INSTITUTE OF MATHEMATICS

THE CZECH ACADEMY OF SCIENCES

On the compressible Navier-Stokes-Cahn-Hilliard equations with dynamic boundary conditions

Laurence Cherfils
Eduard Feireisl
Martin Michálek
Alain Miranville
Madalina Petcu
Dalibor Pražák

Preprint No. 56-2018

PRAHA 2018

ON THE COMPRESSIBLE NAVIER-STOKES-CAHN-HILLIARD EQUATIONS WITH DYNAMIC BOUNDARY CONDITIONS

LAURENCE CHERFILS[#], EDUARD FEIREISL[♭,▽], MARTIN MICHÁLEK[♭], ALAIN MIRANVILLE[♯,*,**], MADALINA PETCU[◇,♯], AND DALIBOR PRAŽÁK[†]

[#]Laboratoire des Sciences de l'Ingénieur pour l'Environnement
Université de La Rochelle, UMR CNRS 7356, F-17042 La Rochelle cedex, France

[♭]Institute of Mathematics of the Academy of Sciences of the Czech Republic
Žitná 25, CZ-115 67 Praha 1, Czech Republic

[▽] Institut für Mathematik, Technische Universitaet Berlin
Strasse des 17. Juni 136, D - 10623 Berlin, Germany

[♯]Laboratoire de Mathématiques et Applications, UMR CNRS 7348 - SP2MI
Université de Poitiers, Boulevard Marie et Pierre Curie - Téléport 2
86962 Chasseneuil, Futuroscope Cedex, France

[*]Fudan University (Fudan Fellow), Shanghai, China

[**]School of Mathematical Sciences, Xiamen University, Xiamen, Fujian, China

[◇]The Institute of Mathematics of the Romanian Academy, Bucharest, Romania
and
The Institute of Statistics and Applied Mathematics of the Romanian Academy
Bucharest, Romania

[†] Department of Mathematical Analysis, Charles University
Sokolovská 83, CZ-186 75 Praha, Czech Republic

Abstract. We consider the compressible Navier-Stokes-Cahn-Hilliard system describing the behavior of a binary mixture of compressible, viscous and macroscopically immiscible fluids. The equations are endowed with dynamic boundary conditions which allows taking into account the interaction between the fluid components and the rigid walls of the physical domain. We establish the existence of global–in–time weak solutions for any finite energy initial data.

1991 *Mathematics Subject Classification.* 35Q30, 35Q35, 76N10, 76T99.

Key words and phrases. Compressible Navier-Stokes-Cahn-Hilliard, two-phase flow, dynamic boundary conditions.

1. Introduction

The motion of an isothermal mixture of two immiscible and incompressible fluids subject to phase separation can be described by the Navier-Stokes equations for the average fluid velocity (nonlinearly) coupled with the convective Cahn-Hilliard equation. The latter describes the evolution of the difference of the relative concentrations of the two fluids. This model is known as model H, but it is also usually called Cahn-Hilliard-Navier-Stokes system (see, e.g., [3], [25], [26], [31]).

In this paper, we are actually interested in a variant of such a model. More precisely, we consider a binary mixture of immiscible compressible fluids occupying a bounded domain $\Omega \subset \mathbb{R}^3$. The time evolution of the mass density $\varrho = \varrho(t,x)$, the bulk velocity $\mathbf{u} = \mathbf{u}(t,x)$ and the concentration difference $c = c(t,x)$ is governed by the system of partial differential equations:

$$\partial_t \varrho + \operatorname{div}_x(\varrho \mathbf{u}) = 0, \tag{1a}$$

$$\partial_t(\varrho \mathbf{u}) + \operatorname{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho, c) = \operatorname{div}_x \mathbb{S}(c, \nabla_x \mathbf{u}) - \operatorname{div}_x \left(\nabla_x c \otimes \nabla_x c - \frac{1}{2} |\nabla_x c|^2 \mathbb{I} \right), \tag{1b}$$

$$\partial_t(\varrho c) + \operatorname{div}_x(\varrho c \mathbf{u}) = \triangle_x \mu, \tag{1c}$$

$$\varrho \mu = \varrho \frac{\partial}{\partial c} f - \triangle_x c, \tag{1d}$$

where μ denotes the chemical potential and $f = f(\varrho, c)$ is the free energy density, the precise form of which is specified later. Here and within the paper we use the classical notation for the scalar product of tensors $\mathbb{A} = \{A_{i,j}\}_{i,j}$ and $\mathbb{B} = \{B_{i,j}\}_{i,j}$:

$$\mathbb{A} : \mathbb{B} = \sum_{i,j} A_{i,j} B_{i,j},$$

while $\mathbf{a} \otimes \mathbf{b}$ denotes the tensor product of two vectors $\mathbf{a}$ and $\mathbf{b}$:

$$\mathbf{a} \otimes \mathbf{b} = \{a_i b_j\}_{i,j}.$$

The equations are supposed to hold for $(t,x) \in (0,T) \times \Omega$.

1.1. Boundary and initial conditions.

We now consider an immiscible two-phase flow in which one fluid displaces the other along the boundary $\Gamma := \partial\Omega$.

It was observed (see, e.g., [9]), in the incompressible case, that the moving contact line, defined as the intersection of the fluid-fluid interface with the solid wall, is incompatible with the no-slip boundary condition (see [10], [34] and the references therein). Indeed, as shown in [10], under the usual hydrodynamic assumptions, namely, incompressible Newtonian fluids, no-slip boundary condition and smooth rigid walls, there is a velocity discontinuity at the moving contact line and the tangential force exerted by the fluids on the solid surface Γ in the vicinity of the contact line becomes infinite. Thus, in immiscible two-phase flows, none of the standard boundary conditions can account for the moving contact line slip velocity profiles obtained from simulations and, therefore, new boundary conditions were required to describe the observed phenomena.

In order to account for moving contact lines a generalization of the Navier boundary conditions has been proposed in [38] (see also [37]) using the laws of thermodynamics and variational principles related to the minimum energy dissipation. These laws state that the entropy associated with the composition diffusion and the work done by the flow to the fluid-fluid interface are conserved.

Following this approach, now in the compressible case, and assuming the boundary Γ sufficiently smooth, we impose the generalized Navier boundary conditions[1]

$$\left.\begin{aligned} \mathbf{u}\cdot\mathbf{n} &= 0 \\ (\mathbb{S}(c,\nabla_x\mathbf{u})\mathbf{n})_\tau + \beta\mathbf{u}_\tau &= \mathcal{L}(c)\nabla_\tau c \end{aligned}\right\} \text{ on } \Gamma, \tag{2a}$$

where $\beta > 0$, together with the Neumann boundary condition for the chemical potential

$$\nabla_x\mu\cdot\mathbf{n} = 0, \text{ on } \Gamma.$$

The main novelty with respect to the existing literature on mixtures of *compressible* fluids are the dynamic boundary conditions for c:

$$\left.\begin{aligned} \partial_t c + \mathbf{u}_\tau\nabla_\tau c &= -\mathcal{L}(c) \\ \mathcal{L}(c) &= -\triangle_\tau c + \xi c + k(c) + \partial_{\mathbf{n}} c \end{aligned}\right\} \text{ on } \Gamma \tag{2b}$$

where $\xi > 0$ is a constant and k a suitable nonlinear function to be specified later. Such boundary condition can be interpreted as a parabolic equation on Γ. Note that viscous fluids are usually supposed to adhere completely to the physical boundary, which gives rise to the standard no-slip condition. Nevertheless, in [35], the so-called generalized Navier boundary conditions were introduced as a more realistic partial slip condition, where $\mathbf{u}_\tau$ is the slip velocity at the boundary measured tangentially to the wall and β is a slip coefficient, see Heida, Málek and Rajagopal [26] for a thorough discussion of this issue.

As mentioned above, it is known that the moving contact line defined at the intersection of the fluid-fluid interface with the wall is incompatible with the standard boundary conditions meaning Neumann boundary condition for the concentration (note that this leads to a static contact line with contact angle $\frac{\pi}{2}$) and no-slip boundary condition for the velocity. In [20] the dynamic boundary conditions that we consider here were introduced and well-posedness for the Cahn-Hilliard-Navier-Stokes incompressible model was proved.

We also mention that different types of dynamic boundary conditions were considered for the numerical study of the incompressible Cahn-Hilliard-Navier-Stokes equations, see, e.g., [6], [30], [40], [41], [44].

Finally, the system is supplemented with the initial conditions:

$$\left.\begin{aligned} \varrho(0,x) &= \varrho_0(x) \\ \varrho\mathbf{u}(0,x) &= \mathbf{m}_0(x) \\ \varrho c(0,x) &= \varrho_0 c_0(x) \end{aligned}\right\} \text{ for } x\in\Omega \tag{3a}$$

[1]Considering $\Gamma\subset\mathbb{R}^3$ a two dimensional surface and $\mathbf{v}$ a vector field, we denote by $\mathbf{v}_\tau(x)$ the orthogonal projection of $\mathbf{v}(x)$ on the tangential plane to Γ at $x\in\Gamma$. As Γ is smooth there exists an outer normal vector $\mathbf{n}(x)$ at each point x. We denote by ∇_τ the tangential gradient operator and by $\triangle_\tau$ the Laplace-Beltrami operator where both make sense at a given point with respect to a two dimensional surface (Riemannian manifold). The surface measure will be denoted by $\mathcal{H}^2$.

and

$$c(0,x) = c_0(x) \text{ for } x \in \Gamma, \tag{3b}$$

where we have introduced the momentum $\mathbf{m} = \varrho\mathbf{u}$.

1.2. **Constitutive relations.** The viscous stress $\mathbb{S}$ is given by Newton's rheological law:

$$\mathbb{S} = \mathbb{S}(c, \nabla_x \mathbf{u}) = \nu(c)\left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3}\operatorname{div}_x \mathbf{u}\,\mathbb{I}\right) + \eta(c)\operatorname{div}_x \mathbf{u}\,\mathbb{I},$$

where the viscosity coefficients $\nu > 0$, $\eta \geq 0$ are continuously differentiable functions of c satisfying:

$$0 < \underline{\nu} \leq \nu(c) \leq \overline{\nu}, \quad 0 \leq \eta(c) \leq \overline{\eta}, \, \forall\, c.$$

The interface between the fluids is modeled via the diffused interface approximation and corresponds to the total free energy in the form:

$$E_{free}(\varrho, c)|_t = \int_\Omega \varrho(t,x) f(\varrho(t,x), c(t,x)) + \frac{1}{2}|\nabla_x c(t,x)|^2 \,\mathrm{d}x. \tag{4}$$

Moreover, we suppose

$$f = f(\varrho, c) = f_e(\varrho) + H(c)\log\varrho + G(c),$$

where H and G are continuously differentiable functions satisfying

$$-H_1 \leq H'(c),\, H(c) \leq H_2,\, c \in \mathbb{R}, \text{ with } H_1,\, H_2 > 0, \tag{5}$$

$$G_1 c - G_2 \leq G'(c) \leq G_3(1+c),\, c \in \mathbb{R}, \text{ with } G_1,\, G_2, G_3 > 0. \tag{6}$$

Note that, due to the assumptions on G and H, $\varrho\log(\varrho)H(c) + \varrho G(c)$ is bounded from below for $(\varrho, c) \in (0,\infty)\times\mathbb{R}$.

The pressure is interrelated to f via

$$p(\varrho, c) = \varrho^2 \frac{\partial f(\varrho, c)}{\partial \varrho} = p_e(\varrho) + \varrho H(c), \tag{7}$$

where

$$p_1 \varrho^{\gamma-1} - p_2 \leq p_e'(\varrho) \leq p_3(1 + \varrho^{\gamma-1}) \tag{8}$$

for a certain $\gamma > 3/2$ and $p_1,\, p_2,\, p_3 > 0$.

Finally, k is a nonlinear function which accounts for the interfacial energy at the mixture-wall interface. The typical choice of k in the context of dynamic boundary conditions for the Cahn-Hilliard-Navier-Stokes model is:

$$k(\vartheta) = -\frac{\sqrt{2}}{6}\cos\vartheta_s \cos(\frac{\pi}{2}\vartheta),$$

where ϑ_s is the static contact angle between the fluid-fluid interface and the wall. We denote by K a fixed primitive function of k satisfying $K \geq 0$.

1.3. **Main result.** To the best of our knowledge, this is the first mathematical study of the compressible Navier-Stokes-Cahn-Hilliard system with dynamic boundary conditions. The existence theory for the system (1a)-(1d) was developed in [1] for the model with "passive" boundary conditions (namely, homogeneous Dirichlet conditions for the velocity field $\mathbf{u}$ and the homogeneous Neumann conditions for the chemical potential μ and concentration c). As the main result, the authors managed to prove the existence of global weak solutions to the model (1-4).

