INSTITUTE OF MATHEMATICS

THE CZECH ACADEMY OF SCIENCES

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Preprint No. 84-2017

PRAHA 2017

WCG SPACES AND THEIR SUBSPACES GRASPED BY PROJECTIONAL SKELETONS

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Dedicated to the memory of Pawel Domański. Requiescat in pace!

Abstract. Weakly compactly generated Banach spaces and their subspaces are characterized by the presence of projectional skeletons with some additional properties. We work with real spaces. However the presented statements can be extended, without much extra effort, to complex spaces.

1. Introduction

Projectional resolutions of the identity in Banach spaces have been an important tool for the theory of nonseparable Banach spaces for decades. Recently, a new related and efficient instrument —*projectional skeletons*— made its successful way into the nonseparable theory. It was introduced by W. Kubiś, in his paper [12]: He proved in particular that the fairly big class of 1-Pličko spaces is exactly that admitting a commutative 1-projectional skeleton. The paper [4] characterized Asplund spaces and Asplund WCG spaces with the help of suitable projectional skeletons. In this note, we characterize WCG spaces and their subspaces in a similar flavor. The more or less already known characterization of weakly Lindelöf determined spaces is also recalled.Throughout this note, we consider only real Banach spaces. The art of how to deal also with complex spaces can be found in [15] and [4].

The notation used here is standard. X (or $(X, \|\cdot\|)$ if we wish to specify the symbol for its norm) will denote a real Banach space, B_X its closed unit ball, and X^* its dual space, endowed with the standard dual norm, also denoted by $\|\cdot\|$. The word "subspace" will always mean a closed linear subset. If Y is a subspace of a Banach space X, we shall denote by B_Y its closed unit ball, i.e., $B_Y = B_X \cap Y$. If M is a subset of X, then $\operatorname{span}(M)$ and $\overline{M}$ mean the linear hull of M and the closure of M, respectively. If M is a subset of X^*, then $\overline{M}^*$ means the weak* closure of M. The action of an element $x^* \in X^*$ on an element $x \in X$ will be denoted by $x^*(x)$ or, alternatively, by $\langle x^*, x\rangle$. If $x \in X$ and $M \subset X^*$, we put $\sup\langle M, x\rangle$ instead of $\sup\{\langle x^*, x\rangle :\ x^* \in M\}$. For a set M in a topological space, $\operatorname{dens} M$ is the smallest cardinal κ such that M has a dense subset of cardinality κ. The weak topology w of a Banach space X is the topology of the pointwise convergence on the elements in X^*, and the weak-star topology w^* is the topology on X^* of the pointwise convergence on points in X. Further concepts are introduced later. For the non-defined ones, the reader is invited to look into, e.g., [9].

1991 *Mathematics Subject Classification.* 46B26, 46B20.

Key words and phrases. Banach space, weakly compactly generated space, weakly Lindelöf determined space, rich family, projectional skeleton, projectional resolution of the identity.

The first author was supported by grants 17-00941S and by RVO: 67985840. The second author was supported in part by MICINN MTM 2014-57838-C2-2-P (Spain).

2. "Dissecting" a nonseparable Banach space

2.1. **Complemented subspaces and projections.** "Dissecting" a Banach space X into pieces —totally or partially ordered "chains" of complemented subspaces— is a tool for, on the one hand, looking into its structure and, on the other, proving results by transfinite induction (the Occam's razor from the scholasticism). To write X as the topological direct sum of two subspaces $X = V \oplus W$ is equivalent to construct a continuous linear projection $P : X \to X$ such that $PX = V$ and $(P^{-1}\{0\} =)\ (I - P)X = W$. In order to carry the construction with no extra effort to other situations —like to dual spaces— it is convenient to consider, from the beginning, projections that are, moreover, $w(X, D)$-continuous, where D is a given r-norming subspace of the dual space X^*, i.e., a subspace D of X^* such that $(1/r)\|x\| \le \sup\{x^*(x) :\ x^* \in D,\ \|x^*\| \le 1\} \le \|x\|$ for all $x \in X$ and a fixed $r \in [1, +\infty)$. The starting point is the following lemma. Its proof is simple; we include it for the sake of completeness. It was formulated in only one direction in, e.g., [11, Lemmata 3.33 and 3.34], [7, Lemma 6.1.1], and [4, Lemma 8].

projection}

Lemma 1. *Let $(X, \|\cdot\|)$ be a Banach space, $r \ge 1$, and two closed subspaces $V \subset X$ and $Y \subset X^*$. Then the conditions* (A$_r$) *and* (B$_r$) *below are mutually equivalent:*

(A$_r$) $\begin{cases} \text{(A1)} & V \textit{ separates points of } \overline{Y}^{w^*} \text{ (i.e., } V^\perp \cap \overline{Y}^{w^*} = \{0\}\text{), and} \\ \text{(A}_r\text{2)} & \textit{for all } v \in V,\ \|v\| \le r.\sup\langle B_Y, v\rangle. \end{cases}$

(B$_r$) *$X = V \oplus Y_\perp$, and the associated projection $P : X \to X$ with range V and kernel $Y_\perp$ satisfies $\|P\| \le r$.*

Before proving Lemma 1, let us show a simple consequence:

ma-adjoint}

Lemma 2. *If for some $r \ge 1$, the projection P satisfies* (B$_r$) *in Lemma* 1 (and so also (A$_r$))*, then $P^*X^* = \overline{Y}^{w^*}$ (and $\|P^*\| \le r$).*

Proof. Observe first that $(Y_\perp)^\perp = \overline{Y}^{w^*}$. Let $x^* \in P^*X^*$, and let $y \in Y_\perp$. We have then $Py = 0$, so $\langle x^*, y\rangle = \langle P^*x^*, y\rangle = \langle x^*, Py\rangle = 0$, hence $x^* \in (Y_\perp)^\perp$ $(= \overline{Y}^{w^*})$. This proves that $P^*X^* \subset \overline{Y}^{w^*}$. To show the reverse inclusion, let $x^* \in \overline{Y}^{w^*}$ $\big(= (Y_\perp)^\perp\big)$. Then, for $x \in X$,

$$\langle x^* - P^*x^*, x\rangle = \langle x^*, x\rangle - \langle P^*x^*, x\rangle = \langle x^*, x\rangle - \langle x^*, Px\rangle = \langle x^*, x - Px\rangle = 0,$$

the last equality being true due to the fact that $x - Px \in Y_\perp$. This shows that $x^* - P^*x^* = 0$, hence $x^* \in P^*X^*$. We proved that $\overline{Y}^{w^*} \subset P^*X^*$. The two inclusions prove the assertion. That $\|P^*\| \le r$ is a general fact whose proof shall be omitted. □

Let us proceed now with the proof of Lemma 1

Proof. Assume (A$_r$). Let $v \in V \cap Y_\perp$. Note that (A$_r$2) implies $v = 0$, hence $V \oplus Y_\perp$ is an algebraic direct sum. If $x^* \in X^*$ vanishes on $V \oplus Y_\perp$, then $x^* \in V^\perp \cap \overline{Y}^{w^*}$ ($= \{0\}$ by (A1)), hence $V \oplus Y_\perp$ is dense in X. Moreover, if $x \in V \oplus Y_\perp$ and $P : V \oplus Y_\perp \to V$ denotes the associated linear (not necessarily continuous) projection onto V, we have

$$\|Px\| \le r\sup\langle B_Y, Px\rangle = r\sup\langle B_Y, Px + (x - Px)\rangle = r\sup\langle B_Y, x\rangle \le r\sup\langle B_X, x\rangle = r\|x\|,$$

where the first inequality is (A$_r$2) and the first equality comes from the fact that $x - Px \in Y_\perp$. Thus, P is continuous (in fact, $\|P\| \le r$), hence $V \oplus Y_\perp$ is closed and so $V \oplus Y_\perp = X$. This shows (B$_r$).

Assume (B_r). Let $x^* \in V^\perp \cap \overline{Y}^{w^*}$. Given $x \in X$ we have $\langle x^*, x\rangle = \langle P^*x^*, x\rangle = \langle x^*, Px\rangle = 0$, where the first equality comes from Lemma 2. It follows that $x^* = 0$. This proves (A1).
Given $x^* \in B_{X^*}$, we have, again by Lemma 2, $P^*x^* \in \overline{Y}^{w^*}$. Since $\|P^*\| \leq r$ we get $P^*x^* \in rB_{\overline{Y}^{w^*}}$, so $P^*B_{X^*} \subset rB_{\overline{Y}^{w^*}}$. Fix $v \in V$. Then

$$\|v\| = \sup\langle B_{X^*}, v\rangle = \sup\langle B_{X^*}, Pv\rangle = \sup\langle P^*B_{X^*}, v\rangle \leq \sup\langle rB_{\overline{Y}^{w^*}}, v\rangle = r\sup\langle B_Y, v\rangle.$$

This proves (A_r2). □

Remark 3. In the rest of the paper, the associated projection $P : X \to X$ with range V and kernel $Y_\perp$ built in Lemma 1 (see (B_r) there) will be denoted by $P_{V\times Y}$.

{subsect-pr

2.2. **Projectional resolutions of the identity and projectional skeletons.** Starting from Lemma 1, and proceeding in a clever way, a "long sequence" (i.e., a projectional resolution of the identity) of norm-1 projections is produced. According to J. Lindenstrauss,

{def-pri}

Definition 4. *A* projectional resolution of the identity *(*PRI*, for short) on a Banach space* $(X, \|\cdot\|)$ *is a family* $(P_\alpha :\ \omega \leq \alpha \leq \operatorname{dens} X)$ *of linear projections on* X *such that* $P_\omega = 0$, $P_{\operatorname{dens} X}$ *is the identity mapping, and for all* $\omega < \alpha \leq \operatorname{dens} X$ *the following hold:*

(i) $\|P_\alpha\| = 1$,
(ii) $\operatorname{dens} P_\alpha X \leq \alpha$,
(iii) $P_\alpha \circ P_\beta = P_\beta \circ P_\alpha = P_\alpha$ *whenever* $\beta \in [\alpha, \operatorname{dens} X]$, *and*
(iv) $\overline{\bigcup_{\beta<\alpha} P_{\beta+1}X} = P_\alpha X$.

