Nonequilibrium processes in systems with memory [1]

Summer School "Nonlinear Analysis and Extremal Problems", Irkutsk

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Contents

Introduction

Forced oscillations of elastoplastic constructions are classical examples of systems with memory far away from equilibrium. In addition to the space and time variables, memory is quantified in terms of a new variable, and the state space has to be extended by a metric space of functions of this memory variable. The thermodynamic quantities then involve also the dependence on the instantaneous memory state. It turns out that in this setting, thermoelastoplasticity ([24]), material fatigue ([11]), phase transitions ([26]), ferromagnetism ([21]), magnetostriction ([25]), and also specific problems in porous media flows ([23]) and magnetohydrodynamics ([10]), can be treated mathematically as systems of partial differential

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equations with hysteresis operators. This text, prepared for the Summer School “Nonlinear Analysis and Extremal Problems 2012” in Irkutsk, is assumed to be an introduction into hysteresis methods of solving nonequilibrium problems in systems with memory. Emphasis will be put on model problems related to energy dissipation and fatigue in oscillating elastoplastic beams and plates.

1 Oscillating lower-dimensional elastoplastic structures

Dynamic problems of multidimensional elastoplasticity lead to quasilinear second order hyperbolic equations, which are difficult from both analytic and computational viewpoints. Apart from the classical results in [9], where the small strain dynamic problem of Prandtl-Reuss plasticity was solved via the method of variational inequalities, the author is not aware of any substantial progress in this direction. Indeed, an abundant literature is devoted to quasistatic problems (see e. g. an overview in [18]), but a satisfactory mathematical theory of oscillating elastoplastic structures still seems to be missing.

In the first two sections, we summarize the results of [16, 28] on derivation of 1D/2D models for oscillations of elastoplastic beams/plates from the general dynamic 3D von Mises plasticity model. Indeed, the dimensional reduction is not an equivalent operation, so that the existence theorem from [9] says nothing about the solvability of the reduced equations, and proofs given in [16, 28] are part of the a posteriori justification of the model.

The dimensional reduction is carried out by the variational method of [6, 30], using scaling hypotheses analogous to the ones in the theory of elastic plates in [6, 8]. As main result, we show that the classical von Mises plasticity criterion with a single-yield condition leads after dimensional reduction to a *multi-yield model* of Prandtl [37] and Ishlinskii [19] type. This can be explained by the fact that in the 1D/2D model only deformations of longitudinal layers parameterized by the transversal coordinate are taken into account, and the individual longitudinal fibers do not switch from the elastic to the plastic regime at the same time. More precisely, the “eccentric” fibers look as if they had higher elasticity modulus and lower yield point than the central ones. Hence, the effect of the existence of plasticized zones (see Fig. 1) is translated into the mathematical language by means of the Prandtl-Ishlinskii combination of elastic-perfectly plastic elements with different yield limits that are not all simultaneously activated, see also Fig. 2 below.

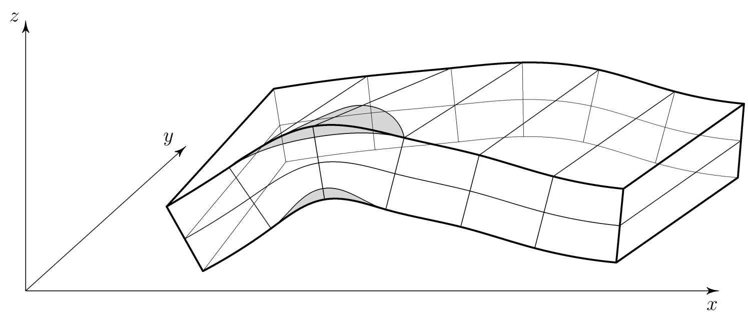

Figure 1. Deformed plate with grey plasticized zone.

This emerging multiyield character of the elastoplastic beam or plate bending problem does not seem to have been taken into consideration in older literature. The multiyield quasistatic model in [1] does not directly refer to plates. In [3, 29, 35], only the quasistatic case is investigated as well, and after dimensional reduction, the yield condition is still described by one sharp surface of plasticity. The relation between dynamic and quasistatic problems in single yield plasticity is studied in [33]. Methods based on Γ-convergence of energy minimizers ([14, 15, 36]) are indeed more rigorous than a simple scaling analysis. More recent applications of Γ-convergence to dimensional reduction in quasistatic elastoplasticity in [31, 32] lead to the multiyield Prandtl-Ishlinskii construction in the Γ-limit as well. The question whether the method can be applied to oscillating systems seems to be open.

1.1 Elastoplastic beam

We consider first the 3D $\to$ 1D transition, and focus on the question how the multi-yield behavior results from the single-yield von Mises model and from the dimensional reduction. This is why we do not look for maximal generality and keep the assumptions as simple as possible. We restrict ourselves to *rectangular beams*, that is, to sets $\Omega \subset \mathbb{R}^3$ of the form $\Omega = (0, L) \times \omega$, where $L > 0$ is the *length* of the beam, and where, with some $h > 0$ and $b > 0$, the set $\omega = (-b, b) \times (-h, h)$ represents its (rectangular) *cross section.* We denote by $x \in (0, L)$ the longitudinal coordinate, by $(y, z) \in \omega$ the transversal coordinates, and by $t \in [0, T]$ the time, where $T > 0$ is given.

In order to compare the resulting equations, we start with the linear elastic isotropic case (Subsection 1.2), and then pass to the elastoplastic model under further simplifying assumptions (Subsection 1.3). We follow the scaling technique of [6, Part A] and [8, Sect. 5.4] in terms of a small parameter $\alpha > 0$ with the intention to keep only lowest order terms in α in the resulting equations. In particular, we assume that

$$h, b = \mathcal{O}(\alpha),\ \ L = \mathcal{O}(1).$$

Let us consider smooth displacements $\mathbf{u} : \Omega \times (0, T) \to \mathbb{R}^3$, decomposed into

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} u_1^L \\ u_2^L \\ u_3^L \end{pmatrix} + \begin{pmatrix} u_1^H \\ u_2^H \\ u_3^H \end{pmatrix} = \mathbf{u}^L + \mathbf{u}^H\,, \tag{1.1}$$

where the superscripts L and H stand for low-order and high-order components with respect to α, respectively. We make the following assumptions:

(B1) The low-order displacement of the midsurface $\mathcal{C} = \{(x, y) \in \mathbb{R}^2;\ (x, y, 0) \in \Omega\}$ is independent of y, that is,

$$\mathbf{u}^L(x, y, 0, t) = \begin{pmatrix} v(x, t) \\ 0 \\ w(x, t) \end{pmatrix} \quad \forall\, (x, y) \in \mathcal{C}\,, \quad \forall\, t \in (0, T)\,, \tag{1.2}$$

with given functions $v, w : (0, L) \times (0, T) \to \mathbb{R}$.

(B2) The low-order deformation

$$\mathbf{F}^L(x,y,z,t) = \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \mathbf{u}^L(x,y,z,t) \tag{1.3}$$

leaves the plane cross sections $\{x\} \times \omega$ perpendicular to the midsurface and straight, that is,

$$\mathbf{F}^L(x,y,z,t) = \mathbf{F}^L(x,y,0,t) + z\,\mathbf{n}(x,y,t) \quad \forall\,(x,y,z,t) \in \Omega \times (0,T)\,, \tag{1.4}$$

where $\mathbf{n}(x,y,t)$ is the unit "upward" normal to the deformed midsurface $\mathcal{C}(t) = \mathcal{C} + \mathbf{F}^L(\mathcal{C},0,t)$ at time t.

(B3) $v_x = \mathcal{O}(\alpha^2)\,,\ w_{xx} = \mathcal{O}(\alpha)\,.$

Under the hypothesis (B3), we can linearize the problem by replacing

$$\mathbf{n}(x,y,t) = \frac{1}{\sqrt{(1+v_x(x,t))^2 + w_x^2(x,t)}} \begin{pmatrix} -w_x(x,t) \\ 0 \\ 1+v_x(x,t) \end{pmatrix}$$

with

$$\tilde{\mathbf{n}}(x,y,t) := \begin{pmatrix} -w_x(x,t) \\ 0 \\ 1 \end{pmatrix}. \tag{1.5}$$

This is justified, since an elementary computation yields that

$$|\,\tilde{\mathbf{n}}(x,y,t) - \mathbf{n}(x,y,t)| \;<\; \left(|v_x(x,t)| + |w_x(x,t)|\right)^2\,,$$

whenever $|v_x(x,t)| < 1$, $|w_x(x,t)| < 1$. This enables us to write for every $(x,y,z,t) \in \Omega \times (0,T)$ the low-order displacement $\mathbf{u}^L(x,y,z,t)$ as

$$\mathbf{u}^L(x,y,z,t) = \begin{pmatrix} v(x,t) - z\,w_x(x,t) \\ 0 \\ w(x,t) \end{pmatrix}. \tag{1.6}$$

The smallness assumptions ensure in particular that the deformation (1.3) is a local homeomorphism. We further compute

$$\nabla\mathbf{u}^L(x,y,z,t) = \begin{pmatrix} v_x(x,t) - z\,w_{xx}(x,t) & 0 & -w_x(x,t) \\ 0 & 0 & 0 \\ w_x(x,t) & 0 & 0 \end{pmatrix}, \tag{1.7}$$

and the low-order strain tensor $\boldsymbol{\varepsilon}^L = (\nabla\mathbf{u}^L + (\nabla\mathbf{u}^L)^T)/2$ becomes

$$\boldsymbol{\varepsilon}^L(x,y,z,t) = \begin{pmatrix} v_x(x,t) - z\,w_{xx}(x,t) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{1.8}$$

1.2 Small elastic deformations

We denote by “ $:$ ” the canonical scalar product in the space $\mathbb{T}^{3\times 3}_{\mathrm{sym}}$ of symmetric (3×3)-tensors, i. e.,

$$\boldsymbol{\xi} : \boldsymbol{\eta} = \sum_{i,j=1}^{3} \xi_{ij}\, \eta_{ij}\,, \quad \forall\, \boldsymbol{\xi} = (\xi_{ij})\,,\ \boldsymbol{\eta} = (\eta_{ij})\,, \quad i,j = 1,2,3\,. \tag{1.9}$$

Moreover, we define for any given $\boldsymbol{\xi} \in \mathbb{T}^{3\times 3}_{\mathrm{sym}}$ its (trace-free) *deviator* $\mathfrak{D}(\boldsymbol{\xi})$ by

$$\mathfrak{D}(\boldsymbol{\xi}) = \boldsymbol{\xi} - \frac{1}{3}\,(\boldsymbol{\xi} : \mathbf{1})\,\mathbf{1}\,, \tag{1.10}$$

where $\mathbf{1} = (\delta_{ij})$ denotes the Kronecker tensor.

To motivate the elastoplastic case treated below, we first study the case of linear isotropic elasticity, in which the strain tensor $\boldsymbol{\varepsilon}$ and the stress tensor $\boldsymbol{\sigma}$ are related to each other through the formula

$$\boldsymbol{\sigma} \;=\; 2\mu\,\boldsymbol{\varepsilon} + \lambda\,(\boldsymbol{\varepsilon} : \mathbf{1})\,\mathbf{1}\,, \tag{1.11}$$

where μ, λ are the Lamé constants. The main issue is to choose a proper scaling of $\boldsymbol{\sigma}$. The component σ_{11} is of the lowest order, which is $\mathcal{O}(\alpha^2)$ due to (1.8) and (1.11). In general, linear scaling cannot be automatically used in nonlinear problems, see [14]. Here, as we only deal with symmetric small strain tensors, the linear scaling hypothesis does not seem to be too controversial, cf. [14, Section 4]. Assuming that the motion is “sufficiently slow” and no volume forces act on the body, we may for scaling purposes refer to the elastostatic equilibrium conditions $\operatorname{div}\boldsymbol{\sigma} = \mathbf{0}$ which, according to the natural scaling of the variables $y, z = \mathcal{O}(\alpha)\,,\ x = \mathcal{O}(1)$ and due to the symmetry of $\boldsymbol{\sigma}$, justify the scaling hypothesis

(B4) $\sigma_{12}, \sigma_{13} = \mathcal{O}(\alpha^3)\,,\ \sigma_{22}, \sigma_{33}, \sigma_{23} = \mathcal{O}(\alpha^4)\,.$

According to (1.11) and Hypothesis (B4), the high-order strain tensor $\boldsymbol{\varepsilon}^H$ is scaled as

(B5) $\varepsilon^H_{12}, \varepsilon^H_{13} = \mathcal{O}(\alpha^3)\,,\ \varepsilon^H_{22}, \varepsilon^H_{33} = \mathcal{O}(\alpha^2)\,,\ \varepsilon^H_{11}, \varepsilon^H_{23} = \mathcal{O}(\alpha^4)\,.$

In terms of the high-order displacements $\mathbf{u}^H$, this corresponds to the scaling $u^H_1 = \mathcal{O}(\alpha^4)$, $u^H_2, u^H_3 = \mathcal{O}(\alpha^3)$, with a vanishing $\mathcal{O}(\alpha^2)$ component of ε^H_{23}.

Let $\bar{\boldsymbol{\sigma}}$, $\bar{\boldsymbol{\varepsilon}}$ denote the stress and strain components of the order $\mathcal{O}(\alpha^2)$ at most. Then

$$\bar{\boldsymbol{\sigma}} : \mathbf{1} = \sigma_{11} = (2\mu + 3\lambda)\,\bar{\boldsymbol{\varepsilon}} : \mathbf{1} = (2\mu + 3\lambda)(\varepsilon^L_{11} + \varepsilon^H_{22} + \varepsilon^H_{33})\,,$$

hence, by (1.11), $\sigma_{11} = 2\mu\varepsilon^L_{11} + \lambda/(2\mu + 3\lambda)\sigma_{11}$. In terms of the Young modulus $E = \mu(2\mu + 3\lambda)/(\mu + \lambda)$ and the Poisson ratio $\nu = \lambda/(2(\mu + \lambda))$, we thus obtain

$$\sigma_{11} = E\varepsilon^L_{11}\,,\ \bar{\varepsilon}_{11} = \varepsilon^L_{11} = v_x - z w_{xx}\,,$$

and

$$\bar{\mathbf{u}} = \mathbf{u}^L\,, \quad \bar{\boldsymbol{\sigma}} \;=\; \begin{pmatrix} E\varepsilon_{11}^L & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \bar{\boldsymbol{\varepsilon}} \;=\; \begin{pmatrix} \varepsilon_{11}^L & 0 & 0 \\ 0 & -\nu\varepsilon_{11}^L & 0 \\ 0 & 0 & -\nu\varepsilon_{11}^L \end{pmatrix}. \tag{1.12}$$

On the upper boundary, we prescribe the boundary condition $\boldsymbol{\sigma}(x,y,h,t)\cdot\boldsymbol{\nu}_3 = \mathbf{f}(x,t)$, where $\boldsymbol{\nu}_3 = (0,0,1)^T$ is the upward normal vector, and $\mathbf{f} = (f_1, 0, f_3)^T$ is a given external surface load. In component form, this boundary condition reads $\sigma_{13} = f_1$, $\sigma_{23} = 0$, $\sigma_{33} = f_3$. In agreement with the scaling hypothesis (B4), we require $f_1 = \mathcal{O}(\alpha^3), f_3 = \mathcal{O}(\alpha^4)$. On the rest of the boundary, we assume the *vanishing normal stress* boundary conditions $\boldsymbol{\sigma}\cdot\boldsymbol{\nu} = 0$, where $\boldsymbol{\nu}$ is the unit outward normal vector. On $\{0\}\times\omega$, this means in particular

$$w_{xx}(0,t) = v_x(0,t) = 0\,, \quad w(0,t) = 0\,, \tag{1.13}$$

where the latter boundary condition is added in order to eliminate possible transversal rigid body displacements and corresponds to a *simply supported beam*. An analogous choice of the boundary conditions is made at the right surface $\{L\}\times\omega$. In accordance with these boundary conditions, we consider the Sobolev space

$$V = \big\{(v,w) \in H^1(0,L)\times H^2(0,L);\; w(0) = w(L) = 0\big\}\,. \tag{1.14}$$

Finally, suppose that the initial conditions

$$v(x,0) = v^0(x)\,,\;\; v_t(x,0) = v^1(x)\,,\;\; w(x,0) = w^0(x)\,,\;\; w_t(x,0) = w^1(x)\,, \tag{1.15}$$

are given. As in [30], we write the *momentum balance equation* in variational form

$$\int_\Omega \rho\, \mathbf{u}_{tt}\cdot\hat{\mathbf{u}}\; dx\,dy\,dz + \int_\Omega \boldsymbol{\sigma} : \hat{\boldsymbol{\varepsilon}}\, dx\,dy\,dz \;=\; \int_{\partial\Omega} (\boldsymbol{\sigma}\cdot\boldsymbol{\nu})\cdot\hat{\mathbf{u}}\; ds\,, \tag{1.16}$$

with the unknown vector $\mathbf{u}$ and tensor $\boldsymbol{\sigma}$, for all admissible displacements $\hat{\mathbf{u}}$ and strains $\hat{\boldsymbol{\varepsilon}}$ of the form (1.12); i. e., we have

$$\hat{\mathbf{u}}(x,y,z) = \begin{pmatrix} \hat{v}(x) - z\,\hat{w}_x(x) \\ 0 \\ \hat{w}(x) \end{pmatrix}, \quad \hat{\boldsymbol{\varepsilon}}(x,y,z) = \begin{pmatrix} \hat{\varepsilon}_{11}(x) & 0 & 0 \\ 0 & -\nu\hat{\varepsilon}_{11}(x) & 0 \\ 0 & 0 & -\nu\hat{\varepsilon}_{11}(x) \end{pmatrix}, \tag{1.17}$$

with $\hat{\varepsilon}_{11} = \hat{v}_x(x) - z\,\hat{w}_{xx}(x)$, where $(\hat{v},\hat{w})$ varies over the space V. It follows from the choice of the boundary conditions that

$$\begin{aligned} \int_{\partial\Omega} (\boldsymbol{\sigma}\cdot\boldsymbol{\nu})\cdot\hat{\mathbf{u}}\; ds \;&=\; 2b\int_0^L \left(f_1\,(\hat{v} - h\,\hat{w}_x) + f_3\,\hat{w}\right)\, dx \\ &=\; 2b\left(\int_0^L f_1\,\hat{v}\, dx + \int_0^L (h(f_1)_x + f_3)\,\hat{w}\, dx\right). \end{aligned}$$

Keeping on the left-hand side of (1.16) only terms of the lowest order in α, we may replace $(\mathbf{u}, \boldsymbol{\sigma})$ by $(\bar{\mathbf{u}}, \bar{\boldsymbol{\sigma}})$ from (1.12). The test functions $\hat{v}, \hat{w}$ are independent of each other, and a straightforward calculation shows that (1.16) decouples into the system

$$\rho \int_0^L v_{tt}(x,t)\,\hat{v}(x)\,dx + E\int_0^L v_x(x,t)\,\hat{v}_x(x)\,dx \;=\; \int_0^L g_1(x,t)\,\hat{v}(x)\,dx\,, \tag{1.18}$$

$$\begin{aligned} &\rho \int_0^L \Big(w_{tt}(x,t)\,\hat{w}(x) \;+\; \frac{h^2}{3}\, w_{xtt}(x,t)\,\hat{w}_x(x)\Big) dx \;+\; \frac{E\,h^2}{3}\int_0^L w_{xx}(x,t)\,\hat{w}_{xx}(x)\,dx \\ &\qquad = \int_0^L g_2(x,t)\,\hat{w}(x)\,dx\,, \end{aligned} \tag{1.19}$$

where we have set

$$g_1(x,t) = \frac{1}{2h} f_1(x,t)\,, \quad g_2(x,t) = \frac{1}{2h}\left(f_3(x,t) + h\,(f_1)_x(x,t)\right)\,. \tag{1.20}$$

The variational system (2.17), (2.18) leads formally to the partial differential equations

$$\rho\, v_{tt} \;-\; E\, v_{xx} \;=\; g_1\,, \tag{1.21}$$

$$\rho\, w_{tt} \;-\; \frac{\rho\, h^2}{3}\, w_{xxtt} \;+\; \frac{E\,h^2}{3}\, w_{xxxx} \;=\; g_2\,, \tag{1.22}$$

which describe the longitudinal (Eq. (1.21)) and transversal (Eq. (1.22)) vibrations of a straight elastic beam.

