OSCILLATION OF SECOND ORDER NEUTRAL DELAY DIFFERENTIAL EQUATIONS

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Abstract. We establish some new oscillation criteria for the second order neutral delay differential equation

$$[r(t)|[x(t)+p(t)x[\tau(t)]]'|^{\alpha-1}[x(t)+p(t)x[\tau(t)]]']' + q(t)f(x[\sigma(t)]) = 0.$$

The obtained results supplement those of Dzurina and Stavroulakis, Sun and Meng, Xu and Meng, Baculíková and Lacková. We also make a slight improvement of one assumption in the paper of Xu and Meng.

Keywords: differential equation, oscillation, second order, delay, neutral type, integral averaging method

MSC 2000: 34C10

1. Introduction

In this paper we deal with the oscillation of the second order neutral delay differential equation

$$(E^+) \qquad [r(t)|[x(t)+p(t)x[\tau(t)]]'|^{\alpha-1}[x(t)+p(t)x[\tau(t)]]']' + q(t)f(x[\sigma(t)]) = 0,$$

where $\alpha > 0$ is a constant, $p, q \in C[t_0,\infty)$, $f \in C(\mathbb{R},\mathbb{R})$.

We suppose throughout the paper that the following hypotheses hold:

(H$_1$) $q(t) \geqslant 0$, $q(t) = 0$ only at isolated points, $0 \leqslant p(t) \leqslant 1$, $p(t) \not\equiv 1$ on any (T,∞);

(H$_2$) $r(t) \in C^1[t_0,\infty)$, $r(t) > 0$, $R(t) := \displaystyle\int_{t_0}^{t} r^{-1/\alpha}(s)\,\mathrm{d}s \to \infty$ as $t \to \infty$;

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(H$_3$) $\dfrac{f(x)}{|x|^{\alpha-1}x} \geqslant \beta > 0$ for $x \neq 0$;

(H$_4$) $\sigma(t) \in C^1[t_0,\infty)$, $\sigma(t) \leqslant t$, $\sigma'(t) \geqslant 0$, $\lim\limits_{t\to\infty} \sigma(t) = \infty$;

(H$_5$) $\tau(t) \in C^1[t_0,\infty)$, $\tau(t) \leqslant t$, $\lim\limits_{t\to\infty} \tau(t) = \infty$.

By a solution of Eq. (E^+) we mean a function $x(t) \in C^1[T_x,\infty)$, $T_x \geqslant t_0$, such that $z(t) = x(t) + p(t)x[\tau(t)]$ has the property $r(t)\left|z'(t)\right|^{\alpha-1} z'(t) \in C^1[T_x,\infty)$ and $x(t)$ satisfies (E^+) on $[T_x,\infty)$. We consider only those solutions $x(t)$ of (E^+) which satisfy $\sup\{|x(t)|\colon\ t \leqslant T\} > 0$ for all $T \geqslant T_x$. We assume that (E^+) possesses such a solution. A nontrivial solution of (E^+) is said to be oscillatory if it has arbitrarily large zeros; otherwise it is called nonoscillatory. Equation (E^+) is oscillatory if all of its solutions are oscillatory.

The oscillatory properties of the corresponding linear equation

$$(r(t)y')' + q(t)y[\tau(t)] = 0$$

have been extended to (E^+) with $p(t) \equiv 0$ and $f(x) = x$ by Mirzov [11], [12], [13], Elbert [5], [6], Kusano et al. [8], [9], Chern et al. [3], Agarwal et al. [1].

Dzurina and Stavroulakis [4] generalized these oscillatory criteria to a particular case of (E^+) when $p(t) \equiv 0$, $f(x) = |x|^{\alpha-1}x$, namely

$$(r(t)|u'(t)|^{\alpha-1}u'(t))' + q(t)|u[\tau(t)]|^{\alpha-1}u[\tau(t)] = 0. \tag{$*$}$$

In [4], Eq. $(*)$ was studied in two separate cases under the assumptions $0 < \alpha < 1$ and $\alpha \geqslant 1$, respectively. Sun and Meng in [14] presented a technique that offers a perfect result for all $\alpha > 0$.

Baculíková and Lacková [2] have studied a particular case of (E^+) of the form

$$[r(t)|[x(t)+p(t)x(\tau(t))]'|^{\alpha-1}[x(t)+p(t)x(\tau(t))]']' + \ q(t)|x[\sigma(t)]|^{\alpha-1}x[\sigma(t)] = 0.$$

Their oscillatory condition obtained by using the integral averaging method requires the restriction $\alpha \geqslant 1$. The technique presented in this paper allows us to drop this restriction.

