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Polyconvexity for Functions of a System of Closed Differential Forms

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Abstract This paper deals with the weakened convexity properties, *mult. ext. quasiconvexity, mult. ext. one convexity, and mult. ext. polyconvexity,*[⋆] for integral functionals of the form

$$I(\omega_1, \dots, \omega_s) = \int_\Omega f(\omega_1, \dots, \omega_s)\, dx$$

where $\omega_1, \dots, \omega_s$ are closed differential forms on a bounded open set $\Omega \subset \mathbb{R}^n$. The main results of the paper are explicit descriptions of mult. ext. quasiaffine and mult ext. polyconvex functions. It turns out that these two classes consist, respectively, of linear and convex combinations of the set of all wedge products of exterior powers of the forms $\omega_1, \dots, \omega_s$. Thus, for example, a function $f = f(\omega_1, \dots, \omega_s)$ is mult. ext. polyconvex if and only if

$$f(\omega_1, \dots, \omega_s) = \Phi(\dots, \omega_1^{q_1} \wedge \dots \wedge \omega_s^{q_s}, \dots)$$

where $q_1, \dots, q_s$ ranges a finite set of integers and Φ is a convex function. An existence theorem for the minimum energy state is proved for mult. ext. polyconvex integrals. The polyconvexity in the *calculus of variations* and *nonlinear elasticity* are particular cases of mult. ext. polyconvexity. Our main motivation comes from *electro-magneto-elastic interactions* in continuous bodies. There the mult. ext. polyconvexity takes the form determined by an involved direct calculation in an earlier paper of the author [34].

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[⋆] I extrapolate the terminology of [4], which introduces the ext. quasiconvexity, ext. one convexity and ext. polyconvexity in the case of a single differential form.

Keywords $\mathscr{A}$-quasiconvexity, $\mathscr{A}$-quasiaffinity, $\mathscr{A}$-polyconvexity, differential forms, electro-magneto-elastic interactions, minimizers of energy, compensated compactness

Contents

1 Introduction

The objective of this paper is to examine the weakened convexity properties for integral functionals depending on several closed differential forms. Such integrals are of interest in continuum mechanics, particularly in nonlinear elasticity and nonlinear electro-magneto-elasticity, as will be outlined below in this introduction and explained in detail in the subsequent exposition. By the weakened convexity properties we mean the triplet of closely related notions of quasiconvexity, rank one convexity, and polyconvexity. The mathematical principles of these were set by Morrey [24–25] and applied in continuum mechanics in the pioneering work by Ball [1]. These works deal with the functionals

$$I(u) = \int_\Omega f(\nabla u(x))\,dx$$

where Ω is a bounded open subset of $\mathbb{R}^n$ ($n = 3$ in nonlinear elasticity) and $u = (u_1, \dots, u_s)$ is an s-tuple of scalar functions on Ω ($s = 3$ in nonlinear elasticity; u is the deformation of the elastic body). It is well-known that the weakened convexity properties of f are directly related to the main qualitative properties of the functional $I(\cdot)$, such as the existence of minima, stability etc.

The framework of the present paper is broader than that of [24–25, 1] inasmuch as we consider integral functionals

$$I(\boldsymbol{\omega}) = \int_\Omega f(\boldsymbol{\omega}(x))\,dx \tag{1.1}$$

where Ω is as before and

$$\boldsymbol{\omega} = (\omega_1, \dots, \omega_s) \tag{1.2}$$

is an s-tuple of differential forms on Ω that are closed in the sense that their exterior derivatives satisfy

$$d\omega_1 = \dots = d\omega_s = 0 \tag{1.3}$$

on Ω. The forms in (1.2) are given by

$$\omega_\alpha = \sum_{1 \le i_1 < \dots < i_{k_\alpha} \le n} \omega_{(\alpha)\, i_1 \cdots i_{k_\alpha}}\, dx_{i_1} \wedge \dots \wedge dx_{i_{k_\alpha}},$$

where k_α is the degree of ω_α and $\omega^\alpha_{i_1\cdots i_{k_\alpha}}$ are the components of ω_α.[⋆] Throughout the paper we assume that the integrand

$$f : \boldsymbol{\Lambda}^{\boldsymbol{k}} \to \bar{\mathbb{R}} := \mathbb{R} \cup \{\infty\} \tag{1.4}$$

is a continuous function defined on the product

$$\boldsymbol{\Lambda}^{\boldsymbol{k}} := \Lambda^{k_1} \times \cdots \times \Lambda^{k_s}, \quad \boldsymbol{k} = (k_1, \ldots, k_s),$$

of the spaces Λ^k of k-vectors on $\mathbb{R}^n$. The reader is referred to Section 2 for the terminology and notation for the exterior algebra and analysis employed here.

Our version of the weakened convexity properties is embodied in the following definition and in Definition 3.5 in Section 3.

1.1 Definitions Let $Q = (0, 1)^n$ and let $C^\infty_{\mathrm{per}}(\mathbb{R}^n, \boldsymbol{\Lambda}^{\boldsymbol{k}})$ be the set of all infinitely differentiable Q-periodic $\boldsymbol{\Lambda}^{\boldsymbol{k}}$-valued maps on $\mathbb{R}^n$. An integrand $f : \boldsymbol{\Lambda}^{\boldsymbol{k}} \to \bar{\mathbb{R}}$ is said to be

(i) mult. ext. quasiconvex if

$$\int_Q f(\boldsymbol{\omega} + \boldsymbol{\psi}(x))\, dx \ge f(\boldsymbol{\omega})$$

for every $\boldsymbol{\omega} \in \boldsymbol{\Lambda}^{\boldsymbol{k}}$ and every $\boldsymbol{\psi} \in C^\infty_{\mathrm{per}}(\mathbb{R}^n, \boldsymbol{\Lambda}^{\boldsymbol{k}})$ such that

$$d\psi_\alpha = 0 \quad \text{on} \quad \mathbb{R}^n \quad \text{and} \quad \int_Q \psi_\alpha(x)\, dx = 0;$$

(ii) mult. ext. quasiaffine if f takes only finite values and both f and $-f$ are mult. ext. quasiconvex;

(iii) mult. ext. polyconvex if there exists a finite number of mult. ext. quasiaffine functions $f_1, \ldots, f_q$ and a convex lower semicontinuous function $\Phi : \mathbb{R}^q \to \bar{\mathbb{R}}$ such that

$$f(\boldsymbol{\omega}) = \Phi\big(f_1(\boldsymbol{\omega}), \ldots, f_q(\boldsymbol{\omega})\big)$$

for each $\boldsymbol{\omega} \in \boldsymbol{\Lambda}^{\boldsymbol{k}}$.