If (i) *is replaced by* $\|P_\alpha\| \leq r$ *for some fixed finite number* $r \geq 1$, *we speak about an r-PRI.*

Instead of a set of projections indexed by an interval of ordinal numbers, W. Kubiś [12] produced from Lemma 1 a set —indexed by a partially ordered set $(\Gamma, \leq)$— of not-necessarily-norm-1 linear and bounded projections with separable range. The index set $(\Gamma, \leq)$ is directed upwards (we shall simply say "directed"), and —to show some continuity property— σ-*complete*, i.e., every increasing sequence (γ_n) in Γ has a "supremum" in Γ, i.e., an element $\gamma \in \Gamma$ such that $\gamma_n \leq \gamma$ for all $n \in \mathbb{N}$, and if $s \in \Gamma$ satisfies $\gamma_n \leq s$ for all $n \in \mathbb{N}$, then $\gamma \leq s$. To be precise, and following W. Kubiś,

{def-ps}

Definition 5. *A* projectional skeleton *in a (rather non-separable) Banach space* $(X, \|\cdot\|)$ *is a family of linear bounded projections* $\big(P_s :\ s \in \Gamma\big)$ *on* X, *indexed by a directed and* σ*-complete set* $(\Gamma, \leq)$ *such that*

(i) P_sX *is separable for every* $s \in \Gamma$,
(ii) $X = \bigcup_{s\in\Gamma} P_sX$,
(iii) $P_t \circ P_s = P_s = P_s \circ P_t$ *whenever* $s, t \in \Gamma$ *and* $s \leq t$, *and*
(iv) *Given a sequence* $s_1 \leq s_2 \leq \cdots$ *in* Γ, *we have* $P_sX = \overline{\bigcup_{n\in\mathbb{N}} P_{s_n}X}$, *where* $s := \sup_{n\in\mathbb{N}} s_n$.

The concept of projectional skeleton has a topological predecessor —retractional skeleton— introduced by W. Kubiś and H. Michalewski in [13].

Remark 6. Observe that (iii) in Definition 5 above is equivalent to the fact that, for all $s \leq t$ in Γ, we have simultaneously $P_sX \subset P_tX$ and $P_s^*X^* \subset P_t^*X^*$.

Definition 7. *For* $r \geq 1$, *we say that* $\big(P_s :\ s \in \Gamma\big)$ *is an* r-projectional skeleton *if it is a projectional skeleton and* $\|P_s\| \leq r$ *for every* $s \in \Gamma$.

Remark 8. For r-skeletons, it is easy to show that the identity in (iv), Definition 5 above, is equivalent to the convergence $P_{s_j}x \to P_s x$ as $j \to \infty$ for every $x \in X$ and every sequence $s_1 \leq s_2 \leq \cdots$ in Γ, with $s := \sup_{n\in\mathbb{N}} s_n$.
Indeed, assume that the identity in (iv) holds. Then, given $x \in X$ and $\varepsilon > 0$ there exists $z \in X$ and $n \in \mathbb{N}$ such that $\|P_s x - P_{s_n} z\| < \varepsilon$. It follows that

$$\begin{aligned}\|P_s x - P_{s_n} x\| &\leq \|P_s x - P_{s_n} z\| + \|P_{s_n} z - P_{s_n} x\| \\ &< \varepsilon + \|P_{s_n} P_{s_n} z - P_{s_n} P_s x\| < \varepsilon + \|P_{s_n}\|.\|P_{s_n} z - P_s x\| < \varepsilon + r\varepsilon = (1+r)\varepsilon. \qquad (1)\end{aligned}$$

For $m \geq n$ we have

criangle-2}

$$\|P_{s_m} x - P_{s_n} x\| = \|P_{s_m} P_s x - P_{s_m} P_{s_n} x\| \leq \|P_{s_m}\|.\|P_s x - P_{s_n} x\| < r(1+r)\varepsilon, \qquad (2)$$

so $\|P_s x - P_{s_m} x\| < (1+r)^2\varepsilon$ for $s := \sup_{n\in\mathbb{N}} s_n$, and we get that $P_{s_n} x \to P_s x$ as $n \to \infty$. The other implication is obvious.

-in-w-star}

Remark 9. Observe, too, that given a sequence $s_1 \leq s_2 \leq \cdots$ in Γ and $x^* \in X^*$, we always have $P^*_{s_n} x^* \xrightarrow{w^*} P^*_s x^*$ where $s = \sup_{n\in\mathbb{N}} s_n$. Indeed, if $x \in X$, we have $\langle (P^*_s - P^*_{s_n})x^*, x\rangle = \langle x^*, (P_s - P_{s_n})x\rangle \to 0$. This shows, in particular, that $P^*_s X^* \subset \overline{\bigcup_{n=1}^{\infty} P^*_{s_n} X^*}^{w^*}$. The other inclusion is obvious, since we mentioned above that $P_{s_n} X^* \subset P^*_s X^*$ for all $n \in \mathbb{N}$ and $P^*_s X^*$ is w^*-closed, due to the w^*-w^*-continuity of P^*_s. We finally get $P^*_s X^* = \overline{\bigcup_{n=1}^{\infty} P^*_{s_n} X^*}^{w^*}$.
It is worth to note that for any $x^* \in X^*$, the net $\{P^*_\gamma x^* :\ \gamma \in \Gamma,\ \leq\}$ is w^*-convergent to x^*. Even more holds: given $x \in X$, there exists $\gamma_0 \in \Gamma$ such that $\langle x^* - P^*_\gamma x^*, x\rangle = 0$ for every $\gamma \geq \gamma_0(x)$ and every $x^* \in X^*$. Indeed, find $\gamma_0 \in \Gamma$ so big that $P_{\gamma_0} X \ni x$. Then for every $\gamma \in \Gamma$, with $\gamma \geq \gamma_0$, and every $x^* \in X^*$ we have

$$\langle x^* - P^*_\gamma x^*, x\rangle = \langle x^*, x\rangle - \langle P^*_\gamma x^*, x\rangle = \langle x^*, x\rangle - \langle x^*, P_\gamma x\rangle = \langle x^*, x - x\rangle = 0$$

Definition 10. *We say that a skeleton* $\big(P_s :\ s \in \Gamma\big)$ *is* commutative *if* $P_s \circ P_t = P_t \circ P_s$, *whenever* $s, t \in \Gamma$ *(no matter if* s, t *are comparable).*

A sufficient condition for the commutativity of a projectional skeleton will be given in Lemma 13 below.

The instrument of PRI served efficiently for half a century in proving many statements for non-separable Banach spaces. As an illustration we recall classical results that *once a Banach space X is weakly compactly generated, then there exist a linear bounded injection from it into $c_0(\operatorname{dens} X)$ and a linear bounded and weak* to weak continuous injection of X^* into $c_0(\operatorname{dens} X)$; moreover, X then admits an equivalent locally uniformly rotund norm whose dual norm is strictly convex.* However, the presence of a PRI itself is not much eloquent about the space in question, provided its density is big enough. For instance, if κ is an ordinal greater than the density of ℓ_∞ (equal to $\mathbf{c}$), then the space $\ell_2(\kappa) \times \ell_\infty$ clearly admits a PRI. But his space does not admit any projectional skeleton because ℓ_∞ is not, by Lindenstrauss, LUR renormable; see [5, pages 120–123]. Note that ℓ_∞ does not admit any projectional skeleton: Otherwise c_0 will be a subspace of $P_\gamma \ell_\infty$ for some $\gamma \in \Gamma$, and so complemented in $P_\gamma \ell_\infty$ by Sobczyk's theorem (see, e.g., [9, Theorem 5. 11]). This will imply that c_0 is complemented in ℓ_∞, and this is false.

2.3. **Consequences of having a projectional skeleton.** By [14, Theorem 12], the existence of an r-projectional skeleton implies that of an r-PRI (but not reversely!).
In contrast with what was said at the end of Subsection 2.2 regarding projectional resolutions, the presence of a projectional skeleton in a Banach space proves to be quite eloquent. We mention the following most striking facta:
If a Banach space X admits a projectional skeleton, then:
- *X admits an r-PRI where $r \geq 1$ is a finite number* [12, Proposition 9 and Theorem 12]*;*
- *X linearly and continuously injects into $c_0(\operatorname{dens} X)$,* [14, Corollary 17.5];
- *X admits a Markushevich basis,* [3]*; and*
- *X admits an equivalent locally uniformly rotund norm,* see [14, Corollary 17.5] modulo S. Troyanski and V. Zizler.

{subsect-bu

2.4. **Building efficiently projectional skeletons: Projectional generators, rich families.** One of the most efficient ways to build a projectional resolution of the identity is to provide a *projectional generator* (M. Valdivia and J. Orihuela): Let X be a Banach space, and let $W \subset X^*$ be a 1-norming subspace. Assume there exists an at most countably valued mapping $\Phi : W \to 2^X$ such that for every nonempty set $B \subset W$, with linear closure, we have $\Phi(B)^\perp \cap \overline{B}^{w^*} = \{0\}$. Then the couple (W, Φ) is called a *projectional generator* on X (see, e.g., [7, page 106] or [11, page 104]). The projectional generator is then used for producing a projectional resolution of the identity on X by a countable "back-and-forth" method. We provide two natural examples of projectional generators:
(PGi) Let us consider a WCG Banach space X, and let $K \subset X$ be a linearly dense and w-compact subset of X. Then, the couple (X^*, Φ), where Φ is the (single-valued) mapping $\Phi : X^* \to X$ that to any $x^* \in X^*$ associates an element in K where x^* attains its supremum on K, is easily seen, by using the Mackey–Arens theorem, or just Lemma 18 below, to be a projectional generator (see, e.g., [11, Proposition 3.43]). The details are given in the proof of Theorem 20 below.
(PGii) Let X be a Banach space, and let M be a linearly dense subset of X. Assume there exists a 1-norming subspace D of X^* such that, for every $x^* \in D$, its support $\operatorname{supp}_M(x^*) := \{m \in M : \ x^*(m) \neq 0\}$ is countable. Define $\Phi(x^*) := \operatorname{supp}_M(x^*)$ for $x^* \in D$. Then (D, Φ) is a projectional generator on X. Indeed, let $B \subset D$ be a nonempty set such that $\overline{B}$ is linear. Pick $x^* \in \Phi(B)^\perp \cap \overline{B}^{w^*}$. Assume $x^* \neq 0$. We can find then $m \in M$ such that $\varepsilon := |x^*(m)| > 0$. Find $b^* \in B$ such that $|\langle x^* - b^*, m\rangle| < \varepsilon/2$. It follows that $|b^*(m)| > \varepsilon/2$, so $m \in \operatorname{supp}_M(b^*)$, hence $m \in \Phi(B)$. Since $|x^*(m)| \neq 0$, we reach a contradiction.