Recently, the dynamic boundary conditions have attracted an interest in the area of phase field models being first introduced in order to take into account the interaction between a binary material and the walls of the physical domain. A large amount of mathematical literature exists on this subject, we refer the interested reader to, e.g., [17], [18], [19], [24], [27], [32], [33], [36], [39], [43]. This kind of dynamic boundary conditions has been coupled to the incompressible Navier-Stokes model and a phase field models in several works, see e.g. [5], [45], [20].

In general, the path to the proof of the existence of weak solutions (especially in the case of compressible Navier-Stokes system) is technical and involves several levels of approximation (see e.g. [1], [14], [15]). Following the approach advocated by J.-L.Lions [28], we establish existence of global-in-time weak solutions performing several steps:

- collecting sufficiently strong *a priori* estimates;
- showing compactness of the set of weak (or strong) solutions in the spaces pertinent to *a priori* bounds;
- constructing a suitable sequence of approximating solutions that converges towards the solution of the problem.

1.3.1. *Weak solutions.* Before stating our main result, we introduce the concept of *weak solution* to the problem (1-4). Here and in what follows we use the following notation:

$$W^{1,2}(\Omega\times\Gamma)=\{f\in W^{1,2}(\Omega)\text{ with trace }f\in W^{1,2}(\Gamma)\}.$$

Definition 1.1. Let $\varrho_0\in L^\gamma(\Omega)$, $\varrho_0>0$, $\mathbf{m}_0$, $|\mathbf{m}_0|^2/\varrho_0\in L^1(\Omega)$, $c_0\in W^{1,2}(\Omega\times\Gamma)$ be the initial data.

We say that $(\varrho,\mathbf{u},c,\mu)$ is a *weak solution* to the initial boundary value problem (1), (2) and (3) if:

- $\varrho\in L^\infty(0,T,L^\gamma(\Omega))$, $\varrho\geq 0$, $\mathbf{u}\in L^2(0,T,W^{1,2}(\Omega;\mathbb{R}^3))$, $c\in L^\infty(0,T,W^{1,2}(\Omega\times\Gamma))$, $\mathcal{L}(c)\in L^2(0,T,L^2(\Gamma))$ and $\mu\in L^2(0,T,W^{1,2}(\Omega))$;
- ϱ is a renormalized solution of the continuity equation (1a), i.e.

(9a)
$$\int_0^T\int_\Omega \varrho B(\varrho)\partial_t\varphi\,\mathrm{d}x+\int_0^T\int_\Omega \varrho B(\varrho)\mathbf{u}\cdot\nabla_x\varphi\,\mathrm{d}x\,\mathrm{d}t=\int_0^T\int_\Omega b(\varrho)\,\mathrm{div}_x\,\mathbf{u}\,\varphi\,\mathrm{d}x\,\mathrm{d}t-\int_\Omega B(\varrho_0)\varphi(0,\cdot)\,\mathrm{d}x,$$

for every $\varphi\in C_c^1([0,T)\times\overline{\Omega})$ and any

(9b)
$$B(\varrho)=B(1)+\int_1^\varrho\frac{b(z)}{z^2}\,\mathrm{d}z$$

where $b\in\mathcal{C}([0,\infty))$ is a bounded function;

- for every $\varphi \in C_c^1([0,T)\times\overline{\Omega};\mathbb{R}^3)$ such that $\varphi\cdot\mathbf{n}=0$ on $(0,T)\times\Gamma$

$$\begin{aligned}&\int_0^T\int_\Omega \varrho\mathbf{u}\cdot\partial_t\varphi\,\mathrm{d}x\,\mathrm{d}t+\int_0^T\int_\Omega \varrho\mathbf{u}\otimes\mathbf{u}\colon\nabla_x\varphi\,\mathrm{d}x\,\mathrm{d}t+\int_0^T\int_\Omega p(\varrho,c)\,\mathrm{div}_x\,\varphi\,\mathrm{d}x\,\mathrm{d}t\\ &-\int_0^T\int_\Omega \mathbb{S}(c,\mathbf{u}):\nabla_x\varphi\,\mathrm{d}x\,\mathrm{d}t+\int_0^T\int_\Omega\left(\nabla_x c\otimes\nabla_x c-\frac{1}{2}|\nabla_x c|^2\mathbb{I}\right):\nabla_x\varphi\,\mathrm{d}x\,\mathrm{d}t\\ &-\int_0^T\int_\Gamma(\beta\mathbf{u}_\tau-(\mathcal{L}(c)-\partial_{\mathbf{n}}c)\nabla_\tau c)\cdot\varphi_\tau\,\mathrm{d}\mathcal{H}^2\,\mathrm{d}t=0-\int_\Omega\mathbf{m}_0\cdot\varphi(0,\cdot)\,\mathrm{d}x;\end{aligned}\tag{9c}$$

- for every $\varphi\in C_c^1([0,T)\times\overline{\Omega})$

$$\int_0^T\int_\Omega \varrho c\partial_t\varphi\,\mathrm{d}x\,\mathrm{d}t+\int_0^T\int_\Omega \varrho c\mathbf{u}\cdot\nabla_x\varphi\,\mathrm{d}x\,\mathrm{d}t-\int_0^T\int_\Omega\nabla_x\mu\cdot\nabla_x\varphi\,\mathrm{d}x\,\mathrm{d}t=-\int_\Omega\varrho_0c_0\varphi(0,\cdot)\,\mathrm{d}x;\tag{9d}$$

-

$$\mu\varrho=\varrho\partial_c f(\varrho,c)-\triangle_x c\quad\text{almost everywhere in }(0,T)\times\Omega;\tag{9e}$$

and

$$\mathcal{L}(c)=-\Delta_\tau c+\xi c+k(c)+\partial_{\mathbf{n}}c\quad\text{almost everywhere in }(0,T)\times\Gamma;\tag{9f}$$

- for every $\eta\in\mathcal{C}_c^1([0,T),H^1(\Gamma))$

$$\begin{aligned}&-\int_0^T\int_\Gamma c\partial_t\eta\,\mathrm{d}\mathcal{H}^2\,\mathrm{d}t+\int_0^T\int_\Gamma\mathbf{u}_\tau\cdot\nabla_\tau c\eta\,\mathrm{d}\mathcal{H}^2\,\mathrm{d}t+\int_0^T\int_\Gamma\nabla_\tau c\cdot\nabla_\tau\eta\,\mathrm{d}\mathcal{H}^2\,\mathrm{d}t\\ &+\int_0^T\int_\Gamma\partial_{\mathbf{n}}c\eta\,\mathrm{d}\mathcal{H}^2\,\mathrm{d}t+\xi\int_0^T\int_\Gamma c\eta\,\mathrm{d}\mathcal{H}^2\,\mathrm{d}t+\int_0^T\int_\Gamma k(c)\eta\,\mathrm{d}\mathcal{H}^2\,\mathrm{d}t=\int_\Gamma c_0\eta(0,\cdot)\,\mathrm{d}\mathcal{H}^2;\end{aligned}\tag{9g}$$

- the energy inequality

$$E_{bulk}(\varrho,\mathbf{u},c)|_t+E_{surf}(c)|_t+\int_0^t\mathcal{D}_{bulk}(\mathbf{u},\mu)|_\tau+\mathcal{D}_{surf}(\mathbf{u},c)|_\tau\,\mathrm{d}\tau\le E_0,\tag{9h}$$

holds for almost every $t\in(0,T)$, where we have set

$$E_{bulk}(\varrho,c,\mathbf{u})|_t=E_{free}(\varrho,c)|_t+\int_\Omega\frac{1}{2}\varrho(t,x)|\mathbf{u}(t,x)|^2\,\mathrm{d}x,\tag{9i}$$

$$\mathcal{D}_{bulk}(\varrho,c,\mathbf{u},\mu)|_t=\int_\Omega\mathbb{S}(t,x)\colon\nabla_x\mathbf{u}(t,x)+|\nabla_x\mu(t,x)|^2\,\mathrm{d}x,\tag{9j}$$

$$E_{surf}(c)|_t=\int_\Gamma\frac{1}{2}|\nabla_\tau c(t,y)|^2+\frac{1}{2}\xi|c(t,y)|^2+K(c(t,y))\,\mathrm{d}\mathcal{H}^2(y)\tag{9k}$$

$$\mathcal{D}_{surf}(\mathbf{u},c)|_t=\int_\Gamma\beta|\mathbf{u}_\tau(t,y)|^2+|\mathcal{L}(c)(t,y)|^2\,\mathrm{d}\mathcal{H}^2(y),\tag{9l}$$

and

$$E_0:=\int_\Omega\frac{1}{2}\frac{|\mathbf{m}_0|^2}{\varrho_0}+\frac{1}{2}|\nabla_x c_0|^2+\varrho_0 f(\varrho_0,c_0)\,\mathrm{d}x+\int_\Gamma\frac{1}{2}|\nabla_\tau c_0|^2+\frac{1}{2}\xi|c_0|^2+K(c_0)\,\mathrm{d}\mathcal{H}^2.$$

1.3.2. *Global–in–time existence.* Having collected all the necessary material, we are ready to state our main result.

Theorem 1.1. *Suppose that $\Omega \subset \mathbb{R}^3$ is a smooth bounded domain. Let the structural hypotheses specified in Section 1.2 be satisfied with $\gamma > \frac{3}{2}$. Let the initial data $\varrho_0 \in L^\gamma(\Omega)$, $\varrho_0 > 0$, $|\mathbf{m}_0|^2/\varrho_0 \in L^1(\Omega)$, $c_0 \in W^{1,2}(\Omega \times \Gamma)$, and $T > 0$ be given.*

Then the Navier-Stokes-Cahn-Hilliard system admits a weak solution in $(0,T) \times \Omega$ in the sense specified in Definition 1.1.

The rest of the paper is devoted to the proof of Theorem 1.1. Following the general strategy delineated above, we first establish the necessary *a priori* bounds in Section 2. In Section 3, we show compactness - any sequence of weak (or strong) solutions contains a subsequence converging to another solution of the same problem. Finally, in Section 4, we propose an approximation scheme and show the main steps in the proof of existence.

2. A priori estimates

We start by collecting *a priori* estimates on hypothetical smooth solutions $(\varrho, \mathbf{u}, c, \mu)$ depending only on the initial conditions ϱ_0, $\mathbf{m}_0$, c_0. We assume that $\varrho_0 > 0$.

2.1. **Continuity equation.** As $\mathbf{u}$ is smooth, the continuity equation (1a) can be solved by the method of trajectories. Indeed, taking into account that the trajectories X of hypothetical fluid particles can be obtained solving the ordinary differential equation

$$\frac{dX}{dt}(t) = \mathbf{u}(t, X(t)),$$

we deduce that ϱ satisfies the ordinary differential equation:

$$\frac{d}{dt}\varrho(t, X(t)) = -\varrho(t, X(t)) \operatorname{div}_x \mathbf{u}(t, X(t)).$$

Hence:

$$\inf_{y \in \Omega} \varrho_0(y) e^{-\int_0^t \| \operatorname{div}_x \mathbf{u}(s,\cdot)\|_{L^\infty} \, \mathrm{d}s} \leq \varrho(t,x) \leq \sup_{y \in \Omega} \varrho_0(y) e^{\int_0^t \| \operatorname{div}_x \mathbf{u}(s,\cdot)\|_{L^\infty} \, \mathrm{d}s}. \tag{10}$$

This relation implies the positivity of the density provided $\operatorname{div}_x \mathbf{u} \in L^1(0,T,L^\infty(\Omega))$ and ϱ_0 is positive. Unfortunately the regularity required is not available for the weak solutions of our problem. Indeed, we will see later on that the available energy estimates imply only that $\mathbf{u} \in L^2(0,T,W^{1,2}(\Omega))$. Nevertheless, this discussion leads us to the conclusion that we may expect $\varrho \geq 0$ for a.a. $(t,x) \in (0,T) \times \Omega$, provided that the initial density ϱ_0 is non-negative.

