ON A GENERALIZED DHOMBRES FUNCTIONAL EQUATION II

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Abstract. We consider the functional equation $f(xf(x)) = \varphi(f(x))$ where $\varphi\colon J \to J$ is a given increasing homeomorphism of an open interval $J \subset (0,\infty)$ and $f\colon (0,\infty) \to J$ is an unknown continuous function. In a previous paper we proved that no continuous solution can cross the line $y = p$ where p is a fixed point of φ, with a possible exception for $p = 1$. The range of any non-constant continuous solution is an interval whose end-points are fixed by φ and which contains in its interior no fixed point except for 1. We also gave a characterization of the class of continuous monotone solutions and proved a sufficient condition for any continuous function to be monotone.

In the present paper we give a characterization of the equations (or equivalently, of the functions φ) which have all continuous solutions monotone. In particular, all continuous solutions are monotone if either (i) 1 is an end-point of J and J contains no fixed point of φ, or (ii) $1 \in J$ and J contains no fixed points different from 1.

Keywords: iterative functional equation, invariant curves, monotone solutions

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1. Introduction

We consider the functional equation

$$f(xf(x)) = \varphi(f(x)), \quad x \in (0,\infty). \tag{1.1}$$

Throughout the paper we assume that $J \subset (0,\infty)$ is an open interval, $\varphi\colon J \to J$ is a given homeomorphism of J, i.e., a continuous, increasing surjective function, and f

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is an unknown function, defined for $x \in (0, \infty)$, with values in J. Thus, any solution of (1.1) is positive.

This equation is a special case of *equations of invariant curves*. A survey of some general results can be found in [5] (cf. also [6]); however, these results cannot be too strong. Concerning the equation (1.1), the case $\varphi(y) = y^2$ was considered by Dhombres in [1]. More general types of φ have been studied, e.g., in a series of our papers [2]–[4], where other references can be found.

We recall the main results from [4]. The range R_f of any non-constant continuous solution f of (1.1) must be a φ-invariant interval (i.e., $\varphi(R_f) = R_f$), and each of the sets $\{x; f(x) < 1\}$, $\{x; f(x) = 1\}$ and $\{x; f(x) > 1\}$ is an interval, possibly empty or degenerate. Moreover, R_f contains no fixed point of φ different from 1, and if $1 \in R_f$ then $\varphi(1) = 1$.

However, to describe the class of continuous solutions of (1.1) it suffices to consider the case $R_f \subset (0, 1]$. This follows easily by the facts given above, since the transformation $f \mapsto \tilde{f}$, where $\widetilde{\varphi}$ is defined by $\widetilde{\varphi}(x) = 1/\varphi(1/x)$, is a bijection between the solutions of (1.1) and the conjugate equation $\tilde{f}(x\tilde{f}(x)) = \widetilde{\varphi}(\tilde{f}(x))$, $x \in (0, \infty)$. This transformation maps constant, increasing, decreasing or continuous functions to functions with the same respective properties (for details, see [4]).

Moreover, if $R_f = (0, 1]$, then $f(x) \in (0, 1)$ on an interval $(0, a)$ or (a, ∞) and $f(x) = 1$ on the complement, and vice versa: If a continuous function g defined on an interval $I \subset (0, \infty)$ satisfies the equation and $R_g = (0, 1)$ then $I = (0, a)$ or (b, ∞), possibly with $a = \infty$, resp. $b = 0$, and g can be uniquely extended to a solution f of (1.1) by $f(x) = 1$ for $x \in (0, \infty) \setminus I$.

