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A Stabilized FE-FV scheme for the compressible Navier-Stokes-Fourier system

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Outline

- Part I: Introduction
- Part II: Numerical method
- Part III: Stability
- Part IV: Test

Navier-Stokes-Fourier system for the compressible, viscous and heat conducting flow.

$$\rho_t + \nabla \cdot (\rho \mathbf{u}) = 0. \tag{1a}$$

$$(\rho \mathbf{u})_t + \mathrm{div}(\rho \mathbf{u} \otimes \mathbf{u}) = -\nabla p + \mathrm{div}\mathbb{S}(\nabla \mathbf{u}). \tag{1b}$$

$$c_v\big((\rho\theta)_t + \mathrm{div}(\rho\theta\mathbf{u})\big) + \mathrm{div}\mathbf{q}(\theta, \nabla\theta) = \mathbb{S}(\nabla \mathbf{u}) : \nabla \mathbf{u} - \theta \frac{\partial p(\rho,\theta)}{\partial \theta}\mathrm{div}\mathbf{u}. \tag{1c}$$

$\rho, p, \mathbf{u}, \theta$ are the fluid density, pressure, velocity and temperature.
$c_v > 0$ is the specific heat per volume.

$$\boxed{\partial_t(\frac{1}{2}\rho\mathbf{u}^2 + \rho e) + \mathrm{div}\left(u(\frac{1}{2}\rho\mathbf{u}^2 + \rho e + p)\right) = \cdots}$$

Boundary condition

$$\mathbf{u}|_{\partial\Omega} = 0, \quad \mathbf{n} \cdot \nabla\theta|_{\partial\Omega} = 0. \tag{1d}$$

Initial values are

$$\rho(\mathbf{x}, 0) = \rho_0 > 0, \quad \mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0, \quad \theta(\mathbf{x}, 0) = \theta_0 > 0. \tag{1e}$$

$$\mathbb{S} = \mu(\nabla \mathbf{u} + \nabla \mathbf{u}^T - \frac{2}{3}\text{div}\mathbf{u}\mathbf{I}) + \eta \text{div}\mathbf{u}\mathbf{I} = 2\mu \mathbf{D}(\mathbf{u}) + \nu \text{div}\mathbf{u}\mathbf{I},$$

$$\text{div}\mathbb{S}(\nabla \mathbf{u}) = 2\mu \text{div}\mathbf{D}(\mathbf{u}) + \nu \nabla \text{div}\mathbf{u}, \quad \mathbf{D}(\mathbf{u}) = \frac{\nabla \mathbf{u} + \nabla \mathbf{u}^T}{2},$$

$$\mathbb{S}(\nabla \mathbf{u}) : \nabla \mathbf{u} = 2\mu |\mathbf{D}(\mathbf{u})|^2 + \nu |\text{div}\mathbf{u}|^2, \quad \mu > 0, \quad \nu = \eta - \frac{2}{3}\mu > 0.$$

The heat flux $\mathbf{q}$ follows Fourier's law

$$\mathbf{q} = -\kappa(\theta)\nabla\theta = -\nabla\mathcal{K}(\theta), \ \text{div}\mathbf{q} = -\Delta\mathcal{K}, \ \mathcal{K}(\theta) = \int_0^\theta \kappa(z)dz.$$

$$p = p^e + p^t, \quad p^e = a\rho^\gamma + b\rho, \quad p^t = \rho\theta, a, b > 0.$$

$$\frac{\partial p}{\partial \theta} = \frac{\partial p^t}{\partial \theta} = \rho.$$

Convergence to weak solution.

E. Feireisl. Dynamics of viscous compressible fluids. Oxford University Press, Oxford, 2004.
E. Feireisl, T. Karper, A. Novotny. A convergent numerical method for the NavierStokesFourier system. Preprint.

$$\mathrm{div}\mathbb{S}(\nabla \mathbf{u}) = \mu \Delta \mathbf{u} + \lambda \nabla \mathrm{div}\mathbf{u}, \quad \mathbb{S}(\nabla \mathbf{u}) : \nabla \mathbf{u} = \mu |\nabla \mathbf{u}|^2 + \lambda |\mathrm{div}\mathbf{u}|^2$$

$$\mathrm{div}\mathbb{S}(\nabla \mathbf{u}) = 2\mu \mathrm{div}\mathbf{D}(\mathbf{u}) + \nu \nabla \mathrm{div}\mathbf{u}, \quad \mathbb{S}(\nabla \mathbf{u}) : \nabla \mathbf{u} = 2\mu |\mathbf{D}(\mathbf{u})|^2 + \nu |\mathrm{div}\mathbf{u}|^2$$