Moreover, if we integrate the continuity equation over Ω and use the fact that the normal velocity vanishes on the boundary, we obtain the conservation of mass:

$$\int_\Omega \varrho(t,x) \, \mathrm{d}x = \int_\Omega \varrho_0(x) \, \mathrm{d}x =: M_0 \quad \text{for all } t \in (0,T).$$

2.2. **Total energy balance.** Taking the scalar product in $\mathbb{R}^3$ of the momentum equation (1b) with $\mathbf{u}$ and combining with the resulting equation obtained from (1a) multiplied by $\frac{1}{2}|\mathbf{u}|^2$, we obtain:

$$\partial_t\left(\frac{1}{2}\varrho|\mathbf{u}|^2\right)+\operatorname{div}_x\left(\frac{1}{2}\varrho|\mathbf{u}|^2\mathbf{u}+p\mathbf{u}-\mathbb{S}\cdot\mathbf{u}\right)+\mathbb{S}\colon\nabla_x\mathbf{u}=p\operatorname{div}_x\mathbf{u}-\triangle_x c\nabla_x c\cdot\mathbf{u}, \tag{11}$$

where we also used equality $\operatorname{div}_x\left(\nabla_x c\otimes\nabla_x c-\frac{1}{2}|\nabla_x c|^2\mathbb{I}\right)=\triangle_x c\nabla_x c$. Next, we multiply (1c) by μ and use (1a) to get

$$\mu\varrho\left(\partial_t c+\mathbf{u}\cdot\nabla_x c\right)=\operatorname{div}_x(\nabla_x\mu\mu)-|\nabla_x\mu|^2. \tag{12}$$

Using (1d), we rearrange the previous equation as follows:

$$\varrho\partial_c f\left(\partial_t c+\mathbf{u}\cdot\nabla_x c\right)=\operatorname{div}_x(\mu\nabla_x\mu)-|\nabla_x\mu|^2+\triangle_x c\partial_t c+\triangle_x c\nabla_x c\cdot\mathbf{u}. \tag{13}$$

By applying the chain rule on the left hand side of (13) we obtain an equivalent form:

$$\begin{aligned}&\partial_t(\varrho f(\varrho,c))+\operatorname{div}_x(\varrho f(\varrho,c)\mathbf{u})-\varrho\partial_\varrho f(\varrho,c)\left(\partial_t\varrho+\nabla_x\varrho\cdot\mathbf{u}\right)\\&\quad=\operatorname{div}_x(\mu\nabla_x\mu)-|\nabla_x\mu|^2+\operatorname{div}_x(\partial_t c\nabla_x c)-\partial_t\left(\frac{1}{2}|\nabla_x c|^2\right)+\triangle_x c\nabla_x c\cdot\mathbf{u}.\end{aligned} \tag{14}$$

As

$$-\varrho\partial_\varrho f(\varrho,c)\left(\partial_t\varrho+\nabla_x\varrho\cdot\mathbf{u}\right)=\varrho^2\partial_\varrho f(\varrho,c)\operatorname{div}_x\mathbf{u}=p(\varrho,c)\operatorname{div}_x\mathbf{u},$$

we can combine (14) with (11) and derive the following pointwise energy balance:

$$\begin{aligned}&\partial_t\left(\frac{1}{2}\varrho|\mathbf{u}|^2+\frac{1}{2}|\nabla_x c|^2+\varrho f(\varrho,c)\right)\\&\quad+\operatorname{div}_x\left(\frac{1}{2}\varrho|\mathbf{u}|^2\mathbf{u}+\varrho f\mathbf{u}+p\mathbf{u}-\mu\nabla_x\mu-\partial_t c\nabla_x c-\mathbb{S}\cdot\mathbf{u}\right)+\mathbb{S}\colon\nabla_x\mathbf{u}+|\nabla_x\mu|^2=0.\end{aligned} \tag{15}$$

Next, we integrate (15) over Ω, use the Stokes theorem, and taking into account boundary conditions $\mathbf{u}\cdot\mathbf{n}=0$ and $\nabla_x\mu\cdot\mathbf{n}=0$, we obtain the following balance law for the bulk energy:

$$\frac{\mathrm{d}}{\mathrm{d}t}E_{bulk}+\mathcal{D}_{bulk}=\int_\Gamma\partial_t c\partial_{\mathbf{n}}c\,\mathrm{d}\mathcal{H}^2+\int_\Gamma\mathbf{u}^t\mathbb{S}\mathbf{n}\,\mathrm{d}\mathcal{H}^2 \tag{16}$$

with the bulk energy E_{bulk} and the bulk dissipation $\mathcal{D}_{bulk}$ defined in (9i) and (9j) above. We remark that the bulk energy is a non-negative quantity, as one can show that

$$\mathbb{S}\colon\nabla_x\mathbf{u}=\frac{\nu(c)}{2}\left|\nabla_x\mathbf{u}+\nabla_x^t\mathbf{u}-\frac{2}{3}\operatorname{div}_x\mathbf{u}\mathbb{I}\right|^2+\eta(c)|\operatorname{div}_x\mathbf{u}|^2.$$

Observe that due to the boundary conditions

$$\begin{aligned}\int_\Gamma\mathbf{u}^t\mathbb{S}\mathbf{n}\,\mathrm{d}\mathcal{H}^2&=\int_\Gamma\mathbf{u}^t\left([\mathbb{S}\mathbf{n}]_\tau+\left(\mathbf{n}^t\mathbb{S}\mathbf{n}\right)\mathbf{n}\right)\mathrm{d}\mathcal{H}^2=\int_\Gamma\mathbf{u}^t[\mathbb{S}\mathbf{n}]_\tau\,\mathrm{d}\mathcal{H}^2\\&=-\beta\int_\Gamma|\mathbf{u}_\tau|^2\,\mathrm{d}\mathcal{H}^2+\int_\Gamma\mathcal{L}(c)\nabla_\tau c\cdot\mathbf{u}_\tau\,\mathrm{d}\mathcal{H}^2\end{aligned} \tag{17}$$

and by (2b), the second integral can be rewritten as follows:

$$\begin{aligned}\int_\Gamma \mathcal{L}(c)\nabla_\tau c\cdot\mathbf{u}_\tau\,\mathrm{d}\mathcal{H}^2 &= -\int_\Gamma |\mathcal{L}(c)|^2\,\mathrm{d}\mathcal{H}^2 - \int_\Gamma \partial_t c\,(-\triangle_\tau c+\xi c+k(c)+\partial_{\mathbf{n}}c)\,\mathrm{d}\mathcal{H}^2\\ &= -\int_\Gamma |\mathcal{L}(c)|^2\,\mathrm{d}\mathcal{H}^2 - \frac{\mathrm{d}}{\mathrm{d}t}\int_\Gamma \frac{1}{2}|\nabla_\tau c|^2+\frac{1}{2}\xi|c|^2+K(c)\,\mathrm{d}\mathcal{H}^2 - \int_\Gamma \partial_t c\partial_{\mathbf{n}}c\,\mathrm{d}\mathcal{H}^2\end{aligned} \tag{18}$$

where due to the Stokes theorem, $\int_\Gamma \mathrm{div}_\tau(c\nabla_\tau c)\,\mathrm{d}\mathcal{H}^2=0$. Therefore back to (17),

$$\int_\Gamma \mathbf{u}^t\mathbb{S}\mathbf{n}\,\mathrm{d}\mathcal{H}^2 = -\frac{\mathrm{d}}{\mathrm{d}t}E_{surf} - \mathcal{D}_{surf} - \int_\Gamma \partial_t c\partial_{\mathbf{n}}c\,\mathrm{d}\mathcal{H}^2 \tag{19}$$

where E_{surf} and $\mathcal{D}_{surf}$ denotes the surface energy and surface dissipation, respectively, cf. (9k), (9l) above. Combining (16) with (19), we obtain the following differential equation for the total energy, which is the sum between the bulk energy and the surface energy:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(E_{bulk}+E_{surf}\right)+\mathcal{D}_{bulk}+\mathcal{D}_{surf}=0. \tag{20}$$

This implies that for $t\in[0,T]$

$$\begin{aligned}&E_{bulk}|_t + E_{surf}|_t + \int_0^t \mathcal{D}_{bulk}|_\tau + \mathcal{D}_{surf}|_\tau\,\mathrm{d}\tau\\ &= \int_\Omega \frac{1}{2}\frac{|\mathbf{m}_0|^2}{\varrho_0}+\frac{1}{2}|\nabla_x c_0|^2+\varrho_0 f(\varrho_0,c_0)\,\mathrm{d}x + \int_\Gamma \frac{1}{2}|\nabla_\tau c_0|^2+\frac{1}{2}\xi|c_0|^2+K(c_0)\,\mathrm{d}\mathcal{H}^2 =: E_0.\end{aligned} \tag{21}$$

Hence, we can straightforwardly derive the following estimates depending only on the initial data:

- **(A1)** $\varrho f\in L^\infty(0,T;L^1(\Omega))$ and also $\varrho\in L^\infty(0,T;L^\gamma(\Omega))$,
  where we remark that the bound for ϱ in $L^\infty(0,T;L^\gamma(\Omega))$ is an immediate consequence of the bound of $\varrho f\in L^\infty(0,T;L^1(\Omega))$ and of the definition of f.
- **(A2)** $\sqrt{\varrho}|\mathbf{u}|\in L^\infty(0,T;L^2(\Omega))$,
- **(A3)** $\mathbb{S}\colon\nabla_x\mathbf{u}\in L^1(0,T;L^1(\Omega))$,
- **(A4)** $\nabla_x c\in L^\infty(0,T;L^2(\Omega;\mathbb{R}^3))$,
- **(A5)** $\nabla_x\mu\in L^2(0,T;L^2(\Omega;\mathbb{R}^3))$,
- **(A6)** $K(c)\in L^\infty(0,T;L^1(\Gamma))$,
- **(A7)** $c\in L^\infty(0,T;W^{1,2}(\Gamma))$,
- **(A8)** $\mathcal{L}(c)\in L^2(0,T;L^2(\Gamma))$,
- **(A9)** $\mathbf{u}\in L^2(0,T;L^2(\Gamma;\mathbb{R}^3))$.

Remark 1. Testing the momentum equation by $\mathbf{u}$ is an essential step in order to derive the presented estimates. We underline that usually for the equations of fluid mechanics, the velocity field $\mathbf{u}$ is not an admissible test function in definition of weak solutions due to its low regularity.

In what follows, let C denote a generic constant which depends only on the hypotheses on the non-linearities, on the initial data and on T.

2.3. **Further a priori estimates.** In order to derive estimates of Sobolev norms of μ, c and $\mathbf{u}$, we have to use a Poincaré type inequality. We need a generalized version of this inequality which applies whenever there is a control over $\varrho\mu$ rather than only μ in the Lebesgue spaces:

Lemma 2.1 (see [13, Lemma 3.1])**.** *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain and let $M_0 > 0$, $K > 0$. Assume that ϱ is a non-negative function such that:*

$$0 < M_0 \leq \int_\Omega \varrho \,\mathrm{d}x, \ \int_\Omega \varrho^\gamma \,\mathrm{d}x \leq K, \ \textit{with}\, \gamma > \frac{6}{5}. \tag{22}$$

Then there exists a positive constant $C = C(M_0, K, \gamma)$ such that the inequality

$$\|\mathbf{v} - \frac{1}{|\Omega|}\int_\Omega \varrho\mathbf{v}\|_{L^2(\Omega)} \leq C\,\|\nabla_x \mathbf{v}\|_{L^2(\Omega;\mathbb{R}^3)} \tag{23}$$

holds for any $\mathbf{v} \in W^{1,2}(\Omega)$.

We will also need an analogous version of Korn-Poincaré inequality:

Lemma 2.2 (see [14, Section 11.10])**.** *Let $\Omega \subset \mathbb{R}^N$, $N > 2$ be a bounded Lipschitz domain, and let $1 < p < \infty$, $M_0 > 0$, $K > 0$, $\gamma > 1$.*

Then there exists a positive constant $C = C(p, M_0, K, \gamma)$ such that the inequality

$$\|\mathbf{v}\|_{W^{1,p}(\Omega;\mathbb{R}^N)} \leq C\left(\left\|\nabla_x \mathbf{v} + \nabla_x^t \mathbf{v} - \frac{2}{N}\,\mathrm{div}_x\,\mathbf{v}\mathbb{I}\right\|_{L^p(\Omega;\mathbb{R}^N)} + \int_\Omega \varrho|\mathbf{v}|\,\mathrm{d}x\right) \tag{24}$$

holds for any $\mathbf{v} \in W^{1,p}(\Omega;\mathbb{R}^N)$ and any non-negative function ϱ such that:

$$0 < M_0 \leq \int_\Omega \varrho \,\mathrm{d}x, \ \int_\Omega \varrho^\gamma \,\mathrm{d}x \leq K. \tag{25}$$

Using the previous estimates together with standard inequalities, we obtain also:

(A10) $\varrho\mathbf{u} = \sqrt{\varrho}(\sqrt{\varrho}\mathbf{u}) \in L^\infty\left(0,T;L^{2\gamma/(\gamma+1)}(\Omega;\mathbb{R}^3)\right)$
which is an immediate consequence of **(A1), (A2)**.