By the above argument, the class $\mathcal{S}(J, \varphi)$ of solutions of (1.1) corresponding to an arbitrary open interval $J \subset (0, \infty)$ and a homeomorphism φ of J is determined by the classes $\mathcal{S}(J, \varphi)$ with J, φ satisfying the conditions

$$J = (p, q),\ 0 \leqslant p < q \leqslant 1,\ \text{and}\ \varphi(y) \neq y \text{ for } x \in J, \tag{1.2}$$

with the reservation that, in the case $q = 1$, the domain of the solution may be a proper subinterval $(0, a)$ or (a, ∞) of $(0, \infty)$. We will assume (1.2) with this reservation throughout the rest of the paper. One of the main recent results reads as follows (φ^k denotes the kth iterate of φ):

1.1 Theorem. (Cf. [4].) *Assume* (1.2). *Then any continuous solution of* (1.1) *with values in J is monotone (i.e., non-decreasing or non-increasing) provided either*

$$\prod_{k=0}^{\infty} \frac{\varphi^k(y)}{\varphi^k(z)} = \infty \quad \textit{for any}\ \ y > z \ \textit{in}\ J, \tag{1.3}$$

or

$$\prod_{k=1}^{\infty} \frac{\varphi^{-k}(y)}{\varphi^{-k}(z)} = \infty \quad \textit{for any} \ \ y > z \ \textit{in} \ J. \tag{1.4}$$

In [4] we conjectured that the condition from the theorem is also necessary. Unfortunately, this is not the case. However, we are able to give the complete solution of the problem. Our main results in this paper are the following three theorems.

1.2 Theorem. *Assume* (1.2) *with* $q = 1$. *Then any continuous solution* f *of* (1.1) *is monotone. In particular,*
(i) *if* $\varphi(y) < y$ *for any* $y \in J$ *then* f *is non-decreasing;*
(ii) *if* $\varphi(y) > y$ *for any* $y \in J$ *then* f *is non-increasing.*

1.3 Theorem. *Assume* (1.2) *with* $q < 1$. *Then any continuous solution of* (1.1) *is non-decreasing if and only if* (1.3) *is satisfied.*

1.4 Theorem. *Assume* (1.2) *with* $q < 1$. *Then any continuous solution of* (1.1) *is non-increasing if and only if* (1.4) *is satisfied.*

1.5 Remark. Assume (1.2). Then neither (1.3) nor (1.4) can be satisfied if $p > 0$. This follows from Lemma 2.2 (ii) below. Thus, by the above theorems, non-monotone continuous solutions of (1.1) do exist if and only if one of the following three conditions is satisfied: (i) $0 < p < q < 1$; (ii) $0 = p < q < 1$, $\varphi(y) < y$ in J, and (1.3) is not true; (iii) $0 = p < q < 1$, $\varphi(y) > y$ in J, and (1.4) is not true.

1.6 Examples. It is not difficult to find functions φ satisfying the conditions (1.3) or (1.4) and hence giving only monotone continuous solutions. For example, take $\varphi(x) = kx^s$, where k, s are positive constants.

Here we provide an example of $\varphi\colon (0, \frac{3}{4}) \to (0, \frac{3}{4})$ allowing non-monotone continuous solutions. For $n \geqslant 0$ put $a_n = 3^{-1}2^{1-n}$, and let $\varphi(a_n) = a_{n+1}$. Moreover, let $\varphi(0) = 0$ and $\varphi(\frac{3}{4}) = \frac{3}{4}$. Let $b_n = \frac{1}{3}a_n + \frac{2}{3}a_{n+1}$. Let φ be linear, with slope 1 on $[a_{n+1}, b_n]$ and $\frac{1}{4}$ on $[b_n, a_n]$, and let φ be linear on $[\frac{2}{3}, \frac{3}{4}]$. Now, by Lemma 2.2 (ii) below, (1.4) is satisfied for no $y > z$. However, we have also $\prod_{k=0}^{\infty} (\varphi^k(a_0)/\varphi^k(b_0)) < \infty$. Indeed, put $b_n = \varphi^n(b_0)$ and let $h_n = a_n - b_n$ for $n \geqslant 0$. Then $h_0 = \frac{2}{9}$ and since $\varphi[b_n, a_n] \subset [b_{n+1}, a_{n+1}]$ and φ has slope $\frac{1}{4}$ on $[b_n, a_n]$, we have $h_n = h_0 4^{-n}$. An easy computation shows that $\prod_{n=0}^{\infty} (a_n/b_n) = \prod_{n=0}^{\infty} (a_n/(a_n - h_n)) = 1/\prod_{n=0}^{\infty} (1 - \frac{1}{3}\frac{1}{2^n}) < \infty$ since $\sum_{n=0}^{\infty} \frac{1}{3}\frac{1}{2^n} < \infty$.