The main results of this paper are the determination of all mult. ext. quasiaffine and mult. ext. polyconvex functions stated in Theorems 1.2 and 1.3, respectively. The proofs apply the $\mathscr{A}$-quasiconvexity [19, 28–29, 35, 11] to the differential constraints (1.3).

We introduce the following notation. If φ is an s-vector and q a nonnegative integer we denote by

$$\varphi^q = \underbrace{\varphi \wedge \cdots \wedge \varphi}_{p \text{ times}}$$

the exterior power of φ, which is an qk-vector. If $\boldsymbol{q} = (q_1, \ldots, q_s)$ is an s-tuple of nonnegative integers and $\boldsymbol{\omega} \in \boldsymbol{\Lambda}^{\boldsymbol{k}}$, we denote by $\boldsymbol{\omega}^{\boldsymbol{q}}$ the covector of degree

[⋆] Greek indices are used to label the components of $\boldsymbol{\omega}$ and similar objects; Latin indices are used to label the components of x and the partial derivatives with respect to them. These two types of indices are tacitly assumed to run from 1 to s and from 1 to n, respectively, unless stated otherwise.

$$\deg(\boldsymbol{q}) := \sum_{\alpha=1}^{s} q_\alpha k_\alpha$$

defined by

$$\boldsymbol{\omega}^{\boldsymbol{q}} = \omega_1^{q_1} \wedge \cdots \wedge \omega_s^{q_s}.$$

Finally, we define the set of admissible exponents by

$$\mathrm{Adm}(\boldsymbol{k}) = \{\boldsymbol{q} : \deg(\boldsymbol{q}) \le n \quad \text{and} \quad q_i \le 1 \quad \text{if} \quad i \in (1, \dots, s) \quad \text{is odd}\}.$$

Since $\varphi^q = 0$ if either k is odd and $q \ge 2$ or if $qk > n$ [see $(2.2)_1$], the power $\boldsymbol{\omega}^{\boldsymbol{q}}$ is nonzero only for $\boldsymbol{q} \in \mathrm{Adm}(\boldsymbol{k})$.

1.2 Theorem *An integrand f is mult. ext. quasiaffine if and only if it is given by*

$$f(\boldsymbol{\omega}) = \sum_{\boldsymbol{q} \in \mathrm{Adm}(\boldsymbol{k})} \boldsymbol{\alpha}_{\boldsymbol{q}} \cdot \boldsymbol{\omega}^{\boldsymbol{q}} \tag{1.5}$$

for each $\boldsymbol{\omega} \in \boldsymbol{\Lambda}^{\boldsymbol{k}}$, where $\boldsymbol{\alpha}_{\boldsymbol{q}} \in \Lambda^{\deg(\boldsymbol{q})}$ for each $\boldsymbol{q} \in \mathrm{Adm}(\boldsymbol{k})$.

The centered dot in (1.5) is the scalar product of covectors defined in Section 2. To determine the set of all independent quasiaffine functions

$$\boldsymbol{\omega} \mapsto \boldsymbol{\omega}^{\boldsymbol{q}}, \quad \boldsymbol{q} \in \mathrm{Adm}(\boldsymbol{k}),$$

one has to form all possible nonvanishing mutual products

$$1, \quad \omega_\alpha, \quad \omega_\beta \wedge \omega_\gamma, \quad \omega_\beta \wedge \omega_\gamma \wedge \cdots,$$

of the elements of the list

$$\omega_1, \quad \omega_2, \quad \dots, \quad \omega_s;$$

the repetitions of the same element being admitted, and the indices $\alpha, \beta, \dots$ ordered in a nondecreasing way to avoid products which differ by a sign.

1.3 Theorem *An integrand f is mult. ext. polyconvex if and only if it is given by*

$$f(\boldsymbol{\omega}) = \Phi(\boldsymbol{\omega}^{\boldsymbol{q}_1}, \dots, \boldsymbol{\omega}^{\boldsymbol{q}_r}) \tag{1.6}$$

for each $\boldsymbol{\omega} \in \boldsymbol{\Lambda}^{\boldsymbol{k}}$, where $\boldsymbol{q}_1, \dots, \boldsymbol{q}_r$ is a fixed collection of distinct, nonzero elements of $\mathrm{Adm}(\boldsymbol{k})$ and $\Phi : \Lambda^{\deg(\boldsymbol{q}_1)} \times \cdots \times \Lambda^{\deg(\boldsymbol{q}_r)} \to \bar{\mathbb{R}}$ is a convex lower semicontinuous function.

1.4 Remark The weakened convexity notions for integrands depending on differential forms have been investigated previously, but not with all fullness.

(i) A recent work [4] of Bandyopadhyay, Dacorogna & Sil deals with the weakened convexity notions in the special case of a function f of a *single* differential form. Their definitions are slightly different but equivalent to the present ones; their terminology is ext. quasiconvexity, ext. quasiaffinity, ext. polyconvexity, ext. one convexity and ext. one affinity. Using different methods, the authors obtain the particular case of the representations (1.5) and (1.6) for a single form.

(ii) The paper by Iwaniec & Lutoborski [21] treats null lagrangians and polyconvex functions of a collection of differential forms $\boldsymbol{\omega}$ as in the present paper. However, there are two major differences. First, while the logic of the present paper and of [4] are similar since they *derive* the representations (1.5) and (1.6) from the quasiaffinity of the integrand, the reference [21] treats the expressions of the type (1.5) and (1.6) as *definitions* of null lagrangians and polyconvex functions. Secondly, the classes of these functions in [21] are smaller than those in the present paper. The difference comes from the fact that [21] uses only expansions that contains simple products of the subsets of the list $\omega_1, \ldots, \omega_s$, but not repetitions, i.e., powers of the forms, cf. [21; Eq. (1.5)].*

The case of several differential forms treated in the present paper has been motivated by the desire to encompass the classical calculus of variations with several unknowns, nonlinear elasticity and electro-magneto-elasticity; areas that remain outside the scope of [4]. Referring to Sections 6 and 7 for details, we now outline these motivations.