(A11) $\mathbf{u} \in L^2(0,T;W^{1,2}(\Omega))$,
which follows from Lemma 2.2 in combination with **(A3)**. Using the Sobolev embedding $W^{1,2}(\Omega) \hookrightarrow L^6(\Omega)$ and **(A1), (A2)**, we also deduce:

(A12) $\varrho\mathbf{u}\otimes\mathbf{u} = \sqrt{\varrho}(\sqrt{\varrho}\mathbf{u})\otimes\mathbf{u} \in L^2\left(0,T;L^{6\gamma/(3+4\gamma)}(\Omega;\mathbb{R}^{3\times 3})\right)$

(A13) $c \in L^\infty(0,T;W^{1,2}(\Omega))$,
this being a consequence of Lemma 2.1 combined with the a priori estimate **(A4)**. From the Sobolev embeddings, we also obtain:

(A14) $\varrho c \in L^\infty\left(0,T;L^{6\gamma/(6+\gamma)}(\Omega)\right)$

(A15) $\varrho c\mathbf{u} = \sqrt{\varrho}(\sqrt{\varrho}\mathbf{u})c \in L^2\left(0,T;L^{6\gamma/(3+4\gamma)}(\Omega;\mathbb{R}^3)\right)$,
which was deduced using **(A1), (A2), (A13)** and the Sobolev embedding $W^{1,2}(\Omega) \hookrightarrow L^6(\Omega)$.

Integrating equation (1d) over Ω, we obtain:

$$\begin{aligned}\int_\Omega \varrho\mu \,\mathrm{d}x &= \int_\Omega \varrho\frac{\partial f}{\partial c}\,\mathrm{d}x - \int_\Gamma \partial_{\mathbf{n}} c \,\mathrm{d}\mathcal{H}^2 \\ &= \int_\Omega \varrho\frac{\partial f}{\partial c}\,\mathrm{d}x - \int_\Gamma (\mathcal{L}(c) - \xi c - k(c)) \,\mathrm{d}\mathcal{H}^2.\end{aligned} \tag{26}$$

Due to our hypotheses about the nonlinearities we know that $|\varrho\partial_c f| \leq |\varrho\log(\varrho)|C(1+|c|)$; hence, taking into account the bounds on ϱ and c, we derive:

(A16) $\varrho\partial_c f \in L^\infty(0,T;L^r(\Omega))$ for any $r < \frac{6\gamma}{6+\gamma}$.

Thus, using **(A7), (A8), (A16)** as well as the form of the nonlinear function k, equation (26) allows us to bound $|\int_\Omega \varrho\mu\,\mathrm{d}x|$. Using Lemma 2.2 in combination with **(A5)**, we conclude that

(A17) $\mu \in L^2(0,T;W^{1,2}(\Omega))$

It immediately follows that:

(A18) $\varrho\mu \in L^2(0,T;L^{6\gamma/(6+\gamma)}(\Omega))$.

Using the elliptic regularity applied to $\triangle_x c$ in (1d), we get:

(A19) $c \in L^2(0,T;W^{2,r}(\Omega))$ for any $r < \frac{6\gamma}{6+\gamma}$.

As $\gamma > 3/2$, it is important to observe that $6\gamma/(6+\gamma) > 6/5$, therefore there exists $r_0 > 6/5$ such that $c \in L^2(0,T;W^{2,r_0}(\Omega)) \hookrightarrow L^2(0,T;W^{1,2+\varepsilon}(\Omega))$ for some $\varepsilon > 0$. By interpolating this estimate with **(A13)** we deduce

(A20) $c \in L^{2+\delta}(0,T;W^{1,2+\delta}(\Omega))$ for some $\delta > 0$ and

(A21) $\nabla_x c \otimes \nabla_x c \in L^{1+\delta/2}((0,T)\times\Omega;\mathbb{R}^{3\times 3})$, $|\nabla_x c|^2 \in L^{1+\delta/2}((0,T)\times\Omega)$.

Taking into account the definition of $\mathbb{S}$ and **(A3)**, we also deduce that:

(A22) $\mathbb{S}(c,\nabla_x \mathbf{u}) \in L^2((0,T)\times\Omega;\mathbb{R}^{3\times 3})$.

2.4. **Dynamic boundary conditions.** Next, we collect estimates based on the elliptic regularity of the Laplace-Beltrami operator (see e.g. [42]). For every $t \in (0,T)$, $c(t,\cdot)$ solves

$$-\triangle_\tau c = \mathcal{L}(c) - k(c) - \xi c - \partial_{\mathbf{n}} c \quad \text{on } \Gamma.$$

Using the previous estimates on $\mathcal{L}(c)$ and on c, we deduce that $\triangle_\tau c \in L^2(0,T;L^2(\Gamma))$ with the norm bounded by a constant depending only on the initial data for system (1). This implies the a priori estimate for

(A23) $c \in L^2(0,T;W^{2,2}(\Gamma))$

therefore we also find:

(A24) $c \in L^2(0,T;\mathcal{C}(\Gamma))$ and $k(c) \in L^2(0,T;\mathcal{C}(\Gamma))$ and

(A25) $\partial_{\mathbf{n}} c$, $\nabla_\tau c \in L^2(0,T;L^q(\Gamma))$ for any $q \in [1,\infty)$.

Due to Sobolev's embeddings, we see that

(A26) $\mathbf{u}_\tau \cdot \nabla_\tau c \in L^2(0,T;L^{4/3}(\Gamma))$,

which implies the bound on the time-derivative of the concentration:

(A27) $\partial_t c \in L^2(0,T;L^{4/3}(\Gamma))$.

2.5. **Pressure estimates.** We realize that all terms occurring in the weak formulation except $p_e(\varrho)$ are a priori bounded at least in a reflexive space L^q, $q > 1$. In order to improve the bounds for the pressure, we test the momentum equation by

$$\begin{aligned} &\chi(t,x) = \psi(t)\varphi(t,x) \\ &\text{with } \varphi = \mathcal{B}\left(\varrho^\beta - \frac{1}{|\Omega|}\int_\Omega \varrho^\beta \,\mathrm{d}x\right) \end{aligned} \tag{27}$$

where $\mathcal{B}$ is the right inverse of the divergence operator, $\psi \in \mathcal{D}((0,T))$ and β is a suitably small positive constant. There are many ways how to define the operator $\mathcal{B}$. Here we adopt the integral formula proposed by Bogovskii and elaborated by Galdi (see [21, Theorem 3.3]). We list some properties of the operator $\mathcal{B}$ (for more details see e.g. [14, Section 11.6], and the references therein).

In what follows, for any function space $X(\Omega)$ we denote $\dot{X}(\Omega)$ its subspace of functions with zero mean. More generally, the functionals whose pairing with constant equals zero.

Theorem 2.3. *Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain. Then:*

(1) There exists a linear operator $\mathcal{B} : \dot{\mathcal{C}}^\infty(\Omega) \to \mathcal{C}_c^\infty(\Omega;\mathbb{R}^N)$ such that $\mathrm{div}_x(\mathcal{B}(f)) = f$ and $\mathcal{B}(f)|_\Gamma = 0$.

(2) We have:

$$\|\mathcal{B}(f)\|_{W^{k+1,p}(\Omega;\mathbb{R}^N)} \le c\|f\|_{W^{k,p}(\Omega)}, \forall 1 < p < \infty,\ k = 0,1,\dots$$

In particular, $\mathcal{B}$ can be extended in a unique way as a bounded linear operator

$$\mathcal{B} : \dot{L}^p(\Omega) \to W_0^{1,p}(\Omega;\mathbb{R}^N).$$

(3) If $f \in \dot{L}^p(\Omega)$ and $g \in W^{1,p}(\Omega)$ such that $f = \mathrm{div}_x \mathbf{g}$, then

$$\|\mathcal{B}(f)\|_{L^q(\Omega;\mathbb{R}^3)} \le c\|\mathbf{g}\|_{L^q(\Omega;\mathbb{R}^3)},$$

with $\mathbf{g} \in E_0^{q,p}(\Omega) = \mathrm{cl}_{E^{q,p}(\Omega)}(\mathcal{C}_c^\infty(\Omega;\mathbb{R}^N))$, where $E^{q,p}(\Omega)$ is the Banach space $E^{q,p}(\Omega) = \{\mathbf{u} \in L^q(\Omega;\mathbb{R}^N), \mathrm{div}\,\mathbf{u} \in L^p(\Omega)\}$ endowed with the norm $\|\mathbf{u}\| := \|\mathbf{u}\|_{L^q(\Omega;\mathbb{R}^N)} + \|\mathrm{div}_x\,\mathbf{u}\|_{L^p(\Omega)}$.

(4) $\mathcal{B}$ can be uniquely extended as a bounded linear operator

$$\mathcal{B} : \dot{W}^{-1,p}(\Omega) \to L^p(\Omega;\mathbb{R}^N)$$

where $\dfrac{1}{p} + \dfrac{1}{p'} = 1$, in such a way that:

$$-\int_\Omega \mathcal{B}(f)\cdot\nabla v \,\mathrm{d}x = \langle f, v\rangle_{(W^{1,p'})^*,W^{1,p'}} \forall v \in W^{1,p'}(\Omega),$$

$$\|\mathcal{B}(f)\|_{L^p(\Omega;\mathbb{R}^N)} \le c\|f\|_{(W^{1,p'}(\Omega))^*}.$$

(5) If $f, \partial_t f \in L^p((0,T)\times\Omega)$, $\int_\Omega f(t,\cdot)\,\mathrm{d}x = 0$ for a. a. $t \in (0,T)$, then:

$$\partial_t\mathcal{B}(f)(t,x) = \mathcal{B}(\partial_t f)(t,x) \text{ a.a. } (t,x) \in (0,T)\times\Omega.$$

Let $h(\varrho) = \varrho^\beta$, then ϱ satisfies the following renormalized equation:

$$\partial_t h(\varrho) + \operatorname{div}_x(h(\varrho)\mathbf{u}) + (\varrho h'(\varrho) - h(\varrho)) \operatorname{div}_x \mathbf{u} = 0. \tag{28}$$

Applying operator $\mathcal{B}$ to equation (28) and using Theorem 2.3, we obtain for φ defined through (27):

$$\partial_t \varphi = -\mathcal{B}(\operatorname{div}_x(h(\varrho)\mathbf{u})) - \mathcal{B}\big[(\varrho h'(\varrho) - h(\varrho)) \operatorname{div}_x \mathbf{u} - \frac{1}{|\Omega|}\int_\Omega (\varrho h'(\varrho) - h(\varrho)) \operatorname{div}_x \mathbf{u}\,\mathrm{d}x\big]. \tag{29}$$

where

$$\|\varphi(t,\cdot)\|_{W^{1,p}(\Omega;\mathbb{R}^3)} \le c(p)\|h(\varrho)\|_{L^p(\Omega)},\ 1 < p < \infty,$$

and

$$\begin{aligned}\|\partial_t \varphi\|_{L^p(\Omega;\mathbb{R}^3)} \le& c\|h(\varrho)\mathbf{u}\|_{L^p(\Omega;\mathbb{R}^3)} \\ &+ c\|(\varrho h'(\varrho) - h(\varrho)) \operatorname{div}_x \mathbf{u} - \frac{1}{|\Omega|}\int_\Omega (\varrho h'(\varrho) - h(\varrho)) \operatorname{div}_x \mathbf{u}\,\mathrm{d}x\|_{(W^{1,p'})^*}.\end{aligned} \tag{30}$$

The last term in (30) is bounded by $\|(\varrho h'(\varrho) - h(\varrho)) \operatorname{div}_x \mathbf{u}\|_{L^{3p/(3+p)}(\Omega)}$ if $3/2 < p < \infty$ and by $\|(\varrho h'(\varrho) - h(\varrho)) \operatorname{div}_x \mathbf{u}\|_{L^s(\Omega)}$, for any $s > 1$, if $1 \le p \le 3/2$.