Numerical implementation

Stability $\frac{d}{dt} E \leq 0.$

Introduction - - Motivation II

$$E = Ke + c_v \mathcal{E} + H.$$

$$Ke = \frac{1}{2}\rho \mathbf{u}^2, \quad \mathcal{E} = \rho\theta = p^t, \quad H = \rho \int_1^\rho \frac{p^e(z)}{z^2}$$

$$H'' \geq 0.$$

$$(\frac{H}{\rho})' = -\frac{H}{\rho^2} + \frac{H'}{\rho}$$

$$p^e = \rho H' - H.$$

Functional spaces

Piecewise linear Crouzeix-Raviart element for velocity.

$$V_{0,h} \equiv \{\mathbf{v}_h \in L^2(\Omega_h); \quad \mathbf{v}_h|_K \in \mathcal{P}^1(K), \forall K \in \Omega_h; \\ \int_\Gamma [\![\mathbf{v}_h]\!] = 0, \forall \Gamma \in \mathcal{E}^{int}; \quad \int_\Gamma \mathbf{v}_h = 0, \forall \Gamma \in \mathcal{E}^{ext}\}.$$

Piecewise constant element for density, pressure and temperature

$$Q_h \equiv \{\phi_h \in L^2(\Omega_h); \phi_h|_K \in \mathcal{P}^0(K), K \in \Omega_h\}.$$

- $\mathcal{E}$ edges
- $\mathcal{E}^{ext} = \mathcal{E} \cap \partial\Omega$ exterior edges
- $\mathcal{E}^{int} = \mathcal{E} \setminus \mathcal{E}^{ext}$ interior edges
- K, L element
- $\Gamma = K \cap L$
- $\mathbf{n}_{\Gamma,K}$ be the outer normal, pointing from K to L

Upwind flux

$$\mathcal{F}^{up}(f,\mathbf{u})|_\Gamma = \begin{cases} f_K & \text{if } s_{\Gamma,K} \geq 0, \\ f_L & \text{else,} \end{cases}$$

where $s^n_{\Gamma,K} = \mathbf{u}^n|_\Gamma \cdot \mathbf{n}_{\Gamma,K} = s^{n,+}_{\Gamma,K} + s^{n,-}_{\Gamma,K}$,

$$s^{n,+}_{\Gamma,K} = \max\{0, s^n_{\Gamma,K}\} \geq 0, \quad s^{n,-}_{\Gamma,K} = \min\{0, s^n_{\Gamma,K}\} \leq 0.$$

$$s^{n,-}_{\Gamma,L} = -s^{n,+}_{\Gamma,K}, \quad s^{n,-}_{\Gamma,K} = -s^{n,+}_{\Gamma,L}, \quad s^n_{\Gamma,L} = -s^n_{\Gamma,K}.$$

Jump

$$[\![f]\!]_\Gamma = f_L - f_K.$$

Average on K

$$\hat{f}_K = \frac{1}{|K|}\int_K f dx.$$

Average on edge

$$\{f\}_\Gamma = \frac{1}{2}(f_K + f_L).$$

Implicit nonlinear FV-FE Scheme

Find $\{(\rho_h^{n+1}, \mathbf{u}_h^{n+1}, \theta_h^{n+1})\}_{n=0}^{n_T-1} \subset (Q_h \times V_{0,h} \times Q_h)$ such that

$$\sum_{K\in\Omega_h} \int_K \frac{\rho_h^{n+1} - \rho_h^n}{\Delta t}\phi_h - \sum_{K\in\Omega_h} \sum_{\Gamma\in\partial K} \int_\Gamma \mathcal{F}^{up}(\rho_h^{n+1}, \mathbf{u}_h^{n+1})[\![\phi_h]\!] + h^\alpha \sum_{K\in\Omega_h} \sum_{\Gamma\in\partial K} \int_\Gamma [\![\rho_h^{n+1}]\!][\![\phi_h]\!] = 0, \quad (2a)$$

for any $\phi_h \in Q_h$.

$$0 < \alpha < 1.$$

$$\sum_{K\in\Omega_h}\int_K \frac{\mathbf{q}_h^{n+1}-\mathbf{q}_h^{n}}{\Delta t}\mathbf{v}_h - \sum_{K\in\Omega_h}\sum_{\Gamma\in\partial K}\int_\Gamma \mathcal{F}^{up}\left(\mathbf{q}_h^{n+1},\mathbf{u}_h^{n+1}\right)[\![\hat{\mathbf{v}}_h]\!]$$

$$-\sum_{K\in\Omega_h}\int_K p_h^{n+1}\mathrm{div}_h\mathbf{v}_h + 2\mu\sum_{K\in\Omega_h}\int_K \mathbf{D}(\mathbf{u}_h^{n+1})\mathbf{D}(\mathbf{v}_h) + \nu\sum_{K\in\Omega_h}\int_K \mathrm{div}_h\mathbf{u}_h^{n+1}\mathrm{div}_h\mathbf{v}_h$$

