f, \varphi_\sigma) \varphi_\sigma(x)}{(\sigma^2 + 1) |(\varphi_\sigma, \overset{\circ}{\varphi}_i)|} d\varrho(\sigma),$$

$$(2.5) \quad \|f\|^2 = \int_{-\infty}^{+\infty} \frac{|(f, \varphi_\sigma)|^2}{(\sigma^2 + 1) |(\varphi_\sigma, \overset{\circ}{\varphi}_i)|} d\varrho(\sigma),$$

with

$$\overset{\circ}{\varphi}_i = \frac{\varphi_i}{\|\varphi_i\|} (\varphi_i \in \mathfrak{N}_{-i}),$$

and

$$(2.6) \quad \varrho(\sigma) = \frac{1}{\pi} \lim_{\tau \rightarrow +0} \int_0^\sigma \Re \frac{1 + \omega(t + i\tau)C(t + i\tau)}{1 - \omega(t + i\tau)C(t + i\tau)} dt,$$

were $\omega(\lambda)$ is an analytical function on the upper half-plane Π^+ with $|\omega(\lambda)| \leq 1$ ($\text{Im } \lambda > 0$) ($\text{Im } \lambda$ the imaginary part of λ) and $C(\lambda)$ is the function

$$C(\lambda) = \frac{[1 - \omega(\lambda)](f, \varphi_{\bar{\lambda}})}{[\omega(\lambda)\chi(\lambda) - 1](\lambda + i)(\varphi_\lambda, \varphi_i)} \quad (\text{Im } \lambda > 0)$$

with $\chi(\lambda)$ the characteristic function of A (see [4], [1]).

Let \mathfrak{P} be the set of all functions $\varrho(\sigma)$ defined by (2.6) (see [5]). We call such a function $\varrho(\sigma)$ the scalar spectral function of the operator A . This function characterizes the spectrum of the quasi-selfadjoint extension A_ω of the operator A associated with the analytic function $\omega(\lambda)$. The spectrum of this extension is the set of points of growth of $\varrho(\sigma)$. We recall here (see [5]) that $\varrho(\sigma)$ is called orthogonal scalar spectral function if it corresponds to a constant function $\omega(\lambda)$ with $|\omega| \equiv 1$.

Now let us look more closely at the function $\varrho(\sigma)$ given by (2.6). It is clear that the homographic function $(1+z)/(1-z)$ transforms the circle $|z|=1$ into the real line \mathbb{R} . So if

$$(2.7) \quad \omega(\lambda) = \varkappa$$

with $|\varkappa|=1$, then

$$\Re \frac{1 + \varkappa C(\sigma)}{1 - \varkappa C(\sigma)} = 0$$

for all $\sigma \in \mathbb{R}$ except at points σ satisfying

$$1 - \varkappa C(\sigma) = 0.$$

We infer that the function $\varrho(\sigma)$ associated with \varkappa has jumps at points of the spectrum of the selfadjoint extension A_\varkappa associated with \varkappa . This spectrum is formed by the zeros of the equation $C(\sigma) = \bar{\varkappa}$.

We denote by \mathfrak{G}_0 the convex hull of these functions:

$$\mathfrak{G}_0 = \left\{ \varrho(\sigma) = \sum_{k=1}^n \alpha_k \varrho_{\varkappa_k}, \alpha_k > 0, \sum_{k=1}^n \alpha_k = 1 \right\},$$

and $\mathfrak{G} = \overline{\mathfrak{G}_0}$, for the convergence at each point of continuity.

For any function $\varrho(\sigma) \in \mathfrak{G}_0$ we have:

$$(2.8) \quad \begin{aligned} \varrho(\sigma) &= \frac{1}{\pi} \lim_{\tau \rightarrow +0} \int_0^\sigma \Re \left[\sum_{k=1}^n \alpha_k \frac{1 + \varkappa_k C(\lambda)}{1 - \varkappa_k C(\lambda)} \right] dt \\ &= \frac{1}{\pi} \lim_{\tau \rightarrow +0} \int_0^\sigma \Re \left[\frac{1 + \omega(\lambda) C(\lambda)}{1 - \omega(\lambda) C(\lambda)} \right] dt \quad (\lambda = \sigma + i\tau) \end{aligned}$$

where $\omega(\lambda)$ is the analytical function corresponding to $\varrho(\sigma) \in \mathfrak{P}$.