Regarding projectional skeletons on a Banach space X, still a projectional generator (W, Φ) on X —if it exists at all— is a useful instrument for building them. Now we allow a slight change in its definition, namely that, for some $r \geq 1$, the subspace W may be r-norming instead of just 1-norming (from now on, this more relaxed requirement and the corresponding concept will be adopted). The set of indices for indexing a projectional skeleton (in the definition, an abstract partially ordered directed and σ-complete set), may quite naturally be particularized in our setting by taking a suitable (partially ordered by inclusion) *rich* family of "rectangles" $V \times Y$ in $X \times D$, where D is a given closed subspace of X^*, and V and Y are separable subspaces of X and D, respectively. We precise this by introducing some useful notation:

Let W be a Banach space. Denote by $\mathcal{S}(W)$ the partially ordered by inclusion family of all separable subspaces of W. A subfamily $\mathcal{R}$ of $\mathcal{S}(W)$ is said to be *rich* (in $\mathcal{S}(W)$ or just in W) if it is *cofinal* (i.e., for every $S \in \mathcal{S}(W)$ there exists $R \in \mathcal{R}$ such that $R \supset S$) and σ*-closed* (i.e., whenever (R_n) is an increasing sequence in $\mathcal{R}$, then $\overline{\bigcup R_n} \in \mathcal{R}$). This concept was introduced by J.M. Borwein and W. Moors in [2].; see also [4].
If $W := X \times Z$, where X and Z are two Banach spaces, we denote by $\mathcal{S}_{\square}(X \times Z)$ the subfamily of $\mathcal{S}(X \times Z)$ consisting of all rectangles $V \times Y$, where V and Y are separable subspaces of X and Z, respectively. Clearly, the family $\mathcal{S}_{\square}(X \times Z)$ is rich in $\mathcal{S}(X \times Z)$. Below, we shall consider rich families $\mathcal{R}$ in $\mathcal{S}_{\square}(X \times Z)$.
For later reference, let us mention here some elementary facta —coming directly from the definition— about rich families $\mathcal{R}$ in $\mathcal{S}_{\square}(X \times Z)$.
• $\bigcup\{V \times Y : \ V \times Y \in \mathcal{R}\} = X \times Z$. Indeed, for $x \in X$ and $z \in Z$ the rectangle $\mathrm{span}\{x\} \times \mathrm{span}\{z\}$ belongs to $\mathcal{S}_{\square}(X \times Z)$, and $\mathcal{R}$ is cofinal there. In particular,

$$X = \bigcup\{V : \text{ there exists } Y \in \mathcal{S}(Z) \text{ such that } V \times Y \in \mathcal{R}\}, \text{ and}$$

$$Z = \bigcup\{Y : \text{ there exists } V \in \mathcal{S}(X) \text{ such that } V \times Y \in \mathcal{R}\}.$$

• The partially ordered set $(\mathcal{R}, \subset)$ is directed and σ-complete. Note that if $(V_n \times Y_n)$ is an increasing sequence in a rich family $\mathcal{R}$, then $\overline{\bigcup(V_n \times Y_n)}$ $(= \overline{\bigcup V_n} \times \overline{\bigcup Y_n})$ is its supremum in $\mathcal{S}_{\square}(X \times Z)$, and it belongs to $\mathcal{R}$ by assumption.
An important feature of rich families is that *the intersection of countably many rich families is not only non-empty but again rich;* see [2].

Suitable rich families give raise to associated projectional skeletons. Indeed, we have the following basic result, whose proof, after Lemma 1 and the previous observations, is now almost obvious:

l-skeleton}

Proposition 11 ([4], Lemma 9)**.** *Let X be a Banach space, and for some $r \geq 1$ let $D \subset X^*$ be a closed r-norming subspace. Assume that there exists a rich family $\Gamma \subset \mathcal{S}_{\square}(X \times D)$ such that for each $\gamma := V \times Y \in \Gamma$, the condition* (A_r) *in Lemma* 1 *is satisfied. Then $(P_\gamma : \ \gamma \in \Gamma)$ is an r-projectional skeleton in X such that $D \subset \bigcup_{\gamma \in \Gamma} P^*_\gamma X^*$.*

Remark All the projectional skeletons in the rest of the paper will adopt the particular form given in Proposition 11.

The existence of a projectional generator on a Banach space X ensures that a rich family with the property in the statement of Proposition 11 does exist. This is the content of the next result:

plies-rich}

Proposition 12 ([4], Proposition 10)**.** *Let $(X, \|\cdot\|)$ be a Banach space with a projectional generator (D, Φ), where D is a closed r-norming subspace of X^* for some $r \geq 1$. Then there is a family Γ rich in $\mathcal{S}_{\square}(X \times D)$ and such that for each $V \times Y \in \Gamma$, the condition* (A_r) *in Lemma* 1 *is satisfied.*

Proof. (Sketch) For every $x \in X$ pick a countable set $\phi(x) \subset B_D$ such that $\|x\| \leq r \sup\{y(x) : y \in \phi(x)\}$. Define Γ as the family of all $V \times Y \in \mathcal{S}_{\square}(X \times D)$ such that there are countable sets $C \subset V$ and $E \subset Y$ satisfying $\overline{C} = V$, $\overline{E} = Y$, $\Phi(E) \subset C$, and $\phi(C) \subset E$. This family satisfies all the requirements for being a rich family in $\mathcal{S}_{\square}(X \times D)$. $\square$

Putting together Propositions 11 and 12, we get that *the existence of a projectional generator* (D, Φ) *on a Banach space* X *ensures the existence of a projectional skeleton in it of the form* $(P_\gamma : \gamma \in \Gamma)$*, where* Γ *is a rich family in* $\mathcal{S}_\Box(X \times D)$.

In the rest of the paper we shall need [4, Lemma 11]. We state it here —with a little complement— for the sake of completeness, and we shall prove only this little addition. Observe that, in Lemma 13(ii) below, the couple (D, Φ) is a projectional generator (see (PGii) in Subsection 2.4 above, where it is proved for the case $r := 1$).

{lemma-rich

Lemma 13. *Let* X *be a Banach space with a linearly dense subset* M *and a subspace* D *of* X^* *such that, for every* $x^* \in D$*, the set* $\Phi(x^*) := \mathrm{supp}_M(x^*)$ *is countable. Then*
(i) [4, Lemma 11] *The family* $\mathcal{R} := \{V \times Y \in \mathcal{S}_\Box(X \times D) : \ M \setminus V \subset Y_\perp\}$ *is rich in* $\mathcal{S}_\Box(X \times D)$.
(ii) *Assume that, moreover, the subspace* D *is* r*-norming for some* $r \geq 1$*. Let* Γ *be the rich family in* $\mathcal{S}_\Box(X \times D)$ *given by the projectional generator* (D, Φ) (Proposition 12). *Then the projectional skeleton provided by Proposition* 11 *via the rich family* $\mathcal{R} \cap \Gamma$ *is commutative.*

Proof. (of (ii)) This follows from the properties of M: Indeed, fix $\gamma := V \times Y \in \mathcal{R} \cap \Gamma$ and $m \in M$. Then, due to the fact that $M \setminus V \subset Y_\perp$, that $P_\gamma(X) = V$ and that $\ker P_\gamma = Y_\perp$, we have

$$P_\gamma(m) = \begin{cases} m, & \text{if } m \in V, \\ 0, & \text{otherwise.} \end{cases}$$

As as consequence, given $\gamma = V \times Y$ and $\gamma' = V' \times Y'$ in $\mathcal{R} \cap \Gamma$, we may easily check that, for $m \in M$,

$$\begin{cases} P_\gamma \circ P_{\gamma'}(m) = P_{\gamma'} \circ P_\gamma(m) = m, \text{ if } m \in V \cap V', \text{ and} \\ P_\gamma \circ P_{\gamma'}(m) = P_{\gamma'} \circ P_\gamma(m) = 0, \text{ otherwise.} \end{cases}$$

Since M is linearly dense in X, we get, then, the commutativity of the projectional skeleton $(P_\gamma : \gamma \in \Gamma)$. $\Box$

3. Classes of nonseparable Banach spaces and projectional skeletons

3.1. **WLD spaces.** A Banach space is called *weakly Lindelöf determined (WLD)* if its closed dual unit ball, provided with the weak* topology, is a Corson compactum, which means that it is homeomorphic to a subset of the Σ-product of real lines. In the next theorem, the equivalence (ii)$\Longleftrightarrow$(iv) can be found in [12, Proposition 21 and Theorem 27].

{thm-wld}

Theorem 14. *For a Banach space* $(X, \|\cdot\|)$ *TFAE:*

(i) X *is weakly Lindelöf determined.*
(ii) *There exists a linearly dense set* $M \subset X$ *which countably supports all elements in* X^**, that is, for every* $x^* \in X^*$ *the set* $\{x \in M : \ x^*(x) \neq 0\}$ *is at most countable.*
(iii) *There exist a set* M *as in* (ii) *and moreover a rich family* $\mathcal{WLD} \subset \mathcal{S}_\Box(X \times X^*)$ *such that every* $V \times Y \in \mathcal{WLD}$ *satisfies* (A_1) *in Lemma* 1*, and* $M \setminus V \subset Y_\perp$.
(iv) *There exists a commutative* 1*-projectional skeleton* $(P_\gamma : \ \gamma \in \Gamma)$ *on* $(X, \|\cdot\|)$ *such that* $\bigcup \{P_\gamma^* X^* : \ \gamma \in \Gamma\} = X^*$.

Proof. (i)$\Longrightarrow$(ii). The existence of the set M goes back to M. Valdivia [16]; see also [8, Theorem 5]. First, a projectional generator is constructed. From it a PRI is found. Having this, the construction of M goes by a transfinite induction over the density of X.