$$+h^\alpha\sum_{K\in\Omega_h}\sum_{\Gamma\in\partial K}\int_\Gamma [\![\rho_h^{n+1}]\!]\{\hat{\mathbf{u}}_h^{n+1}\}[\![\hat{\mathbf{v}}_h]\!] + 2\mu\sum_{K\in\Omega_h}\sum_{\Gamma\in\partial K}\int_\Gamma \frac{1}{h}[\![\mathbf{u}_h^{n+1}]\!][\![\mathbf{v}_h]\!] = 0, \tag{2b}$$

for any $\mathbf{v}_h \in V_{0,h}$.
$\mathbf{q}_h$ is the momentum, piecewise constant for all $K \in \Omega_h$

$$\mathbf{q}_K = \rho_K \hat{\mathbf{u}}_K, \quad \hat{\mathbf{u}}_K = \frac{1}{|K|}\int_K \mathbf{u}_h$$

Numerical Scheme III

$$c_v \sum_{K\in\Omega_h} \int_K \frac{\mathcal{E}_h^{n+1} - \mathcal{E}_h^n}{\Delta t}\phi_h - c_v \sum_{K\in\Omega_h}\sum_{\Gamma\in\partial K}\int_\Gamma \mathcal{F}^{up}(\mathcal{E}_h^{n+1}, \mathbf{u}_h^{n+1})[\![\phi_h]\!]$$
$$+\sum_{K\in\Omega_h}\sum_{\Gamma\in\partial K}\int_\Gamma \frac{1}{d_\Gamma}[\![\mathcal{K}(\theta_h^{n+1})]\!][\![\phi_h]\!] = \sum_{K\in\Omega_h}\int_K (2\mu|\mathbf{D}(\mathbf{u}_h^{n+1})|^2 + \nu|\mathrm{div}\mathbf{u}_h^{n+1}|^2)\phi_h$$
$$+ 2\mu \sum_{K\in\Omega_h}\sum_{\Gamma\in\partial K}\int_\Gamma \frac{1}{h}[\![\mathbf{u}_h^{n+1}]\!]^2 - \sum_{K\in\Omega_h}\int_K \mathcal{E}_h^{n+1}\mathrm{div}_h\mathbf{u}_h^{n+1}\phi_h, \quad (2c)$$

for any $\phi_h \in Q_h$.

Iterative solver, linear or nonlinear ?

- Picard
  Let $\mathbf{w} = (\rho, \mathbf{u}, \theta)$. Given the solution $\mathbf{w}^{n,\ell}(\mathbf{w}^{n,0} = \mathbf{w}^n)$, and solve a linear system to get $\mathbf{w}^{n,\ell+1}$ for $\ell = 1, 2, \cdots$, until $\|\mathbf{w}^{n,\ell+1}\| - \|\mathbf{w}^{n,\ell}\| < \text{tol}$. $\quad \mathbf{w}^{n+1} = \mathbf{w}^{n,\ell+1}$.

$$\sum_{K\in\Omega_h} \int_K \frac{\rho_h^{n,\ell+1} - \rho_h^n}{\Delta t} \phi_h + h^\alpha \sum_{K\in\Omega_h} \sum_{\Gamma\in\partial K} \int_\Gamma [\![\rho_h^{n,\ell+1}]\!] [\![\phi_h]\!]$$
$$= \sum_{K\in\Omega_h} \sum_{\Gamma\in\partial K} \int_\Gamma \mathcal{F}^{up}(\rho_h^{n,\ell}, \mathbf{u}_h^{n,\ell}) [\![\phi_h]\!].$$

- inexact-Newton
  Jacobian Free Newton Krylov(JFNK)

Stability

Positivity preserving

Lemma 1

Suppose that $\rho_h^{n+1} \in Q_h(\Omega_h)$ *satisfies* (2a), *where* $\rho_h^n > 0$ *in* Ω_h *and* $\mathbf{u}_h^{n+1} \in V_{0,h}(\Omega_h)$. *Then*

$$\rho_h^{n+1} > 0, \ \ in\ \Omega_h. \tag{3}$$

Stability of total energy

Theorem 2

Let $(u_h^{n+1}, \rho_h^{n+1}, \theta_h^{n+1})$ *be the solution of the Scheme* (2) *and the density is initially positive. Then the total energy is dissipative in time*

