THE PERIOD OF A WHIRLING PENDULUM

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(Received July 15, 1999)

Abstract. The period function of a planar parameter-depending Hamiltonian system is examined. It is proved that, depending on the value of the parameter, it is either monotone or has exactly one critical point.

Keywords: Hamiltonian system, period function, Picard-Fuchs equations

MSC 2000: 37G15, 34C05

1. Introduction

We will consider the second-order differential equation

$$\ddot{x} = \sin x(\cos x - \gamma), \qquad x \in S^1, \tag{1}$$

which models a motion of a pendulum rotating about its vertical axis. The periodic solutions of this system form two or three one-parameter families (oscillations, rotations and, for $\gamma < 1$, deviated oscillations) separated by homoclinic trajectories.

The question of monotonicity of the period of a one-parameter family of periodic solutions arises in connection with the study of subharmonic bifurcation ([5], [8], [15]), and is in many cases difficult to answer. This difficulty is related to the fact that calculations often lead to elliptic integrals. Some results in particular cases were obtained, for example, by Brunovský and Chow [2], Chicone [3], Chow and Sanders [4], Chow and Wang [6].

In this paper we show that

— if $\gamma \geqslant 4$, then the period function of each family of periodic solutions is monotone;

— if $\gamma < 4$, then the period function of oscillations has exactly one critical point, while rotations and deviated oscillations have a monotone period.

Our proof is based upon Picard-Fuchs equations, the method that has been used by several authors in the study of zeros of abelian integrals, see for example [1], [4], [7], [11], [13].

The paper is organized as follows. First, the dynamics of (1) is shortly described. Then we derive Picard-Fuchs equations and a second order differential equation for the period map T. Also limit properties of T and its derivative are described. Finally, we determine the number of singular points of the period map in the particular regions of the γ-h plane, where h denotes the energy level of (1). In the last section, a brief sketch of numerical computations is given.

2. The phase portrait

The motion of a whirling pendulum is described in [10], p. 272, by the equation

$$\ddot{x} = -\frac{g}{L}\sin x + \omega^2 \sin x \cos x, \qquad x \in S^1, \tag{2}$$

where L is the length of the pendulum, x its angle deviation, and ω is a constant rotation rate. Introducing a new variable $y = \dot{x}$ and then changing the variables $y \to \omega y$, $t \to t/\omega$ converts (2) to an equivalent planar system of first-order equations

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= \sin x(\cos x - \gamma), \end{aligned} \tag{3}$$

where $\gamma = g/L\omega^2 > 0$.

This system is hamiltonian with the energy

$$H(x,y) = \tfrac{1}{2}y^2 - \gamma\cos x + \tfrac{1}{2}\cos^2 x + \gamma - \tfrac{1}{2}. \tag{4}$$

Its levels $H^{-1}(h) = \Gamma_h$ correspond to solutions of (3), where $h \in \langle h_m, \infty)$ with

$$h_m = \begin{cases} -\frac{1}{2}(1-\gamma)^2, & \text{if } \gamma < 1, \\ 0, & \text{if } \gamma \geqslant 1. \end{cases}$$

Depending on γ, we have two qualitatively different dynamics of (3) (see Fig. 1 and Fig. 2).

For all γ, the point $(\pi, 0)$ in the x-y phase plane is a saddle with two homoclinic trajectories $\Gamma^+ = H^{-1}(2\gamma) \cap \{(x,y);\ y > 0\}$ and $\Gamma^- = H^{-1}(2\gamma) \cap \{(x,y);\ y < 0\}$. They form boundaries between two families of periodic trajectories: $\mathcal{P}^0 = \{H^{-1}(h);\ h \in (0, 2\gamma)\}$ corresponding to oscillations of the pendulum, and $\mathcal{P}^+ = \{H^{-1}(h);\ h >$

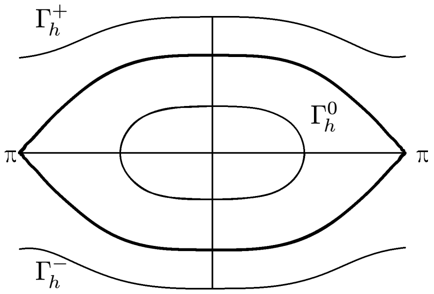

Fig. 1. Phase portrait for $\gamma \geqslant 1$.