Let now \mathfrak{M} be the set of all analytic functions $\varphi(z)$ on the unit disc $K = \{z \in \mathbb{C} : |z| < 1\}$ satisfying, $|\varphi(z)| \leq 1$, $z \in K$ and admitting the representation

$$(2.9) \quad \varphi(z) = \frac{\int_0^{2\pi} e^{it} (1 - ze^{it})^{-1} dS(t)}{\int_0^{2\pi} (1 - ze^{it})^{-1} dS(t)}$$

where $S(t)$ is a monotonic nondecreasing function with total variation equal to one, i.e. $\int_0^{2\pi} dS(t) = 1$. We denote by \mathfrak{M}_0 the set of all functions $\varphi(z) \in \mathfrak{M}$ with $S(t)$ a step function with a finite number of jumps. Consequently, from (2.8) and (2.9), we find easily that

$$(2.10) \quad \begin{aligned} \omega(\lambda) &= \frac{\sum_{k=1}^n \alpha_k \varkappa_k (1 - \varkappa_k C(\lambda))^{-1}}{\sum_{k=1}^n (1 - \varkappa_k C(\lambda))^{-1}} \\ &= \frac{\int_{-\infty}^{+\infty} e^{it} (1 - C(\lambda)e^{it})^{-1} dS(t)}{\int_{-\infty}^{+\infty} (1 - C(\lambda)e^{it})^{-1} dS(t)} = \varphi(C(\lambda)), \end{aligned}$$

with $\varphi(z) \in \mathfrak{M}_0$.

3. DESCRIPTION OF THE SPECTRUM OF QUASI-SELFADJOINT EXTENSIONS OF A CARLEMAN OPERATOR

In this section we will study the spectrum of the quasi-selfadjoint extension A_ω of the Carleman operator A which equals the set of all points of growth of its spectral scalar function $\varrho(t) \in \mathfrak{G}$ (see [1], [19]). We recall ([5], Theorem 2.1) that for all $\varrho_\omega(t) = \varrho(t) \in \mathfrak{G}$ there corresponds an analytic function $\omega(\lambda) = \varphi(C(\lambda))$ with $\varphi(z) \in \mathfrak{M}$.

In the previous section we have observed that the spectrum of a selfadjoint extension A_\varkappa of the Carleman operator A associated with \varkappa ($|\varkappa| = 1$) coincides with the set of all solutions of the equation

$$(3.1) \quad C(\sigma) = \overline{\varkappa}.$$

Let $\Delta_p = [a_p, a_{i(p)}]$ ($p = 1, 2, \dots$) be the interval of the real line \mathbb{R} such that a_p and $a_{i(p)}$ be consecutive (i.e., exist no other a_k between a_p and $a_{i(p)}$). The characteristic function $C(\lambda)$ applies to each interval, namely, for every p, k and $\zeta \in \Delta_p$ there exists a unique $\eta \in \Delta_k$ such that

$$C(\zeta) = C(\eta).$$

We denote by Γ the spectrum of the quasi selfadjoint extension A_ω of the Carleman operator A whose scalar spectral function is

$$\varrho_\omega(t) = \varrho(t).$$

Theorem 1. (1) If $\varrho(t) \in \mathfrak{G}_0$, then for all p ($p = 1, 2, \dots$) Γ contains only a finite number n of points in each interval Δ_p , i.e.

$$\Gamma \cap \Delta_p = \{\sigma_p^1, \sigma_p^2, \dots, \sigma_p^n\}.$$

(2) If $\varrho(t) \in \mathfrak{G}$, we have for all p ($p = 1, 2, \dots$)

$$(3.2) \quad \{z: z = e^{it}, t \in \Gamma \cap \Delta_p\} = \{z: z = e^{it}, t \in \Gamma\}.$$

If $\varrho(t) \in \mathfrak{G}$, we have for all p ($p = 1, 2, \dots$)

$$(3.3) \quad \{z: z = e^{it}, t \in \Gamma \cap \Delta_p\} = \{z: z = e^{it}, t \in \Gamma\}.$$

Proof. Let $\varrho(t) \in \mathfrak{G}_0$. Then $\omega(t)$ associated with $\varrho(t)$ is the rational function (2.10). Therefore the equation

$$\omega(\lambda)C(\lambda) = 1$$

admits only n solutions in each interval $\Delta_p = [a_p, a_{i(p)}]$, ($p = 1, 2, \dots$). Indeed, as noted earlier in this section, for each p, q and $\sigma_q \in \Delta_q$, there is a single point $\sigma_p^q \in \Delta_p$ such that

$$C(\sigma_p^q) = C(\sigma_q).$$

By applying the function φ to this equality we obtain, using (2.10), that

$$\omega(\sigma_p^q) = \omega(\sigma_q).$$

By (2.7), we have also

$$\omega(\sigma_q) = \varkappa_q.$$

Hence

$$\omega(\sigma_p^q) = \varkappa_q.$$

Now by the equality (3.1) it follows that

$$C(\sigma_p^q) = \overline{\varkappa_q}.$$

Then

$$\omega(\sigma_p^q)C(\sigma_p^q) = |\varkappa_q|^2 = 1 \quad (q = 1, 2, \dots, n; p = 1, 2, \dots),$$

and so

$$\Gamma \cap \Delta_p = \{\sigma_p^1, \sigma_p^2, \dots, \sigma_p^n\}.$$

This proves the first assumption. To see the second point, we argue as follows.