1.3 Transversal elastoplastic oscillations

We now turn our interest to elastoplasticity. Following [9, 18], we make further hypotheses.

(M1) The strain tensor $\boldsymbol{\varepsilon}$ is decomposed in elastic and plastic components $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^e + \boldsymbol{\varepsilon}^p$.

(M2) The elastic constitutive law is as in (1.11), that is,

$$\boldsymbol{\sigma} = 2\mu\boldsymbol{\varepsilon}^e + \lambda(\boldsymbol{\varepsilon}^e : \mathbf{1})\mathbf{1}\,. \tag{1.23}$$

As in (1.12), we have

$$\boldsymbol{\sigma} = \begin{pmatrix} E\,\varepsilon^e_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \boldsymbol{\varepsilon}^e = \begin{pmatrix} \varepsilon^e_{11} & 0 & 0 \\ 0 & -\nu\varepsilon^e_{11} & 0 \\ 0 & 0 & -\nu\varepsilon^e_{11} \end{pmatrix}. \tag{1.24}$$

(M3) The plastic deformations are *volume preserving* in the sense that

$$\boldsymbol{\varepsilon}^p : \mathbf{1} \;=\; 0\,. \tag{1.25}$$

The von Mises plastic yield condition is stated in terms of the stress deviator (1.10) in the form

(M4) $\mathfrak{D}(\boldsymbol{\sigma}):\mathfrak{D}(\boldsymbol{\sigma}) \le \frac{2}{3}R^2$, where $R > 0$ is a given *yield limit.*

Note that $\mathfrak{D}(\boldsymbol{\sigma}) = \sigma_{11}\boldsymbol{\eta}$, where

$$\boldsymbol{\eta} = \begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{3} \end{pmatrix},$$

hence (M4) reads

$$|\sigma_{11}| \ \le \ R\,. \tag{1.26}$$

For the plastic strain, we prescribe the *normality flow rule*

(M5) $\boldsymbol{\varepsilon}^p_t : (\boldsymbol{\sigma} - \tilde{\boldsymbol{\sigma}}) \ \ge \ 0 \quad \forall\, \tilde{\boldsymbol{\sigma}} \in \mathbb{T}^{3\times 3}_{\mathrm{sym}} : \quad \mathfrak{D}(\tilde{\boldsymbol{\sigma}}):\mathfrak{D}(\tilde{\boldsymbol{\sigma}}) \le \frac{2}{3}R^2$,

where the subscript $_t$ denotes the time derivative, and we assume for simplicity that

(M6) the motion is only transversal,

that is, the component f_1 of the external surface load vanishes, and

$$\mathbf{u}^L(x,y,z,t) = \begin{pmatrix} -z\,w_x(x,t) \\ 0 \\ w(x,t) \end{pmatrix}, \quad \varepsilon_{11}(x,y,z,t) = \varepsilon^L_{11}(x,y,z,t) \ = \ -z\,w_{xx}(x,t)\,. \tag{1.27}$$

The reason for neglecting the longitudinal displacements here is that the resulting system for w and v would not decouple as in (1.21)–(1.22), and a more complex model would have to be considered.

Introducing the set

$$K = \left\{ \tilde{\boldsymbol{\sigma}} \in \mathbb{T}^{3\times 3}_{\mathrm{sym}}\,;\, \mathfrak{D}(\tilde{\boldsymbol{\sigma}}):\mathfrak{D}(\tilde{\boldsymbol{\sigma}}) \le \frac{2}{3}R^2 \right\}$$

of admissible stresses and using the convex analysis formalism of e. g. [38], we can rewrite (M4)+(M5) in subdifferential form as

$$\boldsymbol{\varepsilon}^p_t \ \in \ \partial I_K(\boldsymbol{\sigma})\,, \tag{1.28}$$

where I_K is the indicator function of K and ∂I_K its subdifferential. For the sake of completeness, we recall other equivalent formulations of the von Mises criterion, cf. also [34].

Proposition 1.1. *Each of the following two conditions is equivalent to (M4)+(M5).*

(i) (multiplier formulation) *Condition (M4) holds, and there exists a multiplier* $\ell_t \ge 0$ *such that* $\ell_t = 0$ *if* $\mathfrak{D}(\boldsymbol{\sigma}):\mathfrak{D}(\boldsymbol{\sigma}) < \frac{2}{3}R^2$, *and*

$$\boldsymbol{\varepsilon}^p_t \ = \ \ell_t\,\mathfrak{D}(\boldsymbol{\sigma})\,; \tag{1.29}$$

(ii) (dissipation formulation) *Let*

$$\Psi(\boldsymbol{\xi}) = \begin{cases} \sqrt{\frac{2}{3}}R\sqrt{\boldsymbol{\xi}:\boldsymbol{\xi}} & \textit{if } \boldsymbol{\xi}:\mathbf{1} = 0\,, \\ +\infty & \textit{if } \boldsymbol{\xi}:\mathbf{1} \neq 0\,, \end{cases}$$

be the pseudopotential of dissipation. *Then*

$$\boldsymbol{\sigma} \;\in\; \partial\Psi(\boldsymbol{\varepsilon}^p_t)\,, \tag{1.30}$$

that is,

$$\boldsymbol{\sigma}:(\boldsymbol{\varepsilon}^p_t - \boldsymbol{\xi}) \geq \Psi(\boldsymbol{\varepsilon}^p_t) - \Psi(\boldsymbol{\xi}) \quad \forall \boldsymbol{\xi} \in \mathbb{T}^{3\times3}_{\mathrm{sym}}\,. \tag{1.31}$$

Sketch of the proof. Typically for rate independent systems, Ψ is homogeneous of degree 1, that is, $\Psi(c\boldsymbol{\xi}) = |c|\Psi(\boldsymbol{\xi})$ for all $\boldsymbol{\xi} \in \mathbb{T}^{3\times3}_{\mathrm{sym}}$ and $c \in \mathbb{R}$. Choosing in (1.31) consecutively $\boldsymbol{\xi} = 2\boldsymbol{\varepsilon}^p_t$ and $\boldsymbol{\xi} = 0$, we see that (1.30) is equivalent to the system

$$\boldsymbol{\sigma}:\boldsymbol{\varepsilon}^p_t - \Psi(\boldsymbol{\varepsilon}^p_t) \;=\; 0\,, \tag{1.32}$$

$$\boldsymbol{\sigma}:\boldsymbol{\xi} - \Psi(\boldsymbol{\xi}) \;\leq\; 0 \quad \forall \boldsymbol{\xi} \in \mathbb{T}^{3\times3}_{\mathrm{sym}}\,. \tag{1.33}$$

The implication (i)$\Rightarrow$(M4)+(M5) is straightforward. Assume now that (M4)+(M5) holds. We obtain (1.33) from (M4) and the Cauchy-Schwarz inequality. Putting in (M5) $\tilde{\boldsymbol{\sigma}} = \boldsymbol{\sigma} \pm \mathbf{1}$, we see that $\boldsymbol{\varepsilon}^p_t:\mathbf{1} = 0$. Identity (1.32) holds automatically if $\boldsymbol{\varepsilon}^p_t = 0$, otherwise we set in (M5) $\tilde{\boldsymbol{\sigma}} = \frac{2}{3}R^2\boldsymbol{\varepsilon}^p_t/\Psi(\boldsymbol{\varepsilon}^p_t)$, and (1.32) follows again from the Cauchy-Schwarz inequality, hence (ii) holds.

It remains to check the implication (ii)$\Rightarrow$(i). To this end, we choose $\boldsymbol{\xi} = \mathfrak{D}(\boldsymbol{\sigma})$ in (1.33), and obtain (M4). This and (1.32) imply in turn that

$$\mathfrak{D}(\boldsymbol{\sigma}):\boldsymbol{\varepsilon}^p_t \;=\; \sqrt{\frac{2}{3}}R\sqrt{\boldsymbol{\varepsilon}^p_t:\boldsymbol{\varepsilon}^p_t} \;\geq\; \sqrt{\mathfrak{D}(\boldsymbol{\sigma}):\mathfrak{D}(\boldsymbol{\sigma})}\sqrt{\boldsymbol{\varepsilon}^p_t:\boldsymbol{\varepsilon}^p_t}\,,$$

and (1.29) follows from the reverse Cauchy-Schwarz inequality. ■

Note that both (1.28) and (1.30) can be interpreted as a kind of *maximal dissipation principle*. In (1.28), for a given stress $\boldsymbol{\sigma}$, the strain rate $\boldsymbol{\varepsilon}^p_t$ is chosen so as to maximize the dissipation rate $\boldsymbol{\sigma}:\boldsymbol{\varepsilon}^p_t$ among all stress values $\tilde{\boldsymbol{\sigma}} \in K$; in (1.31), for a given strain rate $\boldsymbol{\varepsilon}^p_t$, the stress $\boldsymbol{\sigma}$ is required to maximize the reduced dissipation rate $\boldsymbol{\sigma}:\boldsymbol{\varepsilon}^p_t - \Psi(\boldsymbol{\varepsilon}^p_t)$ over the set of all values $\boldsymbol{\xi}$ of the strain rate.

It follows from (M1)–(M4) and (1.29), using the special form of $\boldsymbol{\eta}$, that

$$\boldsymbol{\varepsilon}^p_t(x,y,z,t) \;=\; \frac{3}{2}(\varepsilon^p_{11})_t(x,y,z,t)\,\boldsymbol{\eta} \tag{1.34}$$

for all admissible values of the arguments, and

$$(\varepsilon^p_{11})_t\,\boldsymbol{\eta} = \frac{2}{3}\ell_t E\,(\varepsilon_{11} - \varepsilon^p_{11})\,\boldsymbol{\eta} \tag{1.35}$$

under the constraint $|\varepsilon_{11} - \varepsilon^p_{11}| \le R/E$. This is equivalent to the scalar variational inequality

$$(\varepsilon^p_{11})_t \, (E\,(\varepsilon_{11} - \varepsilon^p_{11}) - \tilde\sigma) \ \ge \ 0 \quad \forall\, \tilde\sigma \in [-R, R]\,. \tag{1.36}$$

We fix for ε^p_{11} the initial condition corresponding to the unperturbed state with no previous loading history, which reads

$$\varepsilon^p_{11}(x,y,z,0) \ = \ \min\left\{\varepsilon_{11}(x,y,z,0) + \frac{R}{E}, \max\left\{0, \varepsilon_{11}(x,y,z,0) - \frac{R}{E}\right\}\right\}. \tag{1.37}$$

This can be justified by assuming that the process begins at time $t = -1$, with

$$\varepsilon_{11}(x,y,z,-1) = \varepsilon^p_{11}(x,y,z,-1) = 0\,, \quad \varepsilon_{11}(x,y,z,t) = (1+t)\,\varepsilon_{11}(x,y,z,0) \ \text{ for } \ t \in [-1,0]\,.$$

Then $\varepsilon^p_{11}(x,y,z,t) = 0$ as long as $(1+t)\,|\varepsilon_{11}(x,y,z,0)| \le R/E$, and $|(1+t)\,\varepsilon_{11}(x,y,z,0) - \varepsilon^p_{11}(x,y,z,t)| = R/E$ otherwise, whence (1.37) follows.

At this point, the notion of hysteresis operators comes into play. Given a function $\varepsilon \in W^{1,1}(0,T)$, the variational inequality

$$\left\{\begin{array}{l} \sigma(t) = E\chi(t)\,, \\ |\sigma(t)| \le R\,, \\ (\varepsilon_t(t) - \chi_t(t))\,(\sigma(t) - y) \ \ge \ 0 \quad \forall\, y \in [-R, R]\,, \\ \sigma(0) \ = \ \min\{R, \max\{\varepsilon(0), -R\}\} \end{array}\right. \tag{1.38}$$

for $t \in [0,T]$ defines a mapping $\mathcal{S}_{R,E} : W^{1,1}(0,T) \to W^{1,1}(0,T) : \varepsilon \mapsto \sigma$, which has been introduced in [20] and is called the *stop with slope E and threshold R*, see Fig. 2. Its mathematical properties will be investigated below in Proposition 3.2.

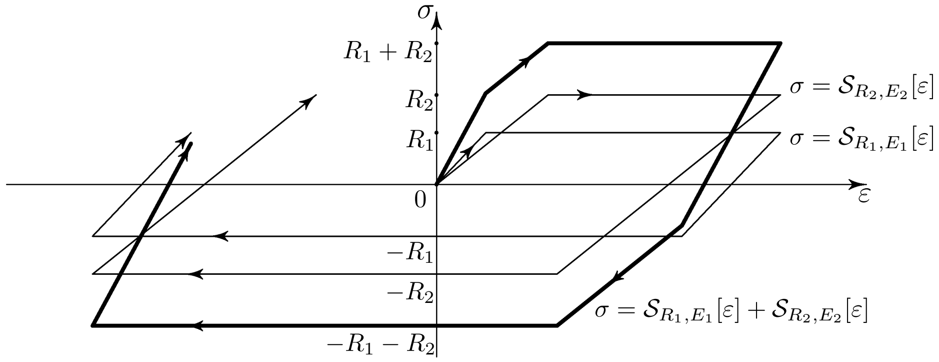

Figure 2. A diagram of the stops.

Each individual stop represents a single elastoplastic element with a sharp transition between the elastic and the plastic regimes. The stops have the following elementary scaling property:

$$\mathcal{S}_{R,E}[\varepsilon] = -\mathcal{S}_{R,E}[-\varepsilon] = c\mathcal{S}_{\frac{R}{c},\frac{E}{c}}[\varepsilon] = \mathcal{S}_{R,\frac{E}{c}}[c\varepsilon] \tag{1.39}$$

for every $R, E, c > 0$ and every $\varepsilon \in W^{1,1}(0,T)$. This enables us to take into account one-parametric stops with unit slope only, and to define

$$\mathfrak{s}_q[\varepsilon] = \mathcal{S}_{q,1}[\varepsilon] \quad \text{for } q > 0\,. \tag{1.40}$$

Then

$$\mathcal{S}_{R,E}[\varepsilon] = \mathfrak{s}_R[E\varepsilon] = E\mathfrak{s}_{\frac{R}{E}}[\varepsilon] = -\mathfrak{s}_R[-E\varepsilon] \tag{1.41}$$

for all admissible arguments. In order to model gradual transitions from pure elasticity to pure plasticity, Prandtl ([37]) and Ishlinskii ([19]) proposed to consider additive superpositions of stops, see Fig. 2 for the case of two stops. The transition becomes smooth, if sums are generalized into a weighted integral

$$\sigma = \mathcal{P}[\varepsilon] := \int_0^\infty \varphi(q)\, \mathfrak{s}_q[\varepsilon]\, dq \tag{1.42}$$

of a continuous family of stops over the whole interval $R \in (0,\infty)$ of admissible thresholds, with a given weight function φ, see Fig. 3. The operator $\mathcal{P}$ is called the *Prandtl-Ishlinskii operator*. We now show that such a superposition of a continuous system of stops with different thresholds and different slopes is precisely what arises spontaneously in our model.

The variational inequality (1.36)–(1.37) is of the form (1.38) with $\varepsilon = \varepsilon_{11}(x,y,z,\cdot)$, $\chi = \varepsilon_{11}^e(x,y,z,\cdot)$, $\sigma = \sigma_{11}(x,y,z,\cdot)$. We thus have by (1.27) that

$$\sigma_{11}(x,y,z,t) = \mathcal{S}_{R,E}[-z\, w_{xx}(x,\cdot)](t)\,, \tag{1.43}$$

and identities (1.39)–(1.41) yield

$$\sigma_{11}(x,y,z,t) = \begin{cases} -\frac{z}{|z|}\,\mathcal{S}_{R,E|z|}[w_{xx}(x,\cdot)](t) = -Ez\,\mathfrak{s}_{\frac{R}{E|z|}}[w_{xx}(x,\cdot)](t) & \text{for } z \neq 0\,, \\ 0 & \text{for } z = 0\,. \end{cases} \tag{1.44}$$

This is the behavior we mentioned in the introduction in connection with Fig. 1. The fibers at distance $|z|$ from the midsurface respond to bending as an elastoplastic element with yield point $\frac{R}{E|z|}$ and slope $E|z|$, that is, the eccentric fibers behave “harder” in elasticity and “softer” in plasticity than the central ones.