1.5 Example (Polyconvexity in nonlinear elasticity) We here derive a well-known result using the present formalism. The nonlinear elastic body is described by the total energy functional

$$I(u) = \int_\Omega W(F)\,dx$$

where $\Omega \subset \mathbb{R}^3$ is the reference configuration of the elastic body, $u : \Omega \to \mathbb{R}^3$ is the deformation function, $F = \nabla u$ is the deformation gradient

$$F = \nabla u = [F_{\alpha i}]_{1 \le \alpha, i \le 3} \quad \text{where} \quad F_{\alpha i} = u_{\alpha, i}$$

and f is the density of the stored energy. The framework of differential forms (1.1)–(1.3) is obtained by putting $s = n = 3$ and writing**

$$\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3) \quad \text{and} \quad f(\boldsymbol{\omega}) = W(F)$$

where

$$\omega_\alpha := F_{\alpha i}\,dx_i. \tag{1.7}$$

The forms ω_α are closed since the definition $F = \nabla u$ gives the integrability condition $\operatorname{curl} F = 0$, which is equivalent to $d\omega_\alpha = 0$.

Referring to Theorem 1.2 and to the rule that follows it, we obtain the complete list of mult. ext. quasiaffine functions by taking all mutual products of ω_1, ω_2, and ω_3 of degree ≤ 3 (since $n = 3$). This gives the constant 0-form equal to 1 and the forms

$$\omega_\alpha, \qquad \omega_\beta \wedge \omega_\gamma, \qquad \text{and} \qquad \omega_1 \wedge \omega_2 \wedge \omega_3, \tag{1.8}$$

where $1 \le \alpha \le 3$ and $1 \le \beta < \gamma \le 3$. Each 1-form and each 2-form has three independent components while each 3-form one component. Thus the forms in (1.8) represent $3 \times 3 + 3 \times 3 + 1 = 19$ independent components.

The function f is mult. ext. polyconvex if and only if it is expressible as a convex function of the elements of the list (1.8). Formulas (6.3) and (6.4) (below) show

* We also mention works by Dacorogna & Fonseca [12] and Bandyopadhyay & Sil [4–5].

** In this and the next subsections, both Greek and Latin indices run from 1 to 3; the summation convention applies to Latin indices.

that the elements of the list (1.8) are in a linear one-to-one correspondence with the elements of the list

$$F, \quad \operatorname{cof} F, \quad \det F. \tag{1.9}$$

Thus *the mult. ext. polyconvexity is equivalent to Ball's polyconvexity* [1]

$$W(F) = \Psi(F, \operatorname{cof} F, \det F)$$

where Ψ is a convex function.

1.6 Example (Electro-magneto-elastostatics) The electro-magneto-elastic interactions have recently received much theoretical attention in view of the technological application of electro- or magneto-sensitive elastomers. These are smart materials whose mechanical properties change instantly by the application of an electric or magnetic fields.

One can model the electro- or magneto-sensitive elastomers as nonlinear electro-magneto-elastic materiakls. Then the total energy is the sum of the energy of the body, the energy of the vacuum electromagnetic field in the exterior of the body, and the term corresponding to the loads. Only the first term is of interest here, which takes the form

$$I(u, D, B) = \int_\Omega W(F, D, B)\, dx$$

where Ω, u, and F are as in Example 1.5 and $D : \Omega \to \mathbb{R}^3$ and $B : \Omega \to \mathbb{R}^3$ are the referential (lagrangean) electric displacement and magnetic induction which satisfy

$$\operatorname{div} D = 0, \quad \operatorname{div} B = 0 \quad \text{in} \quad \Omega.$$

We write

$$\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3, \omega_4, \omega_5) \quad \text{and} \quad f(\boldsymbol{\omega}) = W(F, D, B),$$

where ω_1, ω_2, and ω_3 are as in Example 1.5 and ω_4 and ω_5 are 2-forms given by

$$\omega_4 = \frac{1}{2}\varepsilon_{ijk} D_i\, dx_j \wedge dx_k, \quad \omega_5 = \frac{1}{2}\varepsilon_{ijk} B_i\, dx_j \wedge dx_k \tag{1.10}$$

where ε_{ijk} is the permutation symbol. Both ω_4 and ω_5 are closed since

$$\operatorname{div} D = \operatorname{div} B = 0 \quad \Leftrightarrow \quad d\omega_4 = d\omega_5 = 0$$

as will be shown in Section 7. Thus we have the format (1.1)–(1.3) with $s = 5$. A complete list of quasiaffine functions of $\boldsymbol{\omega}$ consists of the identity term 1, the mechanical terms in (1.8) as before, and the electromagnetic and mechanic-electromagnetic terms

$$\omega_4, \quad \omega_5, \quad \text{and} \quad \omega_\alpha \wedge \omega_4, \quad \omega_\alpha \wedge \omega_5, \tag{1.11}$$

where $1 \le \alpha \le 3$. The list (1.11) involves $2 \times 3 + 3 \times 2 = 12$ scalar components and hence the combination of (1.8) and (1.11) shows that there are $19 + 12 = 31$ independent non-constant mult. ext. quasiaffine functions. It will be explained in Subsection 7 that the forms (1.11) are in a one-to-one correspondence with the terms

$$D, \quad B, \quad FD, \quad FB.$$

Combining this with the results of Subsection 1.5, we conclude that the *mult. ext. polyconvexity for electro-magneto-elasticity* reads

$$W(F, D, B) = \Psi(F, \operatorname{cof} F, \det F, D, B, FD, FB)$$

where Ψ is a convex function. This form of polyconvexity was determined by a direct calculation in [34].

This paper is organized as follows. Section 2 presents an index-free approach to the exterior algebra and analysis. Section 3 introduces the central convexity concepts of the $\mathscr{A}$-quasiconvexity theory. Section 4 is devoted to the proof of the main result of the paper, i.e., Theorem 1.2. Section 5 presents a sample-type existence theorem for minimizers of the total energy under the mult. ext. polyconvexity; in contrast with the analogous result under mult. ext. quasiconvexity, the integrand may take infinite values. Section 6 shows that the classical calculus of variations and nonlinear elasticity may be viewed as particular cases of the present theory. Section 7 describes the mult. ext. polyconvexity for electro-magneto-elastostatics.

2 Preliminaries: exterior calculus

This section summarizes basic notions of exterior algebra and analysis in the extent needed in this paper. Since these needs are purely theoretical, I feel that it that it is preferable to adopt the index-free notation and definitions. I follow [6; Chapter 4] and [18; Chapters One & Four] in taking this abstract attitude, with minor modifications. Bases and components with multiindices are introduced at the end of this section for comparison purposes.