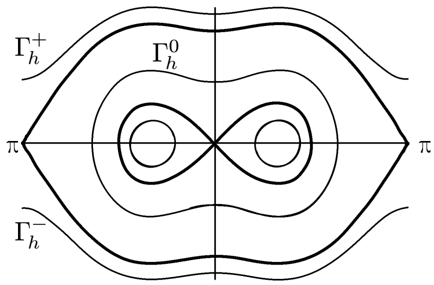

Fig. 2. Phase portrait for $\gamma < 1$

$2\gamma,\ y > 0\}$ and $\mathcal{P}^- = \{H^{-1}(h);\ h > 2\gamma,\ y < 0\}$, corresponding respectively to clockwise and counterclockwise rotations of the pendulum.

The point $(0,0)$ is also a singular point, but its stability depends on γ. If $\gamma \geqslant 1$, then it is a center surrounded by the family $\mathcal{P}^0$. If $\gamma < 1$, then $(0,0)$ is a saddle point with two homoclinic loops (symmetric with respect to the y-axis), $\Gamma^* = H^{-1}(0) \cap \{(x,y),\ x > 0\}$ and $-\Gamma^* = H^{-1}(0) \cap \{(x,y);\ x < 0\}$. Inside each loop, there is a family of periodic solutions (deviated oscillations) $\mathcal{P}^* = \{H^{-1}(h);\ h \in \langle -\frac{1}{2}(1-\gamma)^2, 0),\ x > 0\}$ and $-\mathcal{P}^* = \{H^{-1}(h); h \in \langle -\frac{1}{2}(1-\gamma)^2, 0),\ x < 0\}$, which surround centers $(\arccos\gamma, 0)$ and $(-\arccos\gamma, 0)$, respectively.

In the sequel, we will take into consideration only the families $\mathcal{P}^0$, $\mathcal{P}^*$ and $\mathcal{P}^+$, since, due to symmetry, the results for $-\mathcal{P}^*$ and $\mathcal{P}^-$ are analogous. The superscripts 0, $+$ and $*$ will denote which Γ_h-family is being used; i.e. $T^0(h)$ denotes a function $T(h)$ restricted to $\mathcal{P}^0$.

3. Picard-Fuchs equations for the period function

Let $T(h)$ denote the period of the trajectory Γ_h on the energy level h and let the corresponding solution be $t \mapsto (x(t), y(t))$. Obviously,

$$T(h) = \int_{\Gamma(h)} \frac{\mathrm{d}x}{y}.$$

We define integrals

$$I_n(h) = \int_{\Gamma_h} y(\cos x)^n \mathrm{d}x, \qquad n = 0, 1, 2.$$

Note that $T(h) = I_0'(h)$, where $'$ stands for the derivative with respect to h.

Lemma 1. *Let us denote* $\mathbf{v} = (I_0, I_1, I_2)^\top$. *Then*

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{1}{2} & \frac{1}{2}\gamma & \frac{3}{2} \end{pmatrix} \mathbf{v} = \begin{pmatrix} 2h-2\gamma+1 & 2\gamma & -1 \\ \frac{1}{2}\gamma & h-\gamma & \frac{1}{2}\gamma \\ 0 & \gamma(h-\gamma+1) & h-\gamma+\gamma^2 \end{pmatrix} \mathbf{v}'. \tag{5}$$