We now aim to derive the momentum balance in the same way as in (1.16) to (1.22). To this end, we again make the test functions independent of $\hat{v}$, so that

$$\boldsymbol{\sigma} : \hat{\boldsymbol{\varepsilon}} \;=\; E\,z^2\,\mathfrak{s}_{R/(E|z|)}\,[w_{xx}]\;\hat{w}_{xx}\,.$$

Integrating over ω, we obtain

$$\begin{aligned} \int_\omega z^2\;\mathfrak{s}_{R/(E|z|)}\,[w_{xx}]\;dy\,dz \;&=\; 2b\int_{-h}^{h} z^2\,\mathfrak{s}_{R/(E|z|)}\,[w_{xx}]\;dz \\ =\; 4b\int_0^h z^2\,\mathfrak{s}_{R/(Ez)}\,[w_{xx}]\;dz \;&=\; 4b\left(\frac{R}{E}\right)^3\int_{R/(Eh)}^\infty q^{-4}\,\mathfrak{s}_q\,[w_{xx}]\;dq\,. \end{aligned}$$

For each admissible input function ε, we now put

$$\mathcal{P}[\varepsilon] := \frac{R^3}{E^2\, h} \int_{R/(Eh)}^{\infty} q^{-4}\ \mathfrak{s}_q[\varepsilon]\, dq\,. \tag{1.45}$$

This is a Prandtl-Ishlinskii operator of the form (1.42) with the weight function

$$\varphi(q) = \begin{cases} 0\,, & \text{if} \quad 0 \le q \le \frac{R}{Eh}\,, \\ \frac{R^3}{E^2\, h}\, q^{-4}\,, & \text{if} \quad q > \frac{R}{Eh}\,. \end{cases} \tag{1.46}$$

The counterpart of (1.22) then reads formally

$$\rho\, w_{tt} - \frac{\rho\, h^2}{3}\, w_{xxtt} + \mathcal{P}[w_{xx}]_{xx} = g_2\,. \tag{1.47}$$

Here, we have used the abbreviation

$$\mathcal{P}[w_{xx}]_{xx}(x,t) = \frac{\partial^2}{\partial x^2}\, \mathcal{P}[w_{xx}(x,\cdot)](t)\,. \tag{1.48}$$

Remark 1.2. Note that the Prandtl-Ishlinskii *initial loading curve* $\sigma = \Phi(\varepsilon)$ for the operator (1.45) is bounded and saturation occurs. Indeed, Φ is given by the formula (see [5], [22])

$$\Phi(\varepsilon) = \frac{R^3}{E^2\, h} \int_{R/(Eh)}^{\infty} q^{-4}\ \min\{q, \varepsilon\}\, dq\,, \quad \text{for } \varepsilon \ge 0\,, \tag{1.49}$$

so that

$$\Phi(\varepsilon) = \begin{cases} \frac{E\, h^2}{3}\, \varepsilon\,, & \text{if} \quad \varepsilon \le \frac{R}{Eh}\,, \\ \frac{Rh}{2} \left(1 - \frac{R^2}{3\, E^2 h^2 \varepsilon^2}\right)\,, & \text{if} \quad \varepsilon > \frac{R}{Eh}\,. \end{cases} \tag{1.50}$$

Hence, $\sigma_1 = Rh/3$ is the elasticity limit and $\sigma_2 = Rh/2$ is the saturation bound. The interval between σ_1 and σ_2 is the transition zone from linear elasticity to perfect plasticity, see Fig. 3. More general Prandtl-Ishlinskii initial loading curves describe the cases that the shape of ω is no longer a rectangle, but a domain of the form

$$\omega = \{(y,z) \in \mathbb{R}^2\,;\ -h < z < h\,,\ -b(z) < y < b(z)\}$$

with a positive measurable function b. The same equation with a different Prandtl-Ishlinskii operator also results if we let the Young modulus E depend on z as a model for a layered beam.

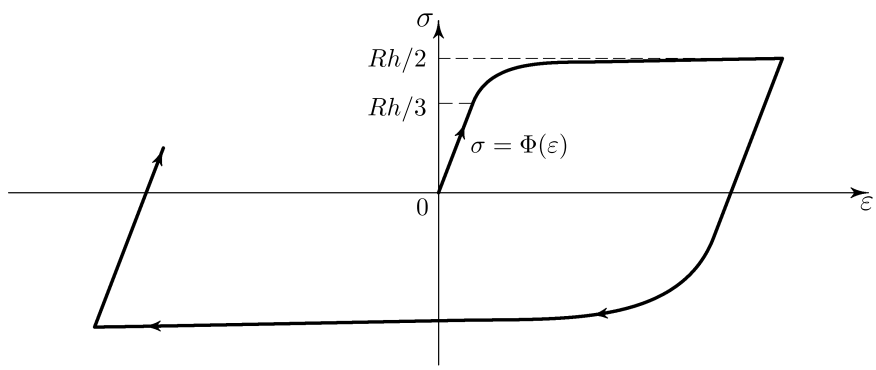

Figure 3. The Prandtl-Ishlinskii operator (1.45).

Remark 1.3. Note that (1.47) reduces to (1.22) if we replace $\mathfrak{s}_q[\varepsilon]$ by ε in the expression (1.45) for $\mathcal{P}[\varepsilon]$ (no plasticity). Also, if we pass to the elastic limit as $R \to \infty$ in (1.47), we recover (1.22) in agreement with natural expectations.

We now formulate the main mathematical result related to the problem of transversal oscillations of an elastoplastic beam. To this end, we normalize all physical constants in (1.47) to unity, which has no bearing on the mathematical analysis. We thus study the following initial-boundary value problem in Q_T, where $Q_t := (0,1) \times (0,t)$ for any $t > 0$:

$$w_{tt} - w_{xxtt} + \mathcal{P}\left[w_{xx}\right]_{xx} = g \quad \text{in } Q_T\,, \tag{1.51}$$

$$w(0,t) = \mathcal{P}[w_{xx}](0,t) = w(1,t) = \mathcal{P}[w_{xx}](1,t) = 0\,, \quad 0 \le t \le T\,, \tag{1.52}$$

$$w(x,0) = z_0(x)\,, \quad w_t(x,0) = z_1(x)\,, \quad 0 \le x \le 1\,. \tag{1.53}$$

We make the following general assumptions on the data of the system:

(H1) $g \in L^2(Q_T)$.

(H2) $z_0 \in H^3(0,1), z_1 \in H^2(0,1)$, and the following compatibility conditions are satisfied:

$$z_0(0) = z_{0,xx}(0) = z_0(1) = z_{0,xx}(1) = 0\,,\ z_1(0) = z_1(1) = 0\,. \tag{1.54}$$

(H3) The weight function $\varphi : (0,\infty) \to [0,\infty)$ of the Prandtl-Ishlinskii operator (1.42) is measurable and satisfies the growth condition

$$\int_0^\infty \left(1 + q^2\right) \varphi(q)\, dq < +\infty\,. \tag{1.55}$$

Remark 1.4. Under condition (1.55) the so-called *clockwise admissible potential* of $\mathcal{P}$, given by the hysteresis operator

$$\mathcal{Q}[\varepsilon] = \frac{1}{2} \int_0^\infty \varphi(q)\, \mathfrak{s}_q^2[\varepsilon]\, dq\,, \tag{1.56}$$

is well defined. It then follows from the variational inequality (1.38) with $\chi = \sigma = \mathfrak{s}_q[\varepsilon]$ that for any input function $\varepsilon \in W^{1,1}(0,T)$, the dissipation rate

$$\mathcal{D}[\varepsilon] := \mathcal{P}[\varepsilon](t)\, \varepsilon_t(t) - \left(\mathcal{Q}[\varepsilon]\right)_t(t) = \int_0^\infty \varphi(q)\, \mathfrak{s}_q[\varepsilon](t)\, \left(\varepsilon - \mathfrak{s}_q[\varepsilon]\right)_t(t)\, dq \tag{1.57}$$

is nonnegative for a. e. $t \in (0,T)$ in agreement with the second principle of thermodynamics.

A crucial difficulty of (1.51) is due to the non-differentiability of hysteresis operators. We therefore replace (1.51)–(1.53) with the following system of initial-boundary value problems

$$u_t = \mathcal{P}\left[w_{xx}\right] \quad \text{in } Q_T\,, \tag{1.58}$$

$$w_t - w_{xxt} = -u_{xx} + f(x,t) \quad \text{in } Q_T\,, \tag{1.59}$$

$$u(0,t) = u(1,t) = 0\,, \quad 0 \le t \le T\,, \tag{1.60}$$

$$w(0,t) = w(1,t) = 0\,, \quad 0 \le t \le T\,, \tag{1.61}$$

$$u(x,0) = z_1(x)\,, \quad 0 \le x \le 1\,, \tag{1.62}$$

$$w(x,0) = z_0(x)\,, \quad 0 \le x \le 1\,, \tag{1.63}$$

which arises from (1.51)–(1.53) if we put

$$u(x,t) \;=\; z_1(x) + \int_0^t \mathcal{P}\,[w_{xx}]\,(x,s)\,ds\,, \quad f(x,t) \;=\; z_1(x) + \int_0^t g(x,s)\,ds\,. \tag{1.64}$$

The main result, proved as Theorem 3.2 in [28], reads as follows. The proof is based on the monotonicity (3.11) and second order energy inequality (3.13) of the Prandtl-Ishlinskii operator.

Theorem 1.5. *Suppose that the conditions (H1)–(H3) are satisfied. Then the system (1.58)–(1.63) has a unique solution pair (u,w) having the following properties:*

(i) $u \in W^{2,\infty}(0,T;L^2(0,1)) \cap L^\infty(0,T;H^2(0,1)) \cap H^1(0,T;H^1(0,1))\,.$

(ii) $w \in W^{1,\infty}(0,T;H^2(0,1)) \cap H^2(0,T;H^1(0,1))\,.$

(iii) *Eq. (1.58) is fulfilled pointwise in $\overline{Q_T}$, and Eq. (1.59) holds almost everywhere in Q_T.*

(iv) *The initial and boundary conditions (1.60)–(1.63) are satisfied pointwise, and it holds*

$$\mathcal{P}[w_{xx}](0,t) = \mathcal{P}[w_{xx}](1,t) = 0 \quad \forall\, t \in [0,T]\,.$$

Remark 1.6. We call (u,w) a strong solution to (1.58)–(1.63), and w a weak solution to (1.51)–(1.53). The meaning of conditions (i), (ii) in Theorem 1.5 is that

$$\left.\begin{array}{rcl} u_{tt}, u_{xx}, w_{xxt} & \in & L^\infty(0,T;L^2(0,1))\,, \\ u_{xt}, w_{xtt} & \in & L^2(Q_T)\,. \end{array}\right\} \tag{1.65}$$

By virtue of the boundary conditions and embedding theorems, we then have

$$u, u_x, u_t, w, w_x, w_t, w_{xt} \;\in\; C(\overline{Q_T})\,. \tag{1.66}$$

2 Plates

We restrict ourselves to plates of constant thickness, that is, to sets $\Omega \subset \mathbb{R}^3$ of the form $\Omega = \Omega_0 \times (-h,h)$, where $\Omega_0 \subset \mathbb{R}^2$ describes the shape of the plate and $2h$ is its thickness. We denote by $(x,y) \in \Omega_0$ the longitudinal coordinates, by $z \in (-h,h)$ the transversal coordinate, and by $t \in [0,T]$ the time, where $T > 0$ is given.

We start again with the linear elastic isotropic case, and then pass to the elastoplastic model. The scaling technique of [6, Part A] and [8, Sect. 5.4] in terms of a small parameter $\alpha > 0$ is now adapted in such a way that

$$h = \mathcal{O}(\alpha),\ \Omega_0 = \mathcal{O}(1).$$

Let us consider smooth displacements $\mathbf{u} : \Omega \times (0,T) \to \mathbb{R}^3$, decomposed as in (1.1). The 2D counterpart of Assumptions (B1)–(B5) now reads:

(P1) The low order displacement of the midsurface $\mathcal{C} = \{(x, y, 0) \in \Omega \, : \, (x, y) \in \Omega_0\}$ is only transversal, that is,

$$\mathbf{u}^L(x, y, 0, t) = \begin{pmatrix} 0 \\ 0 \\ w(x, y, t) \end{pmatrix} \quad \forall\, (x, y) \in \Omega_0 \,, \quad \forall\, t \in (0, T)\,, \tag{2.1}$$

with some function $w : \Omega_0 \times (0, T) \to \mathbb{R}$.

(P2) The low order deformation (1.3) leaves the straight fibers $\{(x, y)\} \times (-h, h)$ perpendicular to the midsurface and straight, that is, (1.4) holds, see Fig. 1.

(P3) $w_{xx}, w_{xy}, w_{yy} = \mathcal{O}(\alpha)$.

Under the hypothesis (P3), we can linearize the problem by replacing

$$\mathbf{n}(x, y, t) = \frac{1}{\sqrt{1 + w_x^2(x, y, t) + w_y^2(x, y, t)}} \begin{pmatrix} -w_x(x, y, t) \\ -w_y(x, y, t) \\ 1 \end{pmatrix}$$

with

$$\tilde{\mathbf{n}}(x, y, t) := \begin{pmatrix} -w_x(x, y, t) \\ -w_y(x, y, t) \\ 1 \end{pmatrix}. \tag{2.2}$$

Here again, we have

$$|\,\tilde{\mathbf{n}}(x, y, t) - \mathbf{n}(x, y, t)| \, < \, |w_x(x, y, t)|^2 + |w_y(x, y, t)|^2 = \mathcal{O}(\alpha^2)\,.$$

This enables us to write for every $(x, y, z, t) \in \Omega \times (0, T)$ the low order displacement $\mathbf{u}^L(x, y, z, t)$ as

$$\mathbf{u}^L(x, y, z, t) = \begin{pmatrix} -z\, w_x(x, y, t) \\ -z\, w_y(x, y, t) \\ w(x, y, t) \end{pmatrix}. \tag{2.3}$$

We further compute

$$\nabla\mathbf{u}^L(x, y, z, t) = \begin{pmatrix} -z\, w_{xx}(x, y, t) & -z\, w_{xy}(x, y, t) & -w_x(x, y, t) \\ -z\, w_{xy}(x, y, t) & -z\, w_{yy}(x, y, t) & -w_y(x, y, t) \\ w_x(x, y, t) & w_y(x, y, t) & 0 \end{pmatrix}, \tag{2.4}$$

and the low order infinitesimal strain tensor $\boldsymbol{\varepsilon}^L = (\nabla\mathbf{u}^L + (\nabla\mathbf{u}^L)^T)/2$ becomes

$$\boldsymbol{\varepsilon}^L(x, y, z, t) = \begin{pmatrix} -z\, w_{xx}(x, y, t) & -z\, w_{xy}(x, y, t) & 0 \\ -z\, w_{xy}(x, y, t) & -z\, w_{yy}(x, y, t) & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{2.5}$$

2.1 Small elastic deformations

With the notation of Subsection 1.2, we consider the components $\sigma_{11}, \sigma_{22}, \sigma_{12}$ to be of the lowest order, which is $\mathcal{O}(\alpha^2)$ due to (P3), (2.5), and (1.11). Arguing as in Subsection 1.2, we assume

(P4) $\sigma_{13}, \sigma_{23} = \mathcal{O}(\alpha^3)\,,\ \sigma_{33} = \mathcal{O}(\alpha^4)\,.$

According to (1.11) and Hypothesis (P4), the high order strain tensor $\boldsymbol{\varepsilon}^H$ is scaled as

(P5) $\varepsilon^H_{13}, \varepsilon^H_{23} = \mathcal{O}(\alpha^3)\,,\ \varepsilon^H_{33} = \mathcal{O}(\alpha^2)\,,\ \varepsilon^H_{11}, \varepsilon^H_{22}, \varepsilon^H_{12} = \mathcal{O}(\alpha^4)\,.$

In terms of the high order displacements $\mathbf{u}^H$, (P5) corresponds to the scaling $u^H_1, u^H_2 = \mathcal{O}(\alpha^4)\,,\ u^H_3 = \mathcal{O}(\alpha^3)\,.$

Let $\bar{\boldsymbol{\sigma}}\,,\ \bar{\boldsymbol{\varepsilon}}$ denote the stress and strain components of the order $\mathcal{O}(\alpha^2)$ at most. Then

$$\bar{\boldsymbol{\sigma}} \;=\; \begin{pmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{12} & \sigma_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \bar{\boldsymbol{\varepsilon}} \;=\; \begin{pmatrix} \varepsilon^L_{11} & \varepsilon^L_{12} & 0 \\ \varepsilon^L_{12} & \varepsilon^L_{22} & 0 \\ 0 & 0 & \varepsilon^H_{33} \end{pmatrix}. \tag{2.6}$$

We compute ε^H_{33} from the relation

$$0 = \sigma_{33} = 2\mu\varepsilon^H_{33} + \lambda(\varepsilon^L_{11} + \varepsilon^L_{22} + \varepsilon^H_{33})\,,$$

that is,

$$\varepsilon^H_{33} = -\frac{\lambda}{2\mu+\lambda}(\varepsilon^L_{11} + \varepsilon^L_{22})\,.$$

Hence,

$$\bar{\boldsymbol{\varepsilon}} : \mathbf{1} = \frac{2\mu}{2\mu+\lambda}(\varepsilon^L_{11} + \varepsilon^L_{22})\,.$$