(ii)$\Longrightarrow$(iii). Assume that there exists a set M as in (ii). As it was proved in Subsection 2.4 (see (PGii) there), the couple (X^*, Φ) is a projectional generator on X. Now, Proposition 12 above, where $D := X^*$ and $r := 1$, provides a rich family $\mathcal{W} \subset \mathcal{S}_{\square}(X \times X^*)$ that satisfies (A_1) in Lemma 1. Put $\mathcal{WLD} := \mathcal{W} \cap \mathcal{R}$, where $\mathcal{R}$ is the rich family defined in Lemma 13(i). It follows that $\mathcal{WLD}$ is also a rich family in $\mathcal{S}_{\square}(X \times X^*)$. And of course, each element in $\mathcal{WLD}$ satisfies (A_1) in Lemma 1.
(iii)$\Longrightarrow$(iv). Letting $D := X^*$, Proposition 11 guarantees that the system $(P_\gamma : \ \gamma \in \Gamma)$ satisfies (iv), except maybe the commutativity statement. However, this follows from (ii) in Lemma 13 above.
(iv)$\Longrightarrow$(ii). This is included in [12, Theorem 27]. In order to get a taste of the proof, let us show this when $\operatorname{dens} X = \omega_1$. Let (iv) hold without "commutative 1-". By [12, Proposition 9], we may and do assume that $\|P_s\| \le r$ for all $s \in \Gamma$ and some fixed r. A shorter reasoning in one's mind reveals that there exists an order homomorphism of the interval $[\omega, \omega_1)$ onto a cofinal subset of Γ such that, when assuming, for simplicity, that $[\omega, \omega_1) \subset \Gamma$, we get that $(P_\alpha : \ \alpha \in [\omega, \omega_1])$, where $P_{\omega_1} := \mathrm{id}_X$, is an r-PRI on X with $\bigcup_{\alpha \in [\omega, \omega_1)} P_\alpha^* X^* = X^*$. For every $\alpha \in [\omega, \omega_1)$ find a countable dense subset C_α in the (separable) subspace $(P_{\alpha+1} - P_\alpha)X$ and put then $M := \bigcup_{\alpha \in [\omega, \omega_1)} C_\alpha$. According to, say, [7, Proposition 6.2.1 (iv)], valid also for r-PRI, the set M is linearly dense in X. We shall show that the support $\mathrm{supp}_M x^* := \{x \in M : \ x^*(x) \neq 0\}$ is at most countable for every $x^* \in X^*$. Funny, $\mathrm{supp}_M x^*$ is such for every $x^* \in P_\omega^* X^*$ $(= \{0\})$. Consider any $\alpha \in (\omega, \omega_1)$. Assume that $\mathrm{supp}_M x^*$ is at most countable for every $x^* \in P_\beta^* X^*$ where $\beta \in [\omega, \alpha)$. Now pick any $x^* \in P_\alpha^* X^*$. If $\alpha - 1$ exists, then $x^* = (P_\alpha^* - P_{\alpha-1}^*)x^* + P_{\alpha-1}^* x^*$, and so $\mathrm{supp}_M x^* \subset C_{\alpha-1} \cup \mathrm{supp}_M P_{\alpha-1}^* x^*$, and the latter set is at most countable; we used here the "orthogonality" of the projections $P_{\beta+1} - P_\beta, \ \beta \in [\omega, \omega_1)$. Second, if α is a limit ordinal, then $x^* = w^*\text{-}\lim_{\beta \uparrow \alpha} P_\beta^* x^*$, and so $\mathrm{supp}_M x^* \subset \bigcup_{\omega \le \beta < \alpha} \mathrm{supp}_M P_\beta^* x^*$; the latter set here being at most countable. We proved (ii). If the density of X is higher than ω_1, we can proceed as in the proof of [12, Theorem 27].
(ii)$\Longrightarrow$(i). If M is as in (ii), then the assignment $B_{X^*} \ni x^* \longmapsto (x^*(m) : \ m \in M) \in \Sigma(M)$ reveals that X is WLD. $\square$

3.2. **WCG spaces.** Let X be a Banach space, and let $A \subset X$ be a non-empty bounded set. We define the pseudo-metric ρ_A on X^* by

$$\rho_A(x_1^*, x_2^*) = \sup |\langle x_1^* - x_2^*, A\rangle|, \ x_1^*, \ x_2^* \in X^*, \tag{3}$$

where $\sup |\langle x^*, A\rangle| := \sup\{|\langle x^*, a\rangle| : \ a \in A\}$ for $x^* \in X^*$. We also denote by $\overline{S}^A$ the closure of a set $S \subset X^*$ in ρ_A. Observe that $\overline{S}^A = \{x^* \in X^* : \ \rho_A(x^*, S) = 0\}$, where $\rho_A(x^*, S) := \inf\{\rho_A(x^*, y^*) : \ y^* \in S\}$.

-shrinking}

Definition 15. *Given an $\varepsilon \ge 0$, a projectional skeleton $(P_s : \ s \in \Gamma)$ in X (if it exists) is called* A-ε-shrinking *if for every sequence $\gamma_1 \le \gamma_2 \le \cdots$ in Γ and for every $x^* \in X^*$ we have*

$$\limsup_{j \to \infty} \rho_A\big(P_{\gamma_j}^* x^*, P_{\sup \gamma_i}^* x^*\big) \le \varepsilon \|x^*\|;$$

if $\varepsilon = 0$, we say just "A-shrinking*".*

Remark 16. Note the following fact: If the set A is w-compact and $P_s(A) \subset A$ for all $s \in \Gamma$, then $(P_s : \ s \in \Gamma)$ is A-shrinking. Indeed, under these requirements for A the argument goes as follows: Let $\gamma_1 \leq \gamma_2 \leq \cdots$ be a sequence in Γ, and let $\gamma := \sup\{\gamma_n : n \in \mathbb{N}\}$. Pick any $x^* \in X^*$. We observed in Subsection 2.2 that $P^*_\gamma x^* \in \overline{\bigcup_{n=1}^\infty P^*_{\gamma_n} X^*}^{w^*}$. If $\mu(X^*, X)$ denotes the Mackey topology on X^* of the uniform convergence on the family of all convex, symmetric and w-compact subsets of X then, by the Mackey–Arens theorem, or just using Lemma 18 below, we have $P^*_\gamma x^* \in \overline{\bigcup_{n=1}^\infty P^*_{\gamma_n} X^*}^{\mu(X^*,X)}$. The sequence $(P^*_{\gamma_n} x^*)$ converges to $P^*_\gamma x^*$ uniformly on A. To show this, we follow the pattern in formulas (1) and (2) above. To be precise, if A° denotes the polar set of A, and $\varepsilon > 0$, then there exists $z^* \in X^*$ and $n \in \mathbb{N}$ such that $P^*_\gamma x^* - P^*_{\gamma_n} z^* \in \varepsilon A^\circ$. We have

$$P^*_\gamma x^* - P^*_{\gamma_n} x^* = (P^*_\gamma x^* - P^*_{\gamma_n} z^*) + (P^*_{\gamma_n} P^*_{\gamma_n} z^* - P^*_{\gamma_n} P^*_\gamma x^*) \in \varepsilon A^\circ + \varepsilon A^\circ \subset 2\varepsilon A^\circ,$$

as $P_{\gamma_n}(A) \subset A$. Hence, for $m \geq n$,

$$P^*_{\gamma_m} x^* - P^*_{\gamma_n} x^* = P^*_{\gamma_m} P^*_\gamma x^* - P^*_{\gamma_m} P^*_{\gamma_n} x^* \in 2\varepsilon A^\circ,$$

so $(P^*_\gamma x^* - P^*_{\gamma_m} x^*) \in 4\varepsilon A^\circ$, and this proves the assertion.

{rem-shrink

Remark 17. Let A be a bounded and linearly dense subset of X, and let $(P_\gamma : \gamma \in \Gamma)$ be an A-shrinking projectional skeleton on X. Then $\bigcup_{\gamma\in\Gamma} P^*_\gamma X^* = X^*$ (in particular, it follows from Theorem 14 that the space X is WLD; Theorem 20 below will provide a more precise result). Indeed, pick any $x^* \in X^*$ and consider the net $\{P^*_\gamma x^* : \gamma \in \Gamma, \leq\}$. Given an arbitrary sequence $\{\gamma_n\}$ in Γ and $\gamma \in \Gamma$ such that $\gamma_n \nearrow \gamma$, it follows from the definition of A-shrinkingness that $P^*_{\gamma_n} x^* \to P^*_\gamma x^*$ uniformly on A. By Proposition 19 we get that the net $\{P^*_\gamma x^* : \gamma \in \Gamma, \leq\}$ converges uniformly on A, say to $z^* \in X^*$, and that, moreover, there exist an increasing sequence (s_n) in Γ and $s_0 \in \Gamma$ such that $s_n \nearrow s_0$ and $P^*_{s_n} x^* \to z^*$ $(= P^*_{s_0} x^*)$ uniformly on A. Since, according to Remark 9, $P^*_\gamma x^* \xrightarrow{w^*} x^*$, given $a \in A$, we have $\langle x^*, a\rangle = \langle P^*_{s_0} x^*, a\rangle$. Due to the fact that A is linearly dense, we get $x^* = P^*_{s_0} x^*$, and this shows that $X^* = \bigcup_{\gamma\in\Gamma} P^*_\gamma X^*$, as we wanted to prove.

A Banach space is called *weakly compactly generated (WCG)* if it contains a linearly dense weakly compact set.

As we mentioned above, the Mackey topology $\mu^* := \mu(X^*, X)$ is the topology on X^* of the uniform convergence on the family of all convex symmetric and weakly compact subsets of X. We need the following particular case of the Mackey–Arens theorem. For the full statement, see, e.g., [9, Theorem 3.41]. We provide here a direct proof.

