

## SET COLORINGS IN PERFECT GRAPHS

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(Received September 1, 2009)

*Abstract.* For a nontrivial connected graph  $G$ , let  $c: V(G) \rightarrow \mathbb{N}$  be a vertex coloring of  $G$  where adjacent vertices may be colored the same. For a vertex  $v \in V(G)$ , the neighborhood color set  $\text{NC}(v)$  is the set of colors of the neighbors of  $v$ . The coloring  $c$  is called a set coloring if  $\text{NC}(u) \neq \text{NC}(v)$  for every pair  $u, v$  of adjacent vertices of  $G$ . The minimum number of colors required of such a coloring is called the set chromatic number  $\chi_s(G)$ . We show that the decision variant of determining  $\chi_s(G)$  is NP-complete in the general case, and show that  $\chi_s(G)$  can be efficiently calculated when  $G$  is a threshold graph. We study the difference  $\chi(G) - \chi_s(G)$ , presenting new bounds that are sharp for all graphs  $G$  satisfying  $\chi(G) = \omega(G)$ . We finally present results of the Nordhaus-Gaddum type, giving sharp bounds on the sum and product of  $\chi_s(G)$  and  $\chi_s(\overline{G})$ .

*Keywords:* set coloring, perfect graph, NP-completeness

*MSC 2010:* 05C15, 05C17, 05C35, 05C70

## 1. INTRODUCTION

In previous work, Chartrand, Okamoto, Rasmussen, and Zhang [2] introduced a vertex coloring called a *set coloring*. We first define

$$\mathbb{N}_k = \{1, 2, \dots, k\}$$

for each positive integer  $k$ . Then, for a nontrivial connected graph  $G$ , let  $c: V(G) \rightarrow \mathbb{N}_k$  be a vertex coloring of  $G$  where adjacent vertices may be colored the same. For a set  $S \subseteq V(G)$ , define the *set  $c(S)$  of colors of  $S$*  by

$$c(S) = \{c(v) : v \in S\}.$$

For a vertex  $v$  in a graph  $G$ , let  $N(v)$  be the neighborhood of  $v$  (the set of all vertices adjacent to  $v$  in  $G$ ). The *neighborhood color set*  $\text{NC}(v) = c(N(v))$  is the set of colors

of the neighbors of  $v$ . The coloring  $c$  is called *set neighbor-distinguishing*, or simply a *set coloring*, if  $\text{NC}(u) \neq \text{NC}(v)$  for every pair  $u, v$  of adjacent vertices of  $G$ .

The minimum number of colors required of such a coloring is called the *set chromatic number* of  $G$  and is denoted by  $\chi_s(G)$ . In this paper we establish the NP-completeness of determining the set chromatic number and initiate the search for classes of graphs for which the set chromatic number can be found efficiently, and in so doing we bring perfect graphs into the discussion. It was reported in [2] that  $\chi_s(G) \leq \chi(G)$  for every graph  $G$ . Here we continue to explore the relationship between these two parameters. In particular, we present new bounds that are sharp for graphs  $G$  satisfying  $\chi(G) = \omega(G)$ , and consequently for all perfect graphs. We show that finding the set chromatic number is apparently hard even for several classes of perfect graphs for which the chromatic number problem is easy, and prove that the set chromatic number can nevertheless be found in polynomial time for threshold graphs. We finally present results of the Nordhaus-Gaddum type, giving sharp bounds on the sum and product of  $\chi_s(G)$  and  $\chi_s(\overline{G})$  and providing realization results.