In terms of the Young modulus E and the Poisson ratio ν, we have

$$\bar{\boldsymbol{\sigma}} = \frac{E}{1-\nu^2}\begin{pmatrix} \varepsilon^L_{11} + \nu\varepsilon^L_{22} & (1-\nu)\varepsilon^L_{12} & 0 \\ (1-\nu)\varepsilon^L_{12} & \nu\varepsilon^L_{11} + \varepsilon^L_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \bar{\boldsymbol{\varepsilon}} = \begin{pmatrix} \varepsilon^L_{11} & \varepsilon^L_{12} & 0 \\ \varepsilon^L_{12} & \varepsilon^L_{22} & 0 \\ 0 & 0 & -\frac{\nu}{1-\nu}(\varepsilon^L_{11} + \varepsilon^L_{22}) \end{pmatrix}, \tag{2.7}$$

with $\boldsymbol{\varepsilon}^L$ given by (2.5). On the upper boundary, we prescribe the boundary condition $\boldsymbol{\sigma}(x,y,h,t)\cdot\boldsymbol{\nu}_3 = \mathbf{f}(x,y,t)\,,$ where $\boldsymbol{\nu}_3 = (0,0,1)^T$ is the upward normal vector, and $\mathbf{f} = (f_1, f_2, f_3)^T$ is a given external surface load. In component form, this boundary condition reads $\sigma_{13} = f_1\,,\ \sigma_{23} = f_2\,,\ \sigma_{33} = f_3\,.$ In agreement with the scaling hypothesis (P4), we require $f_1, f_2 = \mathcal{O}(\alpha^3), f_3 = \mathcal{O}(\alpha^4)\,.$ On the rest of the boundary, we assume the *vanishing normal stress* boundary conditions $\boldsymbol{\sigma}\cdot\boldsymbol{\nu} = 0\,,$ where $\boldsymbol{\nu}$ is the unit outward normal vector. On $\partial\Omega_0 \times (-h,h)\,,$ we add the boundary condition for w

$$w(x,y,t) = 0 \;\text{ for } (x,y) \in \partial\Omega_0\,, \tag{2.8}$$

in order to eliminate possible transversal, rigid body displacements. This corresponds to a *simply supported plate*. In accordance with these boundary conditions, we consider the Sobolev space

$$V = \left\{ w \in H^2(\Omega_0) : \, w\big|_{\partial\Omega_0} = 0 \right\}. \tag{2.9}$$

Finally, suppose that the initial conditions

$$w(x,y,0) = w^0(x,y)\,, \; w_t(x,y,0) = w^1(x,y)\,, \tag{2.10}$$

are given. We now evaluate the momentum balance (1.16) for all admissible displacements $\hat{\mathbf{u}}$ and strains $\hat{\boldsymbol{\varepsilon}}$ of the form (2.3), (2.5), and (2.7), that is,

$$\hat{\mathbf{u}}(x,y,z) = \begin{pmatrix} -z\,\hat{w}_x(x,y) \\ -z\,\hat{w}_y(x,y) \\ \hat{w}(x,y) \end{pmatrix}, \quad \hat{\boldsymbol{\varepsilon}}(x,y,z) = \begin{pmatrix} -z\hat{w}_{xx} & -z\hat{w}_{xy} & 0 \\ -z\hat{w}_{xy} & -z\hat{w}_{yy} & 0 \\ 0 & 0 & \frac{\nu}{1-\nu} z\Delta\hat{w} \end{pmatrix}, \tag{2.11}$$

where $\hat{w}$ varies over the space V. It follows from the choice of the boundary conditions that

$$\begin{aligned} \int_{\partial\Omega} (\boldsymbol{\sigma}\cdot\boldsymbol{\nu})\cdot\hat{\mathbf{u}}\;ds &= \int_{\Omega_0} (-h\,f_1\,\hat{w}_x - h\,f_2\,\hat{w}_y + f_3\,\hat{w})\;dx\,dy \\ &= \int_{\Omega_0} (h(f_1)_x + h(f_2)_y + f_3)\,\hat{w}\,dx\,dy\,. \end{aligned}$$

Keeping on the left hand side of (1.16) only terms of the lowest order in α, we may replace $(\mathbf{u},\boldsymbol{\sigma})$ by $(\bar{\mathbf{u}},\bar{\boldsymbol{\sigma}})$ from (2.3), (2.5), and (2.7), and obtain

$$\begin{aligned} &\rho\int_{\Omega_0}\left(w_{tt}\,\hat{w} + \frac{h^2}{3}\,(w_{xtt}\,\hat{w}_x + w_{ytt}\,\hat{w}_y)\right)dx\,dy \\ &\qquad + \frac{E\,h^2}{3(1+\nu)}\int_{\Omega_0}\left(w_{xx}\,\hat{w}_{xx} + 2w_{xy}\,\hat{w}_{xy} + w_{yy}\,\hat{w}_{yy} + \frac{\nu}{1-\nu}\Delta w\,\Delta\hat{w}\right)dx\,dy \\ &\qquad = \int_{\Omega_0} g\,\hat{w}\,dx\,dy\,, \end{aligned} \tag{2.12}$$

where we have set

$$g(x,y,t) = \frac{1}{2h} f_3(x,y,t) + \frac{1}{2}((f_1)_x + (f_2)_y)(x,y,t)\,. \tag{2.13}$$

The variational equation (2.12) leads formally, on smooth domains, to the partial differential equation describing transversal vibrations of a thin elastic plate

$$\rho\,w_{tt} - \frac{\rho\,h^2}{3}\,\Delta w_{tt} + \frac{E\,h^2}{3(1-\nu^2)}\,\Delta^2 w \;=\; g\,, \tag{2.14}$$

with boundary conditions

$$\left.\begin{aligned} w &= 0 \\ (\boldsymbol{\Delta}_\nu w)\mathbf{n}\cdot\mathbf{n} &= 0 \end{aligned}\right\} \quad \text{on} \;\; \partial\Omega_0\,, \tag{2.15}$$

where $\mathbf{\Delta}_\nu w$ is the matrix

$$\mathbf{\Delta}_\nu w = \begin{pmatrix} w_{xx} + \frac{\nu}{1-\nu}\Delta w & w_{xy} \\ w_{xy} & w_{yy} + \frac{\nu}{1-\nu}\Delta w \end{pmatrix}, \tag{2.16}$$

and $\mathbf{n}$ is the outward normal to Ω_0.

2.2 Elastoplastic oscillations

We still consider here $\bar{\mathbf{u}} = \mathbf{u}^L$, $\bar{\boldsymbol{\sigma}}$, and $\bar{\boldsymbol{\varepsilon}}$ as in (2.3) and (2.6), with $\boldsymbol{\varepsilon}^L$ given by (2.5). We assume the hypotheses (M1)–(M5) from Subsection 1.3 to hold. Here, (M4) reads

$$\sigma_{11}^2 + \sigma_{22}^2 - \sigma_{11}\sigma_{22} + 3\sigma_{12}^2 \leq R^2 . \tag{2.17}$$

Similarly as in (2.7), we obtain

$$\bar{\boldsymbol{\sigma}} = \frac{E}{1-\nu^2}\begin{pmatrix} \varepsilon_{11}^e + \nu\varepsilon_{22}^e & (1-\nu)\varepsilon_{12}^e & 0 \\ (1-\nu)\varepsilon_{12}^e & \nu\varepsilon_{11}^e + \varepsilon_{22}^e & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \boldsymbol{\varepsilon}^e = \begin{pmatrix} \varepsilon_{11}^e & \varepsilon_{12}^e & 0 \\ \varepsilon_{12}^e & \varepsilon_{22}^e & 0 \\ 0 & 0 & -\frac{\nu}{1-\nu}(\varepsilon_{11}^e + \varepsilon_{22}^e) \end{pmatrix}. \tag{2.18}$$

Assume that $\varepsilon_{13}^p = \varepsilon_{23}^p = 0$ at initial time $t = 0$. Then we have by (M3) and (1.29) that

$$\boldsymbol{\varepsilon}^p = \begin{pmatrix} \varepsilon_{11}^p & \varepsilon_{12}^p & 0 \\ \varepsilon_{12}^p & \varepsilon_{22}^p & 0 \\ 0 & 0 & -(\varepsilon_{11}^p + \varepsilon_{22}^p) \end{pmatrix}. \tag{2.19}$$

It is convenient to consider $\bar{\boldsymbol{\sigma}}$, $\boldsymbol{\varepsilon}^e$, and $\boldsymbol{\varepsilon}^p$ as vectors with three components. To this end, we introduce the notation

$$\bar{\boldsymbol{\sigma}}_* = \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{pmatrix}, \quad \boldsymbol{\varepsilon}_*^e = \begin{pmatrix} \varepsilon_{11}^e \\ \varepsilon_{22}^e \\ \varepsilon_{12}^e \end{pmatrix}, \quad \boldsymbol{\varepsilon}_*^p = \begin{pmatrix} \varepsilon_{11}^p \\ \varepsilon_{22}^p \\ \varepsilon_{12}^p \end{pmatrix}, \quad \bar{\boldsymbol{\varepsilon}}_* = \begin{pmatrix} -zw_{xx} \\ -zw_{yy} \\ -zw_{xy} \end{pmatrix}. \tag{2.20}$$

According to (M1) and (2.18), we have

$$\bar{\boldsymbol{\varepsilon}}_* = \boldsymbol{\varepsilon}_*^p + \boldsymbol{\varepsilon}_*^e , \quad \bar{\boldsymbol{\sigma}}_* = \mathbf{C}\boldsymbol{\varepsilon}_*^e , \tag{2.21}$$

where $\mathbf{C}$ is the positive definite matrix

$$\mathbf{C} = \frac{E}{1-\nu^2}\begin{pmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 1-\nu \end{pmatrix}. \tag{2.22}$$

Let $\mathfrak{D}_*, \mathbf{J}$ be the matrices

$$\mathfrak{D}_* = \begin{pmatrix} 1 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \quad \mathbf{J} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}. \tag{2.23}$$

In view of (2.17), condition (M4) can be restated as

$$\bar{\boldsymbol{\sigma}}_* \cdot \mathfrak{D}_* \bar{\boldsymbol{\sigma}}_* \leq R^2\,,$$

and (M5) reads

$$\mathbf{J}(\boldsymbol{\varepsilon}_*^p)_t \cdot (\bar{\boldsymbol{\sigma}}_* - \boldsymbol{\theta}_*) \ \geq \ 0 \quad \forall \boldsymbol{\theta}_* \in K_*\,, \tag{2.24}$$

where

$$K_* = \{\boldsymbol{\theta}_* \in \mathbb{R}^3 \,:\, \boldsymbol{\theta}_* \cdot \mathfrak{D}_* \boldsymbol{\theta}_* \leq R^2\}. \tag{2.25}$$

Alternatively, we can write this variational inequality in the form

$$\begin{aligned} &\boldsymbol{\varepsilon}_*^e \in \mathbf{C}^{-1}(K_*)\,, \\ &\mathbf{JC}(\bar{\boldsymbol{\varepsilon}}_* - \boldsymbol{\varepsilon}_*^e)_t \cdot (\boldsymbol{\varepsilon}_*^e - \boldsymbol{\eta}_*) \ \geq \ 0 \quad \forall \boldsymbol{\eta}_* \in \mathbf{C}^{-1}(K_*)\,. \end{aligned} \tag{2.26}$$

Let us choose in $\mathbb{R}^3$ the scalar product

$$\langle \boldsymbol{\xi}_*, \boldsymbol{\eta}_* \rangle = \mathbf{JC}\boldsymbol{\xi}_* \cdot \boldsymbol{\eta}_*\,. \tag{2.27}$$

This is meaningful, since $\mathbf{JC} = \mathbf{CJ}$ is a symmetric positive definite matrix. We then prescribe the canonical initial condition

$$\boldsymbol{\varepsilon}_*^e(0) = Q_{\mathbf{C}^{-1}(K_*)}(\bar{\boldsymbol{\varepsilon}}_*(0))\,, \tag{2.28}$$

where $Q_{\mathbf{C}^{-1}(K_*)}$ is the orthogonal projection onto $\mathbf{C}^{-1}(K_*)$ with respect to the scalar product (2.27). For every $\bar{\boldsymbol{\varepsilon}}_* \in W^{1,1}(0,T;\mathbb{R}^3)$, Problem (2.26)–(2.28) has the same structure as (1.38), and the solution mapping

$$\mathcal{S}_{\mathbf{C}^{-1}(K_*)} : W^{1,1}(0,T;\mathbb{R}^3) \to W^{1,1}(0,T;\mathbf{C}^{-1}(K_*)) : \ \bar{\boldsymbol{\varepsilon}}_* \mapsto \boldsymbol{\varepsilon}_*^e \tag{2.29}$$

is called the *vectorial stop with characteristic* $\mathbf{C}^{-1}(K_*)$, see Section 3. This concept enables us to rewrite (2.26) as

$$\boldsymbol{\varepsilon}_*^e \ = \ \mathcal{S}_{\mathbf{C}^{-1}(K_*)}[\bar{\boldsymbol{\varepsilon}}_*]\,,$$

or, equivalently,

$$\bar{\boldsymbol{\sigma}}_* \ = \ \mathbf{C}\mathcal{S}_{\mathbf{C}^{-1}(K_*)}[\bar{\boldsymbol{\varepsilon}}_*]\,. \tag{2.30}$$

The stop $\mathcal{S}_Z$ with any symmetric convex closed characteristic Z has the following elementary scaling property:

$$\mathcal{S}_Z[\boldsymbol{\varepsilon}_*] = -\mathcal{S}_Z[-\boldsymbol{\varepsilon}_*] = \frac{1}{c}\mathcal{S}_{cZ}[c\boldsymbol{\varepsilon}_*] \tag{2.31}$$

for every $c > 0$ and every $\boldsymbol{\varepsilon}_* \in W^{1,1}(0,T;\mathbb{R}^3)$, where $cZ = \{\boldsymbol{\theta}_* \in \mathbb{R}^3 \,:\, \frac{1}{c}\boldsymbol{\theta}_* \in Z\}$.

Notice first that we obtain from (2.30), (2.20), and (2.31) that

$$\bar{\boldsymbol{\sigma}}_* \ = \ -z\mathbf{C}\mathcal{S}_{\frac{1}{|z|}\mathbf{C}^{-1}(K_*)} \begin{bmatrix} w_{xx} \\ w_{yy} \\ w_{xy} \end{bmatrix}\,. \tag{2.32}$$

Here again, at distance $|z|$ from the midsurface, the virtual elasticity modulus is $|z|E$ and the virtual yield limit is $R/|z|$. This produces the multiyield effect when integrating over the thickness of the plate.

To derive a counterpart of the partial differential equation (2.14), we consider test functions $\hat{\mathbf{u}}$ and $\hat{\boldsymbol{\varepsilon}}$ as in (2.11), and set in agreement with (2.20)

$$\hat{\boldsymbol{\varepsilon}}_* = \begin{pmatrix} -z\hat{w}_{xx} \\ -z\hat{w}_{yy} \\ -z\hat{w}_{xy} \end{pmatrix} . \tag{2.33}$$

The first and the third integral in (1.16) are evaluated in the same way as in (2.12). The remaining one has to be treated more carefully. Using (2.31), we obtain

$$\begin{aligned}
\int_\Omega \bar{\boldsymbol{\sigma}} : \hat{\boldsymbol{\varepsilon}}\, dx\, dy\, dz &= \int_\Omega \mathbf{J}\bar{\boldsymbol{\sigma}}_* \cdot \hat{\boldsymbol{\varepsilon}}_*\, dx\, dy\, dz \\
&= \int_{-h}^{h}\int_{\Omega_0} \mathbf{JC}\mathcal{S}_{\mathbf{C}^{-1}(K_*)} \begin{bmatrix} -zw_{xx} \\ -zw_{yy} \\ -zw_{xy} \end{bmatrix} \cdot \begin{pmatrix} -z\hat{w}_{xx} \\ -z\hat{w}_{yy} \\ -z\hat{w}_{xy} \end{pmatrix} dx\, dy\, dz \\
&= 2\int_{0}^{h}\int_{\Omega_0} z^2\mathbf{JC}\mathcal{S}_{\frac{1}{z}\mathbf{C}^{-1}(K_*)} \begin{bmatrix} w_{xx} \\ w_{yy} \\ w_{xy} \end{bmatrix} \cdot \begin{pmatrix} \hat{w}_{xx} \\ \hat{w}_{yy} \\ \hat{w}_{xy} \end{pmatrix} dx\, dy\, dz \\
&= \int_{\Omega_0} \mathbf{JC}\left(\int_{1/h}^{\infty} 2q^{-4}\mathcal{S}_{q\mathbf{C}^{-1}(K_*)} \begin{bmatrix} w_{xx} \\ w_{yy} \\ w_{xy} \end{bmatrix} dq\right) \cdot \begin{pmatrix} \hat{w}_{xx} \\ \hat{w}_{yy} \\ \hat{w}_{xy} \end{pmatrix} dx\, dy\, .
\end{aligned}$$

The mapping

$$\mathcal{P} : \begin{pmatrix} w_{xx} \\ w_{yy} \\ w_{xy} \end{pmatrix} \mapsto \int_{1/h}^{\infty} 2q^{-4}\mathcal{S}_{q\mathbf{C}^{-1}(K_*)} \begin{bmatrix} w_{xx} \\ w_{yy} \\ w_{xy} \end{bmatrix} dq \tag{2.34}$$

is called the *vectorial Prandtl-Ishlinskii operator*, and will be investigated in Section 3. The equation for oscillations of an elastoplastic plate can thus be written in the form

$$\begin{aligned}
&\rho\int_{\Omega_0}\left(w_{tt}\hat{w} + \frac{h^2}{3}\left(w_{xtt}\,\hat{w}_x + w_{ytt}\,\hat{w}_y\right)\right) dx\, dy \\
&\quad + \int_{\Omega_0} \mathbf{JC}\mathcal{P}\begin{bmatrix} w_{xx} \\ w_{yy} \\ w_{xy} \end{bmatrix} \cdot \begin{pmatrix} \hat{w}_{xx} \\ \hat{w}_{yy} \\ \hat{w}_{xy} \end{pmatrix} dx\, dy \;=\; \int_{\Omega_0} g\,\hat{w}\, dx\, dy \qquad \forall \hat{w} \in V\, ,
\end{aligned} \tag{2.35}$$

with g as in (2.12)–(2.13).