The main aim of this paper is to extend the integral averaging technique to (E^+) in order to obtain new oscillatory criteria for the general equation (E^+).

2. Main results

We need the following lemma.

Lemma 2.1 (See [7]). *If A and B are nonnegative constants, then*

$$F(A,B) = A^{\lambda} - \lambda A B^{\lambda-1} + (\lambda - 1)B^{\lambda} \geqslant 0, \quad \lambda > 1$$

and the equality holds if and only if $A = B$.

Proof. Note that if $A = 0$ then $F(A,B) = (\lambda-1)B^{\lambda} \geqslant 0$. For $A > 0$ we have

$$F(A,B) = A^{\lambda}[1 - \lambda C^{\lambda-1} + (\lambda-1)C^{\lambda}],$$

where $C = B/A$. Using standard methods of Calculus one can easily verify that

$$f(C) = 1 - \lambda C^{\lambda-1} + (\lambda-1)C^{\lambda} \geqslant 0.$$

The proof is complete. □

We will use a "modified" integral averaging method. Let us consider a function $H(t,s)$ satisfying the following conditions:
(i) $H(t,s) > 0$ for $t > s \geqslant t_0$,
(ii) $H(t,t) = 0$ and $\partial H(t,s)/\partial s < 0$.
Denote for $t > s \geqslant t_0$

$$Q(t,s) = H^{-\alpha}(t,s)\Big(\alpha\sigma'(s)H(t,s) + R[\sigma(s)]r^{1/\alpha}[\sigma(s)] \cdot \frac{\partial H(t,s)}{\partial s}\Big)^{\alpha+1}.$$

Theorem 2.1. *If*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_1)} \int_{t_1}^{t} \Big[H(t,s)R^{\alpha}[\sigma(s)]\beta q(s)(1 - p[\sigma(s)])^{\alpha} - \frac{Q(t,s)}{(\alpha+1)^{\alpha+1}R[\sigma(s)]r^{1/\alpha}[\sigma(s)][\sigma'(s)]^{\alpha}}\Big]\,\mathrm{d}s = \infty, \tag{1}$$

then Eq. (E^{+}) is oscillatory.

Proof. Assume to the contrary that $x(t)$ is a nonoscillatory solution of Eq. (E^{+}). We may assume that $x(t) > 0$. The case of $x(t) < 0$ can be proved by the same arguments. Set

$$z(t) = x(t) + p(t)x[\tau(t)].$$

Then $z(t) \geqslant x(t) > 0$ and

$$[r(t)|z'(t)|^{\alpha-1}z'(t)]' = -q(t)f(x[\sigma(t)]) \leqslant 0.$$

There are two possibilities for $z'(t)$:
(i) $z'(t) > 0$,
(ii) $z'(t) < 0$ for $t \geqslant t_1 \geqslant t_0$.

The condition (ii) implies that for some positive constant M and for all $t \geqslant t_1 \geqslant t_0$

$$r(t)|z'(t)|^{\alpha-1}z'(t) \leqslant -M < 0.$$

Thus

$$-z'(t) \geqslant \Big(\frac{M}{r(t)}\Big)^{1/\alpha}.$$

Integrating the above inequality from t_1 to t, we obtain

$$z(t) \leqslant z(t_1) - M^{1/\alpha}(R(t) - R(t_1)).$$

Letting $t \to \infty$ in the above inequality and using (H_2), we get $z(t) \to -\infty$. This contradiction proves that (i) holds.

For the case (i), we obtain

$$x(t) = z(t) - p(t)x[\tau(t)] \geqslant z(t) - p(t)z[\tau(t)] \geqslant (1 - p(t))z(t). \tag{2}$$

Combining the above inequality and (H_3) with Eq. (E^+), we have

$$[r(t)(z'(t))^\alpha]' + \beta q(t)(1 - p[\sigma(t)])^\alpha z^\alpha[\sigma(t)] \leqslant 0 \tag{3}$$

and

$$[r(t)(z'(t))^\alpha]' \leqslant 0.$$

Therefore

$$r(t)(z'(t))^\alpha \leqslant r[\sigma(t)](z'[\sigma(t)])^\alpha,$$

which implies that

$$\frac{z'[\sigma(t)]}{z'(t)} \geqslant \Big(\frac{r(t)}{r[\sigma(t)]}\Big)^{1/\alpha}. \tag{4}$$

Define

$$w(t) = R^\alpha[\sigma(t)]\frac{r(t)(z'(t))^\alpha}{z^\alpha[\sigma(t)]} > 0 \tag{5}$$

for $t \geqslant t_1$.