In the next Section 2 we recall some technical results, mainly taken from [4]. The proofs of Theorems 1.2–1.4 are in Section 3.

2. Preliminaries

2.1 Theorem. (Cf. [4].) *For $x > 0$ and $y \in J$ put $\Phi(x, y) = (xy, \varphi(y))$. Then Φ is a homeomorphism of $(0,\infty) \times J$ which maps the graph of any continuous solution f of (1.1) onto itself.*

To simplify the notation, throughout the paper we identify any function with its graph. In particular, if $f\colon (0,\infty) \to (0,\infty)$ is a function then $\Phi(f)$ stands for $\{\Phi(x, f(x)); x \in (0,\infty)\}$. The following terminology is standard: a sequence $\{(x_n, y_n)\}_{n=-\infty}^{\infty}$ is the *orbit under* Φ of a point (x_0, y_0) in $(0,\infty) \times J$ provided $\Phi(x_n, y_n) = (x_{n+1}, y_{n+1})$ for any integer n. Induction yields the formulas

$$y_n = \varphi^n(y_0) \quad \text{for any } n, \tag{2.1}$$

$$x_n = x_0 y_0 \varphi(y_0) \dots \varphi^{n-1}(y_0) \quad \text{for } n > 0, \tag{2.2}$$

$$x_{-n} = \frac{x_0}{\varphi^{-1}(y_0)\varphi^{-2}(y_0) \dots \varphi^{-n}(y_0)} \quad \text{for } n > 0. \tag{2.3}$$

2.2 Lemma. (Cf. [4].) *Let $J \subset (0,1)$ be an open interval and let ψ be an increasing continuous function from J onto J such that $\psi(y) \neq y$ in J.*

(i) *If $\prod\limits_{k=1}^{\infty} (\psi^k(y)/\psi^k(z))$ is finite for some $z < y$, then the function*

$$t \mapsto \prod_{k=1}^{\infty} (\psi^k(t)/\psi^k(z))$$

is continuous and strictly increasing in $[z, y]$.

(ii) *If $\lim\limits_{n\to\infty} \psi^n(x) \neq 0$ for some $x \in J$ then, for any $y > z$ in J, the product*

$$\prod_{k=1}^{\infty} (\psi^k(y)/\psi^k(z))$$

is non-zero and finite.

3. Proofs

Proof of Theorem 1.2. (i) Assume, contrary to what we wish to show, that there are $u_0 < v_0$ such that $f(u_0) = y_0 > f(v_0) = z_0$. We may assume that

$$f(x) \in (z_0, y_0) \quad \text{whenever} \quad x \in (u_0, v_0). \tag{3.1}$$

Let $\{(u_n, y_n)\}$ and $\{(v_n, z_n)\}$ be the orbits of (u_0, y_0) and (v_0, z_0), respectively. We will show that

$$u_n < u_{n-1} < v_n \quad \text{for some} \quad n \leqslant 0. \tag{3.2}$$

Indeed, assume $u_{n-1} \geqslant v_n$ for any integer $n \leqslant 0$. Then, by (2.3),

$$\frac{u_{n-1}}{v_n} = \frac{u_0}{v_0} \prod_{k=-1}^{n} \frac{\varphi^k(z_0)}{\varphi^k(y_0)} \cdot \frac{1}{\varphi^{n-1}(y_0)} \geqslant 1,$$

hence

$$\varphi^{n-1}(y_0) \leqslant \frac{u_0}{v_0} \prod_{k=-1}^{n} \frac{\varphi^k(z_0)}{\varphi^k(y_0)} < 1,$$

which is impossible since $\lim_{n \to -\infty} \varphi^{n-1}(y_0) = 1$. The inequality $u_n < u_{n-1}$ follows by (2.3). Thus, we have proved (3.2).