P r o o f. According to (4) we have

$$y^2 = 2h - 2\gamma + 1 + 2\gamma\cos x - \cos^2 x. \tag{6}$$

Then

$$\begin{aligned} I_0 &= \int \frac{y^2}{y}\,\mathrm{d}x \\ &= \int \frac{2h-2\gamma+1+2\gamma\cos x-\cos^2 x}{y}\,\mathrm{d}x \\ &= (2h-2\gamma+1)I_0' + 2\gamma I_1' - I_2', \end{aligned}$$

which is the first equation of (5). To obtain the second, we first integrate I_1 by parts, and then use twice (6):

$$\begin{aligned} I_1 &= -\int \frac{\mathrm{d}y}{\mathrm{d}x}\sin x\,\mathrm{d}x \\ &= \int \frac{\sin^2 x}{y}(\gamma-\cos x)\,\mathrm{d}x \\ &= \int \frac{1-\cos^2 x}{y}(\gamma-\cos x)\,\mathrm{d}x \\ &= \int \frac{1}{y}(y^2-2h+2\gamma-2\gamma\cos x)(\gamma-\cos x)\,\mathrm{d}x \\ &= \gamma I_0 + 2\gamma(\gamma-h)I_0' - I_1 + 2(h-\gamma-\gamma^2)I_1' + 2\gamma I_2'. \end{aligned}$$

Then

$$I_1 = \frac{1}{2}\gamma I_0 + \gamma(\gamma-h)I_0' + (h-\gamma-\gamma^2)I_1' + \gamma I_2', \tag{7}$$

and substituting I_0 into (7) yields the second equation in (5). In a similar way we derive the third equation in (5). First, we use the trigonometrical identity

$$\cos^2 x = \frac{1+\cos 2x}{2}$$

to obtain

$$I_2 = \frac{1}{2} I_0 + \frac{1}{2} \int y \cos(2x)\,\mathrm{d}x. \tag{8}$$

Integrating the second integral by parts gives

$$\begin{aligned}\int y \cos(2x)\,\mathrm{d}x &= -\frac{1}{2} \int \frac{\mathrm{d}y}{\mathrm{d}x} \sin(2x)\,\mathrm{d}x \\ &= \int \frac{1}{y} \cos x(\gamma - \cos x) \sin^2 x\,\mathrm{d}x\end{aligned}$$

and, after using the relation

$$\cos^2 x = 2h - 2\gamma + 1 + 2\gamma \cos x - y^2,$$

we obtain

$$\int y \cos(2x)\,\mathrm{d}x = -\gamma I_1 - I_2 + 2\gamma(h - \gamma + 1) I_1' + 2(h - \gamma + \gamma^2) I_2'.$$

This yields, together with (8), the last equation in (5). □

Lemma 2. *The period map $T(h)$ satisfies the second order differential equation*

$$2aT'' = -bT + cT', \tag{9}$$

where

$$\begin{aligned}a &= 2w(h - \gamma + \gamma^2), \\ b &= h^2 + h\gamma(2.5\gamma - 2) + \gamma^2(0.5\gamma^2 - 2.5\gamma + 1), \\ c &= 2\left(w - w'(h - \gamma + \gamma^2)\right)\end{aligned}$$

with $w = h(h - 2\gamma)\left(2h + (\gamma - 1)^2\right)$ and $w' = 2(h - \gamma)(\gamma - 1)^2 + 2h(3h - 4\gamma)$ being the derivative of w with respect to h.

Proof. The first equation of (5) implies

$$I_2' = (2h - 2\gamma + 1) I_0' + 2\gamma I_1' - I_0. \tag{10}$$

Substituting it into the third equation of (5) gives

$$\begin{aligned}I_2 = {} & \frac{1 - 2(h - \gamma + \gamma^2)}{3} I_0 - \frac{1}{3}\gamma I_1 \\ & + \frac{2}{3}(h - \gamma + \gamma^2)(2h - 2\gamma + 1) I_0' + \frac{2}{3}\gamma(3h - 3\gamma + 2\gamma^2 + 1) I_1'.\end{aligned}$$

If we differentiate the last equation with respect to h and compare it with (10), we obtain

$$\gamma I_1' = I_0 + 2\gamma^2 I_0' + 2vI_0'' + 2\gamma(3h - 3\gamma + 2\gamma^2 + 1)I_1'', \tag{11}$$

where $v = (h - \gamma + \gamma^2)(2h - 2\gamma + 1)$. We now use the first and second equations of (5) to calculate I_1''. First, we eliminate I_2:

$$\frac{1}{2}\gamma I_0 + I_1 = \gamma(h - \gamma + 1)I_0' + (h - \gamma + \gamma^2)I_1',$$

from which

$$I_1 = \gamma(h - \gamma + 1)I_0' + (h - \gamma + \gamma^2)I_1' - \frac{1}{2}\gamma I_0.$$