Our goal is to use the function χ defined in (27) as a test function in the momentum equation. Plugging (27) into the weak formulation (9c) (still assuming that solutions are smooth), we deduce that:

$$\int_0^T \psi(t) \int_\Omega p(\varrho, c)\varrho^\beta \,\mathrm{d}x\,\mathrm{d}t = \sum_{j=1}^6 I_j$$

where[2]

$$\begin{aligned}
I_1 &= \frac{1}{|\Omega|}\int_0^T \psi \int_\Omega \left(\int_\Omega \varrho^\beta(t,y)\,\mathrm{d}y\right) p(\varrho,c)\,\mathrm{d}x\,\mathrm{d}t,\\
I_2 &= -\int_0^T \left(\psi \int_\Omega \varrho\mathbf{u}\partial_t\varphi\,\mathrm{d}x\right)\,\mathrm{d}t,\\
I_3 &= -\int_0^T \left(\psi \int_\Omega \varrho\mathbf{u}\otimes\mathbf{u} \colon \nabla_x\varphi\,\mathrm{d}x\right)\,\mathrm{d}t,\\
I_4 &= \int_0^T \left(\psi \int_\Omega \mathbb{S}(c,\mathbf{u}) \colon \nabla_x\varphi\,\mathrm{d}x\right)\,\mathrm{d}t,\\
I_5 &= -\int_0^T \left(\psi' \int_\Omega \varrho\mathbf{u}\cdot\varphi\,\mathrm{d}x\right)\,\mathrm{d}t,\\
I_6 &= -\int_0^T \psi \left(\int_\Omega \left(\nabla_x c\otimes\nabla_x c - \frac{1}{2}|\nabla_x c|^2\mathbb{I}\right) : \nabla_x\varphi\,\mathrm{d}x\right)\,\mathrm{d}t.
\end{aligned}$$

We need to estimate each of the integrals:

$$|I_1| \le C\|\psi\|_{L^\infty(0,T)}\|\varrho\|_{L^\beta(0,T,L^\beta(\Omega))}\|p(\varrho,c)\|_{L^\infty(0,T,L^1(\Omega))},$$

[2]The boundary terms vanish as φ has zero traces on Γ.

$$|I_2| \le C\|\psi\|_{L^\infty(0,T)}\|\varrho\mathbf{u}\|_{L^\infty(0,T,L^{2\gamma/(\gamma+1)}(\Omega))}\|\partial_t\varphi\|_{L^1(0,T,L^{2\gamma/(\gamma-1)}(\Omega))}$$
$$\le C\|\psi\|_{L^\infty(0,T)}\|\varrho\mathbf{u}\|_{L^\infty(0,T,L^{2\gamma/(\gamma+1)}(\Omega))}(\|\varrho^\beta\mathbf{u}\|_{L^1(0,T,L^{2\gamma/(\gamma-1)}(\Omega))} + \|\varrho^\beta \operatorname{div}_x \mathbf{u}\|_{L^1(0,T,L^{6\gamma/(5\gamma-3)}(\Omega))}),$$

where we can continue estimating the last two terms as:

$$\|\varrho^\beta\mathbf{u}\|_{L^1(0,T,L^{2\gamma/(\gamma-1)}(\Omega)} \le C\|\mathbf{u}\|_{L^2(0,T,L^6(\Omega,\mathbb{R}^3))}\Big(\int_0^T \big(\int_\Omega |\varrho|^{\beta\frac{6\gamma}{2\gamma-3}}\,\mathrm{d}x\big)^{\frac{2\gamma-3}{3\gamma}}\,\mathrm{d}t\Big)^{1/2},$$

and

$$\|\varrho^\beta \operatorname{div}_x \mathbf{u}\|_{L^1(0,T,L^{6\gamma/(5\gamma-3)}(\Omega)} \le C\|\operatorname{div}_x \mathbf{u}\|_{L^2(0,T,L^2(\Omega,\mathbb{R}^3))}\Big(\int_0^T \big(\int_\Omega |\varrho|^{\beta\frac{6\gamma}{2\gamma-3}}\,\mathrm{d}x\big)^{\frac{2\gamma-3}{3\gamma}}\,\mathrm{d}t\Big)^{1/2}.$$

Moreover,

$$|I_3| \le C\|\psi\|_{L^\infty(0,T)}\|\varrho\mathbf{u}\otimes\mathbf{u}\|_{L^2(0,T,L^{6\gamma/(4\gamma+3)}(\Omega))}\|\varrho^\beta\|_{L^2(0,T,L^{6\gamma/(2\gamma-3)}(\Omega))},$$

$$|I_4| \le C\|\psi\|_{L^\infty(0,T)}\|\mathbb{S}(c,\mathbf{u})\|_{L^q((0,T)\times\Omega;\mathbb{R}^{3\times3})}\|\varrho^\beta\|_{L^{q'}((0,T)\times\Omega)},$$

$$|I_5| \le C\|\psi'\|_{L^1(0,T)}\|\varrho\mathbf{u}\|_{L^\infty(0,T,L^{2\gamma/(\gamma+1)}(\Omega,\mathbb{R}^3))}\|\varphi\|_{L^\infty(0,T,L^{2\gamma/(\gamma-1)}(\Omega;\mathbb{R}^3))}$$
$$\le C\|\psi'\|_{L^1(0,T)}\|\varrho\mathbf{u}\|_{L^\infty(0,T,L^{2\gamma/(\gamma+1)}(\Omega,\mathbb{R}^3))}\|\varrho^\beta\|_{L^\infty(0,T,W^{-1,2\gamma/(\gamma-1)}(\Omega))}$$
$$\le C\|\psi'\|_{L^1(0,T)}\|\varrho\mathbf{u}\|_{L^\infty(0,T,L^{2\gamma/(\gamma+1)}(\Omega,\mathbb{R}^3))}\|\varrho^\beta\|_{L^\infty(0,T,L^{6\gamma/(5\gamma-3)}(\Omega))}$$

and

$$|I_6| \le C\|\psi\|_{L^\infty(0,T)}\big(\|\nabla_x c\otimes\nabla_x c\|_{L^{1+\delta/2}((0,T)\times\Omega;\mathbb{R}^{3\times3})} + \||\nabla_x c|^2\|_{L^{1+2/\delta}((0,T)\times\Omega;\mathbb{R}^{3\times3})}\big)\|\nabla_x\varphi\|_{L^{1+2/\delta}((0,T)\times\Omega;\mathbb{R}^{3\times3})}$$
$$\le C\|\psi\|_{L^\infty(0,T)}\big(\|\nabla_x c\otimes\nabla_x c\|_{L^{1+\delta/2}((0,T)\times\Omega;\mathbb{R}^{3\times3})} + \||\nabla_x c|^2\|_{L^{1+2/\delta}((0,T)\times\Omega;\mathbb{R}^{3\times3})}\big)\|\varrho^\beta\|_{L^{1+2/\delta}((0,T)\times\Omega;\mathbb{R}^{3\times3})}.$$

Based on the already generated estimates and on the fact that $\varrho \in L^\infty(0,T,L^\gamma(\Omega))$, all these previous estimates lead us to conclude that there exists $\beta > 0$ small enough such that

$$\Big|\int_0^T \psi\int_\Omega p(\varrho,c)\varrho^\beta\,\mathrm{d}x\,\mathrm{d}t\Big| \le C(1+\|\psi'\|_{L^1(0,T)}).$$

Finally, taking suitable $\psi_n \to 1$ almost everywhere on $(0,T)$ with $\|\psi'\|_{L^1(0,T)} \le C$, we obtain

$$\int_0^T\int_\Omega \varrho^{\gamma+\beta}\,\mathrm{d}x\,\mathrm{d}t \approx \int_0^T\int_\Omega p_e(\varrho,c)\varrho^\beta\,\mathrm{d}x\,\mathrm{d}t \le C.$$

due to (7) and (5). Therefore, we derived our final a priori estimate

(A28) $p \in L^{1+\varepsilon}((0,T)\times\Omega)$ for some $\varepsilon > 0$.

Let us collect the previous estimates in the lemma, which has been just proven:

Lemma 2.4. *Let $(\varrho,\mathbf{u},c,\mu)$ be a smooth solution to (1) satisfying (2) and (3).*
*Then the corresponding norms of functions in **(A1)–(A28)** are uniformly bounded by a constant depending solely on the initial data and T.*

3. Compactness

Pursuing further our strategy delineated in the introductory part we aim to show compactness or sequential stability of the solution set. Accordingly, we assume that $(\mathbf{u}_n,\ \varrho_n,\ c_n,\ \mu_n)$ is a sequence of weak solutions to the problem (1) with the boundary conditions (2) and initial data $\varrho_{0,n}$, $\mathbf{m}_{0,n}$, $c_{0,n}$ satisfying the hypotheses of Theorem 1.1.

We can directly derive uniform bounds **(A10)**–**(A28)** for the renormalized weak solutions due to (9h). Consequently, there exists a subsequence (not relabeled) such that the following holds:

- $\varrho_n \overset{*}{\rightharpoonup} \varrho$ in $L^\infty(0,T;L^\gamma(\Omega))$,
- $\mathbf{u}_n \rightharpoonup \mathbf{u}$ in $L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^3))$,
- $c_n \rightharpoonup c$ in $L^2(0,T;W^{1,2+\varepsilon}(\Omega))$, $L^\infty(0,T;W^{1,2}(\Gamma))$, and $W^{1,2}(0,T;L^{4/3}(\Gamma))$,
- $\mathbb{S}(c_n,\mathbf{u}_n) \rightharpoonup \overline{\mathbb{S}(c,\mathbf{u})}$ in $L^2(0,T;L^2(\Omega;\mathbb{R}^{3\times 3}))$,
- $\varrho_n\mathbf{u}_n \overset{*}{\rightharpoonup} \overline{\varrho\mathbf{u}}$ in $L^\infty\left(0,T;L^{\frac{2\gamma}{\gamma+1}}(\Omega;\mathbb{R}^3)\right)$ and $\varrho_n\mathbf{u}_n \rightharpoonup \overline{\varrho\mathbf{u}}$ in $L^2\left(0,T;L^{\frac{6\gamma}{\gamma+6}}(\Omega;\mathbb{R}^3)\right)$,
- $\varrho_n\mathbf{u}_n\otimes\mathbf{u}_n \rightharpoonup \overline{\varrho\mathbf{u}\otimes\mathbf{u}}$ in $L^2\left(0,T;L^{\frac{6\gamma}{3+4\gamma}}(\Omega;\mathbb{R}^{3\times 3})\right)$,
- $p(\varrho_n,c_n) \rightharpoonup \overline{p(\varrho,c)}$ in $L^{1+\varepsilon}((0,T)\times\Omega)$,
- $\nabla_x c_n\otimes\nabla_x c_n - \frac{1}{2}|\nabla_x c_n|^2\mathbb{I} \rightharpoonup \overline{\nabla_x c\otimes\nabla_x c} - \frac{1}{2}\overline{|\nabla_x c|^2}\mathbb{I} \in L^{1+\delta/2}((0,T)\times\Omega)$,
- $\partial_{\mathbf{n}}c_n\nabla_\tau c_n \overset{*}{\rightharpoonup} \overline{\partial_{\mathbf{n}}c\nabla_\tau c}$ in $L^2(0,T;L^{4/3}(\Gamma))$,
- $\varrho_n c_n \overset{*}{\rightharpoonup} \overline{\varrho c}$ in $L^\infty\left(0,T;L^{\frac{6\gamma}{6+\gamma}}(\Omega)\right)$,
- $\varrho_n c_n\mathbf{u}_n \rightharpoonup \overline{\varrho c\mathbf{u}}$ in $L^2\left(0,T;L^{\frac{6\gamma}{3+4\gamma}}(\Omega;\mathbb{R}^3)\right)$,
- $\mu_n \rightharpoonup \mu$ in $L^2(0,T;W^{1,2}(\Omega))$,
- $\varrho_n\mu_n \rightharpoonup \overline{\varrho\mu}$ in $L^2\left(0,T;L^{6\gamma/(6+\gamma)}(\Omega)\right)$,
- $\varrho_n\partial_c f(\varrho_n,c_n) \rightharpoonup \overline{\varrho\partial_c f(\varrho,c)}$ in $L^\infty(0,T;L^r(\Omega))$, for any $r \le \dfrac{6\gamma}{6+\gamma}$,
- $(\mathbf{u}_n)_\tau\cdot\nabla_\tau c_n \rightharpoonup \overline{\mathbf{u}_\tau\cdot\nabla_\tau c}$ in $L^2(0,T;L^{4/3}(\Gamma))$,
- $k(c_n) \overset{*}{\rightharpoonup} \overline{k(c)}$ in $L^\infty(0,T;L^q(\Gamma))$, for any $q\in[1,\infty)$.

Accordingly, we can pass directly to the limit in the weak formulation in all linear terms. In order to show that $(\varrho,\mathbf{u},c,\mu)$ (or a limit of a subsequence) satisfies also the weak formulation, it is enough to show that all quantities above with the bar are in fact equal to the corresponding quantities without bars.

3.1. Compactness of multilinear terms.

In order to pass to the limit in the multilinear terms, we make use of the following lemma, see e.g. [11]:

Lemma 3.1. *Let $h_n \rightharpoonup h$, $\mathbf{b}_n \rightharpoonup \mathbf{b}$ in $L^1_{\rm loc}((0,T)\times\Omega)$ and a_n be uniformly bounded in $L^\infty(0,T;L^p(\Omega))$. Let*

$$\partial_t a_n + \operatorname{div}_x \mathbf{b}_n = h_n \quad \text{in the sense of distributions on } (0,T)\times\Omega.$$

Then there exists a subsequence of $\{a_n\}_{n\in\mathbb{N}}$ such that $a_n \to a$ in $C_w([0,T],L^p(\Omega))$. If, moreover, $L^p(\Omega) \hookrightarrow\hookrightarrow W^{-1,q}(\Omega)$, then

$$a_n \to a \qquad \text{in } \mathcal{C}([0,T];W^{-1,q}(\Omega)). \tag{31}$$

Due to the previous lemma, **(A1)** and (1a), we get $\varrho_n \to \varrho$ in $\mathcal{C}_w([0,T];L^\gamma(\Omega))$. As $L^\gamma(\Omega) \hookrightarrow\hookrightarrow W^{-1,2}(\Omega)$ provided $\gamma > \frac{6}{5}$, we have:

$$\int_0^T \int_\Omega \varrho_n \mathbf{u}_n \varphi \,\mathrm{d}x\,\mathrm{d}t = \int_0^T \langle \varrho_n, \varphi \mathbf{u}_n \rangle_{W^{-1,2}(\Omega),W^{1,2}(\Omega)} \,\mathrm{d}t$$
$$\to \int_0^T \langle \varrho, \varphi \mathbf{u} \rangle_{W^{-1,2}(\Omega),W^{1,2}(\Omega)} \,\mathrm{d}t = \int_0^T \int_\Omega \varrho \mathbf{u} \varphi \,\mathrm{d}x\,\mathrm{d}t.$$

Hence, $\overline{\varrho \mathbf{u}} = \varrho \mathbf{u}$.