$$E_{\Omega_h}^{n+1} \leq E_{\Omega_h}^{n}. \tag{4}$$

Stability - - Step 1. density scheme I

$$\sum_{K\in\Omega_h}\int_K \frac{\rho_h^{n+1}-\rho_h^n}{\Delta t}\phi_h - \sum_{K\in\Omega_h}\sum_{\Gamma\in\partial K}\int_\Gamma \mathcal{F}^{up}(\rho_h^{n+1},\mathbf{u}_h^{n+1})[\![\phi_h]\!] + h^\alpha \sum_{K\in\Omega_h}\sum_{\Gamma\in\partial K}\int_\Gamma [\![\rho_h^{n+1}]\!][\![\phi_h]\!] = 0,$$

$$\phi_h = -\frac{1}{2}|\hat{\mathbf{u}}_h^{n+1}|^2 \longrightarrow T_{11}+T_{12}+T_{13}=0.$$

$$T_{11} := -\frac{1}{2\Delta t}\sum_{K\in\Omega_h}|K|(\rho_K^{n+1}-\rho_K^n)|\hat{\mathbf{u}}_K^{n+1}|^2,$$

$$T_{12} := -\frac{1}{2}\sum_{K\in\Omega_h}\sum_{\Gamma\in\partial K}\int_\Gamma \mathcal{F}^{up}(\rho_h^{n+1},\mathbf{u}_h^n)|\hat{\mathbf{u}}_K^{n+1}|^2,$$

$$T_{13} := -\frac{1}{2}h^\alpha\sum_{K\in\Omega_h}\sum_{\Gamma\in\partial K}\int_\Gamma [\![\rho_h^{n+1}]\!][\![|\hat{\mathbf{u}}_h^{n+1}|^2]\!].$$

Stability - - Step 1. density scheme II

$$\phi_h = H'(\rho_h^{n+1}) \longrightarrow T_{14} + T_{15} \geq 0.$$

$$T_{14} := \frac{1}{\Delta t} \sum_{K \in \Omega_h} |K| (\rho_K^{n+1} - \rho_K^n) H'(\rho_K^{n+1})$$

$$\geq \frac{1}{\Delta t} \sum_{K \in \Omega_h} |K| \Big(H(\rho_K^{n+1}) - H(\rho_K^n) \Big) = \frac{1}{\Delta t} (H_{\Omega_h}^{n+1} - H_{\Omega_h}^n),$$

$$T_{15} := \sum_{K \in \Omega_h} \sum_{\Gamma \in \partial K} \int_\Gamma \mathcal{F}^{up}(\rho_h^{n+1}, \mathbf{u}_h^n) H'(\rho_h^{n+1}).$$

$$H(\rho^n) = H(\rho^{n+1}) + H'(\rho^{n+1})(\rho^n - \rho^{n+1}) + \frac{H''(\eta_0)}{2}(\rho^n - \rho^{n+1})^2$$

$$\eta_0 \in \mathrm{co}\{\rho^n, \rho^{n+1}\}, \quad \mathrm{co}\{a, b\} = [\min(a, b), \max(a, b)].$$

Stability - - Step 2. momentum scheme

$$\sum_{K\in\Omega_h}\int_K \frac{\mathbf{q}_h^{n+1}-\mathbf{q}_h^n}{\Delta t}\mathbf{v}_h - \sum_{K\in\Omega_h}\sum_{\Gamma\in\partial K}\int_\Gamma \mathcal{F}^{up}(\mathbf{q}_h^{n+1},\mathbf{u}_h^{n+1})[\![\hat{\mathbf{v}}_h]\!]$$

$$-\sum_{K\in\Omega_h}\int_K p_h^{n+1}\mathrm{div}_h\mathbf{v}_h + 2\mu\sum_{K\in\Omega_h}\int_K \mathbf{D}(\mathbf{u}_h^{n+1})\mathbf{D}(\mathbf{v}_h) + \nu\sum_{K\in\Omega_h}\int_K \mathrm{div}_h\mathbf{u}_h^{n+1}\mathrm{div}_h\mathbf{v}_h$$

$$+ h^\alpha\sum_{K\in\Omega_h}\sum_{\Gamma\in\partial K}\int_\Gamma [\![\rho_h^{n+1}]\!]\{\hat{\mathbf{u}}_h^{n+1}\}[\![\hat{\mathbf{v}}_h]\!] + 2\mu\sum_{K\in\Omega_h}\sum_{\Gamma\in\partial K}\int_\Gamma \frac{1}{h}[\![\mathbf{u}_h^{n+1}]\!][\![\mathbf{v}_h]\!] = 0,$$

$$\mathbf{v}_h = \mathbf{u}_h^{n+1} \longrightarrow T_{21}+T_{22}+T_{23}+T_{24}+T_{25}+T_{26} = 0.$$

$$T_{21} := \frac{1}{\Delta t}\sum_{K\in\Omega_h}|K|\left(\rho_K^{n+1}|\hat{\mathbf{u}}_K^{n+1}|^2 - \rho_K^n\hat{\mathbf{u}}_K^n\hat{\mathbf{u}}_K^{n+1}\right),$$