The inner product in $\mathbb{R}^n$ enables us to identify $\mathbb{R}^n$ with its dual; consequently, we do not distinguish between vectors and covectors and between differential forms and multivector fields.

2.1 Alternating maps A map φ from the cartesian product $\underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{k \text{ times}}$ into $\mathbb{R}$ is called k linear if for any i and any collection $v_j \in \mathbb{R}^n$ corresponding to all $j \neq i$, the map on $\mathbb{R}^n$, carrying x into

$$\varphi[v_1, \ldots, v_{i-1}, x, v_{i+1}, \ldots, v_k]$$

is a linear map from $\mathbb{R}^n$ into $\mathbb{R}$. We say that φ is alternating if it has the following three equivalent properties:

- we have
$$\varphi[v_1, \ldots, v_k] = 0 \tag{2.1}$$
whenever the collection $v_1, \ldots, v_k$ contains at least two identical elements;
- we have (2.1) whenever the collection $v_1, \ldots, v_k$ is linearly dependent;
- we have
$$\varphi[v_{\pi(1)}, \ldots, v_{\pi(s)}] = \operatorname{sgn}(\pi)\varphi[v_1, \ldots, v_k]$$
for each $v_1, \ldots, v_k \in \mathbb{R}^n$ and each permutation $\pi : \{1, \ldots, s\} \to \{1, \ldots, s\}$. We denote by $\mathbb{P}_k$ the set of all such permutations and by $\operatorname{sgn}(\pi)$ the sign of $\pi \in \mathbb{P}_k$.

2.2 Theorem *There exists an associative algebra* $\wedge^*$ *with the product* $\wedge$ *having the following properties*:

(i) $\wedge^*$ *has the structure*

$$\wedge^* = \bigoplus_{k=0}^{n} \wedge^k$$

where $\wedge^k \subset \wedge^*$ *are subspaces with the property*

$$\text{if } a \in \wedge^k \text{ and } b \in \wedge^l \text{ then } a \wedge b = \wedge^{k+l}$$

for any $k \geq 0,\ l \geq 0$ *with* $k + l \leq n$;

(ii) *we have*

$$\wedge^1 = \mathbb{R}^n;$$

(iii) *for every alternating k linear map* φ *on* $\mathbb{R}^n$ *there exists a unique linear form L on* $\wedge^k$ *such that*

$$\varphi[v_1, \dots, v_k] = L(v_1 \wedge \cdots \wedge v_k)$$

for each $v_1, \dots, v_k \in \mathbb{R}^n$.

We call $\wedge^*$ the Grassmann algebra over $\mathbb{R}^n$ and $\wedge$ the exterior product. The elements of $\wedge^k$ are called k-vectors or covectors of degree k. It is convenient to put

$$\wedge^k = \{0\} \quad \text{if} \quad k < 0 \quad \text{or} \quad k > n.$$

An alternative approach introduces the spaces $\wedge^k$ first and then defines (often implicitly) $\wedge^*$ as the direct sum.

We say that a k-vector $a \in \wedge^k$ is simple if

$$a = \wedge_{i=1}^{k} v_i := v_1 \wedge \cdots \wedge v_k$$

for some $v_1, \dots, v_k \in \mathbb{R}^n$.

2.3 Properties It is well-known that

- $$\dim \wedge^k = \binom{n}{k}, \quad \dim \wedge^* = 2^n;$$
- for any $a \in \wedge^k,\ b \in \wedge^l$, any $v_1, \dots, v_k \in \mathbb{R}^n$ and any $\pi \in \mathbb{P}_k$ we have
$$a \wedge b = (-1)^{kl} b \wedge a, \quad \wedge_{i=1}^{k} v_{\pi(i)} = \operatorname{sgn}(\pi) \wedge_{i=1}^{k} v_i; \tag{2.2}$$
- the inner product on $\mathbb{R}^n$ extends uniquely to an inner product on $\wedge^*$ with the following two properties:
$$\begin{cases} \wedge^k \perp \wedge^l & \text{if } k \neq l, \\ (\wedge_{i=1}^{k} u_i)\cdot(\wedge_{i=1}^{k} v_i) = \det[u_i \cdot v_j]_{1\leq i,j\leq k} & \text{if } k = l, \end{cases}$$
where $u_i,\ v_j \in \mathbb{R}^n$;
- each alternating scalar k-form f on $\mathbb{R}^n$ has a representation
$$f(v_1, \dots, v_k) = a \cdot (v_1 \wedge \cdots \wedge v_k) \tag{2.3}$$
for all $v_1, \dots, v_k \in \mathbb{R}^n$ and some $a \in \wedge^k$.

One possibility to give an index-free definition of the exterior product is the following.

2.4 Theorem *For each open subset Ω of $\mathbb{R}^n$ there exists a unique linear map d from $C^\infty(\Omega, \wedge^*)$ into itself such that*

$$d(fa) = \nabla f \wedge a,$$

for every $a \in \wedge^k$ and every $f \in C^\infty(\Omega, \mathbb{R})$.

The operation d is called the exterior derivative. Here and below we use the standard notation: if Z is a finite-dimensional vector space then $C^\infty(\Omega, Z)$ denotes the set of all indefinitely differentiable Z-valued maps on Ω and by $C_0^\infty(\Omega, Z)$ the set of all indefinitely differentiable Z-valued maps on $\mathbb{R}^n$ with compact support which is contained in Ω.

A k-form on an open set $\Omega \subset \mathbb{R}^n$ is any map $\omega : \Omega \to \wedge^k$.

2.5 Properties We have:

- $d^2 := d \circ d = 0$;
- the operator d maps smooth k forms into $k+1$ forms; it coincides with the usual gradient on 0-forms, i.e., on scalar functions;
- if $\psi \in C^\infty(\Omega, \wedge^k)$ and $\omega \in C^\infty(\Omega, \wedge^l)$ then

$$d(\psi \wedge \omega) = d\psi \wedge \omega + (-1)^k \psi \wedge d\omega.$$

Alternatively, these the properties can be used as a definition of the exterior derivative, equivalent to that in Theorem 2.4.