{lemma-ma}

Lemma 18. *Let X be a Banach space and let $C \subset X^*$ be a convex set. Then $\overline{C}^{w^*} = \overline{C}^{\mu^*}$.*

Proof. Obviously, $w^* \leq \mu^*$, hence $\overline{C}^{w^*} \supset \overline{C}^{\mu^*}$. Let $x^*_0 \notin \overline{C}^{\mu^*}$. Without loss of generality we may assume $x^*_0 = 0$. Find a convex, symmetric and w-compact subset K of X such that $K^\circ \cap C = \emptyset$. The set K° is a $\|\cdot\|$-neighborhood of 0, so the Separation Theorem gives $\alpha > 0$ and $x^{**} \in X^{**}$ such that $\langle c^*, x^{**}\rangle \geq \alpha > \langle k^*, x^{**}\rangle$ for every $c^* \in C$ and every $k^* \in K^\circ$. Since K° is symmetric we get $|\langle k^*, x^{**}\rangle| \leq \alpha$ for every $k^* \in K$, so $x^{**} \in \alpha K^{\circ\circ}$. The bipolar theorem ensures that $K^{\circ\circ} = K$ $(\subset X)$, so $x^{**} \in X$; thus $\{x^* \in X^* : \langle x^*, x^{**}\rangle \geq \alpha\}$ is w^*-closed (and it contains C), hence $0 \notin \overline{C}^{w^*}$. $\square$

We shall need a short trip to the convergence of nets. Let (M, ρ) be a metric space and let $\mathsf{n} := (x_s)_{s\in\Gamma}$ be a net consisting of elements from M; we recall that Γ is a directed set, with order, "$\leq$", say. We say that n has a *limit* $y \in M$ if for every $\varepsilon > 0$ there is $s \in \Gamma$ such that

$\rho(x_t, y) < \varepsilon$ whenever $t \in \Gamma$ and $t \geq s$. If this is so, we use for y the symbol $\lim_{s\in\Gamma} x_s$. The net n is called *Cauchy* if for every $\varepsilon > 0$ there is $s \in \Gamma$ such that $\rho(x_s, x_{s'}) < \varepsilon$ whenever $s' \in \Gamma$ and $s' \geq s$.

{prop-net}

Proposition 19. *Let (M, ρ) be a metric space, let $(\Gamma, \leq)$ be a directed and σ-complete set, let $T \subset \Gamma$ be a directed subset and let $(x_s)_{s\in\Gamma}$ be a net in M such that $\lim_{j\to\infty} \rho\big(x_{s_j}, x_\gamma\big) = 0$ whenever $s_1 \leq s_2 \leq \cdots$ is a sequence in T and $\gamma := \sup_{j\in\mathbb{N}} s_j$ $(\in \Gamma)$. Then the limit $\lim_{s\in T} x_s$ exists in the metric ρ (and is equal to x_γ where $\gamma := \sup_{i\in\mathbb{N}} t_i$ for some $t_1 \leq t_2 \leq \cdots$ in T).*

Proof. First we show that the net $(x_s)_{s\in T}$ is Cauchy. Assume that this is not true. Find then $\varepsilon > 0$ such that for every $s \in T$ there is $s' \in T$ such that $s' \geq s$ and $\rho\big(x_s, x_{s'}\big) \geq \varepsilon$. Using this, we can construct a sequence $s_1 \leq s_2 \leq s_3 \leq \cdots$ in T such that $\rho\big(x_{s_1}, x_{s_2}\big) \geq \varepsilon$, $\rho\big(x_{s_2}, x_{s_3}\big) \geq \varepsilon$, $\ldots$. Since $(\Gamma, \leq)$ is σ-complete, $s := \sup_{n\in\mathbb{N}} s_n$ exists and belongs to Γ. By the assumption, $\rho\big(x_{s_j}, x_s\big) \longrightarrow 0$ as $j \to \infty$, and so $(\varepsilon \leq)$ $\rho\big(x_{s_j}, x_{s_{j+1}}\big) \longrightarrow 0$ as $j \to \infty$; a contradiction. We proved that our net is Cauchy.
Next, we shall construct a sequence $t_1 \leq t_2 \leq \cdots$ in T as follows. Pick $t_1 \in T$ such that $\rho(x_s, x_{t_1}) < 1$ whenever $s \in T$ and $s \geq t_1$. Consider any $j \in \mathbb{N}$ and assume that $t_j \in T$ was already found. Pick $t \in T$ such that $\rho(x_s, x_t) < \frac{1}{2(j+1)}$ whenever $s \in T$ and $s \geq t$. Find $t_{j+1} \in T$ such that it majorizes both t and t_j. Now, if $s \in T$ is such that $s \geq t_{j+1}$, then

$$\rho(x_s, x_{t_{j+1}}) \leq \rho(x_s, x_t) + \rho(x_t, x_{t_{j+1}}) < 2 \cdot \frac{1}{2(j+1)} = \frac{1}{j+1}\,.$$

Doing so for every $j \in \mathbb{N}$, put $\gamma := \sup_{j\in\mathbb{N}} t_j$. Then we know that $\lim_{j\to\infty} \rho(x_{t_j}, x_\gamma) = 0$. We claim that $\lim_{s\in T} x_s = x_\gamma$. Indeed, take any $\varepsilon > 0$. Pick $j > \frac{2}{\varepsilon}$ so big that $\rho(x_{t_j}, x_\gamma) < \frac{\varepsilon}{2}$. Now, if $s \in T$ and $s \geq t_j$, we have

$$\rho(x_s, x_\gamma) \leq \rho(x_s, x_{t_j}) + \rho(x_{t_j}, x_\gamma) < \tfrac{1}{j} + \tfrac{\varepsilon}{2} < \varepsilon.$$

□

{thm-wcg}

Theorem 20. *For a Banach space $(X, \|\cdot\|)$ TFAE:*

(i) *X is weakly compactly generated.*
(ii) *There exist a bounded closed symmetric convex and linearly dense set $A \subset X$, and a (commutative 1-) projectional skeleton $(P_\gamma :\ \gamma \in \Gamma)$ on $(X, \|\cdot\|)$ (with $\bigcup_{\gamma\in\Gamma} P_\gamma^* X^* = X^*$), which is moreover A-shrinking and satisfies that $P_\gamma(A) \subset A$ for every $\gamma \in \Gamma$.*

Proof. (i)$\Longrightarrow$(ii). Assume that X is WCG. The celebrated Amir-Lindenstrauss theorem provides a set Γ and a linear bounded injective mapping $T : X^* \longrightarrow c_0(\Gamma)$, which is moreover w^*-w-continuous; see [1], [9, Theorem 13.20]. (Its proof may nowadays start from constructing a projectional generator on a reflexive space R, e.g., the simple generator described in (PGi) of Subsection 2.4. Then a PRI on R is constructed from this generator. Then, a transfinite induction argument provides a bounded linear and one-to-one mapping from R into $c_0(\Gamma)$. Finally, combining this with a mapping coming from the factorization theorem [9, Theorem 13.22] yields our $T : X^* \longrightarrow c_0(\Gamma)$.) Once having T, put

$$M := \big\{T^* e_\gamma :\ \gamma \in \Gamma\big\},$$

where e_γ's are the elements of the canonical basis in ℓ_1. It is straightforward to verify that this M is a subset of X (if we identify X with its canonical image in X^{**}), that M is linearly dense in X, and that for every $x^* \in X^*$ the set $\Phi(x^*) := \{m \in M :\ \langle x^*, m\rangle \neq 0\}$ is at

most countable (in addition, $M \cup \{0\}$ is a w-compact set with the only accumulation point 0). According to (PGii) in Subsection 2.4 , (X^*, Φ) is a projectional generator on X (better than that from (PGi)). Let A be the closed convex hull of the set $M \cup (-M)$.
Denote $\|\cdot\|_0 := \|\cdot\|$. For $n \in \mathbb{N}$, let $\|\cdot\|_n$ be the Minkowski functional of the set $\overline{A + (1/n)B_X}$. Then $\|\cdot\|_n$ is an equivalent norm on X. For the norm $\|\cdot\|_n$ the set X^* is obviously 1-norming, and so (X^*, Φ) is a projectional generator on $(X, \|\cdot\|_n)$. Applying Proposition 12, we get a rich family $\mathcal{R}_n$ with property (A$_1$) in Lemma 1. Further put $\mathcal{R}_{-1} := \{V \times Y \in \mathcal{S}_\square(X \times X^*) : M \setminus V \subset Y_\perp\}$; this is a rich family by Lemma 13 (i). Put finally $\mathcal{R} := \bigcap_{n=-1}^\infty \mathcal{R}_n$; this is again a rich family and its elements have the property (A$_1$) in Lemma 1 with respect to all norm $\|\cdot\|_0,\ \|\cdot\|,\ \ldots$ Thus, the 1-projectional skeleton $(P_\gamma :\ \gamma \in \Gamma)$ associated to $\mathcal{R}$ via Proposition 12 is commutative by Lemma 13 (ii). Moreover, for all $\gamma \in \Gamma$ and all $n \in \mathbb{N}$, due to the fact that $\|P_\gamma\|_n = 1$, we have

$$P_\gamma(A) \subset P_\gamma(A + \tfrac{1}{n}B_X) \subset \overline{A + \tfrac{1}{n}B_X} \subset A + \tfrac{2}{n}B_X,$$

and so $P_\gamma(A) \subset A$ for all $\gamma \in \Gamma$. The A-shrinking character of our skeleton comes from Remark 17. And having this, it is easy to check that $\bigcup_{\gamma\in\Gamma} P^*_\gamma X^* = X^*$.

(ii)$\Longrightarrow$(i). Let A and $(P_\gamma :\ \gamma \in \Gamma)$ be as in (ii). We shall prove the bit stronger statement that there exists a weakly compact set $K \subset A$ which is linearly dense in X (and thus X will be a WCG space). By [12, Proposition 9] or [14, Proposition 17.6], there exists a directed and σ-closed subset of Γ, denoted for simplicity again as Γ, such that $r := \sup\{\|P_s\| :\ s \in \Gamma\} < +\infty$. Clearly, this "smaller" $(P_\gamma :\ \gamma \in \Gamma)$ will be an A-shrinking r-projectional skeleton on X.