Differentiating the last equation with respect to h yields

$$I_1'' = -\frac{\gamma}{h - \gamma + \gamma^2}\Big[(h - \gamma + 1)I_0'' + \frac{1}{2}I_0'\Big].$$

Substituting for I_1'' into (11) gives

$$\gamma I_1' = I_0 + 2\gamma^2 I_0' + 2vI_0'' - 2\gamma^2\frac{3h - 3\gamma + 2\gamma^2 + 1}{h - \gamma + \gamma^2}\Big[(h - \gamma + 1)I_0'' + \frac{1}{2}I_0'\Big]. \tag{12}$$

To simplify this equation, we multiply it by $(h - \gamma + \gamma^2)$ and denote

$$w = (h - \gamma + \gamma^2)^2(2h - 2\gamma + 1) - \gamma^2(h - \gamma + 1)(3h - 3\gamma + 1 + 2\gamma^2). \tag{13}$$

Then (12) becomes

$$\gamma(h - \gamma + \gamma^2)I_1' = (h - \gamma + \gamma^2)I_0 - \gamma^2(h - \gamma + 1)I_0' + 2wI_0''.$$

If we differentiate the last equation with respect to h and again substitute for I_1'', we obtain

$$\gamma I_1' = I_0 + \Big(\frac{1}{2}\gamma^2 + h - \gamma\Big)I_0' + 2w'I_0'' + 2wI_0'''.$$

Now, compare this equation with (12) to eliminate I_0 and I_1'. The result is

$$2wI_0''' = 2\Big(\frac{w}{h - \gamma + \gamma^2} - w'\Big)I_0'' + \Big(\frac{3}{2}\gamma^2 - h + \gamma - \gamma^2\frac{3h - 3\gamma + 2\gamma^2 + 1}{h - \gamma + \gamma^2}\Big)I_0',$$

which together with (13) and $I_0' = T$ gives (9). $\square$

Suppose h_0 is a critical point of T, e.g. $T'(h_0) = 0$. It follows from (9) that

$$T''(h_0) = \frac{-b}{2a} T(h_0).$$

Since $T(h_0) > 0$, the following result is obvious:

Corollary 1. *If $T'(h_0) = 0$ for some $h_0 \in (h_m, \infty)$, then*

(14) $$ab > 0 \ (< 0) \text{ at } h = h_0 \Longrightarrow T''(h_0) < 0 \ (> 0).$$

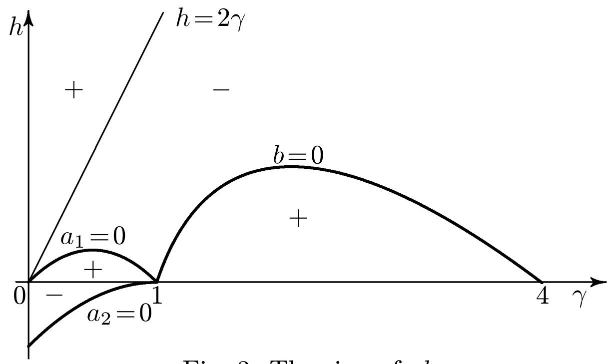

Fig. 3. The sign of ab.

Therefore the curves $a = 0$ and $b = 0$ in the γ-h plane determine the type of the critical points of $T(h)$. The situation is depicted in Fig. 3, where we have denoted $a_1 = h - \gamma + \gamma^2$ and $a_2 = h + \frac{1}{2}(1-\gamma)^2$. There are regions inside which a and b are of constant sign (note that we are interested only in $h \geqslant h_m$). The coefficient a changes its sign when crossing one of the curves $a_1 = 0$, $a_2 = 0$, $h = 2\gamma$ and $h = 0$. The coefficient b vanishes, for given γ, at

$$h^{\pm} = \gamma - \frac{5}{4}\gamma^2 \pm \gamma\sqrt{\frac{17}{16}\gamma^2 - 1}.$$