2.3 Kinematic hardening

In order to model kinematic hardening, we assume that the stress $\bar{\boldsymbol{\sigma}}$ of the form (2.6) is decomposed into the sum $\bar{\boldsymbol{\sigma}} = \boldsymbol{\sigma}^{ep} + \boldsymbol{\sigma}^b$ of a purely elastoplastic stress tensor $\boldsymbol{\sigma}^{ep}$ satisfying hypotheses (M1)–(M6), and the so-called *backstress* $\boldsymbol{\sigma}^b$, which, in the three dimensional representation (2.20), is assumed to obey an elastic constitutive law

$$\boldsymbol{\sigma}^b_* \;=\; \mathbf{JB}\bar{\boldsymbol{\varepsilon}}_*\,, \tag{2.36}$$

where $\mathbf{B}$ is a constant symmetric (3×3)-matrix such that $\mathbf{JB} = \mathbf{BJ}$, and the inequality

$$\mathbf{JB}\boldsymbol{\xi}_*\cdot\boldsymbol{\xi}_* \;\geq\; \beta(\xi_{11}^2 + \xi_{22}^2) \qquad \forall \boldsymbol{\xi}_* = \begin{pmatrix} \xi_{11} \\ \xi_{22} \\ \xi_{12} \end{pmatrix} \tag{2.37}$$

holds with some $\gamma > 0$. Repeating the computation from the previous subsection, we obtain, as a counterpart of (2.35), the equation for w in the form

$$\rho\int_{\Omega_0}\left(w_{tt}\hat{w} \;+\; \frac{h^2}{3}\left(w_{xtt}\,\hat{w}_x + w_{ytt}\,\hat{w}_y + \boldsymbol{\sigma}\cdot\mathbf{D}_2\hat{w}\right)\right)dx\,dy \;=\; \int_{\Omega_0} g\,\hat{w}\,dx\,dy \qquad \forall \hat{w}\in V\,, \tag{2.38}$$

with constitutive equation

$$\boldsymbol{\sigma} = \mathbf{JB}\boldsymbol{\varepsilon} + \mathbf{JC}\mathcal{P}[\boldsymbol{\varepsilon}] \tag{2.39}$$

where

$$\boldsymbol{\varepsilon} = \mathbf{D}_2 w \;=\; \begin{pmatrix} w_{xx} \\ w_{yy} \\ w_{xy} \end{pmatrix}, \tag{2.40}$$

and $\mathcal{P}$ is the Prandtl-Ishlinskii operator (2.34). In order to state the existence and uniqueness result, we first fix the hypotheses and notation. We assume that $\Omega_0 \subset \mathbb{R}^2$ is a Lipschitzian domain, and denote in agreement with Section 2.1

$$\begin{aligned} H &= L^2(\Omega_0)\,, \\ W &= H^1_0(\Omega_0) := \{w \in H^1(\Omega_0)\,:\, w\big|_{\partial\Omega_0} = 0\}\,, \\ V &= H^2(\Omega_0)\cap H^1_0(\Omega_0)\,. \end{aligned}$$

By $\mathbf{n} = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}$ we denote the outward normal vector to Ω_0. For functions $\mathbf{v} : \Omega_0 \to \mathbb{R}^3$ with components (v^1, v^2, v^3), we define the differential operator

$$\mathbf{D}_1\mathbf{v} \;=\; \begin{pmatrix} v^1_x + v^3_y \\ v^3_x + v^2_y \end{pmatrix}. \tag{2.41}$$

For $\hat{w} \in V$ and $\mathbf{v} \in L^2(\Omega_0;\mathbb{R}^3)$ such that $\mathbf{D}_1\mathbf{v} \in L^2(\Omega_0;\mathbb{R}^3)$, we have the following Green/Gauss-type formula

$$\int_{\Omega_0}\left(\mathbf{Jv}\cdot\mathbf{D}_2\hat{w} + \mathbf{D}_1\mathbf{v}\cdot\nabla\hat{w}\right)\,dx\,dy = \int_{\partial\Omega_0}(\mathbf{v}\bullet\mathbf{n})\cdot\nabla\hat{w}\,ds\,, \tag{2.42}$$

where we denote

$$\mathbf{v} \bullet \mathbf{n} \;=\; n_1 \begin{pmatrix} v^1 \\ v^3 \end{pmatrix} + n_2 \begin{pmatrix} v^3 \\ v^2 \end{pmatrix}. \tag{2.43}$$

Note also the formal identity

$$\mathbf{D}_1 \mathbf{D}_2 \hat{w} \;=\; \nabla \Delta \hat{w} \quad \forall \hat{w} \in H^3(\Omega_0)\,. \tag{2.44}$$

We now restate Equation (2.38) in a slightly more general form. Removing the positive constants that have no influence on the existence and uniqueness statement, and keeping the matrices $\mathbf{J}, \mathbf{C}, \mathbf{B}$, we consider the variational problem

$$\begin{aligned} &\int_{\Omega_0} \big(w_{tt}(\hat{w} - \Delta \hat{w}) + \mathbf{J}\,(\mathbf{C}\mathcal{P}\,[\mathbf{D}_2 w] + \mathbf{B}\mathbf{D}_2 w) \cdot \mathbf{D}_2 \hat{w} \big)\, dx\, dy \\ &\qquad = \int_{\Omega_0} \big(g\, \hat{w} + \nabla \mathbf{G} \cdot \nabla \hat{w} \big)\, dx\, dy \qquad \forall \hat{w} \in V\,, \end{aligned} \tag{2.45}$$

where g and $\mathbf{G}$ are given functions, and $\mathcal{P}$ is a Prandtl-Ishlinskii operator as in (3.10) associated with a convex constraint $Z \subset \mathbb{R}^3$ satisfying (3.9). We prescribe initial conditions

$$w(x,y,0) = w^0(x,y)\,, \quad w_t(x,y,0) = w^1(x,y) \quad \text{ for } (x,y) \in \Omega_0, \tag{2.46}$$

and boundary condition

$$w(x,y,t) = 0 \quad \text{ for } (x,y) \in \partial\Omega_0\,. \tag{2.47}$$

Indeed, smooth solutions of the variational equation (2.45) satisfy a second (“no stress”) boundary condition (cf. (2.42), (2.15))

$$((\mathbf{C}\mathcal{P}\,[\mathbf{D}_2 w] + \mathbf{B}\mathbf{D}_2 w) \bullet \mathbf{n}) \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega_0\,. \tag{2.48}$$

Hypothesis 2.1. *The data of Problem (2.45)–(2.47) fulfill the following conditions.*

(i) $w^0, w^1 \in H^3(\Omega_0) \cap V$*, and the compatibility conditions*

$$\big((\mathbf{C}\mathbf{A}_{\mathcal{P}}(\mathbf{D}_2 w^0) + \mathbf{B}\mathbf{D}_2 w^0) \bullet \mathbf{n} \big) \cdot \mathbf{n} = 0\,, \qquad \big(\mathbf{D}_2 w^1 \bullet \mathbf{n} \big) \cdot \mathbf{n} = 0\,,$$

hold a. e. on $\partial\Omega_0$*, where* $\mathbf{A}_{\mathcal{P}}$ *is the initial value mapping (3.15) of the operator* $\mathcal{P}$*;*

(ii) $g \in L^2(0,T;H)$ *and* $\mathbf{G} \in L^2(0,T;H^1(\Omega_0))$ *are such that* $g_t \in L^2(0,T;H)$ *and* $\mathbf{G}_t \in L^2(0,T;H^1(\Omega_0))$

The main result of this section reads as follows. A detailed proof can be found in [16, Theorem 4.2].

Theorem 2.2. *Let Hypothesis 2.1 hold. Then Problem (2.45)–(2.47) admits a unique solution* $w \in L^2(0,T;V)$ *such that*

$$w_t \in L^2(0,T;V) \cap C([0,T];W)\,, \quad w_{tt} \in L^2(0,T;W)\,, \tag{2.49}$$

and Eq. (2.45) holds for a. e. $t \in (0,T)$.

3 Prandtl-Ishlinskii operator

The original Prandtl-Ishlinskii construction ([19, 37]) is one-dimensional as in Subsection 1.3. A vector Prandtl-Ishlinskii model in connection with phase transitions was considered in [27], and we recall it here in an abstract framework. Consider a real, separable Hilbert space X endowed with a scalar product $\langle \cdot, \cdot \rangle$ and norm $|\cdot| = \sqrt{\langle \cdot, \cdot \rangle}$, and assume that a convex closed set $Z \subset X$ containing the origin is given. For each function $v \in W^{1,1}(0,T;X)$, we define $\chi \in W^{1,1}(0,T;X)$ as the unique solution of the variational inequality

$$\left.\begin{aligned} &\chi(t) \in Z \quad \forall t \in [0,T]\,, \\ &\chi(0) = Q_Z(v(0))\,, \\ &\langle \dot{v}(t) - \dot{\chi}(t), \chi(t) - y \rangle \ \geq \ 0 \ \text{ a.e.} \quad \forall y \in Z\,, \end{aligned}\right\} \tag{3.1}$$

where $Q_Z : X \to Z$ is the orthogonal projection onto Z, and the dot denotes differentiation with respect to t. The solution mapping

$$\mathcal{S}_Z : W^{1,1}(0,T;X) \to W^{1,1}(0,T;X) : v \mapsto \chi \tag{3.2}$$

is called the *vectorial stop with characteristic* Z as an abstract counterpart of (2.29). It was introduced in [20] and its analytical properties were studied in detail in [7]. We list here some properties of the stop that are needed in the sequel.

Proposition 3.1. *The mapping $\mathcal{S}_Z$ defined by (3.1)–(3.2) has the following properties.*

(i) *$\mathcal{S}_Z$ is continuous in the strong topology of $W^{1,1}(0,T;X)$;*

(ii) *If the boundary of Z is of class $W^{2,\infty}$ (that is, the outward normal mapping is Lipschitz continuous), then $\mathcal{S}_Z$ is locally Lipschitz continuous in $W^{1,1}(0,T;X)$;*

(iii) *If Z has a nonempty interior, then $\mathcal{S}_Z$ can be extended to a continuous mapping $C([0,T];X) \to C([0,T];X)$;*

(iv) *If Z is uniformly strictly convex, then $\mathcal{S}_Z : C([0,T];X) \to C([0,T];X)$ is $1/2$-Hölder continuous;*

(v) *The mapping $\mathcal{S}_Z$ is monotone in the sense that*

$$\langle \mathcal{S}_Z[v_1](t) - \mathcal{S}_Z[v_2](t), \dot{v}_1(t) - \dot{v}_2(t) \rangle \ \geq \ \frac{1}{2}\frac{d}{dt} \left|\mathcal{S}_Z[v_1](t) - \mathcal{S}_Z[v_2](t)\right|^2 \quad \textit{a. e.} \tag{3.3}$$

for every $v_1, v_2 \in W^{1,1}(0,T;X)$;

(vi) *The mapping $\mathcal{S}_Z$ is locally monotone, i. e.*

$$\left\langle \frac{d}{dt}\mathcal{S}_Z[v](t), \dot{v}(t) \right\rangle \ = \ \left| \frac{d}{dt}\mathcal{S}_Z[v](t) \right|^2 \quad \textit{a. e.} \tag{3.4}$$

for every $v \in W^{1,1}(0,T;X)$;

(vii) *The "second order energy inequality"*

$$\left\langle \frac{d}{dt}\mathcal{S}_Z[v](t), \ddot v(t)\right\rangle \;\ge\; \frac{1}{2}\frac{d}{dt}\left\langle \frac{d}{dt}\mathcal{S}_Z[v](t), \dot v(t)\right\rangle \tag{3.5}$$

holds in the sense of distributions for every $v \in W^{2,1}(0,T;X)$ *.*

Detailed proofs of the above statements are given in [7, Chapter 2] except for the inequality (3.5) which is derived in [22, pp. 37–38]. Notice a certain similarity of (3.5) with the "real" physical energy inequality

$$\langle \mathcal{S}_Z[v](t), \dot v(t)\rangle \;\ge\; \frac{1}{2}\frac{d}{dt}\left|\mathcal{S}_Z[v](t)\right|^2 , \quad \text{a. e.} \tag{3.6}$$

which follows immediately from (3.3) by choosing $v_2 = 0$. In (3.6), the right hand side is the time derivative of the potential energy associated with the stop, and the (nonnegative) difference between the left hand and the right hand sides is the *dissipation rate*. If $\dim X = 1$, then it can be identified with the area of the corresponding hysteresis loops. Instead, the "dissipation" in (3.5) is related to the *curvature* of the hysteresis branches. A detailed discussion on this subject can be found in [22, Section II. 4].

As another consequence of Proposition 3.1 (v) we have

$$\frac{d}{dt}\left|\mathcal{S}_Z[v_1](t) - \mathcal{S}_Z[v_2](t)\right| \;\le\; |\dot v_1(t) - \dot v_2(t)| \quad \text{a. e.}, \tag{3.7}$$

hence

$$|\mathcal{S}_Z[v_1](t) - \mathcal{S}_Z[v_2](t)| \;\le\; |\mathcal{S}_Z[v_1](0) - \mathcal{S}_Z[v_2](0)| + \int_0^t |\dot v_1(\tau) - \dot v_2(\tau)|\, d\tau \tag{3.8}$$

for all $t \in [0,T]$.

We now assume additionally that Z is a bounded, convex, closed set containing 0 in its interior, that is, there exist $C > c > 0$ such that

$$B_c(0) \subset Z \subset B_C(0)\,, \tag{3.9}$$

where $B_\rho(x)$ for $\rho > 0$ and $x \in X$ denotes the open ball centered at x with radius ρ. Given a nonnegative function $\varphi \in L^1(0,\infty)$, we define the *Prandtl-Ishlinskii operator* $\mathcal{P}$ with characteristic Z and density φ by the formula similar to (2.34)

$$\mathcal{P}[v](t) \;=\; \int_0^\infty \mathcal{S}_{qZ}[v](t)\,\varphi(q)\,dq \tag{3.10}$$

for $v \in W^{1,1}(0,T;X)$. The definition is meaningful due to the fact that, setting $v_\infty = \max\{|v(t)| \,:\, t \in [0,T]\}$, we have $\mathcal{S}_{qZ}[v](t) = v(t)$ for all $q > v_\infty/c$ and all $t \in [0,T]$, so that, using the elementary estimate $|\mathcal{S}_{qZ}[v](t)| \le qC$, we have

$$|\mathcal{P}[v](t)| \;\le\; v_\infty\left(1 + \frac{C}{c}\right)\int_0^\infty \varphi(q)\,dq$$

for all $t \in [0,T]$. As a direct consequence of Proposition 3.1, the mapping $\mathcal{P}$ has the following properties.

Proposition 3.2. *Let* (3.9) *hold, let* $\varphi \in L^1(0,\infty)$ *be given,* $\varphi(q) \ge 0$ *a. e., not identically zero, and let* $\mathcal{P}$ *be defined by (3.10). Then we have:*

(i) *Both* $\mathcal{P} : W^{1,1}(0,T;X) \to W^{1,1}(0,T;X)$ *and* $\mathcal{P} : C([0,T];X) \to C([0,T];X)$ *are continuous with respect to the strong topologies;*

(ii) *The mapping* $\mathcal{P}$ *is monotone in the sense that*

$$\langle \mathcal{P}[v_1](t) - \mathcal{P}[v_2](t), \dot v_1(t) - \dot v_2(t)\rangle \ \ge\ \frac{1}{2}\frac{d}{dt}\int_0^\infty \left|\mathcal{S}_Z[v_1](t) - \mathcal{S}_Z[v_2](t)\right|^2 \varphi(q)\,dq \quad a.\ e. \tag{3.11}$$

for every $v_1, v_2 \in W^{1,1}(0,T;X)$ *;*

(iii) *The mapping* $\mathcal{P}$ *is locally monotone, i. e.*

$$|\dot v(t)|^2 \int_0^\infty \varphi(q)\,dq \ \ge\ \left\langle \frac{d}{dt}\mathcal{P}[v](t), \dot v(t)\right\rangle \ \ge\ \left|\frac{d}{dt}\mathcal{P}[v](t)\right|^2 \left(\int_0^\infty \varphi(q)\,dq\right)^{-1} \quad a.\ e. \tag{3.12}$$

for every $v \in W^{1,1}(0,T;X)$ *;*

(iv) *The "second order energy inequality"*

$$\left\langle \frac{d}{dt}\mathcal{P}[v](t), \ddot v(t)\right\rangle \ \ge\ \frac{1}{2}\frac{d}{dt}\left\langle \frac{d}{dt}\mathcal{P}[v](t), \dot v(t)\right\rangle \tag{3.13}$$

holds in the sense of distributions for every $v \in W^{2,1}(0,T;X)$ *.*

The canonical choice of initial conditions in (3.1) makes it possible to evaluate explicitly $\mathcal{P}[v](0)$ at time $t = 0$. We have

$$\mathcal{P}[v](0) \ =\ \int_0^\infty Q_{qZ}(v(0))\,\varphi(q)\,dq\,. \tag{3.14}$$

We thus can define the *initial value mapping*

$$\mathbf{A}_{\mathcal{P}}(v) : X \to X : v \mapsto \int_0^\infty Q_{qZ}(v)\,\varphi(q)\,dq\,. \tag{3.15}$$

Since Q_{qZ} is nonexpansive, we see that $\mathbf{A}_{\mathcal{P}}$ is Lipschitz continuous in X.

We need another important property of the Prandtl-Ishlinskii operator established in [13] and stated below in Proposition 3.3. Let us introduce first some necessary concepts.