In order not to get lost and get a taste of the proof, we first consider the special case when the density of X is ω_1. It is not dramatically difficult to find a subset of Γ which is cofinal, σ-closed, and which is moreover order homeomorphic with the interval $[\omega, \omega_1)$. For simplicity, we will think that this interval is a subset of Γ, and the order relation on it coincides with the order of Γ. (If we let P_{ω_1} to denote the identity mapping on X, then it is easy to check that $(P_\alpha :\ \omega \le \alpha \le \omega_1)$ is a projectional resolution of the identity on $(X, \|\cdot\|)$, with the only exception that $\|P_\alpha\| \le r$ for every $\omega \le \alpha \le \omega_1$.) For every $\omega \le \alpha < \omega_1$ we find a sequence $x_1^\alpha, x_2^\alpha, \ldots$ in the (separable) set $\frac{1}{2}(P_{\alpha+1} - P_\alpha)A$ such that $\|x_n^\alpha\| \to 0$ as $n \to \infty$ and that $\overline{\mathrm{sp}\,\{x_1^\alpha, x_2^\alpha, \ldots\}} = (P_{\alpha+1} - P_\alpha)X$. Now, put $K := \big\{x_n^\alpha :\ \omega \le \alpha < \omega_1 \text{ and } n \in \mathbb{N}\big\} \cup \{0\}$; then clearly $K \subset A$. By a (well known) fact [7, Proposition 6.2.1 (iv)], we immediately get that $\overline{\mathrm{sp}\, K} = X$. We shall show that the set K is weakly compact. Let $\mathcal{U}$ be a family of weakly open sets in X covering K. We have to find a finite subfamily of $\mathcal{U}$ that still covers K. Pick $U \in \mathcal{U}$ so that $U \ni 0$. Find a finite set $F \subset X^*$ so that $U \supset \{x \in X :\ \max\langle F, x\rangle < 1\} \ni 0$. We shall show that $K \setminus U$ is finite and thus the weak compactness of K will be proved. It is actually enough to show that for every $x^* \in F$ the set $M_{x^*} := \{(n, \alpha) \in \mathbb{N} \times (\omega, \omega_1) :\ \langle x^*, x_n^\alpha\rangle \ge 1\}$ is finite. So fix one such $x^* \in F$ and assume that M_{x^*} is infinite. Since $\|x_n^\alpha\| \to 0$ as $n \to \infty$ for every $\alpha \in (\omega, \omega_1)$, the set $\{\alpha \in (\omega, \omega_1) :\ (n, \alpha) \in M_{x^*} \text{ for some } n \in \mathbb{N}\}$ is infinite. Thus, there are an increasing sequence $\alpha_1 < \alpha_2 < \cdots$ in (ω, ω_1) and a sequence (n_j) in $\mathbb{N}$ such that $(n_j, \alpha_j) \in M_{x^*}$ for every $j \in \mathbb{N}$. Put $\alpha := \sup_{j\in\mathbb{N}} \alpha_j$; clearly $\alpha < \omega_1$. Now, for every $j \in \mathbb{N}$ we have (due to the fact that $P_{\alpha_j} x_{n_j}^{\alpha_j} = 0$)

$$\begin{aligned} 1 \ &\le\ \langle x^*, x_{n_j}^{\alpha_j}\rangle = \big\langle x^*, P_\alpha x_{n_j}^{\alpha_j}\big\rangle = \big\langle P_\alpha^* x^*, x_{n_j}^{\alpha_j}\big\rangle \\ &=\ \big\langle P_\alpha^* x^* - P_{\alpha_j}^* x^*, x_{n_j}^{\alpha_j}\big\rangle \le \rho_A\big(P_\alpha^* x^*, P_{\alpha_j}^* x^*\big) \longrightarrow 0 \end{aligned}$$

as $j \to \infty$; a contradiction. We proved that our K is weakly compact. Therefore X is WCG.

Now, we consider the general case, when X is any non-separable space satisfying (ii). We shall prove the existence of a linearly dense and weakly compact set lying (even) in A. Let κ be an uncountable cardinal and assume that such a set was found for all X's with $\operatorname{dens} X < \kappa$. Now, assume that $\operatorname{dens} X = \kappa$. The argument will be split into several steps.
1. Using ideas from the proofs of [14, Theorem 12] and [4, Theorem 15], we find an increasing family T_α, $\omega < \alpha \le \kappa$, of directed subsets of Γ such that $T_\alpha = \bigcup \big\{ T_{\beta+1} :\ \omega < \beta < \alpha \big\}$ and $\operatorname{card} T_\alpha \le \alpha$ for every $\alpha \in (\omega, \kappa]$, and that the set $\bigcup \big\{ P_s X :\ s \in T_\kappa \big\}$ is dense in X.
2. Put $Q_\omega := 0$ and for every $\alpha \in (\omega, \kappa]$ define

$$Q_\alpha x := \lim_{s \in T_\alpha} P_s x, \quad x \in X;$$

see Proposition 19 where $M := X$ and ρ comes from the norm $\|\cdot\|$; then clearly $Q_\alpha X = \overline{\{P_s X :\ s \in T_\alpha\}}$. It is easy to verify that $(Q_\alpha :\ \omega \le \alpha \le \kappa)$ is then an r-PRI on $(X, \|\cdot\|)$; for more details, see the proofs mentioned in Step 1. Moreover, we can easily verify that $P_s \circ Q_\alpha = P_s$ whenever $\alpha \in (\omega, \kappa)$ and $s \in T_\alpha$, and that $Q_\alpha(A) \subset A$ for every $\alpha \in [\omega, \kappa]$.
3. For every $x^* \in X^*$ and every $\alpha \in (\omega, \kappa]$ we have

$$Q_\alpha^* x^* = \rho_A\text{-} \lim_{s \in T_\alpha} P_s^* x^*;$$

the limit here exists according to Proposition 19 where $M := X^*$ and $\rho := \rho_A$. Indeed, for every a from the linearly dense set A we have

$$\begin{aligned} \big\langle Q_\alpha^* x^*, a \big\rangle &= \big\langle x^*, Q_\alpha a \big\rangle = \Big\langle x^*, \lim_{s \in T_\alpha} P_s a \Big\rangle = \lim_{s \in T_\alpha} \big\langle x^*, P_s a \big\rangle \\ &= \lim_{s \in T_\alpha} \big\langle P_s^* x^*, a \big\rangle = \big\langle \rho_A\text{-} \lim_{s \in T_\alpha} P_s^* x^*, a \big\rangle. \end{aligned}$$

4. For every $x^* \in X^*$ and every sequence $\omega < \alpha_1 < \alpha_2 < \cdots < \kappa$, with $\alpha := \sup\{\alpha_1, \alpha_2, \cdots\}$ (which may be equal to κ), we have $\rho_A\big(Q_{\alpha_j}^* x^*, Q_\alpha^* x^*\big) \longrightarrow 0$ as $j \to \infty$. Indeed, take any $\varepsilon > 0$. Find $s_0 \in T_\alpha$ so that $\rho_A\big(P_s^* x^*, Q_\alpha^* x^*\big) < \varepsilon$ whenever $s \in T_\alpha$ and $s \ge s_0$. Find $j_0 \in \mathbb{N}$ so big that $T_{\alpha_{j_0}} \ni s_0$. Now fix any $j \in \mathbb{N}$ greater than j_0. Then $T_{\alpha_j} \supset T_{\alpha_{j_0}} \ni s_0$, and so, by Step 3,

$$Q_{\alpha_j}^* x^* = \rho_A \text{ -} \lim_{s \in T_{\alpha_j}} P_s^* x^* = \rho_A \text{ -} \lim_{s \in T_{\alpha_j},\ s \ge s_0} P_s^* x^*.$$

Hence $\rho_A\big(Q_{\alpha_j}^* x^*, Q_\alpha^* x^*\big) = \lim_{s \in T_{\alpha_j},\ s \ge s_0} \rho_A\big(P_s^* x^*, Q_\alpha^* x^*\big) \le \varepsilon$ for all $j \in \mathbb{N}$ greater than j_0.
5. Fix any $\alpha \in (\omega, \kappa)$. Let Γ_α be the smallest subset of Γ that contains T_α, is directed, and σ-closed. For the construction of such an envelope we refer to the proof of [4, Theorem 15]. The system $\big(H_s^\alpha := P_s\restriction_{Q_\alpha X}\!:\ s \in \Gamma_\alpha\big)$ is then an r-projectional skeleton on $(Q_\alpha X, \|\cdot\|)$ such that $H_s^\alpha(Q_\alpha A) \subset Q_\alpha A$ for every $s \in \Gamma_\alpha$; the verification of this directly follows from the description of Γ_α. Further, our skeleton is $Q_\alpha A$-shrinking. Indeed, consider any sequence $s_1 \le s_2 \le \cdots$ in Γ_α and put $s := \sup\{s_1, s_2, \ldots\}$. Take any $y^* \in (Q_\alpha X)^*$. Find $x^* \in X^*$ such that $x^*\restriction_{Q_\alpha X} = y^*$ and $\|x^*\| = \|y^*\|$. An elementary verification reveals that for every $n \in \mathbb{N}$

$$\rho_{Q_\alpha(A)}\big(H_s^{\alpha *} y^*, H_{s_n}^{\alpha\ *} y^*\big) = \rho_{Q_\alpha(A)}\big(P_s^* x^*, P_{s_n} x^*\big) \le \rho_A\big(P_s^* x^*, P_{s_n} x^*\big)$$

So, knowing that the skeleton $(P_s :\ s \in \Gamma)$ is A-shrinking, we can conclude that the skeleton $\big(H_s^\alpha :\ s \in \Gamma_\alpha\big)$ on $Q_\alpha X$ is $Q_\alpha A$-shrinking.

6. Now we are ready to construct a weakly compact subset of A which is linearly dense in X. Fix any $\alpha \in (\omega, \kappa)$ for a while. The subspace $Q_{\alpha+1}X$ has density less than κ. From Steps 1, 2, 5, by the induction assumption, we find a weakly compact set $K_\alpha \subset \frac{1}{2}Q_{\alpha+1}A$ which is linearly dense in the subspace $Q_{\alpha+1}X$. Define $K := \bigcup_{\alpha\in(\omega,\kappa)}(Q_{\alpha+1} - Q_\alpha)K_\alpha \cup \{0\}$. By [7, Proposition 6.2.1 (iv)], we can easily conclude that K is linearly dense in X. Also, $K \subset A$. As regards the weak compactness of K, let $\mathcal{U}$ be a family of weakly open sets in X covering K. We shall show that $\mathcal{U}$ contains a finite subcover. Find $U \in \mathcal{U}$ so that $U \ni 0$. We shall show that the set of α' from (ω, κ) such that $(Q_{\alpha+1} - Q_\alpha)K_\alpha \setminus U \neq \emptyset$ is finite. Putting together this with the weak compactness of each set $(Q_{\alpha+1} - Q_\alpha)K_\alpha$, we immediately get that $\mathcal{U}$ contains a finite subfamily covering the whole K.