By Theorem 2.1 we have $\Phi(f) = f$, hence

$$\Phi^n(f|[u_0, v_0]) \supset f|[u_n, v_n] \quad \text{for any integer} \quad n, \tag{3.3}$$

since the graph of f is locally connected. By (3.1), $x \in (u_0, v_0)$ implies $\varphi^n(f(x)) \in (z_n, y_n)$ for any integer n. Thus, the projection of the set $\Phi^n(f|[u_0, v_0])$ to the second axis is the interval $[z_n, y_n]$. On the other hand, by (3.2), the range of $f|[u_n, v_n]$ contains $f(u_{n-1}) = y_{n-1}$. Hence, by (3.3), $y_{n-1} \in [z_n, y_n]$, which is impossible since $y_{n-1} = \varphi^{-1}(y_n) > y_n$.

(ii) Assume similarly that there are $u_0 < v_0$ such that $f(u_0) = y_0 < f(v_0) = z_0$ and (3.1) is satisfied. Then

$$u_n < u_{n-1} < v_n \quad \text{for some} \quad n > 0. \tag{3.4}$$

The first inequality follows by (2.2), so to prove (3.4) assume $u_{n-1} \geqslant v_n$ for any $n > 0$. By (2.2), for $n > 1$,

$$\frac{u_{n-1}}{v_n} = \frac{u_0}{v_0} \prod_{k=0}^{n-2} \frac{\varphi^k(y_0)}{\varphi^k(z_0)} \cdot \frac{1}{\varphi^{n-1}(z_0)} \geqslant 1,$$

hence

$$1 = \lim_{n\to\infty} \varphi^{n-1}(z_0) \leqslant \frac{u_0}{v_0} \prod_{k=0}^{\infty} \frac{\varphi^k(y_0)}{\varphi^k(z_0)} < 1,$$

which is a contradiction. The rest of the argument is similar to the first case. Note that if $u_{n-1} \in [u_n, v_n]$ for some $n > 0$, then $y_{n-1} = f(u_{n-1}) \notin [y_n, z_n]$ since $y_n > y_{n-1}$. □

To simplify the notation, we will use in the sequel the following abbreviations:

$$P(u,v) := \prod_{k=0}^{\infty} \frac{\varphi^k(u)}{\varphi^k(v)}, \quad Q(u,v) := \prod_{k=1}^{\infty} \frac{\varphi^{-k}(u)}{\varphi^{-k}(v)}.$$

Since φ is increasing the functions $P(u,v), Q(u,v)$ for $u, v \in J$ are well-defined.

Proof of Theorem 1.3. We have $\lim\limits_{k\to\infty} \varphi^{-k}(z) = q \in (0,1)$ for $z \in J$. Hence, by Lemma 2.2 (ii), $Q(y_0, z_0) < \infty$. Since $\varphi(y) < y$ in J, any monotone solution must be non-decreasing. This follows easily by (2.1)–(2.3). So, by Theorem 1.1, it suffices to assume that $P(y_0, z_0) < \infty$ for some $y_0 > z_0$ in J, and to show that there is a continuous solution which is not monotone. By Lemma 2.2 (i) we may assume that y_0 is close enough to z_0 so that

$$1 < P(y_0, z_0) Q(y_0, z_0) \leqslant \frac{1}{q}. \tag{3.5}$$

Then there are positive reals $u_0 < v_0$ such that

$$q Q(y_0, z_0) \leqslant \frac{u_0}{v_0} \leqslant P(z_0, y_0). \tag{3.6}$$

Let $\{(u_n, y_n)\}$ and $\{(v_n, z_n)\}$ be the orbits of (u_0, y_0) and (v_0, z_0), respectively. We will show that there is a decreasing continuous function g from $[u_0, v_0]$ onto $[z_0, y_0]$ such that

$$\frac{u}{v} \leqslant P(g(v), g(u)) \quad \text{for any} \quad u < v \quad \text{in} \quad [u_0, v_0]. \tag{3.7}$$