## 2. NP-COMPLETENESS OF THE SET CHROMATIC NUMBER

From a result in [2], we know that  $\chi_s(G) = 2$  if and only if  $G$  is bipartite and so  $\chi_s(G) = \chi(G)$  for every bipartite graph. Since bipartite graphs can be recognized in polynomial time, it follows that then  $\chi_s(G)$  can be computed in polynomial time for bipartite graphs  $G$ . We are concerned in this section with the identification of additional classes of graphs for which the set chromatic number can be computed efficiently. This, of course, is of no particular interest if computation of the set chromatic number is sufficiently hard. In [2] it was shown that not only is the set chromatic number bounded above by the chromatic number but that the difference between the set chromatic and chromatic numbers can be arbitrarily large. The perceived difficulty of finding the set chromatic number suggests that computation of the set chromatic number is no less complex than that of the chromatic number, and we prove here that this is the case. We first state the decision variant of the optimization problem *set chromatic number*, which we call *graph set  $k$ -colorability*.

**Instance.** A graph  $G = (V, E)$ , and  $k \in \mathbb{Z}^+$ .

**Question.** Does there exist a mapping  $c: V \rightarrow \mathbb{N}_k$  with the property that  $c$  is a set  $k$ -coloring of  $G$ , i.e., a labeling such that for every edge  $e = uv \in E$  the sets  $\text{NC}(u)$  and  $\text{NC}(v)$  of colors assigned by  $c$  are distinct?

The theorem itself follows from viewing  $k$ -colorability as a restricted case of set  $k$ -colorability.

**Theorem 2.1.** *Graph set  $k$ -colorability is NP-complete.*

PROOF. We know that every instance of *graph set  $k$ -colorability* (GSkC) can be regarded as an instance of *graph  $k$ -colorability* in which the labeling  $c$  is itself a proper coloring. It follows that the proper  $k$ -colorings of an instance  $(G, k)$  of *graph  $k$ -colorability* are in one-to-one correspondence with the set  $k$ -colorings of  $G$  that satisfy this additional restriction. Since every instance of the restricted version of GSkC is an instance of *graph  $k$ -colorability*, the transformation required is trivial.  $\square$

In the literature on graph algorithms, perfect graphs occupy a place of distinction. Recall that a graph  $G$  is *perfect* if  $\chi(H) = \omega(H)$  for all induced subgraphs  $H$  of  $G$ . The bipartite graphs constitute only one of the many classes of perfect graphs. For a comprehensive introduction to perfect graphs, their characterizations, and key results on the computational properties of various subclasses, see Golumbic [6]. The chromatic number  $\chi(G)$  is computable in polynomial time for many classes of perfect graphs. If  $G$  is a member of such a class, and if it can be shown that  $\chi_s(G) = \chi(G)$  for every member of the class, then  $\chi_s(G)$  can be efficiently computed for every graph in the class by the use of an off-the-shelf algorithm for computing  $\chi(G)$ .

Intuitively, computation of  $\chi_s(G)$  is harder than computation of  $\chi(G)$ , so it should come as no surprise that there are perfect classes to which the preceding remark does not apply in general. We illustrate this with a member of two such classes, the chordal graphs and the split graphs. Recall that a graph  $G$  is *chordal* if  $G$  contains no induced  $k$ -cycle for  $k > 3$ , and  $G$  is a *split graph* if  $V(G)$  can be partitioned into an independent set  $S$  and a set  $K$  that induces a clique. It is not difficult to see that every split graph is chordal. The graph  $G$  in Figure 1 is a split graph, whose vertex labels represent a set 3-coloring  $c$ , demonstrating that  $\chi_s(G) < \chi(G) = 4$ .

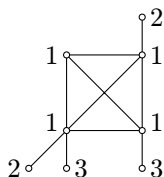


Figure 1. Split graph  $G$ , with  $\chi_s(G) = 3 < 4 = \chi(G)$ .