$$T_{22} := \sum_{K\in\Omega_h}\sum_{\Gamma\in\partial K}\int_\Gamma \mathcal{F}^{up}(\mathbf{q}_h,\mathbf{u}_h)^{n+1}\hat{\mathbf{u}}_K^{n+1},\ T_{23} := -\sum_{K\in\Omega_h}\sum_{\Gamma\in\partial K}\int_\Gamma (p^e)_K^{n+1}s_\Gamma^{n+1}$$

$$T_{24} := \sum_{K\in\Omega_h}\int_K 2\mu|\mathbf{D}(\mathbf{u}_h^{n+1})|^2 + \nu|\mathrm{div}\mathbf{u}_h^{n+1}|^2 + 2\mu\sum_{K\in\Omega_h}\sum_{\Gamma\in\partial K}\int_\Gamma \frac{1}{h}[\![\mathbf{u}_h^{n+1}]\!]^2,$$

$$T_{25} := h^\alpha\sum_{K\in\Omega_h}\sum_{\Gamma\in\partial K}\int_\Gamma [\![\rho_h^{n+1}]\!]\{\hat{\mathbf{u}}_h^{n+1}\}[\![\hat{\mathbf{u}}_h^{n+1}]\!],\qquad T_{26} := -\sum_{K\in\Omega_h}\int_K (p^t)_h^{n+1}\mathrm{div}\mathbf{u}_h^{n+1}$$

Term $T_{11} + T_{21}$

$$
\begin{aligned}
&T_{11} + T_{21} \\
&= -\frac{1}{2\Delta t} \sum_{K \in \Omega_h} |K| (\rho_K^{n+1} - \rho_K^n) |\hat{\mathbf{u}}_K^{n+1}|^2 + \frac{1}{\Delta t} \sum_{K \in \Omega_h} |K| \left(\rho_K^{n+1} |\hat{\mathbf{u}}_K^{n+1}|^2 - \rho_K^n \hat{\mathbf{u}}_K^n \hat{\mathbf{u}}_K^{n+1} \right) \\
&= \frac{1}{\Delta t} \sum_{K \in \Omega_h} |K| \left(\frac{1}{2} \rho_K^{n+1} |\hat{\mathbf{u}}_K^{n+1}|^2 - \frac{1}{2} \rho_K^n |\hat{\mathbf{u}}_K^n|^2 + \frac{1}{2} \rho_K^n (\hat{\mathbf{u}}_K^{n+1} - \hat{\mathbf{u}}_K^n)^2 \right) \\
&\geq \frac{1}{\Delta t} (Ke_{\Omega_h}^{n+1} - Ke_{\Omega_h}^n).
\end{aligned}
$$

Term $T_{12}+T_{22}$

$$T_{12}:=-\frac{1}{2}\sum_{K\in\Omega_h}\sum_{\Gamma\in\partial K}\int_\Gamma \mathcal{F}^{up}(\rho_h^{n+1},\mathbf{u}_h^n)|\hat{\mathbf{u}}_K^{n+1}|^2$$

Contribution of T_{12} at an arbitrary edge $\Gamma=K\cap L$

$$-\frac{1}{2}\rho_K^{n+1}s_{\Gamma,K}^{n,+}|\hat{\mathbf{u}}_K^{n+1}|^2-\frac{1}{2}\rho_L^{n+1}s_{\Gamma,K}^{n,-}|\hat{\mathbf{u}}_K^{n+1}|^2-\frac{1}{2}\rho_L^{n+1}s_{\Gamma,L}^{n,+}|\hat{\mathbf{u}}_L^{n+1}|^2-\frac{1}{2}\rho_K^{n+1}s_{\Gamma,L}^{n,-}|\hat{\mathbf{u}}_L^{n+1}|^2$$
$$=-\frac{1}{2}\rho_K^{n+1}s_{\Gamma,K}^{n,+}(|\hat{\mathbf{u}}_K^{n+1}|^2-|\hat{\mathbf{u}}_L^{n+1}|^2)-\frac{1}{2}\rho_L^{n+1}s_{\Gamma,L}^{n,+}(|\hat{\mathbf{u}}_L^{n+1}|^2-|\hat{\mathbf{u}}_K^{n+1}|^2).$$

Contribution of T_{22} at an arbitrary edge $\Gamma=K\cap L$

$$(\rho_K\hat{\mathbf{u}}_K)^{n+1}s_{\Gamma,K}^{n,+}(\hat{\mathbf{u}}_K^{n+1}-\hat{\mathbf{u}}_L^{n+1})+(\rho_L\hat{\mathbf{u}}_L)^{n+1}s_{\Gamma,L}^{n,+}(\hat{\mathbf{u}}_L^{n+1}-\hat{\mathbf{u}}_K^{n+1}).$$