Depending on γ, there are several cases:

1. $\gamma < 4/\sqrt{17}$. Then $b > 0$ for all $h \geqslant h_m$.
2. $4/\sqrt{17} \leqslant \gamma < 1$ or $\gamma > 4$. Then $h^+ \leqslant h_m$, which means that $b > 0$ for all $h > h_m$.
3. $\gamma = 1$ or $\gamma = 4$. Then $h^- < h^+ = 0$, which implies that b is positive for all $h > 0$, and $b = 0$ at $h = 0$.
4. $1 < \gamma < 4$. Then $h^- < 0 < h^+$, and so $b = 0$ only at the point h^+.

The following lemma will be helpful for determining the sign of $T'(h)$:

Lemma 3.

$$\lim_{h\to h_m^+} T(h) = \begin{cases} \frac{2\pi}{\sqrt{1-\gamma^2}}, & \text{if } \gamma < 1,\\ \infty, & \text{if } \gamma = 1,\\ \frac{2\pi}{\sqrt{\gamma-1}}, & \text{if } \gamma > 1, \end{cases}$$

and

$$\lim_{h\to h_m^+} T'(h) = \begin{cases} \pi\frac{1+2\gamma^2}{(1-\gamma^2)^{\frac{5}{2}}}, & \text{if } \gamma < 1,\\ -\infty, & \text{if } \gamma = 1,\\ \frac{\pi}{(\gamma-1)^{\frac{5}{2}}}\left(\frac{1}{4}\gamma - 1\right), & \text{if } \gamma > 1. \end{cases}$$

P r o o f. We examine three cases separately.

1. $\gamma < 1$. It is easily seen that in $\mathcal{P}^*$

$$T(h) = 2\int_{x_h^+}^{x_h^-} \frac{\mathrm{d}x}{y}$$

with $x_h^{+,-} = \arccos(\gamma \pm \sqrt{(1-\gamma)^2 + 2h})$ and $y = \sqrt{2h - 2\gamma + 1 - \cos^2 x + 2\gamma\cos x}$. Let us define new coordinates (h, φ) by

$$\begin{aligned} x &= \arccos s \\ y &= \sin\varphi\sqrt{(1-\gamma)^2 + 2h}, \end{aligned}$$

where h is the level of the energy $H(x, y)$, $\varphi \in [0, \pi]$ is the angle between the x-axis and the line connecting the points $(\arccos\gamma, 0)$ and (x, y), and

$$s = \gamma - \cos\varphi\sqrt{(1-\gamma)^2 + 2h}.$$

Then

$$\frac{1}{2}T(h) = -\int_0^{\pi} \frac{1}{y}\frac{\mathrm{d}x}{\mathrm{d}\varphi}\,\mathrm{d}\varphi = \int_0^{\pi} \frac{\mathrm{d}\varphi}{\sqrt{1-s^2}}.$$

Since $\lim\limits_{h\to h_m} s = \gamma$, we have

$$\lim_{h\to h_m} T(h) = \frac{2\pi}{\sqrt{1-\gamma^2}}.$$

We now compute the derivative of $T(h)$:

$$\begin{aligned} \frac{1}{2}T'(h) &= \int_0^{\pi} \frac{ss'}{(1-s^2)^{\frac{3}{2}}}\,\mathrm{d}\varphi \\ &= \frac{-1}{\sqrt{(1-\gamma)^2 + 2h}}\int_0^{\pi} \frac{\cos\varphi\left(\gamma - \cos\varphi\sqrt{(1-\gamma)^2 + 2h}\right)}{(1-s^2)^{\frac{3}{2}}}\,\mathrm{d}\varphi. \end{aligned}$$