Using now the same argument with c_n and then μ_n, we also obtain $\overline{\varrho c} = \varrho c$, $\overline{\varrho \mu} = \varrho \mu$, and $\overline{\varrho c^2} = \varrho \overline{c^2}$.

Similarly, as $L^{2\gamma/(\gamma+1)}(\Omega) \hookrightarrow\hookrightarrow W^{-1,2}(\Omega)$ whenever $\gamma > \frac{3}{2}$, we obtain that $\varrho_n \mathbf{u}_n \to \varrho u$ in $\mathcal{C}([0,T];W^{-1,2}(\Omega))$ which leads to $\overline{\varrho \mathbf{u} \otimes \mathbf{u}} = \varrho \mathbf{u} \otimes \mathbf{u}$ and $\overline{\varrho \mathbf{u} c} = \varrho \mathbf{u} c$.

Finally, $\overline{\varrho c^2} = \varrho c^2$, as $\varrho_n c_n \to \overline{\varrho c} = \varrho c$ in $\mathcal{C}([0,T];L^{6\gamma/(6+\gamma)}_\omega(\Omega))$ and $L^{6\gamma/(6+\gamma)}_\omega(\Omega) \hookrightarrow\hookrightarrow W^{-1,2}(\Omega)$ if $\gamma > \frac{3}{2}$.

3.2. **Compactness of the dynamic boundary condition.** Directly from the Aubin-Lions lemma, we obtain:

$$W^{1,2}(0,T;L^{4/3}(\Gamma)) \cap L^2(0,T;W^{2,2}(\Gamma)) \hookrightarrow\hookrightarrow L^p(0,T;W^{1,q}(\Gamma))$$

for some suitable p and q. This implies the strong convergence of c_n and $\nabla_\tau c_n$ on $(0,T) \times \Gamma$. Consequently, $\overline{k(c)} = k(c)$, $\overline{\mathbf{u}_\tau \cdot \nabla_\tau c} = \mathbf{u}_\tau \cdot \nabla_\tau c$ and $\overline{\partial_{\mathbf{n}} c \nabla_\tau c} = \partial_{\mathbf{n}} c \nabla_\tau c$.

3.3. **Strong convergence of concentrations in** $(0,T) \times \Omega$**.** We borrow the idea from [1] to show that

$$\nabla_x c_n \to \nabla_x c \quad \text{in } L^2((0,T) \times \Omega), \tag{32}$$

and:

$$c_n \to c \quad \text{in } L^2(0,T;W^{1,2}(\Omega)). \tag{33}$$

First, we observe that

$$\int_0^T \int_\Omega \varrho c_n^2 \,\mathrm{d}x\,\mathrm{d}t \to \int_0^T \int_\Omega \varrho c^2 \,\mathrm{d}x\,\mathrm{d}t. \tag{34}$$

Indeed, in Section 3.1 we proved that $\varrho_n c_n^2 \rightharpoonup \overline{\varrho c^2} = \varrho c^2$. Using (A20), we can also deduce that $(\varrho_n - \varrho) c_n^2 \rightharpoonup 0$ in $L^r((0,T) \times \Omega)$ for some $r > 1$ and thus we get (34). Therefore

$$c_n \to c \quad \text{strongly in } L^2((0,T) \times \Omega, \varrho \,\mathrm{d}x). \tag{35}$$

It is sufficient to show the convergence in L^1. To this end, we compute for an arbitrary $\varepsilon > 0$:

$$\begin{aligned}\limsup_{n\to\infty} \int_{\varrho>0} |c_n - c| \,\mathrm{d}x &= \limsup_{n\to\infty} \left(\int_{\varrho>\varepsilon} |c_n - c| \,\mathrm{d}x + \int_{\varrho\in(0,\varepsilon)} |c_n - c| \,\mathrm{d}x \right) \\ &\leq \limsup_{n\to\infty} \frac{1}{\varepsilon} \int_\Omega |c_n - c| \varrho \,\mathrm{d}x + \limsup_{n\to\infty} \|c_n - c\|_{L^p(X)} |\{\varrho \in (0,\varepsilon)\}|^{\frac{p-1}{p}} \\ &\leq C |\{\varrho \in (0,\varepsilon)\}|^{\frac{p-1}{p}},\end{aligned}$$

which vanishes as $\varepsilon \to 0$.

Equation (1d) implies that for every $\varphi \in \mathcal{D}((0,T) \times (\bar{\Omega}))$

$$\int_0^T \int_\Omega \nabla_x c_n \nabla_x \varphi \,\mathrm{d}x \,\mathrm{d}t - \int_0^T \int_\Gamma \partial_{\mathbf{n}} c_n \varphi \,\mathrm{d}\mathcal{H}^2 \,\mathrm{d}t = \int_0^T \int_\Omega \varrho_n \big(\mu_n - \frac{\partial f}{\partial c}(\varrho_n, c_n)\big) \varphi \,\mathrm{d}x \,\mathrm{d}t. \tag{36}$$

Since we know that $\varrho_n \mu_n \rightharpoonup \overline{\varrho\mu} = \varrho\mu$ and $\varrho_n \partial_c f(\varrho_n, c_n) \rightharpoonup \overline{\varrho \partial_c f(\varrho, c)}$ in $L^2(0,T; L^{6\gamma/(6+\gamma)}(\Omega))$, we can pass to the limit in (36) and then substitute φ to c in order to obtain:

$$\int_0^T \int_\Omega |\nabla_x c|^2 \,\mathrm{d}x - \int_\Gamma c \partial_{\mathbf{n}} c \,\mathrm{d}\mathcal{H}^2 \,\mathrm{d}t = \int_0^T \int_\Omega (\varrho\mu - \overline{\varrho \partial_c f(\varrho, c)}) c \,\mathrm{d}x \,\mathrm{d}t.$$

Using a density argument, we can also take $\varphi = c_n$ and then passing to the limit and taking into account the convergences proven previously, we get:

$$\lim_{n\to\infty} \int_0^T \int_\Omega |\nabla_x c_n|^2 \,\mathrm{d}x \,\mathrm{d}t - \int_0^T \int_\Gamma c \partial_{\mathbf{n}} c \,\mathrm{d}\mathcal{H}^2 \,\mathrm{d}t = \int_0^T \int_\Omega \varrho\mu c \,\mathrm{d}x \,\mathrm{d}t - \int_0^T \int_\Omega \overline{\varrho \partial_c f(\varrho, c) c} \,\mathrm{d}x \,\mathrm{d}t.$$

It is now sufficient to show that $\overline{\varrho \partial_c f(\varrho, c) c} = \overline{\varrho \partial_c f(\varrho, c)} c$, in order to conclude that

$$\nabla_x c_n \to \nabla_x c \text{ strongly in } L^2(0,T; L^2(\Omega; \mathbb{R}^3)).$$

To this end, we observe that:

$$\int_0^T \int_\Omega \varrho_n \partial_c f(\varrho_n, c_n) c_n \varphi \,\mathrm{d}x \,\mathrm{d}t = \int_0^T \int_\Omega \varrho_n \partial_c f(\varrho_n, c_n) c \varphi \,\mathrm{d}x \,\mathrm{d}t + \int_0^T \int_\Omega \varrho_n \partial_c f(\varrho_n, c_n)(c_n - c) \varphi \,\mathrm{d}x \,\mathrm{d}t,$$

where the last integral converges to zero. Indeed, a straightforward computation yields

$$\begin{aligned}\int_0^T \int_\Omega \varrho_n \partial_c f(\varrho_n, c_n)(c_n - c) \varphi \,\mathrm{d}x \,\mathrm{d}t &= \int_{\varrho>0} \varrho_n \partial_c f(\varrho_n, c_n)(c_n - c) \varphi \,\mathrm{d}x \,\mathrm{d}t \\ &\quad + \int_{\varrho=0} \varrho_n \partial_c f(\varrho_n, c_n)(c_n - c) \varphi \,\mathrm{d}x \,\mathrm{d}t\end{aligned}$$

and both integrals converge to zero due to argument above and the fact that $\varrho_n \chi_{\{\varrho=0\}} \to 0$ strongly in $L^q(0,T; L^q(\Omega))$ for all $1 \leq q < \gamma$.

Finally, convergence (33) follows from the strong convergence of $\nabla_x c_n$, Lemma 2.2 and (35). Whence, $\overline{\mathbb{S}(c, \mathbf{u})} = \mathbb{S}(c, \mathbf{u})$, $\overline{\nabla_x c \otimes \nabla_x c} - \frac{1}{2}\overline{|\nabla_x c|^2} = \nabla_x c \otimes \nabla_x c - \frac{1}{2}|\nabla_x c|^2$.

3.4. **Compactness of the pressure.** There are last two nonlinear terms remaining for which we have to check the weak convergence to the "right limit", namely $p(\varrho, c)$ and $\varrho\partial_c f(\varrho, c)$. To conclude the proof of Theorem 1.1, it is sufficient to show almost everywhere convergence of the densities. We recall that we are dealing with a similar system to the one in [1] except for the difference in the boundary conditions for the momentum equation and the Cahn–Hilliard equation. However, the main argument in the cited paper depends only locally on the momentum equation. The crucial information coming from the momentum equation is called the effective viscous flux identity:

$$\overline{p(\varrho,c)b(\varrho)} - \overline{p(\varrho,c)}\,\overline{b(\varrho)} = \left(\frac{4}{3}\nu(c)+\eta(c)\right)\left(\overline{b(\varrho)\,\mathrm{div}_x\,\mathbf{u}} - \overline{b(\varrho)}\,\mathrm{div}_x\,\mathbf{u}\right) \quad \text{in } (0,T)\times\Omega \tag{37}$$

for any $b \in \mathcal{C}^1([0,\infty))$ with b' compactly supported in $[0,\infty)$. It is sufficient to show that (37) holds in K for any compact $K \subset (0,T)\times\Omega$ which is accomplished by using localized test functions. The rest of the proof of $\varrho_n \to \varrho$ a.e. (at least for a subsequence) follows from (37) and the renormalization property of the continuity equation (1a), which has exactly the same boundary condition as in [1]. Therefore, with reference to [1] (or also [12], [14]) we conclude the proof of Theorem 1.1.

Finally, let us mention that the energy inequality (9h) follows from the lower semi-continuity of norms. The weak continuity and the fulfillment of the initial and boundary conditions follow from Lemma 3.1 and Subsection 3.2.

4. Approximation scheme

We complete the existence proof by introducing the approximation scheme following the lines of [16]. Let H_m be an m-dimensional space spanned over the m first vectors of a basis of the space $W^{1,2}(\Omega;\mathbb{R}^N)$ of functions with zero normal trace. As the boundary Γ is smooth, we may assume that $H_m \subset C^2(\overline{\Omega};\mathbb{R}^N)$.