Term $T_{12} + T_{22}$

$$
\begin{aligned}
T_{12} + T_{22} &= \sum_{\Gamma \in \mathcal{E}^{int}} \int_\Gamma s_{\Gamma,K}^{n,+} \rho_K^{n+1} \Big(\frac{1}{2}|\hat{\mathbf{u}}_L^{n+1}|^2 + \frac{1}{2}|\hat{\mathbf{u}}_K^{n+1}|^2 - \hat{\mathbf{u}}_L^{n+1}\hat{\mathbf{u}}_K^{n+1}\Big) \\
&\quad + \sum_{\Gamma \in \mathcal{E}^{int}} \int_\Gamma s_{\Gamma,L}^{n,+} \rho_L^{n+1} \Big(\frac{1}{2}|\hat{\mathbf{u}}_L^{n+1}|^2 + \frac{1}{2}|\hat{\mathbf{u}}_K^{n+1}|^2 - \hat{\mathbf{u}}_L^{n+1}\hat{\mathbf{u}}_K^{n+1}\Big) \\
&= \frac{1}{2} \sum_{\Gamma \in \mathcal{E}^{int}} \int_\Gamma |\mathbf{u}_\Gamma^n \cdot \mathbf{n}| \rho^{n+1,up} (\hat{\mathbf{u}}_K^{n+1} - \hat{\mathbf{u}}_L^{n+1})^2 \\
&\geq 0.
\end{aligned}
$$

Term $T_{13} + T_{25}$

$$
\begin{aligned}
T_{13} + T_{25} \\
&= -\frac{1}{2} h^{\alpha} \sum_{K \in \Omega_h} \sum_{\Gamma \in \partial K} \int_{\Gamma} [\![\rho_h^{n+1}]\!] [\![|\hat{\mathbf{u}}_h^{n+1}|^2]\!] + h^{\alpha} \sum_{K \in \Omega_h} \sum_{\Gamma \in \partial K} \int_{\Gamma} [\![\rho_h^{n+1}]\!] \{ \hat{\mathbf{u}}_h^{n+1} \} [\![\hat{\mathbf{u}}_h^{n+1}]\!] \\
&= 0,
\end{aligned}
$$

as for all $\Gamma = K \cap L$

$$
\begin{aligned}
\{ \hat{\mathbf{u}}_h^{n+1} \}_{\Gamma} [\![\hat{\mathbf{u}}_h^{n+1}]\!]_{\Gamma} &= \frac{\hat{\mathbf{u}}_L^{n+1} + \hat{\mathbf{u}}_K^{n+1}}{2} (\hat{\mathbf{u}}_L^{n+1} - \hat{\mathbf{u}}_K^{n+1}) = \frac{|\hat{\mathbf{u}}_L^{n+1}|^2 - |\hat{\mathbf{u}}_K^{n+1}|^2}{2} \\
&= \frac{1}{2} [\![|\hat{\mathbf{u}}_h^{n+1}|^2]\!]_{\Gamma}.
\end{aligned}
$$

Term $T_{23} + T_{15}$

$$T_{23} + T_{15} = \sum_{K\in\Omega_h} \sum_{\Gamma\in\partial K} \int_\Gamma -(p^e)_K^{n+1} s_\Gamma^{n+1} + \mathcal{F}^{up}(\rho_h^{n+1}, \mathbf{u}_h^n) H'(\rho_h^{n+1})$$

$$= \sum_{K\in\Omega_h} \sum_{\Gamma\in\partial K} \int_\Gamma s_\Gamma^{n+1} \left(H(\rho_K^{n+1}) - \rho_K^{n+1} H'(\rho_K^{n+1}) + \rho^{n+1,up} H'(\rho_K^{n+1}) \right)$$

$$= \sum_{\Gamma\in\mathcal{E}^{int}} \int_{\Gamma=K\cap L} s_{\Gamma,K}^{n+1,+} \left(H(\rho_K^{n+1}) - H(\rho_L^{n+1}) + H'(\rho_L^{n+1})(\rho_L^{n+1} - \rho_K^{n+1}) \right)$$

$$+ \sum_{\Gamma\in\mathcal{E}^{int}} \int_{\Gamma=K\cap L} s_{\Gamma,L}^{n+1,+} \left(H(\rho_L^{n+1}) - H(\rho_K^{n+1}) + H'(\rho_K^{n+1})(\rho_K^{n+1} - \rho_L^{n+1}) \right)$$