Nevertheless, we can provide one nontrivial example of a class of perfect graphs for which the chromatic number can be computed in polynomial time and possessing the additional property that  $\chi_s(G) = \chi(G)$  for every member of the class. Properly contained within the split graphs are the threshold graphs. The following characterization is due to Chvátal and Hammer [4]. Let  $\delta_0 = 0$ , and let  $\delta_1 < \delta_2 < \dots < \delta_k$  be

the distinct positive vertex degrees found in  $G$ . For each  $i$ , where  $0 \leq i \leq k$ , define  $V_i = \{v \in V : \deg(v) = \delta_i\}$ . Then  $G$  is a *threshold graph* if and only if for all  $u \in V_i$  and  $v \in V_j$ ,  $uv \in E$  if and only if  $i + j > k$ . The following consequence will be useful to us: if  $u \in V_i$ ,  $v \in V_j$ , and  $i \leq j$ , then  $N(u) - \{v\} \subseteq N(v) - \{u\}$ , with equality if and only if  $i = j$ .

For an example, see Figure 2, in which  $\delta_i = i$  for  $1 \leq i \leq 6$ ,  $V_1 = \{v_7\}$ ,  $V_2 = \{v_6\}$ ,  $V_3 = \{v_4, v_5\}$ ,  $V_4 = \{v_3\}$ ,  $V_5 = \{v_2\}$ , and  $V_6 = \{v_1\}$ . The threshold in this example is  $t = 6$ . Note that the mapping  $c: V(G) \rightarrow \mathbb{N}_4$  in Figure 2 is both a proper coloring and a set 4-coloring of  $G$ .

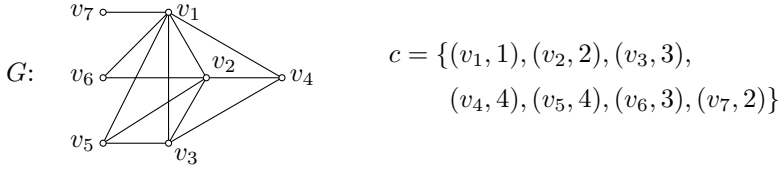


Figure 2. A threshold graph  $G$ , with  $t = 6$  and  $\chi_s(G) = 4 = \chi(G)$ , and a coloring  $c$  that is both proper and a set 4-coloring.

**Theorem 2.2.** *For every threshold graph  $G$ ,  $\chi_s(G) = \chi(G)$ .*

**Proof.** Suppose that  $G = (V, E)$  is a threshold graph. Since we already know that  $\chi_s(G) \leq \chi(G)$ , we need only to show that  $\chi_s(G) \geq \chi(G)$ . Since  $G$  is a split graph,  $V$  can be partitioned as  $V = \{K, S\}$ , where  $K$  induces a largest clique and  $S$  is an independent set. Since  $G$  is perfect,  $\chi(G) = \omega(G) = |K|$ . Let  $|K| = k$  and assume, to the contrary, that there exists a set coloring  $c: V \rightarrow \mathbb{N}_{k-1}$ . Since  $c$  uses fewer than  $k$  colors, there exists a nonempty subset  $X$  of  $K$  such that for each vertex  $v \in K$ ,  $v$  belongs to  $X$  if and only if there exists a vertex  $w \in K - \{v\}$  with  $c(v) = c(w)$ . Let  $X = \{x_1, x_2, \dots, x_l\}$ , where  $\deg x_1 \leq \deg x_2 \leq \dots \leq \deg x_l$ , and observe that  $2 \leq l \leq k$  and  $|c(K)| \geq k - l + 1$ .

Since  $c(K) \subseteq \text{NC}(x) \subseteq \mathbb{N}_{k-1}$  for every  $x \in X$  and no two vertices in  $X$  have the same neighborhood color set, it follows that  $N(x) \cap S \neq N(y) \cap S$  for every two vertices  $x, y \in X$ . Therefore,  $N(x) - \{y\} \neq N(y) - \{x\}$ , that is, no two vertices in  $X$  have the same degree and so  $N(x_i) \cap S \subset N(x_j) \cap S$  if  $i < j$ . This in turn implies that

$$c(K) \subseteq \text{NC}(x_1) \subset \text{NC}(x_2) \subset \dots \subset \text{NC}(x_l) \subseteq \mathbb{N}_{k-1},$$

which is impossible since  $|c(K)| \geq k - l + 1$ . Hence, such a set coloring  $c$  does not exist and so  $\chi_s(G) \geq k = \chi(G)$ .  $\square$

By Hammer and Simeone [7], if  $G$  is a split graph, then  $\omega(G)$  can be found in polynomial time. Since split graphs are perfect, and since every threshold graph is

a split graph, it follows from Theorem 2.2 that  $\chi_s(G)$  can be computed in polynomial time on threshold graphs.