Following [16], we propose the approximate problem in the form:

$$\partial_t\varrho + \mathrm{div}(\varrho\mathbf{u}) = \epsilon\Delta\varrho,\ \ \varrho(0,\cdot) = \varrho_{0,\epsilon,\delta},\ \partial_{\mathbf{n}}\varrho = 0 \text{ on } \Gamma. \tag{38}$$

The approximate velocity field $\mathbf{u}$ belongs to the space $C^1([0,T];H_m)$, and the integral identity

(39)

$$\begin{aligned}
&\int_0^T\int_\Omega \varrho\mathbf{u}\cdot\partial_t\varphi\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_\Omega \varrho\mathbf{u}\otimes\mathbf{u}\colon\nabla_x\varphi\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_\Omega (p(\varrho,c)+\delta\varrho^G)\,\mathrm{div}_x\,\varphi\,\mathrm{d}x\,\mathrm{d}t\\
&-\varepsilon\int_0^T\int_\Omega \nabla(\varrho_\varepsilon\mathbf{u}_\varepsilon)\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t - \int_0^T\int_\Omega \mathbb{S}(c,\mathbf{u}):\nabla_x\varphi\,\mathrm{d}x\,\mathrm{d}t\\
&+\int_0^T\int_\Omega\left(\nabla_x c\otimes\nabla_x c - \frac{1}{2}|\nabla_x c|^2\mathbb{I}\right):\nabla_x\varphi\,\mathrm{d}x\,\mathrm{d}t - \int_0^T\int_\Gamma(\beta\mathbf{u}_\tau - (\mathcal{L}(c)-\partial_{\mathbf{n}}c)\nabla_\tau c)\cdot\varphi_\tau\,\mathrm{d}\mathcal{H}^2\,\mathrm{d}t\\
&= -\int_\Omega \mathbf{m}_{0,\varepsilon,\delta}\cdot\varphi(0,\cdot)\,\mathrm{d}x,
\end{aligned}$$

holds for every $\varphi \in C_c^1([0;T);H_m)$.

$$\partial_t c + \mathbf{u}\cdot\nabla c = \frac{1}{\varrho}\Delta\mu,\ \mu = \frac{\partial f(\varrho,c)}{\partial c} - \frac{1}{\varrho}\Delta c,\ c(0,\cdot) = c_{0,\epsilon,\delta}, \tag{40}$$

where (39) and (40) are coupled via the boundary conditions (2).

Here $\epsilon > 0$, $\delta > 0$ are positive parameters and $\varrho_{0,\epsilon,\delta}$, $\mathbf{m}_{0,\epsilon,\delta}$, $c_{0,\epsilon,\delta}$ are smooth approximations of the initial data and $G > 6$.

4.1. **Solvability of the approximate problem.** For a given $c \in L^\infty(0,T;W^{1,2}(\Omega))\cap L^2(0,T;W^{3,2}(\Omega))$, the system of equations (38), (39) admits a solution $[\varrho,\mathbf{u}]$ unique in the class

$$\mathbf{u} \in C^1([0,T];H_m),\ \varrho,\ \frac{1}{\varrho} \in C([0,T];C^{2+\nu}(\overline{\Omega})) \cap C^1([0,T];C^{\nu}(\overline{\Omega})), \tag{41}$$

see [11, Chapter 7]. Moreover, it is straightforward to check that the mapping

$$c \in L^\infty(0,T;W^{3,2}(\Omega)) \cap L^2(0,T;W^{2,2}(\Omega)) \mapsto [\varrho,\mathbf{u}] \in C([0,T];C(\Omega)) \times C([0,T];H_m)$$

is continuous.

Following [16], we obtain the approximate solutions as a fixed point of the mapping

$$\mathcal{M} : c \mapsto [\varrho,\mathbf{u}] \mapsto \mathcal{M}[c],$$

where $c = \mathcal{M}[c]$ is the unique solution of

$$\begin{aligned} \partial_t c + \mathbf{u}\cdot\nabla c = \frac{1}{\varrho}\Delta\mu,\ \mu = \frac{\partial f(\varrho,c)}{\partial c} - \frac{1}{\varrho}\Delta c,\\ \nabla_x\mu\cdot\mathbf{n} = 0,\ \partial_t c + \mathbf{u}_\tau\nabla_\tau c = -\triangle_\tau c + \xi c + k(c) + \partial_{\mathbf{n}} c \ \text{ on } \Gamma,\\ c(0,\cdot) = c_{0,\epsilon,\delta}, \end{aligned} \tag{42}$$

with ϱ and $\mathbf{u}$ *given*.

In order to solve (42) we use the following auxiliary result. For the sake of simplicity, we omit the terms that can be treated as a lower order perturbation.

Lemma 4.1. *Let ϱ and $\mathbf{u}$ be given such that ϱ, $\partial_t\varrho$, $D_x^2\varrho \in \mathcal{C}([0,T]\times\bar{\Omega})$, $\varrho > \underline{\varrho} > 0$ and $\mathbf{u} \in \mathcal{C}^1([0,T]\times\bar{\Omega})$, $D_x^2\mathbf{u} \in \mathcal{C}([0,T]\times\bar{\Omega})$, with $\mathbf{u}\cdot\mathbf{n} = 0$ on Γ. Let also consider $c_0 \in \mathcal{C}(\bar{\Omega})$. Then the linear problem:*

$$\partial_t c = -\frac{1}{\varrho}\Delta(\frac{1}{\varrho}\Delta c) - \mathbf{u}\cdot\nabla c, \tag{43}$$

endowed with the boundary conditions:

$$\begin{aligned} \partial_{\mathbf{n}}(\frac{1}{\varrho}\Delta c) = 0 \ \text{on } \Gamma,\\ \partial_t c + \mathbf{u}_\tau\cdot\nabla_\tau c = \Delta_\tau c - \partial_{\mathbf{n}} c \ \text{on } \Gamma, \end{aligned} \tag{44}$$

admits a strong solution c unique in the class $W^{1,p}(0,T,L^p(\Omega)) \cap L^p(0,T,W^{4,p}(\Omega))$ with $c_{|\Gamma}$ in $W^{1-\frac{1}{4p},p}(0,T,L^p(\Gamma)) \cap L^p(0,T,W^{4-\frac{1}{p},p}(\Gamma))$.

Proof. We proceed by a fixed point argument. Let us consider c_Γ be given such that

$$c_\Gamma \in X_\Gamma, \text{ with } X_\Gamma = W^{1-\frac{1}{4p},p}(0,T,L^p(\Gamma)) \cap L^p(0,T,W^{4-\frac{1}{p},p}(\Gamma))$$

and $c_\Gamma(0) = c_0$ on Γ. Then according to [7], the following initial boundary value problem:

$$\begin{aligned} &\partial_t c = -\frac{1}{\varrho}\Delta(\frac{1}{\varrho}\Delta c) - \mathbf{u}\cdot\nabla c, \\ &\partial_{\mathbf{n}}(\frac{1}{\varrho}\Delta c) = 0 \text{ on } \Gamma, \\ &c = c_\Gamma \text{ on } \Gamma, \\ &c(0,\cdot) = c_0, \end{aligned} \tag{45}$$

admits a unique solution $c \in X_\Omega$, where $X_\Omega = W^{1,p}(0,T,L^p(\Omega)) \cap L^p(0,T,W^{4,p}(\Omega))$, with $1 < p < \infty$. Moreover,

$$\|c\|_{X_\Omega} \leq C(\|c_0\|_{\mathcal{C}^2(\bar{\Omega})} + \|c_\Gamma\|_{X_\Gamma}),$$

where C is a positive constant.

Given c a solution of (45), we first note that $\partial_{\mathbf{n}} c \in L^p(0,T,W^{3-\frac{1}{p}}(\Gamma))$ and

$$\|\partial_n c\|_{L^p(0,T,W^{3-\frac{1}{p}}(\Gamma))} \leq C(\|c_0\|_{\mathcal{C}^2(\bar{\Omega})} + \|c_\Gamma\|_{X_\Gamma}).$$

Now, let us consider the following parabolic problem on the boundary Γ of the domain:

$$\begin{aligned} &\partial_t \chi - \Delta_\tau \chi = -\mathbf{u}_\tau \cdot \nabla_\tau \chi - \partial_{\mathbf{n}} c, \\ &\chi(0) = c_0|_\Gamma. \end{aligned} \tag{46}$$

According to the maximal L^p regularity for parabolic problems (see [2]), we obtain:

$$\partial_t \chi \in L^p(0,T,W^{3-\frac{1}{p},p}(\Gamma)), \quad \chi \in L^p(0,T,W^{5-\frac{1}{p},p}(\Gamma)), \tag{47}$$

with

$$\|\partial_t \chi\|_{L^p(0,T,W^{3-\frac{1}{p},p}(\Gamma))} + \|\chi\|_{L^p(0,T,W^{5-\frac{1}{p},p}(\Gamma))} \leq C(\mathbf{u},p)(\|c_0\|_{\mathcal{C}^2(\bar{\Omega})} + \|\partial_n c\|_{L^p(0,T,W^{3-\frac{1}{p}}(\Gamma))}). \tag{48}$$

Consequently, the mapping $\mathcal{T} : c_\Gamma \to \chi$ is a compact mapping on X_Γ. Our ultimate goal is to apply Schaefer's fixed point theorem. To this end, we have to establish suitable bounds on the set of c_Γ's satisflying $\lambda\mathcal{T}(c_\Gamma) = c_\Gamma$ with $\lambda \in [0,1]$. Thus, using the equations:

$$\begin{aligned} &\partial_t c = -\frac{1}{\varrho}\Delta(\frac{1}{\varrho}\Delta c) - \mathbf{u}\cdot\nabla c, \\ &\partial_{\mathbf{n}}(\frac{1}{\varrho}\Delta c) = 0, \quad c = c_\Gamma \text{ on } \Gamma, \\ &\partial_t c_\Gamma - \Delta_\tau c_\Gamma = -\mathbf{u}_\tau \cdot \nabla_\tau c_\Gamma - \lambda\partial_{\mathbf{n}} c, \end{aligned} \tag{49}$$

with $\lambda \in [0,1]$, we obtain estimates on c by multiplying equation $(49)_1$ by $-\Delta c$ and integrating over the domain. We get:

$$\begin{aligned}(50)\qquad &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big\{\int_\Omega |\nabla c|^2\,\mathrm{d}x + \frac{1}{\lambda}\int_\Gamma |\nabla_\tau c_\Gamma|^2\,\mathrm{d}\mathcal{H}^2\Big\} + \frac{1}{\lambda}\int_\Gamma |\partial_t c_\Gamma|^2\,\mathrm{d}\mathcal{H}^2 + \int_\Omega |\nabla(\frac{1}{\varrho}\Delta c)|^2\,\mathrm{d}x\\ &= -\frac{1}{\lambda}\int_\Gamma \partial_t c_\Gamma \mathbf{u}_\tau\cdot\nabla_\tau c_\Gamma\,\mathrm{d}\mathcal{H}^2 + \int_\Omega \mathbf{u}\cdot\nabla c\Delta c\,\mathrm{d}x.\end{aligned}$$

We need to estimate the terms on the right hand side of (50). We have:

$$\begin{aligned}(51)\qquad |\frac{1}{\lambda}\int_\Gamma \partial_t c_\Gamma \mathbf{u}_\tau\cdot\nabla_\tau c_\Gamma\,\mathrm{d}\mathcal{H}^2| &\le \frac{1}{\lambda}\|\partial_t c_\Gamma\|_{L^2(\Gamma)}\|\mathbf{u}_\tau\|_{L^\infty(\Gamma)}\|\nabla_\tau c_\Gamma\|_{L^2(\Gamma)}\\ &\le \frac{1}{4\lambda}\|\partial_t c_\Gamma\|^2_{L^2(\Gamma)} + \frac{C}{\lambda}\|\mathbf{u}_\tau\|^2_{L^\infty(\Gamma)}\|\nabla_\tau c_\Gamma\|^2_{L^2(\Gamma)},\end{aligned}$$

and also:

$$(52)\qquad |\int_\Omega \mathbf{u}\cdot\nabla c\Delta c\,\mathrm{d}x| \le \|\mathbf{u}\|_{L^\infty(\Omega)}\|\varrho\|_{L^\infty(\Omega)}\|\nabla c\|_{L^2(\Omega)}\|\frac{1}{\varrho}\Delta c\|_{L^2(\Omega)}.$$

Using Lemma 2.1, we also have:

$$(53)\qquad \|\frac{1}{\varrho}\Delta c\|_{L^2(\Omega)} \le C\big(\|\nabla(\frac{1}{\varrho}\Delta c)\|_{L^2(\Omega)} + |\int_\Omega \Delta c\,\mathrm{d}x|\big).$$

The last term in (53) is estimated as follows:

$$\begin{aligned}(54)\qquad |\int_\Omega \Delta c\,\mathrm{d}x| &= |\int_\Gamma \partial_{\mathbf{n}} c\,\mathrm{d}\mathcal{H}^2| = |\frac{1}{\lambda}\int_\Gamma (\partial_t c_\Gamma - \Delta_\tau c_\tau + \mathbf{u}_\tau\cdot\nabla_\tau c_\Gamma)\,\mathrm{d}\mathcal{H}^2|\\ &\le C(\frac{1}{\lambda}\|\partial_t c\|_{L^2(\Gamma)} + \frac{1}{\lambda}\|\mathbf{u}_\tau\|_{L^2(\Gamma)}\|\nabla_\tau c_\Gamma\|_{L^2(\Gamma)}).\end{aligned}$$

Returning to (52), we get:

(55)

$$\begin{aligned}|\int_\Omega \mathbf{u}\cdot\nabla c\Delta c\,\mathrm{d}x| &\le \|\mathbf{u}\|_{L^\infty(\Omega)}\|\varrho\|_{L^\infty(\Omega)}\|\nabla c\|\big\{\|\nabla(\frac{1}{\varrho}\Delta c))\|_{L^2(\Omega)} + \frac{1}{\lambda}\|\partial_t c\|_{L^2(\Gamma)} + \|\mathbf{u}_\tau\|_{L^2(\Gamma)}\|\nabla_\tau c_\Gamma\|_{L^2(\Gamma)}\big\}\\ &\le \frac{1}{2}\|\nabla(\frac{1}{\varrho}\Delta c))\|^2_{L^2(\Omega)} + \frac{1}{4\lambda}\|\partial_t c\|^2_{L^2(\Gamma)} + C\|\mathbf{u}\|^2_{L^\infty(\Omega)}\|\varrho\|^2_{L^\infty(\Omega)}\|\nabla c\|^2_{L^2(\Omega)}\\ &+ C\|\mathbf{u}\|_{L^\infty(\Omega)}\|\varrho\|_{L^\infty(\Omega)}\|\mathbf{u}_\tau\|_{L^2(\Gamma)}\|\nabla c\|^2_{L^2(\Omega)} + \frac{C}{\lambda}\|\mathbf{u}\|_{L^\infty(\Omega)}\|\varrho\|_{L^\infty(\Omega)}\|\mathbf{u}_\tau\|_{L^2(\Gamma)}\|\nabla_\tau c_\Gamma\|^2_{L^2(\Gamma)}.\end{aligned}$$

We finally obtain:

$$\begin{aligned}(56)\qquad &\frac{\mathrm{d}}{\mathrm{d}t}\Big\{\int_\Omega |\nabla c|^2\,\mathrm{d}x + \frac{1}{\lambda}\int_\Gamma |\nabla_\tau c_\Gamma|^2\,\mathrm{d}\mathcal{H}^2\Big\} + \frac{1}{\lambda}\int_\Gamma |\partial_t c_\Gamma|^2\,\mathrm{d}\mathcal{H}^2 + \int_\Omega |\nabla(\frac{1}{\varrho}\Delta c)|^2\,\mathrm{d}x\\ &\le C\|\mathbf{u}\|_{L^\infty(\Omega)}\|\varrho\|_{L^\infty(\Omega)}\|\mathbf{u}_\tau\|_{L^2(\Gamma)}\|\nabla c\|^2_{L^2(\Omega)} + \frac{C}{\lambda}\|\mathbf{u}\|_{L^\infty(\Omega)}\|\varrho\|_{L^\infty(\Omega)}\|\mathbf{u}_\tau\|_{L^2(\Gamma)}\|\nabla_\tau c_\Gamma\|^2_{L^2(\Gamma)}.\end{aligned}$$

Using the Gronwall lemma, we have:

$$\int_\Gamma |\nabla_\tau c_\Gamma(x,t)|^2 \,\mathrm{d}\mathcal{H}^2 + \int_0^T \int_\Gamma |\partial_t c_\Gamma|^2 \,\mathrm{d}\mathcal{H}^2 \,\mathrm{d}t \le C(c_0, \|\mathbf{u}\|_{\mathcal{C}([0,T]\times\bar\Omega)}, \|\varrho\|_{\mathcal{C}([0,T]\times\bar\Omega)}). \tag{57}$$

It follows immediately that c_Γ is bounded in $L^\infty(0,T,H^1(\Gamma)) \cap H^1(0,T,H^1(\Gamma))$, which is compactly embedded in X_Γ. From Schaefer's fixed point theorem, the result follows. □

In view of Lemma 4.1, the approximate solutions for fixed parameters m, ϵ, and δ can be obtained by means of a fixed point argument, exactly as in [16]. The rest of the existence proof consists in performing successively the limit processes $m \to \infty$, $\epsilon \to 0$, and, finally $\delta \to 0$. This is definitely a very lengthy and technical procedure which consists in applying the formal compactness arguments, discussed in Section 3, to the family of approximate solutions. This process has been described in detail e.g. in [1] and we therefore omit it here to keep the presentation concise.

Acknowledgements: This work was supported by the mobility project PHC Barrande 2017 3070YE/7AMB17FR053 of collaboration between France and Czech Republic.

References

[1] H. Abels, E. Feireisl. *On a diffuse interface model for a two-phase flow of compressible viscous fluids*, *Indiana University Mathematics Journal*, 57(2), pp. 659–698, 2008

[2] H. Amann. *Linear and Quasilinear Parabolic problems*, Vol. 1, Abstract linear theory. *Monographs in Mathematics*, 89, Birkhäuser Boston, Inc. Boston, MA, 1995

[3] D. M. Anderson, G. B. McFadden and A. A. Wheeler. *Diffuse-interface methods in fluid mechanics*, *Annual review of fluid mechanics*, vol. 30, pp. 139–165, Annual Reviews, Palo Alto, CA, 1998

[4] D. Bresch, P.-E. Jabin. *Global weak solutions of PDEs for compressible media: A compactness criterion to cover new physical situations*, *Shocks, singularities and oscillations in nonlinear optics and fluid mechanics*, pp. 33–54, Springer INdAM Ser., 17, Springer, Cham, 2017

[5] L. Cherfils, M. Petcu. *On the viscous Cahn-Hilliard-Navier-Stokes equations with dynamic boundary conditions*, Comm. Pure Appl. Anal., 14, pp. 1419–1449, 2016

[6] M. Doi. *Dynamics of domains and textures*, Theoretical Challenges in the Dynamics of Complex Fluids, pp. 293–314, 1996

[7] R. Denk, M. Hieber, J. Prüss. *Optimal L^p-L^q-estimates for parabolic boundary value problems with inhomogeneous data*, Math. Z., 257, pp. 193–224, 2007

[8] S. Dong. *On imposing dynamic contact-angle boundary conditions for wall-bounded liquid-gas flows*, Comput. Methods Appl. Mech. Engrg., pp. 179–200, 2012

[9] E.B. Dussan V. *The moving contact line: the slip boundary condition*, J. Fluid Mech., 77, pp. 665–684, (1976)

[10] E.B. Dussan V, S.H. Davis. *On the motion of a fluid-fluid interface along a solid surface*, J. Fluid Mech., 65, pp. 71–95, (1974)

[11] E. Feireisl. *Dynamics of Viscous Compressible Fluids*, Oxford University Press, Oxford, 2014

[12] E. Feireisl. *On the motion of a viscous, compressible, and heat conducting fluid*, *Indiana Univ. Math. J.*, 53(6), pp. 1705–1738, 2004

[13] E. Feireisl, P. Laurençot. *Non-isothermal Smoluchowski-Poisson equations as a singular limit of the Navier-Stokes-Fourier-Poisson system*, *J. Math. Pures Appl.*, 88, pp. 325–349, 2007

[14] E. Feireisl, A. Novotný. *Singular Limits in Thermodynamics of Viscous Fluids*, Springer International Publishing, 2017

[15] E. Feireisl, A. Novotný and H. Petzeltová. *On the existence of globally defined weak solutions to the Navier-Stokes equations*, *J. Math. Fluid Mech.*, 3(4), pp. 358–392, 2001

[16] E. Feireisl, M. Petcu. *An interface model of a two-phase flow with thermal fluctuations*, preprint 2018

[17] H.P. Fischer, J. Reinhard, W. Dieterich, J.-F. Gouyet, P. Maass, A. Majhofer, D. Reinel. *Time-dependent density functional theory and the kinetics of lattice gas systems in contact with a wall*, J. Chem. Phys., 108, pp. 3028–3037, 1998

[18] C. Gal. *A Cahn-Hilliard model in bounded domains with permeable walls*, Math. Methods Appl. Sci., 29 (2006), pp. 2009–2036.

[19] C. Gal. *Exponential attractors for a Cahn-Hilliard model in bounded domains with permeable walls*, Electron. J. Differential Equations, (2006), pp. 1–23 (electronic).

[20] C. Gal, M. Grasselli, A. Miranville. *Cahn-Hilliard-Navier-Stokes system with moving contact lines*, *Calc. Var. Partial Differential Equations*, 55(33), 2016.

[21] G. P. Galdi. *An introduction to the mathematical theory of the Navier-Stokes equations, Vol. I.* Springer-Verlag, New York, 1994

[22] G. Gilardi, A. Miranville, G. Schimperna. *On the Cahn-Hilliard equation with irregular potentials and dynamic boundary conditions*, Commun. Pure Appl. Anal., 8, pp. 881–912, 2009

[23] G. Gilardi, A. Miranville, G. Schimperna. *Long time behavior of the Cahn-Hilliard equation with irregular potentials and dynamic boundary conditions*, Chinese Ann. Math., Ser. B, 31, pp. 679–712, 2010

[24] G. Ruiz Goldstein, A. Miranville, G. Schimperna. *A Cahn-Hilliard equation in a domain with non-permeable walls*, Phys. D, 240, pp. 754–766, 2011

[25] M. Heida. *On the derivation of thermodynamically consistent boundary conditions for the Cahn-Hilliard-Navier-Stokes system*, Internat. J. Engrg. Sci., 62, pp. 126–156, 2013

[26] M. Heida, J. Málek, K. R. Rajagopal. *On the development and generalizations of Cahn-Hilliard equations within a thermodynamic framework*, Z. Angew. Math. Phys., 63, pp. 145–169, 2012

[27] R. Kenzler, F. Eurich, Ph. Maass, B. Rinn, J. Schropp, E. Bohl, W. Dieterich. *Phase separation in confined geometries: solving the Cahn-Hilliard equation with generic boundary conditions*, Comput. Phys. Comm., 133 (2001), pp. 139-157.

[28] J. L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires*, DUNOD, 2002

[29] P.-L. Lions. *Mathematical topics in fluid mechanics. Vol. 2*, volume 10 of *Oxford Lecture Series in Mathematics and its Applications*. The Clarendon Press, Oxford University Press, New York, 1998. Compressible models, Oxford Science Publications.

[30] C. Liu, J. Shen. *A phase field model for the mixture of two incompressible fluids and its approximation by a Fourier-spectral method*, Phys. D, 179, pp. 211–228, 2003

[31] J. Lowengrub, L. Truskinovsky. *Quasi–incompressible Cahn–Hilliard fluids and topological transitions*, *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, 454(1978), pp. 2617–2654, 1998.

[32] A. Miranville, S. Zelik. *Exponential attractors for the Cahn-Hilliard equation with dynamic boundary conditions*, Math. Methods Appl. Sci., 28, pp. 709–735, 2005

[33] A. Miranville, S. Zelik. *The Cahn-Hilliard equation with singular potentials and dynamic boundary conditions*, Discrete Contin. Dynam. Systems, 28, pp. 275–310, 2010

[34] H.K. Moffatt. *Viscous and resistive eddies near a sharp corner*, J. Fluid Mech., 18, pp. 1–18 (1964)

[35] C. L. Navier. *Sur les lois de l'équilibre et du mouvement des corps élastiques*, *Mem. Acad. R. Sci. Inst. France*, 6(369), 1827.

[36] J. Prüss, R. Racke, S. Zheng. *Maximal regularity and asymptotic behavior of solutions for the Cahn-Hilliard equation with dynamic boundary conditions*, Ann. Mat. Pura Appl. (4), **185** (2006), pp. 627–648.

[37] T. Qian, X. P. Wang, P. Sheng. *Molecular scale contact line hydrodynamics of immiscible flows*, Physical Reviews E, 68, 016306, 2003

[38] T. Qian, X. P. Wang, P. Sheng. *A variational approach to moving contact line hydrodynamics*, *J. Fluid Mech.*, 564, pp. 333–360, 2006.

[39] R. Racke, S. Zheng. *The Cahn-Hilliard equation with dynamic boundary conditions*, Adv. Differential Equations, **8** (2003), pp. 83–110.

[40] J. Shen, X. Yang. *Decoupled, energy stable schemes for phase-field models of two-phase incompressible flows*, SIAM J. Num. Anal., 53, pp. 279–296, 2015

[41] J. Shen, X. Yang, H. Yu. *Efficient energy stable numerical schemes for a phase field moving contact line model*, J. Comput. Physics, 284, pp. 617–630, 2015

[42] Michael E. Taylor. *Partial differential equations I. Basic theory*, volume 115 of *Applied Mathematical Sciences*. Springer, New York, second edition, 2011.

[43] H. Wu, S. Zheng. *Convergence to equilibrium for the Cahn-Hilliard equation with dynamic boundary conditions*, J. Differential Equations, **204** (2004), pp. 511–531.

[44] X. Yang, H. Yu. *Linear, second order and unconditionally energy stable schemes for a phrase-field moving contact line model*, arXiv:1703.01311v1, 2017

[45] B. You, F. Li, C. Zhang. *Finite dimensional global attractor of the Cahn-Hilliard-Navier-Stokes system with dynamic boundary conditions*, Commun. Math. Sci., 1, pp. 53–76,2018