With the convex closed set Z satisfying (3.9), we associate the *Minkowski functional* $M_Z : X \to \mathbb{R}^+$ defined by the formula

$$M_Z(\chi) = \inf\left\{ s > 0 : \frac{1}{s}\chi \in Z\right\}, \tag{3.16}$$

and the polar set Z^* to Z

$$Z^* = \{\eta \in X : \langle \chi, \eta\rangle \le 1 \ \forall \chi \in Z\}. \tag{3.17}$$

We then obviously have

$$B_{\frac{1}{C}}(0) \subset Z^* \subset B_{\frac{1}{c}}(0),$$

and the inequalities

$$\frac{|\chi|}{C} \le M_Z(\chi) \le \frac{|\chi|}{c}, \qquad c|\eta| \le M_{Z^*}(\eta) \le C|\eta| \tag{3.18}$$

hold for every $\chi, \eta \in X$.

The Minkowski functional of a convex closed set containing 0 is proper, convex, and lower semicontinuous. In addition to (3.9), we assume that the unit outward normal mapping is Lipschitz continuous on ∂Z. Then, by [4, Lemma 3.1], the subdifferential $\partial M_Z(\chi)$ for all $\chi \ne 0$ contains a single vector parallel to the unit outward normal vector $n_Z(\chi/M_Z(\chi))$ taken at the point $\chi/M_Z(\chi)$ on the boundary of Z. We define the *duality mapping* $J_Z : X \to X$ by the formula

$$J_Z(\chi) = M_Z(\chi)\partial M_Z(\chi)\,. \tag{3.19}$$

Still by [4, Lemma 3.1], there exists $L > 0$ such that

$$|J_Z(\chi) - J_Z(\eta)| \le L|\chi - \eta|, \qquad \forall \chi, \eta \in X\,. \tag{3.20}$$

The Minkowski functionals M_Z and M_{rZ} for $r > 0$ are related through a simple scaling formula. Indeed,

$$M_{rZ}(\chi) = \inf\left\{s > 0 : \frac{1}{s}\chi \in rZ\right\} = \inf\left\{s > 0 : \frac{1}{rs}\chi \in Z\right\} = \frac{1}{r}M_Z(x)\,, \tag{3.21}$$

and

$$\begin{aligned} \xi \in \partial M_{rZ}(\chi) &\Longleftrightarrow \langle \xi, \chi - \eta\rangle \ge M_{rZ}(\chi) - M_{rZ}(\eta) \quad \forall \eta \in X \\ &\Longleftrightarrow \langle r\xi, \chi - \eta\rangle \ge M_Z(\chi) - M_Z(\eta) \quad \forall \eta \in X \\ &\Longleftrightarrow r\xi \in \partial M_Z(\chi)\,, \end{aligned} \tag{3.22}$$

hence $\partial M_{rZ} = \frac{1}{r}\partial M_Z$. We thus conclude that

$$J_{rZ} = \frac{1}{r^2} J_Z\,. \tag{3.23}$$

Formula [7, (3.35)] can be written here in the form

$$M_{Z^*}(J_Z(\chi)) = M_Z(\chi) \quad \forall \chi \in X\,, \tag{3.24}$$

hence, by (3.18),

$$\frac{|\chi|}{|J_Z(\chi)|} = \frac{|\chi|}{M_Z(\chi)}\frac{M_{Z^*}(J_Z(\chi))}{|J_Z(\chi)|} \le C^2 \quad \forall \chi \in X \setminus \{0\}\,. \tag{3.25}$$

In terms of the Minkowski functional, putting $\xi(t) = v(t) - \chi(t)$, we can represent the variational inequality (3.1) by the differential inclusion $\chi(t) \in \partial M_{Z^*}(\dot\xi(t))$, or, equivalently, by the identity

$$\langle \dot v(t), \chi(t)\rangle = \frac{d}{dt}\left(\frac{1}{2}|\chi(t)|^2\right) + M_{Z^*}(\dot\xi(t)) \quad a.e. \tag{3.26}$$

This is the so-called *energetic formulation*, see [34]. The energetic interpretation of (3.26) is that $\langle \dot{v}(t), \chi(t)\rangle$ is the power supplied to the system, part of which is used for the potential increase $\frac{d}{dt}\left(\frac{1}{2}|\chi(t)|^2\right)$, and the other part $\left\langle \dot{\xi}(t), \chi(t)\right\rangle = M_{Z^*}(\dot{\xi}(t))$ is dissipated.

We now state and prove the result announced above, namely the Lipschitz continuity of the dissipation functional.

Proposition 3.3. *Let* $v_1, v_2 \in W^{1,1}(0,T;X)$ *and* $r > 0$ *be given, and let* $\chi_r^{(i)} = \mathcal{S}_{rZ}[v_i]$, $i = 1, 2$. *Set* $\xi_r^{(i)} = v_i - \chi_r^{(i)}$. *Then we have*

$$
\begin{aligned}
&|\left\langle \chi_r^{(1)}, \dot{\xi}_r^{(1)}\right\rangle - \left\langle \chi_r^{(2)}, \dot{\xi}_r^{(2)}\right\rangle|(t) + r^2 C^2 \frac{d}{dt}\left|\frac{1}{2}M_{rZ}^2(\chi_r^{(1)}(t)) - \frac{1}{2}M_{rZ}^2(\chi_r^{(2)}(t))\right| \\
&\quad \le r\frac{C^2}{c}|\dot{v}_1 - \dot{v}_2|(t) + (1+2LC^2)|\dot{v}_2|(t)\int_0^t |\dot{v}_1 - \dot{v}_1|(\tau)\,d\tau \quad a.e.,
\end{aligned}
\tag{3.27}
$$

where c, C, L *are the constants in* (3.9), (3.20).

Proof. Set

$$
N(t) = \left|\frac{1}{2}M_{rZ}^2(\chi_r^{(1)}(t)) - \frac{1}{2}M_{rZ}^2(\chi_r^{(2)}(t))\right|. \tag{3.28}
$$

By virtue of [4, Eqs. (47) and (51)], we have

$$
\begin{aligned}
&\dot{\xi}_r^i(t) \ne 0 \\
&\Rightarrow \left\langle J_{rZ}(\chi_r^i(t)), \dot{\xi}_r^i(t)\right\rangle > 0,\ \dot{\xi}_r^i(t) = \frac{\left\langle J_{rZ}(\chi_r^i(t)), \dot{\xi}_r^i(t)\right\rangle}{|J_{rZ}(\chi_r^i(t))|^2} J_{rZ}(\chi_r^i(t)) \quad a.e.,
\end{aligned}
\tag{3.29}
$$

$$
\begin{aligned}
&|\left\langle J_{rZ}(\chi_r^{(1)}), \dot{\xi}_r^{(1)}\right\rangle - \left\langle J_{rZ}(\chi_r^{(2)}), \dot{\xi}_r^{(2)}\right\rangle| + \frac{d}{dt}N(t) \\
&\quad \le |\langle J_{rZ}(\chi_r^{(1)}), \dot{v}_{(1)}\rangle - \langle J_{rZ}(\chi_r^{(2)}), \dot{v}_{(2)}\rangle| \quad a.e.
\end{aligned}
\tag{3.30}
$$

The substantial step consists in proving that

$$
\begin{aligned}
\left|\left\langle \chi_r^{(1)}, \dot{\xi}_r^{(1)}\right\rangle - \left\langle \chi_r^{(2)}, \dot{\xi}_r^{(2)}\right\rangle\right| &\le r^2C^2\left|\left\langle J_{rZ}(\chi_r^{(1)}), \dot{\xi}_r^{(1)}\right\rangle - \left\langle J_{rZ}(\chi_r^{(2)}), \dot{\xi}_r^{(2)}\right\rangle\right| \\
&\quad + (1+LC^2)|\dot{v}_{(2)}||\chi_r^{(1)} - \chi_r^{(2)}| \quad a.e.
\end{aligned}
\tag{3.31}
$$

This is obvious if $\dot{\xi}_r^{(1)}(t) = \dot{\xi}_r^{(2)}(t) = 0$. If for instance $\dot{\xi}_r^{(1)}(t) \ne 0$, $\dot{\xi}_r^{(2)}(t) = 0$, then

$$
\left\langle \chi_r^{(1)}, \dot{\xi}_r^{(1)}\right\rangle = \langle J_{rZ}(\chi_r^{(1)}), \chi_r^{(1)}\rangle \frac{\left\langle J_{rZ}(\chi_r^{(1)}), \dot{\xi}_r^{(1)}\right\rangle}{|J_{rZ}(\chi_r^{(1)})|^2} \le r^2C^2\left\langle J_{rZ}(\chi_r^{(1)}), \dot{\xi}_r^{(1)}\right\rangle
$$

by virtue of (3.23), (3.25), and we obtain (3.31) in a straightforward way. It remains to consider the case $\dot\xi_r^{(1)}(t)\neq 0$, $\dot\xi_r^{(2)}(t)\neq 0$. Then

$$\begin{aligned}
&\left|\left\langle \chi_r^{(1)},\dot\xi_r^{(1)}\right\rangle-\left\langle \chi_r^{(2)},\dot\xi_r^{(2)}\right\rangle\right|\\
&=\left|\langle J_{rZ}(\chi_r^{(1)}),\chi_r^{(1)}\rangle\,\frac{\left\langle J_{rZ}(\chi_r^{(1)}),\dot\xi_r^{(1)}\right\rangle}{|J_{rZ}(\chi_r^{(1)})|^2}-\langle J_{rZ}(\chi_r^{(2)}),\chi_r^{(2)}\rangle\,\frac{\left\langle J_{rZ}(\chi_r^{(2)}),\dot\xi_r^{(2)}\right\rangle}{|J_{rZ}(\chi_r^{(2)})|^2}\right|\\
&\le\frac{\left\langle J_{rZ}(\chi_r^{(1)}),\chi_r^{(1)}\right\rangle}{|J_{rZ}(\chi_r^{(1)})|^2}\left|\left\langle J_{rZ}(\chi_r^{(1)}),\dot\xi_r^{(1)}\right\rangle-\left\langle J_{rZ}(\chi_r^{(2)}),\dot\xi_r^{(2)}\right\rangle\right|\\
&\quad+\left|\left\langle J_{rZ}(\chi_r^{(2)}),\dot\xi_r^{(2)}\right\rangle\right|\left|\frac{\left\langle J_{rZ}(\chi_r^{(1)}),\chi_r^{(1)}\right\rangle}{|J_{rZ}(\chi_r^{(1)})|^2}-\frac{\left\langle J_{rZ}(\chi_r^{(2)}),\chi_r^{(2)}\right\rangle}{|J_{rZ}(\chi_r^{(2)})|^2}\right|\\
&\qquad\le\frac{|\chi_r^{(1)}|}{|J_{rZ}(\chi_r^{(1)})|}\left|\left\langle J_{rZ}(\chi_r^{(1)}),\dot\xi_r^{(1)}\right\rangle-\left\langle J_{rZ}(\chi_r^{(2)}),\dot\xi_r^{(2)}\right\rangle\right|\\
&\qquad\quad+\left|\left\langle J_{rZ}(\chi_r^{(2)}),\dot\xi_r^{(2)}\right\rangle\right|\frac{|\left\langle J_{rZ}(\chi_r^{(2)}),\chi_r^{(1)}-\chi_r^{(2)}\right\rangle|}{|J_{rZ}(\chi_r^{(2)})|^2}\\
&\qquad\quad+\left|\left\langle J_{rZ}(\chi_r^{(2)}),\dot\xi_r^{(2)}\right\rangle\right|\,|\chi_r^{(1)}|\left|\frac{J_{rZ}(\chi_r^{(1)})}{|J_{rZ}(\chi_r^{(1)})|^2}-\frac{J_{rZ}(\chi_r^{(2)})}{|J_{rZ}(\chi_r^{(2)})|^2}\right|.
\end{aligned}\tag{3.32}$$

We now estimate the three terms on the right hand side of (3.32) as follows. Notice first that by Proposition 3.1 (ii), we have $|\dot\xi_r^{(2)}|\le|\dot v_{(2)}|$ a.e. Furthermore, we have the elementary vector identity

$$\left|\frac{z}{|z|^2}-\frac{z'}{|z'|^2}\right|=\frac{1}{|z|\,|z'|}|z-z'|,\qquad\forall z,z'\in X\setminus\{0\}\,.$$

Hence,

$$\left|\left\langle J_{rZ}(\chi_r^{(2)}),\dot\xi_r^{(2)}\right\rangle\right|\frac{|\left\langle J_{rZ}(\chi_r^{(2)}),\chi_r^{(1)}-\chi_r^{(2)}\right\rangle|}{|J_{rZ}(\chi_r^{(2)})|^2}\le|\dot v_{(2)}||\chi_r^{(1)}-\chi_r^{(2)}|\,,$$

and

$$\begin{aligned}
&\left|\left\langle J_{rZ}(\chi_r^{(2)}),\dot\xi_r^{(2)}\right\rangle\right|\,|\chi_r^{(1)}|\left|\frac{J_{rZ}(\chi_r^{(1)})}{|J_{rZ}(\chi_r^{(1)})|^2}-\frac{J_{rZ}(\chi_r^{(2)})}{|J_{rZ}(\chi_r^{(2)})|^2}\right|\\
&\qquad\le\frac{|\chi_r^{(1)}|}{|J_{rZ}(\chi_r^{(1)})|}|\dot v_{(2)}||J_{rZ}(\chi_r^{(1)})-J_{rZ}(\chi_r^{(2)})|\,.
\end{aligned}$$

In view of (3.23) and (3.20), we thus obtain (3.31) from (3.32).

Combining (3.31) with (3.30), we obtain

$$\begin{aligned}
&\left|\left\langle \chi_r^{(1)},\dot\xi_r^{(1)}\right\rangle-\left\langle \chi_r^{(2)},\dot\xi_r^{(2)}\right\rangle\right|+r^2C^2\frac{d}{dt}N(t)\\
&\qquad\le r^2C^2|J_{rZ}(\chi_r^{(1)})||\dot v_{(1)}-\dot v_{(2)}|+(1+2LC^2)|\dot v_{(2)}||\chi_r^{(1)}-\chi_r^{(2)}|.
\end{aligned}\tag{3.33}$$

We now refer to (3.24) which yields

$$rc|J_{rZ}(\chi_r^{(1)})| \le M_{(rZ)^*}(J_{rZ}(\chi_r^{(1)})) = M_{rZ}(\chi_r^{(1)}) \le 1\,,$$

and (3.27) follows from (3.33) and (3.8). ■

4 A model for cyclic fatigue

We now propose a system to model important experimental features of elastoplastic oscillations subject to material fatigue, such as *material softening*, *heat release*, and *material failure in finite time*. The analysis of the so-called rainflow method of cyclic damage evaluation carried out in [2] has shown a qualitative and quantitative correspondence between the damage accumulation rule and the energy dissipation. On the other hand, experimental measurements at the point of material failure confirm strong temperature increase, which manifests an energy dissipation peak. In fact, temperature tests are regularly used in engineering practice for damage analysis in high frequency regimes (e.g. in aircraft industry). Our substantial modeling hypothesis thus consists in introducing a scalar fatigue parameter m, assuming that its time derivative (the fatigue rate) is proportional to the dissipation rate, and that the material parameters depend on m. We believe that this assumption is realistic. Plastic deformations are driven by moving dislocations and ruptures of interatomic connections, which at the same time dissipate energy, and reduce the cohesion of the solid. Note that in the Gurson model for void nucleation and growth in elastoplastic materials, see [17], the elasticity domain shrinks, being parameterized by the plastic dissipation rate. For a more detailed discussion about the modeling issues, see [11].

The PDE system of momentum and energy balance equations for transversal oscillations of an elastoplastic plate under fatigue is derived in Section 4.1. The unknowns of the full problem are w (the transversal displacement), θ (absolute temperature), and m (fatigue). We do not prove the well-posedness of the complete system resulting from a thermodynamic analysis. We only make a first step in this direction and solve the momentum balance equation coupled with the fatigue accumulation equation, assuming that the temperature history is known. An existence and uniqueness theory for the full system will be the subject of further research.

It cannot be expected that solutions of the system with fatigue exist globally in time. The material failure in finite time is an integral part of the model. We give an efficient lower bound for the existence time.