So, assume that set of α's as above is infinite. Find a finite set $F \subset X^*$ so that $U \supset \{x \in X : \max\langle F, x\rangle < 1\} \ni 0$. Then, for sure, there is $x^* \in F$ such that the set $\big\{x \in (Q_{\alpha+1} - Q_\alpha)K_\alpha : \langle x^*, x\rangle \geq 1\big\}$ is nonemtpy for infinitely many $\alpha \in (\omega, \kappa)$. Then there exists an infinite increasing sequence $\alpha_1 < \alpha_2 < \cdots$ in (ω, κ) and for every $j \in \mathbb{N}$ a point $x_j \in (Q_{\alpha_j+1} - Q_{\alpha_j})K_{\alpha_j}$ such that $\langle x^*, x_j\rangle \geq 1$. Put $\alpha := \sup_{j\in\mathbb{N}} \alpha_j$; clearly $\alpha \leq \kappa$. Now, for every $j \in \mathbb{N}$ (due to the fact that $Q_{\alpha_j}x_j = 0$)

$$\begin{aligned} 1 &\leq \langle x^*, x_j\rangle = \big\langle x^*, Q_\alpha x_j\big\rangle = \big\langle Q_\alpha^* x^*, x_j\big\rangle \\ &= \big\langle Q_\alpha^* x^* - Q_{\alpha_j}^* x^*, x_j\big\rangle \leq \rho_A\big(Q_\alpha^* x^*, Q_{\alpha_j}^* x^*\big) \longrightarrow 0 \end{aligned}$$

as $j \to \infty$; a contradiction. We proved that our K is weakly compact, and so X is WCG. □

Remark 21. (Important) If we do not care about the commutativity of skeletons constructed on WCG spaces, we may take, in the proof above, for the projectional generator the simple one constructed in Subsection 2.4 (PGi); thus avoiding the use of Amir-Lindenstrauss theorem.

3.3. **SWCG spaces.** Before attacking the problem of characterizing the class of subspaces of WCG spaces via projectional skeletons, we recall the following criterion going back to V. Farmaki [10].

{farma}

Theorem 22. [8, Theorem 3] *A Banach space X is a subspace of a weakly compactly generated space if and only if there exists a linearly dense set $\Delta \subset B_X$ such that for every $\varepsilon > 0$ there is a decomposition $\Delta = \bigcup_{n=1}^\infty \Delta_n^\varepsilon$ such that*

$$\forall n \in \mathbb{N} \quad \forall x^* \in X^* \quad \#\big\{\delta \in \Delta_n^\varepsilon : \langle x^*, \delta\rangle > \varepsilon\|x^*\|\big\} < \omega.$$

{swcg}

Theorem 23. *For a Banach space $(X, \|\cdot\|)$ TFAE:*

(i) *X is a subspace of a weakly compactly generated space.*

(ii) *There exist a (commutative 1-) projectional skeleton $(P_\gamma : \gamma \in \Gamma)$ on X and a countable family $\mathcal{A}$ of convex closed symmetric subsets of B_X such that*

(a) *$P_\gamma(A) \subset A$ for every $A \in \mathcal{A}$ and every $\gamma \in \Gamma$,*

(b) *for every $A \in \mathcal{A}$ there is $\varepsilon_A > 0$ such that the skeleton $(P_\gamma : \gamma \in \Gamma)$ is A-ε_A-shrinking, and*

(c) *$\bigcup\{A \in \mathcal{A} : \varepsilon_A < \varepsilon\} = B_X$ for every $\varepsilon > 0$.*

Proof. (i)$\Longrightarrow$(ii). Assume that X is a subspace of a WCG space $(W, \|\cdot\|)$. Assume that the norm on W is an extension of the norm on X. We shall proceed similarly as in the proof of [4, Proposition 10]. The space W being WCG, here exists a projectional generator (W^*, Φ)

constructed as at the begining of the proof of Theorem 20. Now we define a multivalued mapping $\psi : W \longrightarrow [W^*]^{\leq\omega}$ as follows. For $w \in W$ we find a countable set $\psi(w) \subset W^*$ such that

$$\|w\| = \sup\big\{|\langle w^*, w\rangle| : \ w^* \in \psi(w) \ \text{ and } \ \|w^*\| \leq 1\big\}$$

$$\|w\|_m = \sup\big\{|\langle w^*, w\rangle| : \ w^* \in \psi(w) \ \text{ and } \ \|w^*\|_m \leq 1\big\} \quad m \in \mathbb{N},$$

$$\|w\|_{n,m} = \sup\big\{|\langle w^*, w\rangle| : \ w^* \in \psi(w) \ \text{ and } \ \|w^*\|_{n,m} \leq 1\big\} \quad m,n \in \mathbb{N},$$

where $\|\cdot\|_m$ and $\|\cdot\|_{m,n}$ are the Minkowski functionals of the sets $B_X + \frac{1}{m}B_W$ and $(nK + \frac{1}{m}B_W) \cap B_X$, respectively, $B_W := \{w \in W : \ \|w\| \leq 1\}$ and $B_X := \{x \in X : \ \|x\| \leq 1\}$. (Here we always use the convention that a norm and the dual norm to it is denoted by the same symbol.) Proposition 12 and Lemma 13 above yields a rich family $\Gamma \subset \mathcal{S}_{\square}(W \times W^*)$ and a commutative projectional skeleton $(Q_\gamma : \ \gamma \in \Gamma)$ on W such that $\|Q_\gamma\| = \|Q_\gamma\|_m = \|Q_\gamma\|_{m,n} = 1$ for every $\gamma \in \Gamma$ and every $n, m \in \mathbb{N}$. Now, given any $\gamma \in \Gamma$, for every $m \in \mathbb{N}$ we have

$$Q_\gamma(B_X) \subset Q_\gamma\big(B_X + \tfrac{1}{m}B_W\big) \subset \overline{B_X + \tfrac{1}{m}B_W} \subset B_X + \tfrac{2}{m}B_W,$$

and hence $Q_\gamma(B_X) \subset B_X$, and so $Q_\gamma X \subset X$. Thus, putting $P_\gamma := Q_\gamma\restriction_X, \ \gamma \in \Gamma$, we immediately get that $(P_\gamma : \ \gamma \in \Gamma)$ is a commutative 1-projectional skeleton on the space $(X, \|\cdot\|)$. We define

$$A^m_n := \overline{nK + \tfrac{1}{2m}B_W} \cap B_X \quad \text{for} \quad m,n \in \mathbb{N}$$

and put $\mathcal{A} := \{A^m_n : \ m,n \in \mathbb{N}\}$. Clearly, $A^m_1 \cup A^m_2 \cup \cdots = B_X$ for every $m \in \mathbb{N}$. Further, for every $\gamma \in \Gamma$ and every $n, m \in \mathbb{N}$ we have $P_\gamma(A^m_n) = Q_\gamma(A^m_n) \subset A^m_n$ as $\|Q_\gamma\|_{n,m} = 1$. Finally, fix any $m, n \in \mathbb{N}$, fix any $x^* \in X^*$, consider any sequence $\gamma_1 \leq \gamma_2 \leq \cdots$ in Γ, and put $\gamma := \sup\{\gamma_1, \gamma_2, \ldots\}$. Find a $w^* \in W^*$ such that $w^*\restriction_X = x^*$ and $\|w^*\| = \|x^*\|$. From Theorem 20 we know that

$$\sup\big|\big\langle Q^*_{\gamma_j}w^* - Q^*_\gamma w^*, K\big\rangle\big| \longrightarrow 0 \quad \text{as} \quad j \to \infty. \tag{4}$$

Also, for every $x \in A^m_n$ we have

$$\big\langle P^*_{\gamma_j}x^* - P^*_\gamma x^*, x\big\rangle = \big\langle x^*, P_{\gamma_j}x\big\rangle - \big\langle x^*, P_\gamma x\big\rangle = \big\langle w^*, Q_{\gamma_j}x\big\rangle - \big\langle w^*, Q_\gamma x\big\rangle = \big\langle Q^*_{\gamma_j}w^* - Q^*_\gamma w^*, x\big\rangle.$$

Therefore

$$\begin{aligned}
\limsup_{j\to\infty} \sup\big|\big\langle P^*_{\gamma_j}x^* - P^*_\gamma x^*, A^m_n\big\rangle\big| &= \limsup_{j\to\infty} \sup\big|\big\langle Q^*_{\gamma_j}w^* - Q^*_\gamma w^*, A^m_n\big\rangle\big| \\
&= \limsup_{j\to\infty} \sup\big|\big\langle Q^*_{\gamma_j}w^* - Q^*_\gamma w^*, \overline{nK + \tfrac{1}{2m}B_W} \cap B_X\big\rangle\big| \\
&\leq \tfrac{1}{2m} \limsup_{j\to\infty} \sup\big|\big\langle Q^*_{\gamma_j}w^* - Q^*_\gamma w^*, B_W\big\rangle\big| \\
&\leq \tfrac{1}{2m} \cdot 2\|w^*\| = \tfrac{1}{m}\|x^*\|
\end{aligned}$$

where the first inequality comes from (4). We proved that the skeleton $(P_\gamma; \ \gamma \in \Gamma)$ is A^m_n-$\frac{1}{m}$-shrinking with respect to the norm $\|\cdot\|$.

(ii)$\Longrightarrow$(i). If X is separable, then it is even WCG. Further, let κ be any uncountable cardinal and assume that the implication holds for all Banach spaces with density less than κ. Now consider any X, with density κ, and satisfying (ii). A reasoning as in the proof of Theorem 20 authorizes us to assume that $(P_\gamma : \ \gamma \in \Gamma)$ is an r-projectional skeleton with some finite $r \geq 1$. (Again, we do not need that this skeleton is commutative nor that it is a 1-skeleton.)