### 3. BOUNDS FOR THE SET CHROMATIC NUMBER IN TERMS OF THE CHROMATIC NUMBER IN PERFECT GRAPHS

In this section we present bounds for  $\chi_s(G)$  in terms of  $\chi(G)$  in perfect graphs. We first summarize some results presented in [2]. For integers  $a$  and  $b$  with  $a < b$ , let

$$[a..b] = \{x \in \mathbb{Z} : a \leq x \leq b\}.$$

In particular,  $[1..b] = \mathbb{N}_b$ .

**Observation 3.1** ([2]). If  $u$  and  $v$  are two adjacent vertices in a graph  $G$  such that  $N(u) - \{v\} = N(v) - \{u\}$ , then  $c(u) \neq c(v)$  for every set coloring  $c$  of  $G$ . Furthermore, if  $S = N(u) - \{v\} = N(v) - \{u\}$ , then  $\{c(u), c(v)\} \not\subseteq c(S)$ .

**Theorem 3.1** ([2]). For every graph  $G$ ,

$$(3.1) \quad \chi_s(G) \geq 1 + \lceil \log_2 \omega(G) \rceil.$$

Observe that Theorem 3.1 gives us the following: If  $G$  is a graph with  $\chi_s(G) = a \geq 2$ , then  $\omega(G) \leq 2^{a-1}$ .

As an immediate corollary of Theorem 3.1 we have the following result for graphs satisfying  $\chi(G) = \omega(G)$ .

**Corollary 3.1.** Let  $G$  be a graph satisfying  $\chi(G) = \omega(G)$ . If  $\chi_s(G) = a \geq 2$ , then  $\chi(G) \leq 2^{a-1}$ .

Note that Corollary 3.1 applies to the entire class of perfect graphs. The following realization result summarizes the pairs  $(a, b)$  for which there exist perfect graphs  $G$  satisfying  $\chi_s(G) = a$  and  $\chi(G) = b$ , establishing the sharpness of the bound given in Corollary 3.1.

**Theorem 3.2.** For each pair  $a, b$  of integers with  $2 \leq a \leq b \leq 2^{a-1}$ , there exists a perfect graph  $G$  with  $\chi_s(G) = a$  and  $\chi(G) = b$ .

**Proof.** If  $a = b$ , then  $\chi_s(K_b) = \chi(K_b) = b$ . Hence we may assume that  $3 \leq a < b \leq 2^{a-1}$ . Consider the real-valued function  $f_a: [1, a-1] \rightarrow [a, 2^{a-1}]$  defined by  $f_a(x) = 2^x - x + a - 1$ . Note that  $f_a$  is strictly increasing on  $[1, a-1]$ . Consequently, there exists an integer  $p \in [2, a-1]$  such that

$$a = f_a(1) \leq f_a(p-1) < b \leq f_a(p) \leq f_a(a-1) = 2^{a-1}$$

and so

$$2^{p-1} + a - p + 1 \leq b \leq 2^p + a - p - 1.$$

Let  $S_1, S_2, \dots, S_{2^p}$  be the  $2^p$  subsets of  $\mathbb{N}_p$ , where  $|S_1| \leq |S_2| \leq \dots \leq |S_{2^p}|$ . (Hence  $S_1 = \emptyset$  and  $S_{2^p} = \mathbb{N}_p$ .)