$$= \sum_{\Gamma\in\mathcal{E}^{int}} \int_{\Gamma=K\cap L} \left(s_{\Gamma,K}^{n+1,+} \frac{H''(\eta_1)}{2} + s_{\Gamma,L}^{n+1,+} \frac{H''(\eta_2)}{2} \right) (\rho_K^{n+1} - \rho_L^{n+1})^2 \geq 0$$

$$\eta_1, \eta_2 \in \mathrm{co}\{\rho_K^{n+1}, \rho_L^{n+1}\}$$

Stability - - Kinetic and elastic energy

$$\boxed{0 = \sum_{i=1}^{3} T_{1i} + \sum_{i=1}^{6} T_{2i} \geq T_{24} + T_{26} + \frac{1}{\Delta t}\left(Ke_{\Omega_h}^{n+1} - Ke_{\Omega_h}^{n} + H_{\Omega_h}^{n+1} - H_{\Omega_h}^{n}\right)}$$

$$T_{24} := \sum_{K\in\Omega_h} \int_K 2\mu|\mathbf{D}(\mathbf{u}_h^{n+1})|^2 + \nu|\mathrm{div}\mathbf{u}_h^{n+1}|^2 + 2\mu \sum_{K\in\Omega_h} \sum_{\Gamma\in\partial K} \int_\Gamma \frac{1}{h}[\![\mathbf{u}_h^{n+1}]\!]^2,$$

$$T_{26} := -\sum_{K\in\Omega_h} \int_K \mathcal{E}_h^{n+1}\mathrm{div}\mathbf{u}_h^{n+1}$$

$$E_h = Ke_h + H_h + c_v\mathcal{E}_h$$

$$T_{24} + T_{26} + \frac{c_v}{\Delta t}\sum_{K\in\Omega_h}\int_K (\mathcal{E}_h^{n+1} - \mathcal{E}_h^{n}) \quad = \quad ?$$

Stability - - Step 3. $\phi_h = 1$ for temperature scheme

$$
c_v \sum_{K\in\Omega_h} \int_K \frac{\mathcal{E}_h^{n+1} - \mathcal{E}_h^n}{\Delta t} \phi_h - c_v \sum_{K\in\Omega_h} \sum_{\Gamma\in\partial K} \int_\Gamma \mathcal{F}^{up}(\mathcal{E}_h^{n+1}, \mathbf{u}_h^{n+1}) [\![\phi_h]\!]
$$

$$
+ \sum_{K\in\Omega_h} \sum_{\Gamma\in\partial K} \int_\Gamma \frac{1}{d_\Gamma} [\![\mathcal{K}(\theta_h^{n+1})]\!] [\![\phi_h]\!] = \sum_{K\in\Omega_h} \int_K \left(2\mu|\mathbf{D}(\mathbf{u}_h^{n+1})|^2 + \nu|\mathrm{div}\mathbf{u}_h^{n+1}|^2\right)\phi_h
$$

$$
+ 2\mu \sum_{K\in\Omega_h} \sum_{\Gamma\in\partial K} \int_\Gamma \frac{1}{h} [\![\mathbf{u}_h^{n+1}]\!]^2 - \sum_{K\in\Omega_h} \int_K \mathcal{E}_h^{n+1} \mathrm{div}_h \mathbf{u}_h^{n+1} \phi_h,
$$

$$
T_{31} + T_{32} + T_{33} = 0.
$$

$$
T_{31} := \frac{c_v}{\Delta t} \sum_{K\in\Omega_h} \int_K (\mathcal{E}_h^{n+1} - \mathcal{E}_h^n) = \frac{c_v}{\Delta t} (\mathcal{E}_{\Omega_h}^{n+1} - \mathcal{E}_{\Omega_h}^n),
$$

$$
T_{32} := -\sum_{K\in\Omega_h} \int_K \left(2\mu|\mathbf{D}(\mathbf{u}_h^{n+1})|^2 + \nu|\mathrm{div}\mathbf{u}_h^{n+1}|^2\right) - 2\mu \sum_{K\in\Omega_h} \sum_{\Gamma\in\partial K} \int_\Gamma \frac{1}{h} [\![\mathbf{u}_h^{n+1}]\!]^2,
$$

$$
T_{33} := \sum_{K\in\Omega_h} \int_K \mathcal{E}_h^{n+1} \mathrm{div}_h \mathbf{u}_h^{n+1}.
$$

$$\mu = \nu = 1.0, a = 1.0, b = 1.0, \gamma = 3.0, c_v = 1.4, \alpha = 0.83.$$

Time step is

$$\Delta t = \mathrm{CFL}\frac{\min(h_K)}{\max(|U|) + c}, \; c = \sqrt{\frac{\partial p}{\partial \rho}} = \sqrt{a\gamma\rho^{\gamma-1} + b + \theta}, \mathrm{CFL} = 0.6.$$

Boundary conditions is set as periodic.