2.6 Definition Let $1 \le p, q \le \infty$ and $\omega \in L^q(\Omega, \wedge^k)$. We say that ω has the weak interior derivative in L^q if there exists a $\operatorname{div}\omega \in L^q(\Omega, \wedge^{k-1})$ such that

$$\int_\Omega \operatorname{div}\omega \cdot \chi \, dx = -\int_\Omega \omega \cdot d\chi \, dx$$

for every $\chi \in C_0^\infty(\Omega, \wedge^{k-1})$. We say that ω has the weak exterior derivative in L^q if there exists a $d\omega \in L^q(\Omega, \wedge^{k+1})$ such that

$$\int_\Omega \operatorname{div}\psi \cdot \omega \, dx = -\int_\Omega d\omega \cdot \psi \, dx$$

for every $\psi \in C_0^\infty(\Omega, \wedge^{k+1})$.

Alternatively, the divergence (or its multiple by a factor ± 1) is denoted by δ and called the interior derivative or codifferential. We use the standard notation: if Z is a finite-dimensional vector space and $1 \le p \le \infty$ then $L^p(\Omega, Z)$ is the space of all Z-valued maps on Ω that are Lebesgue integrable with power p.

2.7 Coordinate expressions If $e_1, \dots, e_n$ is an orthonormal basis in $\mathbb{R}^n$ then

$$d\omega = \sum_{i=1}^n e_i \wedge \omega_{,i}. \tag{2.4}$$

Further, the k-vectors defined by

$$e_{i_1,\dots,i_k} = e_{i_1} \wedge \cdots \wedge e_{i_k}, \quad 1 \le i_1 < \dots < i_k \le n,$$

form an orthonormal basis in $\wedge^k$. A k-form ω on Ω has an expansion

$$\omega = \sum_{1 \le i_1 < \ldots < i_k \le n} \omega_{i_1, \ldots, i_k} e_{i_1, \ldots, i_k}, \tag{2.5}$$

where $\omega_{i_1, \ldots, i_k}$ are real-valued functions on Ω. Denote by $x_i : \mathbb{R}^n \to \mathbb{R}$ the i-th coordinate function, i.e., the function which associates with any $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ the number x_i. Noting that $dx_i = e_i$, one can rewrite (2.5) more standardly as

$$\omega = \sum_{1 \le i_1 < \cdots < i_k \le n} \omega_{i_1 \cdots i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$

One has

$$d\omega = \sum_{j=1}^{n} \sum_{1 \le i_1 < \cdots < i_k \le n} \omega_{i_1 \cdots i_k, j} dx_j \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}. \tag{2.6}$$

3 $\mathscr{A}$-quasiconvexity theory

The purpose of this section is to discuss the notions of $\mathscr{A}$-quasiaffinity, $\mathscr{A}$-polyconvexity and Λ-convexity. The $\mathscr{A}$-quasiconvexity theory has been introduced in [11] and further developed in many papers, most notably in [19]. Closely related is the compensated compactness theory [28–29]. The reader is referred to [7, 22–23] for more recent developments and additional literature.

3.1 The differential operator $\mathscr{A}$ The following dimensions will be needed in the subsequent discussion:

$$\begin{aligned} n &= \text{the number of independent variables, } x = (x_1, \ldots, x_n), \\ d &= \text{the number of dependent variables, } z = (z_1, \ldots, z_d), \\ l &= \text{the number of differential constrains.} \end{aligned}$$

Let $C^\infty_{\rm per}(\mathbb{R}^n, \mathbb{R}^d)$ denote the set of all infinitely differentiable Q-periodic maps $z : \mathbb{R}^n \to \mathbb{R}^d$. We shall consider the first–order differential constraint

$$\mathscr{A} z = 0 \tag{3.1}$$

on a map $z \in C^\infty(\mathbb{R}^n, \mathbb{R}^d)$ where

$$\mathscr{A} z = \sum_{i=1}^{n} A^{(i)} z_{,i} \tag{3.2}$$

with $A^{(i)} \in \operatorname{Lin}(\mathbb{R}^d, \mathbb{R}^l)$, $i = 1, \ldots, n$. For each $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$ define

$$\mathbb{A}(\xi) = \sum_{i=1}^{n} \xi_i A^{(i)},$$

which is an element of $\operatorname{Lin}(\mathbb{R}^d, \mathbb{R}^l)$, and make the standing assumption that the rank of $\mathbb{A}(\xi)$ is the same for all $\xi \neq 0$. The following set will play an important role:

$$\Lambda = \{ z \in \mathbb{R}^d : \mathbb{A}(\xi) z = 0 \quad \text{for some } \xi \in \mathbb{R}^n, \ \xi \neq 0 \}.$$

3.2 Definition A continuous function $f : \mathbb{R}^d \to \bar{\mathbb{R}}$ is said to be
(i) $\mathscr{A}$-quasiconvex if

$$\int_Q f(z + w(x))\, dx \geq f(z)$$

for every $z \in \mathbb{R}^d$ and every $w \in C^\infty_{\mathrm{per}}(\mathbb{R}^n, \mathbb{R}^d)$ such that

$$\mathscr{A} w = 0 \quad \text{on} \quad \mathbb{R}^n \quad \text{and} \quad \int_Q w\, dx = 0;$$

(ii) $\mathscr{A}$-quasiaffine if f takes only finite values and both f and $-f$ are $\mathscr{A}$-quasiconvex;
(iii) Λ-convex if

$$f((1-t)z_1 + tz_2) \leq (1-t)f(z_1) + tf(z_2)$$

for every $t \in (0, 1)$ and $z_1, z_2 \in \mathbb{R}^d$ such that $z_2 - z_1 \in \Lambda$;
(iv) Λ-affine if it takes only finite values and both f and $-f$ are Λ-convex;
(v) $\mathscr{A}$-polyconvex if there exists a finite number of $\mathscr{A}$-quasiaffine functions $f_1, \ldots, f_q$ and a convex lower semicontinuous function $\Phi : \mathbb{R}^q \to \bar{\mathbb{R}}$ such that

$$f(z) = \Phi(f_1(z), \ldots, f_q(z))$$

for each $z \in \mathbb{R}^d$.

The following result is standard but central for the development that follows.