Suppose first that  $p = a - 1$ . Let  $U = \{u_1, u_2, \dots, u_b\}$  be the vertex set of a complete graph  $K_b$ . A graph  $G$  is constructed from  $K_b$  by adding the vertices in the set  $X = \{x_1, x_2, \dots, x_p\}$  and joining  $x_i$  to  $u_j$  if and only if  $i \in S_j$  for  $1 \leq i \leq p$  and  $2 \leq j \leq b$  ( $\leq 2^{a-1}$ ). Thus  $\chi(G) = b$ , while  $\chi_s(G) \geq a$  by Theorem 3.1. On the other hand, the coloring that assigns the color  $a$  to every vertex in  $U$  and the color  $i$  to the vertex  $x_i$  for  $1 \leq i \leq a - 1$  is a set  $a$ -coloring of  $G$ . Therefore,  $\chi_s(G) = a$ .

We now assume that  $p \leq a - 2$ . Let the vertex set of a complete graph  $K_b$  be partitioned into two sets  $U$  and  $W$ , where  $U = \{u_1, u_2, \dots, u_{b-a+p+1}\}$  and  $W = \{w_1, w_2, \dots, w_{a-p-1}\}$ . We construct a graph  $G$  from  $K_b$  by adding the vertices in the set  $X = \{x_1, x_2, \dots, x_p\}$  and joining  $x_i$  to  $u_j$  if and only if  $i \in S_j$  for  $1 \leq i \leq p$  and  $1 \leq j \leq b - a + p + 1$  ( $\leq 2^p$ ).

By Observation 3.1, every set coloring must assign a distinct color to each of the  $a - p$  vertices in  $W \cup \{u_1\}$ . Assume, to the contrary, that there exists a set  $(a - 1)$ -coloring of  $G$  that uses the colors in  $\mathbb{N}_{a-1}$ . Without loss of generality, suppose that  $c(W \cup \{u_1\}) = \mathbb{N}_{a-p}$ . Then  $\mathbb{N}_{a-p} \subseteq \text{NC}(v)$  for every vertex  $v$  in  $U - \{u_1\}$ . However, there are only  $2^{p-1}$  subsets of  $\mathbb{N}_{a-1}$  that contain  $\mathbb{N}_{a-p}$  as a subset, while  $|U - \{u_1\}| = b - a + p > 2^{p-1}$ . This is impossible. Therefore,  $\chi_s(G) \geq a$ . To verify that  $\chi_s(G) \leq a$ , consider the  $a$ -coloring  $c: V(G) \rightarrow \mathbb{N}_a$  defined by

$$c(v) = \begin{cases} i & \text{if } v = x_i \ (1 \leq i \leq p), \\ p + 1 & \text{if } v \in U, \\ p + 1 + i & \text{if } v = w_i \ (1 \leq i \leq a - p - 1) \end{cases}$$

and observe that this is a set coloring of  $G$ . Therefore,  $\chi_s(G) = a$ .  $\square$

In each of the preceding cases, the graph  $G$  is a split graph and is consequently perfect.

We conclude this section with the following conjecture, which, if true, would generalize Corollary 3.1.

**Conjecture 3.1.** *Let  $G$  be a connected graph. If  $\chi_s(G) = a \geq 2$ , then  $\chi(G) \leq 2^{a-1}$ .*

Note that if this generalization holds, then the condition that  $G$  is perfect can be dropped from Theorem 3.2 and the construction used in the proof can be employed without alteration.

#### 4. NORDHAUS-GADDUM TYPE INEQUALITIES

We now leave the perfect graphs behind and turn our attention to Nordhaus-Gaddum type inequalities. The proof of Lemma 4.1 is adapted from the proof of the analogous result for chromatic number by Chartrand and Polimeni [3]. The proof of Theorem 4.1 is adapted from the proof by H. V. Kronk (see [1]) of the Nordhaus-Gaddum result for chromatic number.