The last expression is of type "0/0" if $h = h_m$. To find its limit at the point $h = h_m$, we use L'Hospital's rule:

$$\begin{aligned}
\lim_{h\to h_m} T'(h) &= -2 \lim_{h\to h_m} \frac{\int_0^{\pi} \frac{\cos\varphi\left(\gamma - \cos\varphi\sqrt{(1-\gamma)^2+2h}\right)}{(1-s^2)^{\frac{3}{2}}}\,\mathrm{d}\varphi}{\sqrt{(1-\gamma)^2+2h}} \\
&= 2 \lim_{h\to h_m} \int_0^{\pi} \frac{\cos^2\varphi(1+2s^2)}{(1-s^2)^{\frac{5}{2}}}\,\mathrm{d}\varphi \\
&= \pi\frac{1+2\gamma^2}{(1-\gamma^2)^{\frac{5}{2}}}.
\end{aligned}$$

2. $\gamma = 1$. In this case,

$$T(h) = 4\int_0^{x_h} \frac{\mathrm{d}x}{y}$$

with $x_h = \arccos(1-\sqrt{2h})$. The new coordinates are of the form

$$\begin{aligned}
x &= \arccos s \\
y &= \sqrt{2h}\sin\varphi,
\end{aligned}$$

where $\varphi \in \left[0, \frac{\pi}{2}\right]$ is the angle between the x-axis and the line connecting the points $(0,0)$ and (x,y), and

$$s = 1-\sqrt{2h}\cos\varphi.$$

Easy computations yield

$$T(h) = 4\int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\varphi}{\sqrt{1-s^2}}$$

and then

$$T'(h) = \frac{-4}{(2h)^{\frac{5}{4}}}\int_0^{\frac{\pi}{2}} \frac{1-\sqrt{2h}\cos\varphi}{\sqrt{\cos\varphi}(2-\sqrt{2h}\cos\varphi)}\,\mathrm{d}\varphi,$$

which means that $\lim\limits_{h\to 0^+} T'(h) = -\infty$.

3. $\gamma > 1$. Again, in $\mathcal{P}^0$ we have

$$T(h) = 4\int_0^{x_h} \frac{\mathrm{d}x}{y}$$

with $x_h = \arccos(\gamma - \sqrt{(\gamma-1)^2+2h})$. This integral can be arranged (see, e.g. [9], [14]) into the form

$$T(h) = \frac{4}{\sqrt[4]{(\gamma-1)^2+2h}}K(k). \tag{15}$$

Here

$$K(k)=\int_0^1 \frac{\mathrm{d}s}{\sqrt{1-s^2}\sqrt{1-k^2s^2}}$$

is the complete elliptic integral of the first kind with the elliptic modulus k, where

$$k^2=\frac{1}{2}\Big(1+\frac{h-\gamma+1}{\sqrt{(\gamma-1)^2+2h}}\Big).$$

With h increasing on $\langle 0,2\gamma\rangle$, the elliptic modulus k increases on $\langle 0,1\rangle$. The integral $K(k)$ can be expressed via the infinite series

$$K(k)=\frac{\pi}{2}\Big(1+\frac{1}{4}k^2+\mathcal{O}(k^4)\Big)$$

which is increasing for $k\in\langle 0,1\rangle$, and

$$\lim_{k\to 0^+}K(k)=K(0)=\frac{\pi}{2},\qquad \lim_{k\to 1^-}K(k)=+\infty.$$

It follows immediately that

$$\lim_{h\to 0^+}T(h)=\frac{2\pi}{\sqrt{\gamma-1}}.$$

Differentiating (15) with respect to h gives

$$T'(h)=\frac{\pi}{((\gamma-1)^2+2h)^{\frac{5}{4}}}\Big\{\frac{\gamma^2-\gamma+h}{\sqrt{(\gamma-1)^2+2h}}\Big(\frac{1}{4}+\mathcal{O}(k^2)\Big)-1-\mathcal{O}(k^2)\Big\}.$$

Now, it is easy to check the last limit of the lemma, provided we realize that $k\to 0$ as $h\to 0$. □