Differentiating $w(t)$, we obtain

$$w'(t) = \alpha R^{\alpha-1}[\sigma(t)]\frac{\sigma'(t)r(t)\big(z'(t)\big)^{\alpha}}{r^{1/\alpha}[\sigma(t)]z^{\alpha}[\sigma(t)]} + R^{\alpha}[\sigma(t)]\frac{\big[r(t)\big(z'(t)\big)^{\alpha}\big]'}{z^{\alpha}[\sigma(t)]} - \alpha R^{\alpha}[\sigma(t)]\frac{r(t)\big(z'(t)\big)^{\alpha}z'[\sigma(t)]\sigma'(t)}{z^{\alpha+1}[\sigma(t)]}. \tag{6}$$

Using (3), (4) and (5), we have

$$\begin{aligned} w'(t) &\leqslant \frac{\alpha\sigma'(t)}{R[\sigma(t)]r^{1/\alpha}[\sigma(t)]}w(t) - R^{\alpha}[\sigma(t)]\beta q(t)\big(1-p[\sigma(t)]\big)^{\alpha} \\ &\quad - \frac{\alpha\sigma'(t)}{R[\sigma(t)]r^{1/\alpha}[\sigma(t)]}\cdot\frac{R^{\alpha+1}[\sigma(t)]r^{(\alpha+1)/\alpha}(t)\big(z'(t)\big)^{\alpha+1}}{z^{\alpha+1}[\sigma(t)]}, \\ w'(t) &\leqslant \frac{\alpha\sigma'(t)}{R[\sigma(t)]r^{1/\alpha}[\sigma(t)]}w(t) - \frac{\alpha\sigma'(t)}{R[\sigma(t)]r^{1/\alpha}[\sigma(t)]}w^{(\alpha+1)/\alpha}(t) \\ &\quad - R^{\alpha}[\sigma(t)]\beta q(t)\big(1-p[\sigma(t)]\big)^{\alpha}. \end{aligned}$$

Multiplying this inequality with $H(t,s) > 0$ and then integrating from t_1 to t we have

$$\begin{aligned} \int_{t_1}^{t} H(t,s)R^{\alpha}[\sigma(s)]\beta q(s)\big(1-p[\sigma(s)]\big)^{\alpha}\,\mathrm{d}s &\leqslant \int_{t_1}^{t} H(t,s)\frac{\alpha\sigma'(s)}{R[\sigma(s)]r^{1/\alpha}[\sigma(s)]}w(s)\,\mathrm{d}s \\ &\quad - \int_{t_1}^{t} H(t,s)\frac{\alpha\sigma'(s)}{R[\sigma(s)]r^{1/\alpha}[\sigma(s)]}w^{(\alpha+1)/\alpha}(s)\,\mathrm{d}s - \int_{t_1}^{t} H(t,s)w'(s)\,\mathrm{d}s. \end{aligned}$$

Now integrating (by parts) from t_1 to t we arrive at

$$\begin{aligned} &\int_{t_1}^{t} H(t,s)R^{\alpha}[\sigma(s)]\beta q(s)\big(1-p[\sigma(s)]\big)^{\alpha}\,\mathrm{d}s \\ &\quad\leqslant H(t,t_1)w(t_1) + \int_{t_1}^{t}\frac{\alpha\sigma'(s)H(t,s)}{R[\sigma(s)]r^{1/\alpha}[\sigma(s)]} \\ &\quad\quad\times\Big[w(s)\Big(1+\frac{R[\sigma(s)]r^{1/\alpha}[\sigma(s)]}{\alpha\sigma'(s)H(t,s)}\cdot\frac{\partial H(t,s)}{\partial s}\Big) - w^{(\alpha+1)/\alpha}(s)\Big]\,\mathrm{d}s. \end{aligned} \tag{7}$$