**Lemma 4.1.** *Let  $G$  be a graph of order  $n$ , with complement  $\overline{G}$ . Then  $\chi_s(G) \times \chi_s(\overline{G}) \geq n$ .*

*Proof.* Let  $c$  and  $\bar{c}$  be a  $\chi_s(G)$ -coloring of  $G$  and a  $\chi_s(\overline{G})$ -coloring of  $\overline{G}$ , respectively. Assign the color  $(a_i, b_i)$  to vertex  $v_i \in V(K_n)$ , where  $c_i$  is the color assigned to  $v_i$  in  $G$  and  $b_i$  is the color assigned to  $v_i$  in  $\overline{G}$  for  $1 \leq i \leq n$ . We then define the neighborhood color set of  $v_i$  in  $K_n$  as the set of ordered pairs given by  $\text{NC}(v_i) = \{(a_j, b_j) : v_j \in V(K_n)\}$ . This forms a set coloring of  $K_n$  with at most  $\chi_s(G) \cdot \chi_s(\overline{G})$  colors. Since  $\chi_s(K_n) = n$ , the result follows.  $\square$

**Theorem 4.1.** *If a graph  $G$  has order  $n$ , then we have the following bounds:*

- (a)  $2\sqrt{n} \leq \chi_s(G) + \chi_s(\overline{G}) \leq n + 1$ , and
- (b)  $n \leq \chi_s(G) \cdot \chi_s(\overline{G}) \leq \left(\frac{1}{2}(n + 1)\right)^2$ .

*Proof.* The lower bound in (b) follows from Lemma 4.1. Also, since the arithmetic mean of two positive numbers is at least as large as their geometric mean, it follows that

$$\sqrt{\chi_s(G) \cdot \chi_s(\overline{G})} \leq \frac{\chi_s(G) + \chi_s(\overline{G})}{2}$$

and so the lower bound in (a) follows. The upper bounds in both (a) and (b) are straightforward results of the Nordhaus-Gaddum inequalities [8] for the chromatic numbers of a graph and its complement and of the fact that  $\chi_s(G) \leq \chi(G)$ .  $\square$

Sharpness of the bounds will follow from Proposition 4.4. For the chromatic number, Stewart [9] and Finck [5] showed that no improvement in the Nordhaus-Gaddum Theorem is possible (without employing additional conditions). We state this theorem as follows.

**Theorem 4.2** ([5], [9]). *Let  $n$  be a positive integer. For every two integers  $a$  and  $b$  such that  $2\sqrt{n} \leq a + b \leq n + 1$  and  $n \leq ab \leq \left(\frac{1}{2}(n + 1)\right)^2$ , there is a graph  $G$  of order  $n$  such that  $\chi(G) = a$  and  $\chi(\overline{G}) = b$ .*

We now present the analogue of Theorem 4.2 for set colorings, providing realization results for Theorem 4.1.

**Proposition 4.1.** *Let  $n$  be a positive integer. For every two integers  $a$  and  $b$  such that  $2\sqrt{n} \leq a + b \leq n + 1$  and  $n \leq ab \leq (\frac{1}{2}(n + 1))^2$ , there is a graph  $G$  of order  $n$  such that  $\chi_s(G) = a$  and  $\chi_s(\overline{G}) = b$ .*

**Proof.** Since  $a + b - 1 \leq n \leq ab$ , there exists a partition of  $n$  into  $a$  positive integers  $n_1, n_2, \dots, n_a$  such that  $1 \leq n_1 \leq n_2 \leq \dots \leq n_a = b$ . Thus  $n = \sum_{i=1}^a n_i$  and

$$a + b - 1 \leq \sum_{i=1}^{a-1} n_i + n_a = \sum_{i=1}^a n_i \leq an_a = ab.$$

Let  $G = K_{n_1, n_2, \dots, n_a}$ . Then  $\overline{G} = K_{n_1} \cup K_{n_2} \cup \dots \cup K_{n_a}$ . It follows that  $\chi_s(G) = a$ , and  $\chi_s(\overline{G}) = n_a = b$ .  $\square$

Note that, by the construction given in the proof of Proposition 4.4, the bounds given in Theorem 4.1 are sharp.

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