4. Monotonicity of the period

We are now ready to examine the monotonicity of the period map T of (3). Recall that there are two (for $\gamma\geqslant 1$) or three (for $\gamma<1$) one-parameter families of periodic solutions with periods T^*, T^0 and T^+ defined on $(-\frac{1}{2}(1-\gamma)^2,0)$, $(0,2\gamma)$, and $(2\gamma,\infty)$, respectively. It is not difficult to see that

$$\begin{aligned}
T^*(h)\to\infty &\quad\text{as}\quad h\to 0^-,\\
T^0(h)\to\infty &\quad\text{as}\quad h\to 0^+ \text{ and } \gamma\leqslant 1,\\
T^0(h)\to\infty &\quad\text{as}\quad h\to 2\gamma^-,\\
T^+(h)\to\infty &\quad\text{as}\quad h\to 2\gamma^+.
\end{aligned}$$

Theorem 1. *Let γ be a positive real number. Then*

1. $T^+(h)$ *is strictly decreasing;*
2. $T^0(h)$
   (a) *is strictly increasing, if $\gamma \geqslant 4$ and*
   (b) *has exactly one critical point which is its global minimum point, if $\gamma < 4$;*
3. $T^*(h)$ *is strictly increasing.*

Proof. In Fig. 4, the domains of the particular period functions are bounded by the curves $h = 0$, $h = 2\gamma$, and $h = -\frac{1}{2}(1-\gamma)^2$. They also, together with $h - \gamma + \gamma^2 = 0$ and $b = 0$, form the boundaries of the regions where ab does not change its sign (compare with Fig. 3). We now consider particular cases.

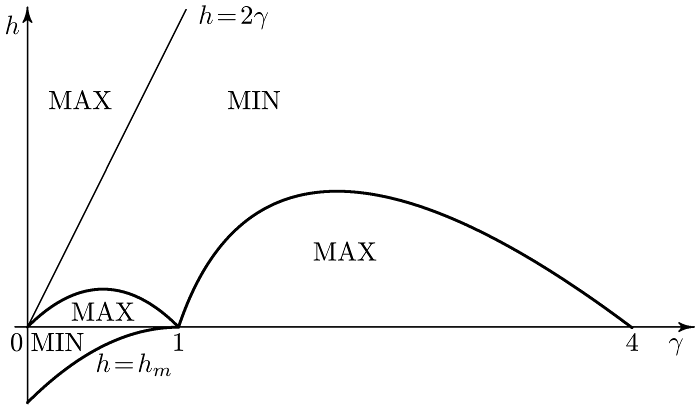

Fig. 4. Possible types of critical points.

1. $T^+(h)$.

Since neither $a = 0$ nor $b = 0$ intersect the region above the line $h = 2\gamma$, $T''(h_0)$ is, according to (9), of one sign at any critical point $h_0 > 2\gamma$ of T. Namely, $ab > 0$, which implies, by (14), that every critical point should be a local maximum. But $T(h) \to \infty$ as $h \to 2\gamma$. Thus, there is no critical point of $T^+(h)$, and $T'(h) < 0$ for all $h > 2\gamma$.

2. $T^0(h)$.

Between the lines $h = 0$ and $h = 2\gamma$, there are two subcases depending on the value of the parameter γ:

(i) $\gamma \geqslant 4$:

Fig. 4 shows that any critical point in the interval $(0, 2\gamma)$ should be a minimum point. Since $\lim\limits_{h\to 0^+} T'(h) > 0$, we can conclude that there is no critical point of $T^0(h)$.

(ii) $\gamma < 4$:

By Lemma 3, $\lim_{h\to 0^+} T'(h) < 0$, which together with $\lim_{h\to 2\gamma^-} T(h) = \infty$ implies that there is at least one minimum point. Consulting Fig. 4 we obtain that $T(h)$ has exactly one minimum point $\bar{h}$, particularly

$$\begin{aligned} &\text{if } \gamma \in (1,4) \ \text{ then } \bar{h} \in (h^+, 2\gamma); \\ &\text{if } \gamma \in (0,1) \ \text{ then } \bar{h} \in \left(\gamma - \gamma^2, 2\gamma\right). \end{aligned}$$

3. $T^*(h)$.