3.3 Theorem ([19; Proposition 3.4]) *If $f : \mathbb{R}^d \to \mathbb{R}$ is a continuous $\mathscr{A}$-quasiconvex function then f is Λ-convex; consequently, if f is continuous and $\mathscr{A}$-quasiaffine then f is Λ-affine.*

3.4 Application to closed differential forms Naturally, to apply the $\mathscr{A}$-quasiconvexity theory to the situation described in Section 1, one identifies the variable z with $\boldsymbol{\omega}$ and the differential constraint (3.1) with (1.3). Explicitly, if we define the sets $\boldsymbol{\Lambda}^{\boldsymbol{k}+1}$ and $\boldsymbol{\Lambda}^{\boldsymbol{k}-1}$ by

$$\boldsymbol{\Lambda}^{\boldsymbol{k}\pm 1} := \Lambda^{k_1 \pm 1} \times \cdots \times \Lambda^{k_s \pm 1},$$

then $\mathscr{A}$ is a map from $C^\infty(\mathbb{R}^n, \boldsymbol{\Lambda}^{\boldsymbol{k}})$ into $C^\infty(\mathbb{R}^n, \boldsymbol{\Lambda}^{\boldsymbol{k}+1})$ defined by

$$\mathscr{A}\boldsymbol{\omega} = (d\omega_1, \ldots, d\omega_s)$$

for any $\boldsymbol{\omega} \in C^\infty(\mathbb{R}^n, \boldsymbol{\Lambda}^{\boldsymbol{k}})$. Referring to (2.4), we find that the transformations $A^{(i)}$ occurring in (3.2) are linear maps from $\boldsymbol{\Lambda}^{\boldsymbol{k}}$ into $\boldsymbol{\Lambda}^{\boldsymbol{k}+1}$ given by

$$A^{(i)}\boldsymbol{\sigma} = (e_i \wedge \sigma_1, \ldots, e_i \wedge \sigma_s)$$

for any $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_s) \in \boldsymbol{\Lambda}^{\boldsymbol{k}}$. Consequently, if $\xi \in \mathbb{R}^n$, then $\mathbb{A}(\xi)$ is the transformation from $\boldsymbol{\Lambda}^{\boldsymbol{k}}$ into $\boldsymbol{\Lambda}^{\boldsymbol{k}+1}$ given by

$$\mathbb{A}(\xi)\boldsymbol{\sigma} = \xi \wedge \boldsymbol{\sigma} := (\xi \wedge \sigma_1, \ldots, \xi \wedge \sigma_s).$$

The equation $\mathbb{A}(\xi)z = 0$ occurring in the definition of the cone Λ reduces to $\xi \wedge \boldsymbol{\sigma} = 0$ for any $\boldsymbol{\sigma} \in \boldsymbol{\Lambda}^{\boldsymbol{k}}$ where $\xi \in \mathbb{R}^n$, $\xi \neq 0$. It is easy to see that the element $\boldsymbol{\sigma}$ solves that equation if and only if it is of the form $\boldsymbol{\sigma} = \boldsymbol{\lambda} \wedge \xi$ for some $\boldsymbol{\lambda} \in \boldsymbol{\Lambda}^{\boldsymbol{k}-1}$ where

$$\boldsymbol{\lambda} \wedge \xi := (\xi \wedge \lambda_1, \ldots, \xi \wedge \lambda_s)$$

for any $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_s) \in \boldsymbol{\Lambda}^{\boldsymbol{k}-\mathbf{1}}$. Hence

$$\Lambda = \{\boldsymbol{\lambda} \wedge \xi \in \boldsymbol{\Lambda}^{\boldsymbol{k}} : \boldsymbol{\lambda} \in \boldsymbol{\Lambda}^{\boldsymbol{k}-\mathbf{1}},\ \xi \in \mathbb{R}^n\}.$$

The $\mathscr{A}$ quasiconvexity, quasiaffinity and polyconvexity in the sense of the general theory (Definitions 3.2(i), (ii), (v)) reduce to the mult. ext. quasiconvexity, quasiaffinity and polyconvexity introduced in Definitions 1.1. The Λ convexity and affinity take the following forms.

3.5 Definition An integrand f of type $\boldsymbol{k}$ is said to be

(i) mult. ext. one convex if

$$f(\boldsymbol{\omega} + t\boldsymbol{\lambda} \wedge \xi) \le (1-t)f(\boldsymbol{\omega}) + tf(\boldsymbol{\omega} + \boldsymbol{\lambda} \wedge \xi) \tag{3.3}$$

for every $t \in (0, 1)$ and every $\boldsymbol{\omega} \in \boldsymbol{\Lambda}^{\boldsymbol{k}}$, $\xi \in \mathbb{R}^n$, and $\boldsymbol{\lambda} \in \boldsymbol{\Lambda}^{\boldsymbol{k}-\mathbf{1}}$.

(ii) mult. ext. one affine if f takes only finite values and both f and $-f$ are mult. ext. one convex, i.e., if (3.3) is replaced by

$$f(\boldsymbol{\omega} + t\boldsymbol{\lambda} \wedge \xi) = (1-t)f(\boldsymbol{\omega}) + tf(\boldsymbol{\omega} + \boldsymbol{\lambda} \wedge \xi). \tag{3.4}$$

4 Characterization of mult. ext. affine functions (proof)

The objective of this section is to prove Theorem 1.2, in the following slightly more detailed form. Recall the standing assumption that the integrand in (1.4) is a continuous function.

4.1 Theorem *For an integrand f the following three conditions are equivalent*:

(i) *f is mult. ext. quasiaffine*;

(ii) *f is mult. ext. one affine*;

(iii) *f is given by* (1.5) *where $\boldsymbol{\alpha}_{\boldsymbol{q}} \in \wedge^{\deg(\boldsymbol{q})}$ for each $\boldsymbol{q} \in \mathrm{Adm}(\boldsymbol{k})$.*

Here (i) ⇒ (ii) is a general assertion, Theorem 3.3, while (iii) ⇒ (i) will follow by a simple application of the Stokes theorem. The main task is the proof of (ii) ⇒ (iii), which will be divided into a sequence of lemmas. The proof starts in Lemma 4.3 which asserts certain antisymmetry of partial derivatives and shows that f is a polynomial in $\boldsymbol{\omega}$. Then we decompose the polynomial f into a sum of homogeneous polynomials $f_{\boldsymbol{q}}$ of various degrees in the components of $\boldsymbol{\omega}$. Each $f_{\boldsymbol{q}}$ inherits the mult. ext. one affinity from f. Next, we determine the forms of a homogeneous mult. ext. one affine polynomials in one variable form ω or two variables ω_1, ω_2. Putting these two particular results together in an iterative way, we obtain Item (iii) of Theorem 4.1.