- ▷ First, if $\varrho(t) \in \mathfrak{G}_0$, then equality (3.3) follows from the bijection established by the characteristic function $C(t)$ between Δ_p 's.
- ▷ Now if $\varrho(t) \in \mathfrak{G}$ and $\varrho(t) \notin \mathfrak{G}_0$, then there is a sequence of scalar spectral functions $\varrho_n(t) \in \mathfrak{G}_0$ which converges to $\varrho(t)$. Since equality (3.3) is true for $\varrho_n(t)$ for any n , it is also true for $\varrho(t)$. □

Theorem 2. Let E be a closed set contained in the interval $\Delta_p = [a_p, a_{i(p)}]$. Then there is $\varrho(t) \in \mathfrak{G}$ such that the spectrum Γ of the quasi-selfadjoint extension A_ω of the Carleman operator A having $\varrho(t)$ as the scalar spectral function satisfies the equality

$$\Gamma \cap \Delta_p = E.$$

Proof. We choose a countable set

$$\Omega = \{\sigma_p^1, \sigma_p^2, \dots\} \subset \Delta_p,$$

dense in E . It is clear that if we denote

$$\Omega_n = \{\sigma_p^1, \sigma_p^2, \dots, \sigma_p^n\},$$

then

$$\Omega = \bigcup_{n=1}^{\infty} \Omega_n.$$

Let

$$C(\sigma_p^k) = \overline{\varkappa_k} \quad (k = 1, 2, \dots, n),$$

and for all n ($n = 1, 2, \dots$), let us form the spectral function by setting

$$\varrho_n(t) = \sum_{k=1}^{n-1} \frac{1}{2^k} \varrho_{\varkappa_k}(t) + \frac{1}{2^{n-1}} \varrho_{\varkappa_n}(t),$$

where $\varrho_{\varkappa_k}(t)$ denotes the orthogonal spectral function associated with \varkappa_k ($k = 1, 2, \dots, n$).

Clearly, $\varrho_n(t) \in \mathfrak{G}_0$. We will show that $\varrho_n(t)$ converges pointwise as n tends to ∞ . We start by introducing the function

$$S_n(t) = \int_{-\infty}^t \frac{d\varrho_n(\sigma)}{\sigma^2 + 1}.$$

According to the formula (2.5) $S_n(t)$ is a distribution function, i.e.,

$$S_n(+\infty) = \lim_{t \rightarrow +\infty} S_n(t) = \int_{-\infty}^{+\infty} \frac{d\varrho_n(\sigma)}{\sigma^2 + 1} = \|\varphi_i^\circ\|^2 = 1$$

and

$$S_n(-\infty) = \lim_{t \rightarrow -\infty} S_n(t) = 0.$$

Since

$$S_n(t) = \sum_{k=1}^{n-1} \frac{1}{2^k} S_{\varkappa_k}(t) + \frac{1}{2^{n-1}} S_{\varkappa_n}(t),$$

we have

$$\begin{aligned} |S_{n+n_0}(t) - S_n(t)| &= \left| \sum_{k=n}^{n+n_0-1} \frac{1}{2^k} S_{\varkappa_k}(t) + \frac{1}{2^{n+n_0-1}} S_{\varkappa_{n+n_0}}(t) - \frac{1}{2^{n-1}} S_{\varkappa_n}(t) \right| \\ &\leq \sum_{k=n}^{n+n_0-1} \frac{1}{2^k} + \frac{1}{2^{n+n_0-1}} - \frac{1}{2^{n-1}}. \end{aligned}$$

It is clear that this quantity tends to 0 as n tends to ∞ . Therefore, at each point t , $S_n(t)$ converges to a limit, denoted by $S(t)$.

Thus $\varrho_n(t)$ converges to $\varrho(t)$ as n tends to ∞ and

$$S(t) = \int_{-\infty}^t \frac{d\varrho(\sigma)}{\sigma^2 + 1}.$$

The spectrum of $\varrho_n(t)$ is $\{\sigma_\varrho^1, \sigma_\varrho^2, \dots, \sigma_\varrho^n\} = \Omega_n$, consequently the spectrum of $\varrho(t)$ is $\overline{\Omega} = E$. □

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