Set $A = w(s)$ and

$$B = \Big[\frac{1}{\lambda}\Big(1+\frac{R[\sigma(s)]r^{1/\alpha}[\sigma(s)]}{\alpha\sigma'(s)H(t,s)}\cdot\frac{\partial H(t,s)}{\partial s}\Big)\Big]^{1/(\lambda-1)},$$

where $\lambda = (\alpha+1)/\alpha > 1$. Then

$$(\lambda-1)B^{\lambda} = \frac{(\alpha\sigma'(s)H(t,s) + R[\sigma(s)]r^{1/\alpha}[\sigma(s)]\partial H(t,s)/\partial s)^{\alpha+1}}{\alpha(\alpha+1)^{\alpha+1}H^{\alpha+1}(t,s)[\sigma'(s)]^{\alpha+1}}. \tag{8}$$

Applying Lemma 2.1 to (7) and using (8) and the definition of the function $Q(t,s)$, we conclude that

$$\frac{1}{H(t,t_1)}\int_{t_1}^{t}\Big[H(t,s)R^{\alpha}[\sigma(s)]\beta q(s)\big(1-p[\sigma(s)]\big)^{\alpha} - \frac{Q(t,s)}{(\alpha+1)^{\alpha+1}R[\sigma(s)]r^{1/\alpha}[\sigma(s)][\sigma'(s)]^{\alpha}}\Big]\,\mathrm{d}s \leqslant w(t_1).$$

Letting $t\to\infty$ we get a contradiction with (1), since the left hand side of the previous inequality tends to ∞. This completes the proof of Theorem 2.1. □

3. Concluding remarks

Remark 1. Note that if $p(t)\equiv 1$ then (1) is never fulfilled. This is due to the fact that (2) gives in this case only $x(t)\geqslant 0$ and our arguments of the proof of Theorem 2.1 fail. So condition (H_1) must hold and this assumption has to be added also to Theorem 1 in [15].

Setting $H(t,s)=(t-s)^n$, n being a positive integer, Theorem 2.1 reduces to

Theorem 3.1. *If*

$$\limsup_{t\to\infty}\frac{1}{(t-t_1)^n}\int_{t_1}^{t}\Big[(t-s)^n R^{\alpha}[\sigma(s)]\beta q(s)\big(1-p[\sigma(s)]\big)^{\alpha} - \frac{Q(t,s)}{(\alpha+1)^{\alpha+1}R[\sigma(s)]r^{1/\alpha}[\sigma(s)][\sigma'(s)]^{\alpha}}\Big]\,\mathrm{d}s=\infty,$$

where

$$Q(t,s)=(t-s)^n\Big(\alpha\sigma'(s)-\frac{nR[\sigma(s)]r^{1/\alpha}[\sigma(s)]}{t-s}\Big)^{\alpha+1},$$

then Eq. (E^+) is oscillatory.

For the particular case of (E^+), namely for

$$[|x'(t)|^{\alpha-1}x'(t)]'++q(t)|x[\sigma(t)]|^{\alpha-1}x[\sigma(t)]=0, \tag{9}$$

we have

Corollary 3.1. *If*

$$\limsup_{t\to\infty} \frac{1}{(t-t_1)^n} \int_{t_1}^{t} (t-s)^n \times \Big[[\sigma(s)]^\alpha q(s) - \Big(\frac{\alpha}{\alpha+1}\Big)^{\alpha+1} \frac{\sigma'(s)}{\sigma(s)} \Big(1 - \frac{n\sigma(s)}{\alpha(t-s)\sigma'(s)}\Big)^{\alpha+1}\Big] \, \mathrm{d}s = \infty$$

then the equation (9) *is oscillatory.*

Recently, W. T. Li [Theorem 2.2 in [10]] presented the following oscillatory criterion for

$$[r(t)|x'(t)|^{\alpha-1}x'(t)]' + q(t)|x[\sigma(t)]|^{\alpha-1}x[\sigma(t)] = 0. \tag{10}$$