4.2 Polynomials in $\boldsymbol{\omega}$ A function $f : \boldsymbol{\Lambda}^{\boldsymbol{k}} \to \mathbb{R}$ is said to be a polynomial of degree m if it is of the form

$$f(\boldsymbol{\omega}) = \sum_{j=0}^{m} f_j(\boldsymbol{\omega}), \tag{4.1}$$

$\boldsymbol{\omega} \in \boldsymbol{\Lambda}^{\boldsymbol{k}}$, where

$$f_j(\boldsymbol{\omega}) = C_j[\underbrace{\boldsymbol{\omega}, \ldots, \boldsymbol{\omega}}_{j \text{ times}}],$$

with C_j a symmetric j-linear form on $\boldsymbol{\Lambda}^{\boldsymbol{k}}$ for $j = 0, \ldots, m$. Each f_j is called a homogeneous polynomial of degree j.

4.3 Lemma *Any mult. ext. one affine integrand $f : \bigwedge^{\boldsymbol{k}} \to \mathbb{R}$ is a polynomial of degree $\leq n$; moreover, for any integer $j \geq 2$ and any $\boldsymbol{\omega} \in \bigwedge^{\boldsymbol{k}}$ the derivative $\mathrm{D}^j f(\omega) = \mathrm{D}^j f$ satisfies*

$$\mathrm{D}^j f[\boldsymbol{\lambda}_1 \wedge \xi_{\tau(1)}, \dots, \boldsymbol{\lambda}_j \wedge \xi_{\tau(j)}] = \operatorname{sgn}(\tau)\, \mathrm{D}^j f[\boldsymbol{\lambda}_1 \wedge \xi_1, \dots, \boldsymbol{\lambda}_j \wedge \xi_j]. \quad (4.2)$$

for any $\boldsymbol{\lambda}_1, \dots, \boldsymbol{\lambda}_j \in \bigwedge^{\boldsymbol{k-1}}$, $\xi_1, \dots, \xi_j \in \mathbb{R}^n$, and $\tau \in \mathbb{P}_j$.

By Subsection 2.1, the alternating property (4.2) is equivalent to the condition

$$\mathrm{D}^j f(\boldsymbol{\lambda}_1 \wedge \xi_1, \dots, \boldsymbol{\lambda}_j \wedge \xi_j) = 0 \quad (4.3)$$

whenever $\xi_1, \dots, \xi_j$ are linearly dependent. In the broader context of the theory of compensated compactness, the analog of (4.3) is a necessary, and under the constant rank assumption also sufficient condition for the weak continuity of the composite function $u \mapsto f(u)$ (see [35; Theorem 18], [29; Theorem 3.4]). Alternatively, the analog of (4.3) is necessary for f to be quasiaffine in the context of higher-order variational problems (see [2; Theorem 3.4]) and of $\mathscr{A}$-quasiconvexity [11; Section 1.2]. Nevertheless we give complete proof of Lemma 4.3, if only since our proof is shorter.

Proof (Cf. [32; Proof of Propositions 13.5.2 and 13.5.3].) Let f be a mult. ext. one affine integrand of type $\boldsymbol{k}$. Prove first the assertion of the lemma under the additional assumption that f is infinitely differentiable.

Differentiating the mult. ext. one affinity condition (3.4) we obtain

$$\mathrm{D}^2 f[\boldsymbol{\lambda}_1 \wedge \xi, \boldsymbol{\lambda}_2 \wedge \xi] = 0$$

for any $\xi \in \mathbb{R}^n$ and $\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2 \in \bigwedge^{\boldsymbol{k-1}}$. As an easy consequence of the polarization identity (or by Subsection 2.1) we obtain

$$\mathrm{D}^2 f[\boldsymbol{\lambda}_1 \wedge \xi_1, \boldsymbol{\lambda}_2 \wedge \xi_2] + \mathrm{D}^2 f[\boldsymbol{\lambda}_1 \wedge \xi_2, \boldsymbol{\lambda}_2 \wedge \xi_1] = 0.$$

Differentiating the last identity $j - 2$ times in the directions $\boldsymbol{\lambda}_3 \wedge \xi_3, \dots, \boldsymbol{\lambda}_j \wedge \xi_j$, we obtain

$$\begin{aligned} &\mathrm{D}^j f[\boldsymbol{\lambda}_1 \wedge \xi_1, \boldsymbol{\lambda}_2 \wedge \xi_2, \dots, \boldsymbol{\lambda}_j \wedge \xi_j] \\ &\qquad + \mathrm{D}^j f[\boldsymbol{\lambda}_1 \wedge \xi_2, \boldsymbol{\lambda}_2 \wedge \xi_1, \dots, \boldsymbol{\lambda}_j \wedge \xi_j] = 0. \end{aligned}$$

This establishes (4.2) for the special case of the permutation τ which interchanges the first two indices in $\{1, 2, \dots, p\}$, and hence any two indices by the symmetry of partial derivatives. Since any permutation is a composition of these special permutations, one concludes that (4.2) holds generally.

Applying (4.2) with $j = n + 1$ and fixing $\boldsymbol{\lambda}_1, \dots, \boldsymbol{\lambda}_{n+1}$, we see that the $n + 1$ form

$$(\xi_1, \dots, \xi_{n+1}) \mapsto \mathrm{D}^{n+1} f[\boldsymbol{\lambda}_1 \wedge \xi_1, \dots, \boldsymbol{\lambda}_{n+1} \wedge \xi_{n+1}]$$

on $\mathbb{R}^n$ is alternating and hence in vanishes. Since

$$\operatorname{span}\{\boldsymbol{\lambda} \wedge \xi : \boldsymbol{\lambda} \in \textstyle\bigwedge^{\boldsymbol{k-1}},\ \xi \in \mathbb{R}^n\} = \textstyle\bigwedge^{\boldsymbol{k}},$$

we deduce that

$$\mathrm{D}^{n+1} f[\boldsymbol{\sigma}_1, \dots, \boldsymbol{\sigma}_{n+1}] = 0$$

for any $\boldsymbol{\sigma}_1, \ldots, \boldsymbol{\sigma}_{n+1} \in \Lambda^k$ by the $n+1$-linearity of $\mathrm{D}^{n+1} f$. Thus f is a polynomial of degree at most n.