4.1 The model

We first simplify the notation and write the constitutive relation (2.39)–(2.40) in the form

$$\boldsymbol{\sigma} = \mathbf{B}\boldsymbol{\varepsilon} + \int_0^\infty \gamma(r)\,\mathbf{C}\boldsymbol{\chi}_r dr\,, \quad \boldsymbol{\varepsilon} = \mathbf{D}_2 w\,, \tag{4.1}$$

where $\boldsymbol{\chi}_r$ for $r > 0$ are solutions of the family of variational inequalities

$$\left.\begin{array}{ll} \boldsymbol{\chi}_r(x,t) \in rZ & \text{for all } \ x \in \Omega,\ t \in (0,T),\ r > 0\,, \\ \frac{\partial}{\partial t}(\boldsymbol{\varepsilon} - \boldsymbol{\chi}_r)\cdot\, \mathbf{C}(\boldsymbol{\chi}_r - \mathbf{z}) \geq 0 & \text{for all } \ \mathbf{z} \in rZ \ \ a.e., \\ \boldsymbol{\chi}_r(x,0) = Q_{rZ}(\boldsymbol{\varepsilon}(x,0)), & \end{array}\right\} \tag{4.2}$$

with a given closed convex set $Z \subset \mathbb{R}^3$ containing 0 in its interior. In other words, we replace $\mathbf{JB}$ by $\mathbf{B}$, $\mathbf{JC}$ by $\mathbf{C}$, and $\mathbf{C}^{-1}(K_*)$ by Z. We include the fatigue and temperature dependence into the model by introducing a scalar fatigue parameter $m(x,t) \geq 0$, assuming that both the matrix $\mathbf{B}$ and the function γ depend on m, and complement the constitutive law (4.1) with viscosity and thermal expansion terms to obtain

$$\boldsymbol{\sigma} = \mathbf{B}(m)\boldsymbol{\varepsilon} + \int_0^\infty \gamma(m,r)\, \mathbf{C}\boldsymbol{\chi}_r dr + \mathbf{R}\boldsymbol{\varepsilon}_t - \beta(\theta-\theta_0)\mathbf{1}\,, \tag{4.3}$$

where $\theta > 0$ is the absolute temperature, $\theta_0 > 0$ is a given referential temperature, $\mathbf{1}$ is the vector $(1,1,0)$, β is the thermal expansion coefficient, $\mathbf{R}$ is the viscosity matrix, and $\mathbf{B}(m), \gamma(m,r)$ are functions specified below in Hypothesis 4.1. By analogy to [11, 16], we associate with (4.3) the free energy $\mathcal{F}$ defined by the formula

$$\mathcal{F}[\boldsymbol{\varepsilon},\theta] = c_0\theta(1-\log(\theta/\theta_0)) + \frac{1}{2}\mathbf{B}(m)\boldsymbol{\varepsilon}\cdot\boldsymbol{\varepsilon} + \frac{1}{2}\int_0^\infty \gamma(m,r)\, \mathbf{C}\boldsymbol{\chi}_r\cdot\boldsymbol{\chi}_r dr - \beta(\theta-\theta_0)\boldsymbol{\varepsilon}\cdot\mathbf{1}\,, \tag{4.4}$$

where c_0 is the constant specific heat capacity. The internal energy $\mathcal{U}$ and the entropy $\mathcal{S}$ thus have the form

$$\mathcal{U}[\boldsymbol{\varepsilon},\theta] = c_0\theta + \frac{1}{2}\mathbf{B}(m)\boldsymbol{\varepsilon}\cdot\boldsymbol{\varepsilon} + \frac{1}{2}\int_0^\infty \gamma(m,r)\, \mathbf{C}\boldsymbol{\chi}_r\cdot\boldsymbol{\chi}_r dr + \beta\theta_0\boldsymbol{\varepsilon}\cdot\mathbf{1}\,, \tag{4.5}$$

$$\mathcal{S}[\boldsymbol{\varepsilon},\theta] = c_0\log(\theta/\theta_0) + \beta\boldsymbol{\varepsilon}\cdot\mathbf{1}\,. \tag{4.6}$$

The equations for the state variables θ and m are derived from the first and the second principles of thermodynamics in the form

$$\frac{\partial}{\partial t}\mathcal{U}[\boldsymbol{\varepsilon},\theta] + \operatorname{div}\mathbf{q} = \boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}_t\,, \qquad \text{(energy conservation)} \tag{4.7}$$

$$\frac{\partial}{\partial t}\mathcal{S}[\boldsymbol{\varepsilon},\theta] + \operatorname{div}\frac{\mathbf{q}}{\theta} \geq 0\,, \qquad \text{(Clausius-Duhem inequality)} \tag{4.8}$$

where $\mathbf{q}$ is the heat flux vector that we assume in the form

$$\mathbf{q} = -\kappa\nabla\theta\,, \tag{4.9}$$

with a constant heat conductivity coefficient $\kappa > 0$. Then (4.7) reads

$$\begin{aligned} c_0\theta_t - \kappa\Delta\theta = \mathbf{R}\boldsymbol{\varepsilon}_t\cdot\boldsymbol{\varepsilon}_t - \frac{1}{2}m_t\left(\mathbf{B}'(m)\boldsymbol{\varepsilon}\cdot\boldsymbol{\varepsilon} + \int_0^\infty \gamma_m(m,r)\, \mathbf{C}\boldsymbol{\chi}_r\cdot\boldsymbol{\chi}_r dr\right) \\ + \int_0^\infty \gamma(m,r)\frac{\partial}{\partial t}(\boldsymbol{\varepsilon}-\boldsymbol{\chi}_r)\cdot\, \mathbf{C}\boldsymbol{\chi}_r dr - \beta\theta\boldsymbol{\varepsilon}_t\cdot\mathbf{1}\,. \end{aligned} \tag{4.10}$$

The notation is slightly ambiguous, and we hope that the reader will not get confused. For simplicity, we denote by $_t$ and $_m$ partial derivatives with respect to the corresponding variables. The index $_r$ is not a partial derivative. There is no differentiation with respect to r in the paper.

In view of (4.9), we see that the Clausius-Duhem inequality (4.8) is certainly satisfied if

$$\theta\frac{\partial}{\partial t}\mathcal{S}[\boldsymbol{\varepsilon},\theta] + \boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}_t - \frac{\partial}{\partial t}\mathcal{U}[\boldsymbol{\varepsilon},\theta] \ge 0\,, \tag{4.11}$$

that is,

$$\begin{aligned}\mathbf{R}\boldsymbol{\varepsilon}_t\cdot\boldsymbol{\varepsilon}_t - \frac{1}{2}m_t\left(\mathbf{B}'(m)\boldsymbol{\varepsilon}\cdot\boldsymbol{\varepsilon} + \int_0^\infty \gamma_m(m,r)\,\mathbf{C}\boldsymbol{\chi}_r\cdot\boldsymbol{\chi}_r dr\right)&\\ + \int_0^\infty \gamma(m,r)\frac{\partial}{\partial t}(\boldsymbol{\varepsilon}-\boldsymbol{\chi}_r)\cdot\,\mathbf{C}\boldsymbol{\chi}_r dr \ge 0\,.&\end{aligned} \tag{4.12}$$

The last integral term in (4.12) is nonnegative by virtue of (4.2). The assumption that the fatigue accumulation rate m_t is nonnegative (that is, fatigue can only increase in time) is therefore compatible with the second principle provided

$$\mathbf{B}'(m)\ \text{ is negative semidefinite}\,,\quad \gamma_m(m,r)\le 0\ \ a.e. \tag{4.13}$$

In other words, *material softening takes place under increasing fatigue* in agreement with experimental evidence similarly as in [17].

We close the system by assuming that the fatigue accumulation rate m_t at a point $x\in\Omega$ is proportional to the plastic dissipation rate averaged over a neighborhood of the point x, that is,

$$\begin{aligned}m_t(x,t) &= \int_\Omega \lambda(x-y)\int_0^\infty \gamma(m,r)\frac{\partial}{\partial t}(\boldsymbol{\varepsilon}-\boldsymbol{\chi}_r)\cdot\,\mathbf{C}\boldsymbol{\chi}_r(y,t)dydr \\ &-\frac{1}{2}\int_\Omega \lambda(x-y)m_t(y,t)\left(\mathbf{B}'(m)\boldsymbol{\varepsilon}\cdot\boldsymbol{\varepsilon}+\int_0^\infty\gamma_m(m,r)\,\mathbf{C}\boldsymbol{\chi}_r\cdot\boldsymbol{\chi}_r dr\right)(y,t)dy\,,\end{aligned} \tag{4.14}$$

where $\lambda\in L^\infty(\mathbb{R}^2)$ is a given nonnegative function with compact support. The full system then consists of equations

$$\int_\Omega \big(\varrho w_{tt}(x,t)(\varphi - \frac{h^2}{12}\Delta\varphi)(x) + \boldsymbol{\sigma}(x,t)\cdot\mathbf{D}_2\varphi(x)\big)dx = \int_\Omega g(x,t)\varphi(x)dx \tag{4.15}$$

$$\forall\varphi\in W^{2,2}(\Omega)\cap W_0^{1,2}(\Omega)\,,$$

$$\boldsymbol{\varepsilon} = \mathbf{D}_2 w, \tag{4.16}$$

coupled with (4.2), (4.3), (4.10), (4.14), and its solutions satisfy the thermodynamic principles (4.7), (4.8).

4.2 Statement of the problem

As a first step towards the full system (4.15), (4.16), (4.2), (4.3), (4.10), (4.14), we assume that a function $\theta : \Omega \times (0,T) \to \mathbb{R}^3$ describing a combined action of thermal expansion and external load is given, set the physical constants to 1 for simplicity, and consider the problem

$$\int_\Omega \big(w_{tt}(x,t)(\varphi - \Delta\varphi)(x) + \mathbf{R}\mathbf{D}_2 w_t(x,t)\cdot \mathbf{D}_2\varphi(x)\big)dx$$
$$= \int_\Omega \big(\theta - \mathcal{P}[m,\boldsymbol{\varepsilon}]\big)(x,t)\cdot \mathbf{D}_2\varphi(x)dx \qquad \forall \varphi \in W^{2,2}(\Omega)\cap W_0^{1,2}(\Omega)\,, \tag{4.17}$$

$$\boldsymbol{\varepsilon} = \mathbf{D}_2 w = \begin{pmatrix} w_{xx} \\ w_{yy} \\ w_{xy} \end{pmatrix}, \tag{4.18}$$

$$\mathcal{P}[m,\boldsymbol{\varepsilon}] = \mathbf{B}(m)\boldsymbol{\varepsilon} + \int_0^\infty \gamma(m,r)\,\mathbf{C}\boldsymbol{\chi}_r dr\,, \tag{4.19}$$

$$m_t(x,t) = \int_\Omega \lambda(x-y)\int_0^\infty \gamma(m,r)\,\mathbf{C}\boldsymbol{\chi}_r\cdot(\boldsymbol{\xi}_r)_t dr dy \tag{4.20}$$
$$-\frac{1}{2}\int_\Omega \lambda(x-y)m_t(y,t)\Big(\mathbf{B}'(m)\boldsymbol{\varepsilon}\cdot\boldsymbol{\varepsilon} + \int_0^\infty \gamma_m(m,r)\,\mathbf{C}\boldsymbol{\chi}_r\cdot\boldsymbol{\chi}_r dr\Big)(y,t)dy$$

$$w(x,0) = w_t(x,0) = m(x,0) = 0 \quad \forall x \in \Omega\,, \tag{4.21}$$

where we denote $\boldsymbol{\xi}_r = \boldsymbol{\varepsilon} - \boldsymbol{\chi}_r$. The following hypotheses are assumed to hold.

Hypothesis 4.1. *We fix a Lipschitzian domain $\Omega \subset \mathbb{R}^2$, and denote by $|\cdot|_p$ the $L^p(\Omega)$ norm for $p \ge 1$. We assume that there exists a constant $\nu > 0$ such that for all $\varphi \in W^{2,2}(\Omega)\cap W_0^{1,2}(\Omega)$ we have*

$$|\mathbf{D}_2\varphi|_2 \le \nu|\Delta\varphi|_2\,, \tag{4.22}$$

and for $T > 0$ we denote $\Omega_T = \Omega\times(0,T)$. Furthermore,

(i) $\mathbf{R}$, $\mathbf{C}$ *are symmetric positive definite 3×3 matrices, and there exist constants $c^*, c_* > 0$ such that $\mathbf{R}\boldsymbol{\varepsilon}\cdot\boldsymbol{\varepsilon} \ge c_*|\boldsymbol{\varepsilon}|^2$, $\mathbf{C}\boldsymbol{\varepsilon}\cdot\boldsymbol{\varepsilon} \ge c_*|\boldsymbol{\varepsilon}|^2$, $|\mathbf{C}\boldsymbol{\varepsilon}| \le c^*|\boldsymbol{\varepsilon}|$ for all $\boldsymbol{\varepsilon} \in \mathbb{R}^3$;*

(ii) $\lambda \in L^\infty(\mathbb{R}^2)$ *is a given function with compact support, and $\Lambda > 0$ is a constant such that $0 \le \lambda \le \Lambda$ a.e.;*

(iii) $\mathbf{B}(m)$ *for $m \ge 0$ is a symmetric positive semidefinite 3×3 matrix, and there exists $b_* > 0$ such that $|\mathbf{B}(m)\boldsymbol{\varepsilon}| \le b_*|\boldsymbol{\varepsilon}|$ for all $\boldsymbol{\varepsilon} \in \mathbb{R}^3$. Moreover, $\mathbf{B}'(m)$ is negative semidefinite and depends Lipschitz continuously on m, $\mathbf{B}'(0) = \mathbf{0}$;*

(iv) $\gamma : [0,\infty)\times(0,\infty) \to [0,\infty)$ *is a given C^2-function such that $\gamma_m(m,r) \le 0$ a.e., $\gamma_m(0,r) = 0$ a.e., and there exists a constant $\Gamma > 0$ such that*

$$\int_0^\infty (\gamma(m,r) + \gamma_m(m,r) + \gamma_{mm}(m,r))(1+r^2)dr \;\le\; \Gamma \quad \forall m > 0;$$

(v) $\theta \in L^2(\Omega_T;\mathbb{R}^3)$ *is a given function for some* $T>0$*;*

(vi) $Z \subset \mathbb{R}^3$ *is a bounded convex closed domain with boundary of class* $W^{2,\infty}$, $0 \in \operatorname{Int} Z$.

We prove in the next sections the following existence and uniqueness results.

Theorem 4.2. *Let Hypothesis 4.1 hold and let* $R>0$ *be given. Then there exists* $T^R \in (0,T]$ *and a unique* w *such that* $w_{tt} \in L^2(\Omega_{T^R})$, $w_t \in L^2(0,T^R;W^{2,2}(\Omega)\cap W_0^{1,2}(\Omega))$, *equations* (4.2), (4.17)–(4.19), *and* (4.21) *are satisfied almost everywhere, and* (4.20) *is replaced by*

$$\begin{aligned} m_t(x,t) = Q_R\bigg(\int_\Omega \lambda(x-y)\int_0^\infty \gamma(m,r)\,\mathbf{C}\boldsymbol{\chi}_r\cdot(\boldsymbol{\xi}_r)_t\,dr\,dy\bigg) \\ -\frac{1}{2}\int_\Omega \lambda(x-y)m_t(y,t)\Big(\mathbf{B}'(m)\boldsymbol{\varepsilon}\cdot\boldsymbol{\varepsilon}+\int_0^\infty \gamma_m(m,r)\,\mathbf{C}\boldsymbol{\chi}_r\cdot\boldsymbol{\chi}_r\,dr\Big)(y,t)\,dy, \end{aligned} \tag{4.23}$$

where $Q_R(z)=\min\{R,\max\{z,-R\}\}$ *is the projection of* $\mathbb{R}$ *onto* $[-R,R]$.

Theorem 4.3. *In addition to Hypothesis 4.1, assume that* $\theta_t \in L^2(\Omega_T;\mathbb{R}^3)$. *Then there exists* $T^* \in (0,T]$ *and a unique solution* w *to* (4.2), (4.17)–(4.21) *with the additional regularity* $w_{tt} \in L^2(0,T^*;W_0^{1,2}(\Omega))$, $w_t \in L^\infty(0,T^*;W^{2,2}(\Omega)\cap W_0^{1,2}(\Omega))$. *Furthermore, there exists a constant* $C>0$ *such that if* $\theta^{(1)},\theta^{(2)}$ *are two functions satisfying the hypotheses and if* $w^{(1)},w^{(2)}$ *are the corresponding solutions, then the differences* $\bar w = w^{(1)}-w^{(2)}, \bar\theta = \theta^{(1)}-\theta^{(2)}$ *satisfy the inequality*

$$|\bar w_t|_2^2(t)+|\nabla \bar w_t|_2^2+\int_0^t |\mathbf{D}_2\bar w_t|_2^2(\tau)\,d\tau \le C\int_0^t |\bar\theta|_2^2(\tau)\,d\tau \qquad \forall t\in[0,T^*]. \tag{4.24}$$

4.3 Oscillating plate

In this section, we investigate first the decoupled problem (4.2), (4.17), (4.19), (4.21), (4.23), with the goal to obtain (4.18) by means of a fixed point argument.

Here, we assume that $\boldsymbol{\varepsilon}$ is known, and $\boldsymbol{\varepsilon}_t \in L^2(\Omega_T;\mathbb{R}^3)$. More specifically, under Hypothesis 4.1, we define for $t\in(0,T)$ the functions

$$p(t) = (|\theta(t)|_2+\Gamma)^2, \tag{4.25}$$

$$q(t) = \frac{2}{c_*^2}\int_0^t \mathrm{e}^{b_*^2(t^2-\tau^2)/c_*^2}p(\tau)\,d\tau, \tag{4.26}$$

and for $\hat T \in (0,T]$ consider the set

$$\begin{aligned} E_{\hat T} = \Big\{\boldsymbol{\varepsilon}\in L^2(\Omega_{\hat T};\mathbb{R}^3) : \boldsymbol{\varepsilon}_t \in L^2(\Omega_{\hat T};\mathbb{R}^3), \\ \boldsymbol{\varepsilon}(x,0)=0, \int_0^t |\boldsymbol{\varepsilon}_t(\tau)|_2^2\,d\tau \le q(t) \text{ for } t\in(0,\hat T]\Big\}. \end{aligned} \tag{4.27}$$

The definition is meaningful, as by Hypothesis 4.1, $p \in L^1(0,T)$. We fix the number

$$A := \sup \left| \frac{1}{2} \int_\Omega \lambda(x-y)\Big(\mathbf{B}''(m)\boldsymbol{\varepsilon} \cdot \boldsymbol{\varepsilon} + \int_0^\infty \gamma_{mm}(m,r)\, \mathbf{C}\boldsymbol{\chi}_r \cdot \boldsymbol{\chi}_r dr\Big)(y,t)dy \right|, \tag{4.28}$$

where the supremum is taken over all $\boldsymbol{\varepsilon} \in E_T$ and $m \in L^\infty(\Omega_T)$ (it is indeed finite), and define $\mu(t)$ as the solution of the ODE

$$\dot\mu(t) = A\mu(t)\dot\mu(t) + R\,, \quad \mu(0) = 0\,, \tag{4.29}$$

that is,

$$\mu(t) = \frac{1}{A} - \sqrt{\frac{1}{A^2} - \frac{2R}{A}t} \quad \text{for } t \in [0, 1/(2AR)]\,. \tag{4.30}$$

We see that $\dot\mu(t)$ blows up to $+\infty$ as $t \nearrow 1/(2AR)$. We choose a small $\delta \in (0,1)$ that we keep fixed throughout the paper, and set

$$T^R = \min\left\{T, \frac{1-\delta^2}{2AR}\right\}. \tag{4.31}$$

Eq. (4.23) cannot be expected to have global solutions for the same reason as in [12]. We state the intermediate result in the following form.