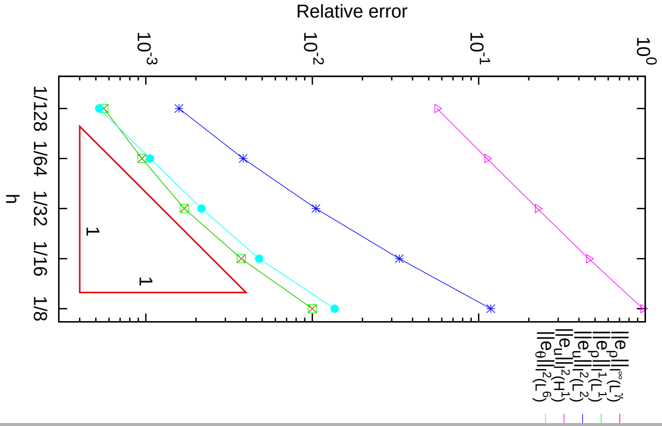
density

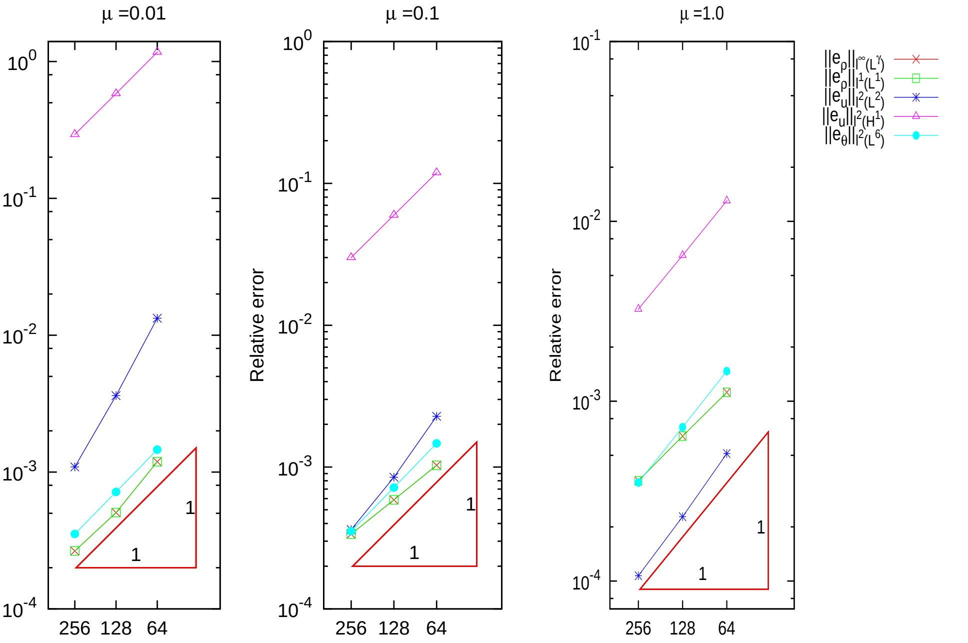
temperature

velocity

Table : Convergence results of Poiseulle flow

h	$\Vert\rho\Vert_{l^\infty(L^\gamma)}$	EOC	$\Vert\rho\Vert_{l^1(L^1)}$	EOC	$\Vert\mathbf{u}\Vert_{l^2(L^2)}$	EOC	$\Vert\mathbf{u}\Vert_{l^2(H^1)}$	EOC	$\Vert\theta\Vert_{l^2(L^2)}$	EOC
1/8	1.87e-02	–	1.00e-02	–	1.18e-01	–	9.70e-01	–	1.36e-02	–
1/16	8.07e-03	1.21	3.74e-03	1.42	3.33e-02	1.83	4.58e-01	1.08	4.79e-03	1.51
1/32	3.89e-03	1.05	1.70e-03	1.14	1.05e-02	1.67	2.25e-01	1.03	2.16e-03	1.15
1/64	2.03e-03	0.94	9.47e-04	0.84	3.84e-03	1.45	1.12e-01	1.01	1.06e-03	1.03
1/128	1.07e-03	0.92	5.60e-04	0.76	1.58e-03	1.28	5.61e-02	1.00	5.24e-04	1.02




μ =0.01
μ =0.1
μ =1.0
Relative error
Relative error
10^0
10^{-1}
10^{-2}
10^{-3}
10^{-4}
256 128 64
1
1
$\|e_\rho\|_{l^\infty(L^\gamma)}$
$\|e_\rho\|_{l^1(L^1)}$
$\|e_u\|_{l^2(L^2)}$
$\|e_u\|_{l^2(H^1)}$
$\|e_\theta\|_{l^2(L^6)}$