Denote $P(z_0, y_0) = \alpha$. Thus $\alpha < 1$ and there is an increasing continuous function ϱ on $[\alpha, 1]$ such that $\varrho(x)/x$ is non-decreasing, $\varrho(1) = v_0$ and $\varrho(\alpha) = u_0$. Put $\tau(x) = \varrho(P(z_0, x))$. Then τ maps $\{z_0, y_0\}$ onto $\{u_0, v_0\}$ and, by Lemma 2.2 (i), it is continuous and decreasing. Then, for $z < y$ in $[z_0, y_0]$,

$$\frac{\tau(y)}{\tau(z)} = \frac{\varrho(P(z_0, y))}{\varrho(P(z_0, z))} \leqslant \frac{P(z_0, y)}{P(z_0, z)} = P(z, y).$$

The inequality follows since $\varrho(x)/x$ is non-decreasing, the equality since $P(x_0, y)$ and $P(z_0, z)$ are non-zero and finite. Put $g = \tau^{-1}$ to get (3.7).

Next we show that there is an increasing continuous function h from $[v_1, u_0]$ onto $[z_1, y_0]$ such that

$$\frac{u}{v} \leqslant Q(h(u), h(v)) \quad \text{for any} \quad u < v \quad \text{in} \quad [v_1, u_0]. \tag{3.8}$$

Actually, we have $v_1 < u_0$ since $z_0 < q$ and (3.6) implies $z_0 < u_0/v_0$, hence $v_1 = v_0 z_0 < u_0$. Put $Q(y_0, z_0) = \beta$. Similarly to the previous case, define $\tau(x) = \varrho(Q(x, z_0))$ for $x \in [z_1, y_0]$. Here ϱ is an increasing continuous function such that $\varrho(z_0/q) = v_1$, $\varrho(\beta) = u_0$ (we have $z_0/q < 1 < \beta$), and $\varrho(x)/x$ is non-increasing. By Lemma 2.2 (i), $\tau(x)$ is continuous, increasing, with $\tau(z_1) = v_1$ and $\tau(y_0) = u_0$. Since $\varrho(x)/x$ is non-increasing, for any $z < y$ in $[z_1, y_0]$ we have

$$\frac{\tau(y)}{\tau(z)} = \frac{\varrho(Q(y, z_0))}{\varrho(Q(z, z_0))} \leqslant \frac{Q(y, z_0)}{Q(z, z_0)} = Q(y, z),$$

which proves (3.8) when putting $h = \tau^{-1}$.

Define $f_0\colon [v_1, v_0] \to [z_1, y_0]$ by $f_0(x) = h(x)$ for $x \in [v_1, u_0]$ and $f_0(x) = g(x)$ otherwise. Set

$$\tau_n(x) = x \prod_{k=0}^{n-1} \varphi^k(f_0(x)) \quad \text{for} \quad n > 0 \quad \text{and} \quad x \in [v_1, v_0]. \tag{3.9}$$

We will show that τ_n is increasing. For $x < y$ in $[u_0, v_0]$ we have $f_0(x) > f_0(y)$ hence, by (3.7),

$$\frac{\tau_n(y)}{\tau_n(x)} = \frac{y}{x} \prod_{k=0}^{n-1} \frac{\varphi^k(f_0(y))}{\varphi^k(f_0(x))} > \frac{y}{x} \prod_{k=0}^{\infty} \frac{\varphi^k(f_0(y))}{\varphi^k(f_0(x))} \geqslant 1.$$

This proves that τ_n is increasing in $[u_0, v_0]$. For $x \in [v_1, u_0]$, $f_0(x)$ is increasing, hence τ_n as a composition of increasing functions is increasing as well. Consequently, τ_n is increasing in $[u_0, v_0]$ and by (2.2), τ_n maps $[u_0, v_0]$ onto $[u_n, v_n]$ for $n > 0$.