Proposition 4.4. *Let $\boldsymbol{\varepsilon} \in E_{T^R}$ be given. Then system (4.2), (4.17), (4.19), (4.21), (4.23) has a unique solution with the regularity as in Theorem 4.2, and we have $\mathbf{D}_2 w \in E_{T^R}$.*

Proof. We first check by a fixed point argument that (4.23) has a unique solution m in $[0, T^R]$. We define the set

$$M_R = \{m \in L^\infty(\Omega_{T^R}) : m_t \in L^\infty(\Omega_{T^R}), m(x,0) = 0, 0 \le m_t(x,t) \le \dot\mu(t) \;\; a.e.\}, \tag{4.32}$$

and for $m \in M_R$ define $\tilde m(x,0) = 0$, and

$$\begin{aligned}\tilde m_t(x,t) := Q_R \Bigg(&\Big(\int_\Omega \lambda(x-y) \int_0^\infty \gamma(m,r)\, \mathbf{C}\boldsymbol{\chi}_r \cdot (\boldsymbol{\xi}_r)_t dr dy\Big) \\ &- \frac{1}{2}\int_\Omega \lambda(x-y) m_t(y,t)\Big(\mathbf{B}'(m)\boldsymbol{\varepsilon}\cdot\boldsymbol{\varepsilon} + \int_0^\infty \gamma_m(m,r)\,\mathbf{C}\boldsymbol{\chi}_r\cdot\boldsymbol{\chi}_r dr\Big)(y,t)dy\end{aligned} \tag{4.33}$$

for $t \in (0, T^R]$. Note that $\mathbf{B}'(0) = \mathbf{0}, \gamma_m(0,r) = 0$. Hence,

$$\tilde m_t(x,t) \le A\mu(t)\dot\mu(t) + R = \dot\mu(t)\,,$$

that is, $\tilde m_t \in M_R$. Furthermore, for $m^{(1)}, m^{(2)} \in M_R$, we have

$$\begin{aligned}|\tilde m_t^{(1)}(t) - \tilde m_t^{(2)}(t)|_\infty &\le A\mu(t)|m_t^{(1)}(t) - m_t^{(2)}(t)|_\infty + A\dot\mu(t)|m^{(1)}(t) - m^{(2)}(t)|_\infty \\ &+ \tilde C|\boldsymbol{\varepsilon}_t(t)|_1 |m^{(1)}(t) - m^{(2)}(t)|_\infty\,,\end{aligned} \tag{4.34}$$

where $\tilde{C} > 0$ is a constant which comes out from the following computation:

$$
\begin{aligned}
&\Big| Q_R\left(\int_\Omega \lambda(x-y)\int_0^\infty \gamma(m^{(1)},r)\,\mathbf{C}\boldsymbol{\chi}_r\cdot(\boldsymbol{\xi}_r)_t drdy\right)\\
&\qquad - Q_R\left(\int_\Omega \lambda(x-y)\int_0^\infty \gamma(m^{(2)},r)\,\mathbf{C}\boldsymbol{\chi}_r\cdot(\boldsymbol{\xi}_r)_t drdy\right)\Big|\\
&\leq \int_\Omega \lambda(x-y)\int_0^\infty |\gamma(m^{(1)},r)-\gamma(m^{(2)},r)|\,\mathbf{C}\boldsymbol{\chi}_r\cdot(\boldsymbol{\xi}_r)_t drdy\\
&\leq \int_\Omega \lambda(x-y)\left|\int_{m^{(1)}(y,t)}^{m^{(2)}(y,t)}\int_0^\infty |\gamma_m(m,r)|rCc^* drdm\right||\boldsymbol{\varepsilon}_t|dy\\
&\leq \Gamma c^* C\int_\Omega \lambda(x-y)|m^{(1)}-m^{(2)}||\boldsymbol{\varepsilon}_t|(y,t)dy \ \leq\ \Lambda\Gamma c^* C|\boldsymbol{\varepsilon}_t(t)|_1|m^{(1)}(t)-m^{(2)}(t)|_\infty,
\end{aligned}
$$

that is, $\tilde{C} = \Lambda\Gamma c^* C$. On $[0, T^R]$, we have $A\mu(t) \leq 1-\delta$. Inequality (4.34) is thus of the form

$$\dot{v}(t) \leq (1-\delta)\dot{u}(t) + \alpha(t)u(t)\,, \tag{4.35}$$

where we set $v(t) = \int_0^t |\tilde{m}_t^{(1)}(\tau) - \tilde{m}_t^{(2)}(\tau)|_\infty d\tau$, $u(t) = \int_0^t |m_t^{(1)}(\tau) - m_t^{(2)}(\tau)|_\infty d\tau$, $\alpha(t) = A\dot{\mu}(t) + \tilde{C}|\boldsymbol{\varepsilon}_t(t)|_1$.

Put

$$\hat{\alpha}(t) = \frac{1}{\delta}\int_0^t \alpha(\tau)d\tau\,, \tag{4.36}$$

and test (4.35) by $\mathrm{e}^{-2\hat{\alpha}(t)}$. This yields

$$\int_0^{T^R} \mathrm{e}^{-2\hat{\alpha}(t)}\dot{v}(t)dt \leq \Big(1-\frac{\delta}{2}\Big)\int_0^{T^R} \mathrm{e}^{-2\hat{\alpha}(t)}\dot{u}(t)dt\,, \tag{4.37}$$

that is,

$$\int_0^{T^R} \mathrm{e}^{-2\hat{\alpha}(t)}|\tilde{m}_t^{(1)}(t) - \tilde{m}_t^{(2)}(t)|_\infty dt \leq \Big(1-\frac{\delta}{2}\Big)\int_0^{T^R} \mathrm{e}^{-2\hat{\alpha}(t)}|m_t^{(1)}(t) - m_t^{(2)}(t)|_\infty dt\,. \tag{4.38}$$

We see that the mapping that with m associates $\tilde{m}$ is a contraction on M_R, hence Eq. (4.23) has a unique solution m for every $\boldsymbol{\varepsilon} \in E_{T^R}$.

Eq. (4.17) with a given $\boldsymbol{\varepsilon} \in E_{T^R}$ is linear in w and the existence and uniqueness of a solution can be easily proved e.g. by Galerkin approximations. The required regularity follows by testing (4.17) successively by $\varphi = w_t$ and $\varphi = (I-\Delta)^{-1}w_{tt}$, using the assumption (4.22). To complete the proof, it remains to check that $\mathbf{D}_2 w \in E_{T^R}$. Choosing again $\varphi = w_t$ in (4.17) and using Hypothesis 4.1, we obtain

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\big(|w_t|_2^2 + |\nabla w_t|^2\big) + c_*|\mathbf{D}_2 w_t|_2^2 &\leq (|\theta|_2 + b_*|\boldsymbol{\varepsilon}|_2 + \Gamma)|\mathbf{D}_2 w_t|_2\\
&\leq \frac{1}{c_*}\big(p(t) + b_*^2|\boldsymbol{\varepsilon}|_2^2\big) + \frac{c_*}{2}|\mathbf{D}_2 w_t|_2^2,
\end{aligned} \tag{4.39}
$$

hence

$$\int_0^t |\mathbf{D}_2 w_t(\tau)|_2^2 d\tau \le \frac{2}{c_*^2} \int_0^t \big(p(\tau) + b_*^2 |\boldsymbol{\varepsilon}(\tau)|_2^2\big) d\tau. \tag{4.40}$$

We have $|\boldsymbol{\varepsilon}(x,t)| \le \int_0^t |\boldsymbol{\varepsilon}_t(x,\tau)| d\tau$ a.e., so that $|\boldsymbol{\varepsilon}(t)|_2^2 \le t \int_0^t |\boldsymbol{\varepsilon}_t(\tau)|_2^2 d\tau \le tq(t)$. From (4.40) it follows that

$$\int_0^t |\mathbf{D}_2 w_t(\tau)|_2^2 d\tau \le \frac{2}{c_*^2} \int_0^t \big(p(\tau) + b_*^2 \tau q(\tau)\big) d\tau = q(t), \tag{4.41}$$

which we wanted to prove. ∎

4.4 The coupled system

This section is devoted to the proof of Theorem 4.2. We start with an auxiliary result on the solution mapping of (4.2), (4.19), (4.23) which with a given $\boldsymbol{\varepsilon} \in E_{T^R}$ associates $m \in M_R$.

Lemma 4.5. *There exists a constant C_2 depending only on R and on the data of the problem such that for all $\boldsymbol{\varepsilon}^{(1)}, \boldsymbol{\varepsilon}^{(2)} \in E_{T^R}$, the corresponding solutions $m^{(1)}, m^{(2)} \in M_R$ to* (4.19)–(4.23) *satisfy the inequality*

$$\int_0^{T^R} |m_t^{(1)} - m_t^{(2)}|_\infty(t) dt \le C_2 \int_0^{T^R} |\boldsymbol{\varepsilon}_t^{(1)} - \boldsymbol{\varepsilon}_t^{(2)}|_2(t) dt\,. \tag{4.42}$$

Proof. With the notation of Section 4.3 we have

$$\begin{aligned} \delta |m_t^{(1)}(t) - m_t^{(2)}(t)|_\infty &\le S\dot{\mu}(t)\Big(|\boldsymbol{\varepsilon}^{(1)}(t) - \boldsymbol{\varepsilon}^{(2)}(t)|_2 + \int_0^t |\boldsymbol{\varepsilon}_t^{(1)} - \boldsymbol{\varepsilon}_t^{(2)}|_2(\tau) d\tau\Big) \\ &+ \alpha(t)|m^{(1)}(t) - m^{(2)}(t)|_\infty + \Lambda \int_0^\infty \gamma(0,r) \Big| \mathbf{C}\boldsymbol{\chi}_r^{(1)} \cdot \big(\boldsymbol{\xi}_r^{(1)}\big)_t - \mathbf{C}\boldsymbol{\chi}_r^{(2)} \cdot \big(\boldsymbol{\xi}_r^{(2)}\big)_t \Big|_1 (t) dr\,, \end{aligned} \tag{4.43}$$

where $S > 0$ is a constant, and where we have used the fact that $\gamma(m_2, r) \le \gamma(0,r)$ by Hypothesis 4.1 (iv). Testing (4.43) by $\mathrm{e}^{-\hat{\alpha}(t)}$, with $\hat{\alpha}$ from (4.36), yields that

$$\begin{aligned} &\delta \frac{d}{dt}\Big(\mathrm{e}^{-\hat{\alpha}(t)} \int_0^t |m_t^{(1)} - m_t^{(2)}|_\infty(\tau) d\tau\Big) \le \mathrm{e}^{-\hat{\alpha}(t)}\Big(S\dot{\mu}(t)\Big(|\boldsymbol{\varepsilon}^{(1)}(t) - \boldsymbol{\varepsilon}^{(2)}(t)|_2 \\ &+ \int_0^t |\boldsymbol{\varepsilon}_t^{(1)} - \boldsymbol{\varepsilon}_t^{(2)}|_2(\tau) d\tau\Big) + \Lambda \int_0^\infty \gamma(0,r) \Big| \mathbf{C}\boldsymbol{\chi}_r^{(1)} \cdot \big(\boldsymbol{\xi}_r^{(1)}\big)_t - \mathbf{C}\boldsymbol{\chi}_r^{(2)} \cdot \big(\boldsymbol{\xi}_r^{(2)}\big)_t \Big|_1 (t) dr\Big). \end{aligned} \tag{4.44}$$

Integrating from 0 to t and using (3.27) we obtain the assertion. ∎

Lemma 4.6. *The mapping defined in Proposition 4.4 that with $\boldsymbol{\varepsilon} \in E_{T^R}$ associates $\mathbf{D}_2 w \in E_{T^R}$ is a contraction with respect to a suitable norm.*

Proof. We test the difference of Eqs. (4.17) written for $\boldsymbol{\varepsilon}^{(1)}, \boldsymbol{\varepsilon}^{(2)}$ and the corresponding solutions $w^{(1)}, w^{(2)}$ by $\bar{w}_t = w_t^{(1)} - w_t^{(2)}$, and obtain

$$\frac{d}{dt}\big(|\bar{w}_t|_2^2 + |\nabla \bar{w}_t|^2\big) + c_* |\mathbf{D}_2 \bar{w}_t|_2^2 \le \frac{1}{c_*}\big|\mathcal{P}[m^{(1)}, \boldsymbol{\varepsilon}^{(1)}] - \mathcal{P}[m^{(2)}, \boldsymbol{\varepsilon}^{(2)}]\big|_2^2. \tag{4.45}$$

We have

$$|\mathcal{P}[m^{(1)}, \boldsymbol{\varepsilon}^{(1)}] - \mathcal{P}[m^{(2)}, \boldsymbol{\varepsilon}^{(2)}]|_2(t) \leq C_3\Big(|m_1 - m_2|_\infty(t) + |\boldsymbol{\varepsilon}^{(1)} - \boldsymbol{\varepsilon}^{(2)}|_2(t) + \int_0^t |\boldsymbol{\varepsilon}_t^{(1)} - \boldsymbol{\varepsilon}_t^{(2)}|_2(\tau)d\tau\Big), \tag{4.46}$$

hence, by Lemma 4.5,

$$\frac{d}{dt}\big(|\bar{w}_t|_2^2 + |\nabla \bar{w}_t|^2\big) + |\mathbf{D}_2 \bar{w}_t|_2^2 \leq C_4 \int_0^t |\bar{\boldsymbol{\varepsilon}}_t|_2^2(\tau)d\tau \tag{4.47}$$

with a constant $C_4 > 0$, and with the notation $\bar{\boldsymbol{\varepsilon}} = \boldsymbol{\varepsilon}^{(1)} - \boldsymbol{\varepsilon}^{(2)}$. Testing (4.47) by $\mathrm{e}^{-2C_4 t}$ and integrating from 0 to T^R, we get the inequality

$$\int_0^{T^R} \mathrm{e}^{-2C_4 t}|\mathbf{D}_2 \bar{w}_t|_2^2 dt \leq \frac{1}{2}\int_0^{T^R} \mathrm{e}^{-2C_4 t}|\bar{\boldsymbol{\varepsilon}}_t|_2^2(t)dt\,, \tag{4.48}$$

which completes the proof. ■

We are now ready to finish the proof of Theorem 4.2.

Proof of Theorem 4.2. It suffices to combine Proposition 4.4 with Lemma 4.6 and apply the contraction principle. ■

4.5 Proof of Theorem 4.3

The main goal of this section is to remove the cut-off function Q_R in (4.23). This will be done by establishing additional estimates, where the dependence on R is explicitly taken into account. The constants $C_5, \dots, C_{10}$ which appear in the formulas below are independent of R.

We assume here the additional regularity $\theta_t \in L^2(\Omega_T)$ of the right hand side of (4.17). Testing Eq. (4.17) by w_{tt} (for the Galerkin approximations first, and then passing to the limit) and integrating by parts we obtain

$$\begin{aligned}\int_0^t \big(|w_{tt}|_2^2 + |\nabla w_{tt}|_2^2\big)(\tau)d\tau + |\mathbf{D}_2 w_t(t)|_2^2 \qquad\qquad\qquad\qquad\qquad\qquad \\ \leq C_5\left(|\theta(t)|_2^2 + |\mathcal{P}[m,\boldsymbol{\varepsilon}](t)|_2^2 + \Big(\int_0^t \big(|\theta_t|_2^2 + |\mathcal{P}[m,\boldsymbol{\varepsilon}]_t|_2^2\big)(\tau)d\tau\Big)^{1/2}\right).\end{aligned} \tag{4.49}$$

Using the inequality

$$|\mathcal{P}[m,\boldsymbol{\varepsilon}]_t|_2 \leq C_6\big(|m_t|_\infty + |\boldsymbol{\varepsilon}_t|_2\big),$$

we infer from (4.49) that

$$|\mathbf{D}_2 w_t(t)|_2^2 \leq C_7(1+R) \tag{4.50}$$

for all $t \in [0, T^R]$. On the other hand, we have

$$\Big|\int_\Omega \lambda(x-y)\int_0^\infty \gamma(m,r)\,\mathbf{C}\boldsymbol{\chi}_r \cdot (\boldsymbol{\xi}_r)_t dr dy\Big| \leq C_8|\mathbf{D}_2 w_t(t)|_1 \leq C_9\sqrt{1+R}\,.$$

Choosing $R = R^*$ sufficiently large, we see that the truncation in (4.23) is never active in the interval $[0, T^{R^*}] =: [0, T^*]$, and the solution to (4.17)–(4.19), (4.21) (4.23) in fact satisfies (4.17)–(4.21) as well. The additional regularity follows from (4.49).

To complete the proof of Theorem 4.3, it remains to prove inequality (4.24). As in the proof of Lemma 4.6, we take the differences of Eqs. (4.17) written for $\boldsymbol{\varepsilon}^{(1)}, \boldsymbol{\varepsilon}^{(2)}$ and the corresponding solutions $w^{(1)}, w^{(2)}$, this time with possibly different $\theta^{(1)}$ and $\theta^{(2)}$. We obtain

$$\frac{d}{dt}\left(|\bar{w}_t|_2^2 + |\nabla \bar{w}_t|^2\right) + c_*|\mathbf{D}_2\bar{w}_t|_2^2 \le \frac{1}{c_*}\left(|\bar{\theta}|_2^2 + |\mathcal{P}[m^{(1)}, \boldsymbol{\varepsilon}^{(1)}] - \mathcal{P}[m^{(2)}, \boldsymbol{\varepsilon}^{(2)}]|_2^2\right).$$

Exploiting (4.46) and Lemma 4.5, we obtain, for a constant $C_{10} > 0$, that

$$\begin{aligned}\frac{d}{dt}\left(|\bar{w}_t|_2^2 + |\nabla \bar{w}_t|^2\right)(t) + c_*|\mathbf{D}_2\bar{w}_t|_2^2(t) \;&\le\; C_{10}\left(|\bar{\theta}|_2^2(t) + \int_0^t |\bar{\boldsymbol{\varepsilon}}_t|_2^2(\tau)d\tau\right)\\ &\le\; C_{10}\left(|\bar{\theta}|_2^2(t) + \int_0^t |\mathbf{D}_2\bar{w}_t|_2^2(\tau)\,d\tau\right),\end{aligned}$$

where in the last line we used (4.18). The Gronwall lemma now yields the assertion.

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