We copy here Steps 1 and 2 from the proof of Theorem 20. The notation follows the notation there.
3. *Given any* $0 \neq x^* \in X^*$, *any* $\alpha \in (\omega, \kappa]$, *and any* $A \in \mathcal{A}$, *there exists* $\gamma_{x^*,\alpha,A} \in T_\alpha$ *such that* $\rho_A\big(P_s^* x^*, Q_\alpha^* x^*\big) < 5\varepsilon_A \|x^*\|$ *whenever* $s \in T_\alpha$ *and* $s \geq \gamma_{x^*,\alpha,A}$. Indeed, first we observe that the net $\big(P_s^* x^*\big)_{s \in T_\alpha}$ is "$2\varepsilon_A\|x^*\|$-Cauchy". Assume this is not so. Pick some $s_1 \in T_\alpha$. Find then $s_2 \in T_\alpha$ such that $s_2 \geq s_1$ and $\rho_A\big(P_{s_1}^* x^*, P_{s_2}^* x^*\big) \geq 2\varepsilon_A\|x^*\|$. Find then $s_3 \in T_\alpha$ such that $s_3 \geq s_2$ and $\rho_A\big(P_{s_2}^* x^*, P_{s_3}^* x^*\big) \geq 2\varepsilon_A\|x^*\|, \ldots$. Put $s := \sup\{s_1, s_2, \ldots\}$ $(\in \Gamma)$. Then we know that $\limsup_{j\to\infty} \rho_A\big(P_{s_j}^* x^*, P_s^* x^*\big) < \varepsilon_A\|x^*\|$, and so $\rho_A\big(P_{s_j}^* x^*, P_{s_{j+1}}^* x^*\big) < 2\varepsilon_A\|x^*\|$ for all $j \in \mathbb{N}$ big enough, a contradiction. We proved that there exists $\gamma_{x^*,\alpha,A} \in T_\alpha$ such that $\rho_A\big(P_{\gamma_{x^*,\alpha,A}}^* x^*, P_s^* x^*\big) < 2\varepsilon_A\|x^*\|$ whenever $s \in T_\alpha$ and $s \geq \gamma_{x^*,\alpha,A}$, and hence

$$\big|\big\langle P_{s'}^* x^* - P_s^* x^*, a\big\rangle\big| < 4\varepsilon_A\|x^*\| \quad \text{whenever } a \in A,\ s, s' \in T_\alpha \text{ and } s, s' \geq \gamma_{x^*,\alpha,A}\,.$$

Applying here $\lim_{s' \in T_\alpha,\ s' \geq \gamma_{x^*,\alpha,A}}$, we get that $\big|\big\langle Q_\alpha^* x^*, a\big\rangle - \big\langle P_s^* x^*, a\big\rangle\big| \leq 4\varepsilon_A\|x^*\|$ whenever $a \in A,\ s \in T_\alpha$ and $s \geq \gamma_{x^*,\alpha,A}$, and so our claim is proved.
4. *Given any* $0 \neq x^* \in X^*$, *any* $A \in \mathcal{A}$, *and any sequence* $\alpha_1 < \alpha_2 < \cdots$ *in* (ω, κ), *with* $\alpha := \sup_{j\in\mathbb{N}} \alpha_j$, *then*

$$\limsup_{j\to\infty} \rho_A\big(Q_{\alpha_j}^* x^*, Q_\alpha^* x^*\big) \leq 10\,\varepsilon_A\|x^*\|.$$

Indeed, from Step 3, find $s_\alpha \in T_\alpha$ so big that $\rho_A\big(P_s^* x^*, Q_\alpha^* x^*\big) \leq 5\varepsilon_A\|x^*\|$ whenever $s \in T_\alpha$ and $s \geq s_\alpha$. Find $j_0 \in \mathbb{N}$ so big that $T_{\alpha_{j_0}} \ni s_\alpha$. Now, take any $j \in \mathbb{N}$ greater than j_0. From Step 3 find $s_{\alpha_j} \in T_{\alpha_j}$ so big that $\rho_A\big(P_s^* x^*, Q_{\alpha_j}^* x^*\big) < 5\varepsilon_A\|x^*\|$ whenever $s \in T_{\alpha_j}$ and $s \geq s_{\alpha_j}$. Pick some $\bar{s} \in T_{\alpha_j}$ so that $\bar{s} \geq s_\alpha$ and $\bar{s} \geq s_{\alpha_j}$. Then $\bar{s} \in T_\alpha$ and so $\rho_A\big(P_{\bar{s}}^* x^*, Q_\alpha^* x^*\big) \leq 5\varepsilon_A\|x^*\|$. Also, $\rho_A\big(P_{\bar{s}}^* x^*, Q_{\alpha_j}^* x^*\big) < 5\varepsilon_A\|x^*\|$. Therefore $\rho_A\big(Q_\alpha^* x^*, Q_{\alpha_j}^* x^*\big) < 10\,\varepsilon_A\|x^*\|$ for every $j \in \mathbb{N}$ greater than j_0.
5. This is just a tiny generalization of Step 5 from the proof of Theorem 20. *For every* $\alpha \in (\omega, \kappa)$ *the system* $\big(H_s^\alpha := P_s\restriction_{Q_\alpha X}\colon\ s \in \Gamma_\alpha\big)$ *is an r-projectional skeleton in* $Q_\alpha X$ *such that for every* $A \in \mathcal{A}$ *we have* $H_s^\alpha(Q_\alpha A) \subset Q_\alpha A$ *whenever* $s \in \Gamma_\alpha$, *this skeleton is* $Q_\alpha(A)$-ε_A*-shrinking, and* $\bigcup\{Q_\alpha(A):\ A \in \mathcal{A},\ \varepsilon_A < \varepsilon\} = B_{Q_\alpha X}$ *for every* $\varepsilon > 0$. Recall that Γ_α is the smallest subset of Γ that contains T_α, is directed, and σ-closed. The verification of all these statements is very similar to that from Step 5 in the proof of Theorem 20, and hence it is omitted.
6. Now, given any $\alpha \in (\omega, \kappa)$, we have at hand the validity of the assertion (ii) from our Theorem where X is replaced by $Q_\alpha X$ and $\mathcal{A}$ is replaced by the family $\{Q_\alpha(A):\ A \in \mathcal{A}\}$. Hence, by the induction assumption, $Q_\alpha X$ is a subspace of a WCG space. Thus for every $\alpha \in (\omega, \kappa)$ the subspace $Q_{\alpha+1}X$, and hence also $(Q_{\alpha+1} - Q_\alpha)X$, is a subspace of a WCG space. Therefore Theorem 22 yields a set $\Delta_\alpha \subset (Q_{\alpha+1} - Q_\alpha) \cap B_X$, linearly dense in $(Q_{\alpha+1} - Q_\alpha)X$, such that for every $\varepsilon > 0$ we have a decomposition $\Delta_\alpha = \bigcup_{n=1}^\infty \Delta_{\alpha,n}^\varepsilon$ such that

$$\forall n \in \mathbb{N} \quad \forall y^* \in \big((Q_{\alpha+1} - Q_\alpha)X\big)^* \quad \#\big\{\delta \in \Delta_{\alpha,n}^\varepsilon:\ \langle y^*, \delta\rangle > \varepsilon\|y^*\|\big\} < \omega. \qquad (5)$$ {14}

Now put $\Delta := \bigcup\{\Delta_\alpha:\ \alpha \in (\omega, \kappa)\}$. According to, [7, Proposition 6.2.1 (iv)], this set is linearly dense in the whole X. Fix any $\varepsilon > 0$. We shall verify the criterion from Theorem 22 for the space X. We have by (c) in (ii) of our theorem

$$\Delta = \bigcup \big\{\bigcup\{\Delta_{\alpha,n}^\varepsilon:\ \alpha \in (\omega, \kappa)\} \cap A:\ n \in \mathbb{N},\ A \in \mathcal{A},\ \varepsilon_A < \tfrac{\varepsilon}{10}\big\};$$

note that the "bigger" union consists of countably many pieces. Fix for a while any $n \in \mathbb{N}$ and any $A \in \mathcal{A}$. Consider any $x^* \in X^*$. We have to verify that the set

$$\big\{\delta \in \bigcup\{\Delta^\varepsilon_{\alpha,n} :\ \alpha \in (\omega,\kappa)\} \cap A :\ \langle x^*, \delta\rangle > \varepsilon\|x^*\|\big\}$$

is finite. Arguing by contradiction, assume that this is not so; thus $x^* \neq 0$. Find then a one-to-one sequence $\delta_1, \delta_2, \ldots$ in $\bigcup\{\Delta^\varepsilon_{\alpha,n} :\ \alpha \in (\omega,\kappa)\} \cap A$ such that $\langle x^*, \delta_j\rangle > \varepsilon\|x^*\|$ for every $j \in \mathbb{N}$. For every $j \in \mathbb{N}$ find $\alpha_j \in (\omega,\kappa)$ so that $\delta_j \in \Delta^\varepsilon_{\alpha_j,n} \cap A$. From (5) we easily deduce that the set $\{\alpha_1, \alpha_2, \ldots\}$ is infinite. When going to subsequences, we may and do assume that $\alpha_1 < \alpha_2 < \cdots$. Put $\alpha := \sup\{\alpha_1, \alpha_2, \ldots\}$ (which may be also equal to κ). For every $j \in \mathbb{N}$ we have (as $Q_{\alpha_j}\delta_j = 0$)

$$\varepsilon\|x^*\| < \langle x^*, \delta_j\rangle = \big\langle Q^*_\alpha x^* - Q^*_{\alpha_j} x^*, \delta_j\big\rangle \leq \rho_A\big(Q^*_\alpha x^*, Q^*_{\alpha_j} x^*\big),$$

and letting $j \to \infty$, Step 4 guarantees that

$$\varepsilon\|x^*\| \leq \limsup_{j\to\infty} \rho_A\big(Q^*_\alpha x^*, Q^*_{\alpha_j} x^*\big) \leq 10\,\varepsilon_A\|x^*\| < 10 \cdot \tfrac{\varepsilon}{10}\|x^*\| = \varepsilon\|x^*\|,$$

a contradiction. Hallelujah! We thus verified the criterion from Theorem 22 and therefore X is a subspace of a WCG space. $\square$

CHALLENGE. To characterize weakly $\mathcal{K}$-analytic and Vašák, i.e. weakly $\mathcal{K}$-countably determined Banach spaces, via skeletons.
To characterize Banach spaces which are simultaneously Asplund and 1-Plička (W. Kubiś) via skeletons.

Acknowledgments. The authors thank Marek Cúth for discussions related to the topic of this note.

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