The discussed region is bounded by the lines $\gamma = 0$, $\gamma = 1$, $h = 0$ and the curve $h = -\frac{1}{2}(1-\gamma)^2$. Fig. 4 shows that any critical point in this region should be a minimum point. Since $\lim_{h\to 0^-} T^*(h) = \infty$, and the derivative of $T^*(h)$ is positive near the point $h = h_m$ (see Lemma 3), we can conclude that there is no critical point of $T^*(h)$. $\square$

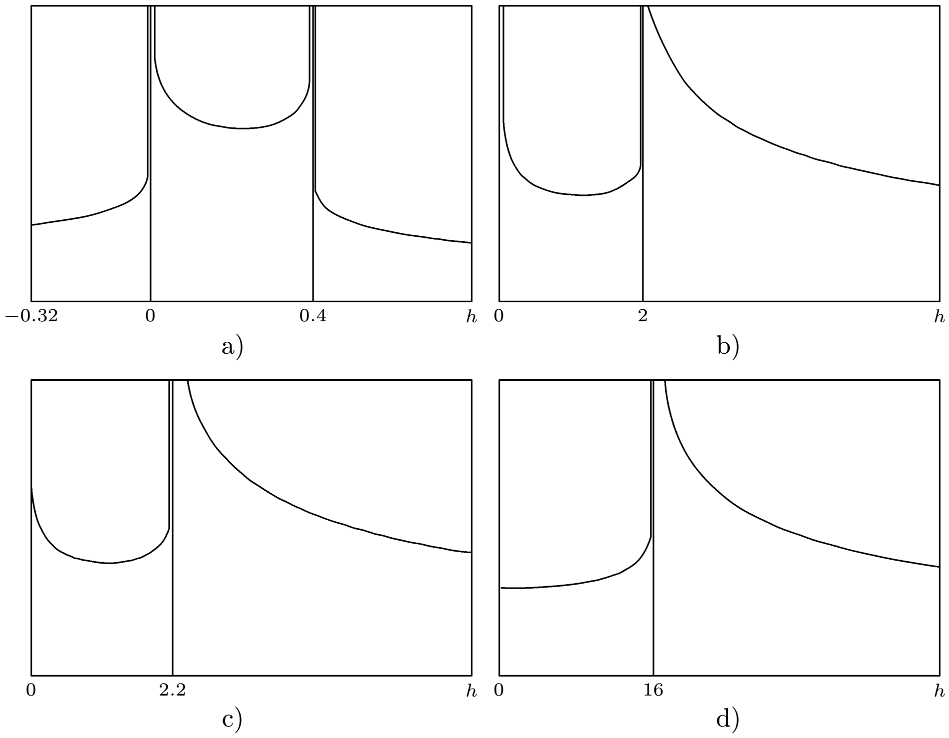

Fig. 5. Graphs of $T(h)$: a) $\gamma = 0.2$, b) $\gamma = 1$, c) $\gamma = 1.1$, d) $\gamma = 8$.

5. Numerical computations of $T(h)$

The graphs of the period function in the particular cases are in Fig. 5. The data for the graphs were computed in two ways. For $\gamma \geqslant 1$ and $h \in (0, 2\gamma)$ we have used the relation (15) where we have substituted the infinite series for $K(k)$. In the other cases we computed

$$T(h) = \int_{\Gamma_h} \frac{\mathrm{d}x}{y}$$

numerically using the Simpson rule with slightly modified boundaries to avoid the situation $y = 0$. However, both methods have not been applicable near the points $h = h_m$ and $h = 2\gamma$ because of great numerical errors. To complete the picture we used the results of Lemma 3:

— if $\gamma \neq 1$ then the limit at $h = h_m$ is finite;
— if $\gamma = 1$ then $T(h)$ is (near $h = 0$) approximately $2\pi(2h)^{-1/4}$;
— near $h = 2\gamma$ we applied (15) with use of the inequality (see [12])

$$1 + \frac{k'^2}{8} < \frac{K(k)}{\log(4/k')} < 1 + \frac{k'^2}{4}.$$

A c k n o w l e d g m e n t s. I thank Prof. M. Medveď, Comenius University, and B. Rudolf, Slovak University of Technology, for their helpful comments which have led to the correction of results of Lemma 3. I also thank Ing. R. Balogh, Slovak University of Technology, for his great help with figures in the article.

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