Similarly, let

$$\tau_n(x) = \frac{x}{\prod\limits_{k=-1}^{n} \varphi^k(f_0(x))} \quad \text{for} \quad n < 0 \quad \text{and} \quad x \in [v_1, v_0]. \tag{3.10}$$

Then again τ_n is increasing. This is clearly true for $x \in [u_0, v_0]$ since $f_0(x)$ is decreasing there. So let $x < y$ in $[v_1, u_0]$. Since $f_0(x) < f_0(y)$ we have, by (3.8),

$$\frac{\tau_n(y)}{\tau_n(x)} = \frac{y}{x} \prod_{k=-1}^{n} \frac{\varphi^k(f_0(y))}{\varphi^k(f_0(x))} > \frac{y}{x} \prod_{k=1}^{\infty} \frac{\varphi^{-k}(f_0(y))}{\varphi^{-k}(f_0(x))} \geqslant 1.$$

Thus, $\tau_n(x) < \tau_n(y)$. Hence, τ_n is increasing on $[v_1, u_0]$ and consequently, on $[v_1, v_0]$.

Finally, for any integer n put

$$f_n(x) = \varphi^n(f_0(\tau_n^{-n}(x)) \quad \text{for} \quad x \in [v_{n+1}, v_n],$$

letting τ_0 be the identity function. By (3.9), (3.10) and Theorem 2.1, and since τ_n is increasing and maps $[v_1, v_0]$ onto $[v_{n+1}, v_n]$, we get that $f = \bigcup\limits_n f_n$ is a continuous solution of (1.1) which is not monotone (e.g., on $[v_1, v_0]$). This completes the proof. □

P r o o f o f T h e o r e m 1.4. It is similar to that of Theorem 1.3, so we give here only a brief outline, emphasizing the differences. There are $u_0 < v_0$ in $(0, \infty)$ and z_0 *greater than* y_0 in J such that

$$q\alpha = qP(z_0, y_0) \leqslant \frac{u_0}{v_0} \leqslant Q(y_0, z_0) = \beta.$$

We need a *decreasing* continuous g from $[v_1, u_0]$ onto $[y_0, z_1]$ such that

$$\frac{u}{v} \leqslant P(g(v), g(u)) < \infty \quad \text{for any} \quad u < v \quad \text{in} \quad [v_1, u_0]. \tag{3.11}$$

Put $g = \tau^{-1}$ where $\tau(x) = \varrho(P(y_0, x))$, $\varrho(z_0/(q\alpha)) = v_1 = u_0 z_0$, $\varrho(1) = u_0$, and $\varrho(x)/x$ is non-decreasing.

Similarly find an increasing continuous function h from $[u_0, v_0]$ onto $[y_0, z_0]$ such that

$$\frac{u}{v} \leqslant Q(h(u), h(v)) < \infty \quad \text{for any} \quad u < v \quad \text{in} \quad [u_0, v_0]. \tag{3.12}$$

To do this we set $\tau(x) = \varrho(Q(x, z_0))$ for $x \in [y_0, z_0]$ so that $\varrho(x)/x$ is non-decreasing, $\varrho(\beta) = u_0$ and $\varrho(1) = v_0$, and put $h = \tau^{-1}$.

The rest of the proof is similar to that of Theorem 1.3. We only replace (3.7) and (3.8) by (3.11) and (3.12), respectively. □

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