Denote

$$\frac{\partial H}{\partial s} = -h_2(t,s)\sqrt{H(t,s)}.$$

Theorem 3.2. *If there exists a positive nondecreasing function $\varrho(t) \in C^1[t_0,\infty)$ such that*

$$\limsup_{t\to\infty} \int_{t_1}^{t} \Big[H(s,t_1)q(s) - \frac{r[\sigma(s)]\varrho(s)\big(h_2(s,t_1) + \frac{\varrho'(s)}{\varrho(s)}\sqrt{H(s,t_1)}\big)^{\alpha+1}}{(\alpha+1)^{\alpha+1}(\sigma'(s))^\alpha [H(s,t_1)]^{(\alpha-1)/2}}\Big] \, \mathrm{d}s > 0 \tag{11}$$

and

$$\limsup_{t\to\infty} \int_{t_1}^{t} \Big[H(t,s)q(s) - \frac{r[\sigma(s)]\varrho(s)\big(h_2(t,s) + \frac{\varrho'(s)}{\varrho(s)}\sqrt{H(t,s)}\big)^{\alpha+1}}{(\alpha+1)^{\alpha+1}(\sigma'(s))^\alpha [H(t,s)]^{(\alpha-1)/2}}\Big] \, \mathrm{d}s > 0, \tag{12}$$

then the equation (10) *is oscillatory.*

On the other hand, Theorem 2.1 for (10) reduces to

Corollary 3.2. *If*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_1)} \int_{t_1}^{t} \Big[H(t,s)R^\alpha[\sigma(s)]q(s) - \frac{(\alpha\sigma'(s)H(t,s) + R[\sigma(s)]r^{1/\alpha}[\sigma(s)] \cdot \partial H(t,s)/\partial s)^{\alpha+1}}{(\alpha+1)^{\alpha+1} H^\alpha(t,s) R[\sigma(s)] r^{1/\alpha}[\sigma(s)][\sigma'(s)]^\alpha}\Big] \, \mathrm{d}s = \infty, \tag{13}$$

then the equation (10) *is oscillatory.*

Corollary 3.2 supplements Theorem 3.2 and reduces the conditions (11) and (12) to one condition (13).

References

[1] *R. P. Agarwal, S. L. Shieh, C. C. Yeh*: Oscillation criteria for second-order retarded differential equations. Math. Comput. Modelling *26* (1997), 1–11.

[2] *B. Baculíková, D. Lacková*: Oscillation criteria for second order retarded differential equations. Proc. of CDDEA 2006, Studies of the University of Žilina *20* (2006), 11–18.

[3] *J. L. Chern, W. Ch. Lian, C. C. Yeh*: Oscillation criteria for second order half-linear differential equations with functional arguments. Publ. Math. Debrecen *48* (1996), 209–216.

[4] *J. Džurina, I. P. Stavroulakis*: Oscillation criteria for second-order delay differential equations. Appl. Math. Comput. *140* (2003), 445–453.

[5] *A. Elbert*: A half-linear second order differential equation. Colloq. Math. Soc. Bolyai 30: Qualitative Theory of Differential Equations (Szeged) (1979), 153–180.

[6] *A. Elbert*: Oscillation and nonoscillation theorems for some nonlinear ordinary differential equations. Lecture Notes in Math., Ordinary and Partial Differential Equations *964* (1982), 187–212.

[7] *G. H. Hardy, J. E. Littlewood, G. Polya*: Inequalities. Second ed., Cambridge University Press, Cambridge, 1952.

[8] *T. Kusano, Y. Naito, A. Ogata*: Strong oscillation and nonoscillation of quasilinear differential equations of second order. Differ. Equ. Dyn. Syst. *2* (1994), 1–10.

[9] *T. Kusano, N. Yoshida*: Nonoscillation theorems for a class of quasilinear differential equations of second order. J. Math. Anal. Appl. *189* (1995), 115–127.

[10] *W. T. Li*: Interval oscillation of the second-order half-linear functional differential equations. Appl. Math. Comput. *155* (2004), 451–468.

[11] *D. D. Mirzov*: On the oscillation of a system of nonlinear differential equations. Diferencialnye Uravnenija *9* (1973), 581–583. (In Russian.)

[12] *D. D. Mirzov*: On some analogs of Sturm's and Kneser's theorems for nonlinear systems. J. Math. Anal. Appl. *53* (1976), 418–425.

[13] *D. D. Mirzov*: On the oscillation of solutions of a system of differential equations. Mat. Zametki *23* (1978), 401–404. (In Russian.)

[14] *Y. G. Sun, F. W. Meng*: Note on the paper of Džurina and Stavroulakis. Appl. Math. Comput. *174* (2006), 1634–1641.

[15] *R. Xu, F. W. Meng*: Some new oscillation criteria for second order quasi-linear neutral delay differential equations. Appl. Math. Comput. *182* (2006), 797–803.

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