This proves the lemma under the assumption that f is infinitely differentiable. If f is merely continuous, we approximate it by the sequence f_ρ, $\rho > 0$, of mollifications of f. Clearly, the functions f_ρ are mult. ext. one affine also. Thus each f_ρ is a polynomial of degree $\leq n$ and hence the limit f is again a polynomial of degree $\leq n$. □

4.4 Decomposition of f into homogeneous polynomials The notion of a polynomial as introduced in Subsection 4.2 applies to a function on any vector space V in place of Λ^k. We now use the fact that the arguments $\boldsymbol{\omega} \in \Lambda^k$ are s-tuples of objects $\omega_1, \ldots, \omega_s$ which belong to vector spaces. This allows to decompose each homogeneous polynomial into smaller blocks of degree q_1 in ω_1, of degree q_2 in ω_2, … If $\boldsymbol{q} = (q_1, \ldots, q_s)$ is an s-tuple of nonnegative integers, we say that a polynomial $g : \Lambda^k \to \mathbb{R}$ is homogeneous of degree $\boldsymbol{q}$ if

$$g(t_1\omega_1, \ldots, t_s\omega_s) = t_1^{q_1} \cdots t_s^{q_s} g(\omega_1, \ldots, \omega_s) \tag{4.4}$$

for any $(\omega_1, \ldots, \omega_s) \in \Lambda^k$ and any $t_1, \ldots, t_s \in \mathbb{R}$. Combining (4.4) with the assumption that g is a polynomial, one finds that g has a representation

$$g(\boldsymbol{\omega}) = A[\underbrace{\omega_1, \ldots, \omega_1}_{q_1 \text{ times}}, \ldots, \underbrace{\omega_s, \ldots, \omega_s}_{q_s \text{ times}}] \tag{4.5}$$

with A a multilinear form on

$$\underbrace{\Lambda^{k_1} \times \cdots \times \Lambda^{k_1}}_{q_1 \text{ times}} \times \cdots \times \underbrace{\Lambda^{k_s} \times \cdots \times \Lambda^{k_s}}_{q_s \text{ times}}$$

of degree

$$j := \sum_{i=1}^{s} q_i \tag{4.6}$$

that is symmetric under permutations of the arguments belonging to the same brace in the right-hand side of (4.5). An easy argument based on a multinomial theorem shows that any homogeneous polynomial of degree j can be decomposed into a sum of homogeneous polynomials of degrees $\boldsymbol{q}$ satisfying (4.6).

Thus by Lemma 4.3, any mult. ext. one affine integrand f has a representation

$$f(\boldsymbol{\omega}) = \sum_{\boldsymbol{q} \in \mathrm{P}(n)} f_{\boldsymbol{q}}(\boldsymbol{\omega}) \tag{4.7}$$

where each $f_{\boldsymbol{q}}$ is a $\boldsymbol{q}$-homogeneous polynomial, and the sum extends over the set $\mathrm{P}(n)$ of all s-tuples $\boldsymbol{q} = (q_1, \ldots, q_s)$ of nonnegative integers satisfying

$$\sum_{i=1}^{s} q_i \leq n.$$

It is easy to see that each $f_{\boldsymbol{q}}$ is mult. ext. one affine separately. Indeed, if $\boldsymbol{t} = (t_1, \ldots, t_s)$ is an s-tuple of real numbers and f is mult. ext. one affine then the function $f^{(\boldsymbol{t})}$, given by

$$f^{(\boldsymbol{t})}(\omega_1, \ldots, \omega_s) = f(t_1\omega_1, \ldots, t_s\omega_s),$$

$\boldsymbol{\omega} \in \boldsymbol{\Lambda}^k$, is mult. ext. one affine also. Indeed, the mult. ext. one affinity condition for $f^{(t)}$ reads

$$f^{(t)}(\boldsymbol{\omega} + t\boldsymbol{\lambda} \wedge \xi) = (1-t)f^{(t)}(\boldsymbol{\omega}) + tf^{(t)}(\boldsymbol{\omega} + \boldsymbol{\lambda} \wedge \xi)$$

for every t, $\boldsymbol{\omega}$, ξ, and $\boldsymbol{\lambda}$ as in Definition 3.5. The scaling (4.4) provides

$$\sum_{\boldsymbol{q} \in \mathrm{P}(n)} t_1^{q_1} \cdots t_s^s f_{\boldsymbol{q}}(\boldsymbol{\omega} + t\boldsymbol{\lambda} \wedge \xi) = \sum_{\boldsymbol{q} \in \mathrm{P}(n)} t_1^{q_1} \cdots t_s^s \big((1-t)f_{\boldsymbol{q}}(\boldsymbol{\omega}) + tf_{\boldsymbol{q}}(\boldsymbol{\omega} + \boldsymbol{\lambda} \wedge \xi)\big).$$

This is an equality of two polynomials in the scalar variables $t_1, \dots, t_s$. Equating the coefficients, we obtain the mult. ext. one convexity of $f_{\boldsymbol{q}}$:

$$f_{\boldsymbol{q}}(\boldsymbol{\omega} + t\boldsymbol{\lambda} \wedge \xi) = (1-t)f_{\boldsymbol{q}}(\boldsymbol{\omega}) + tf_{\boldsymbol{q}}(\boldsymbol{\omega} + \boldsymbol{\lambda} \wedge \xi).$$

Thus it suffices to examine the mult. ext. one convexity of each $f_{\boldsymbol{q}}$ separately. However, we shall split the considerations into subcases further. We realize that the specific form on the mult. ext. one convexity in the s-tuple $\boldsymbol{\omega} = (\omega_1, \dots, \omega_s)$ implies the mult. ext. one convexity in any subgroup of the group of variables $\omega_1, \dots, \omega_s$.

We shall start the proof from the mult. ext. one convexity in a single homogeneous polynomial in one variable.

4.5 Lemma *If $f : \wedge^k \to \mathbb{R}$ is a mult. ext. one affine function given by*

$$f(\omega) = A[\underbrace{\omega, \dots, \omega}_{q \text{ times}}]$$

where A is a q-linear form on $\underbrace{\wedge^k \times \cdots \times \wedge^k}_{q \text{ times}}$, then there exists an $\alpha \in \wedge^{kq}$ such that

$$f(\omega) = \alpha \cdot \omega^q \tag{4.8}$$

for every $\omega \in \wedge^k$.

Proof We shall prove this by evaluating the second derivative of f and employing the antisymmetry condition (4.2) with $j = 2$. One finds that the second derivative $\mathrm{D}^2 f(\omega)[\sigma_1, \sigma_2]$ at $\omega \in \wedge^k$ corresponding to the increments $\sigma_1, \sigma_2 \in \wedge^k$ is given by

$$\mathrm{D}^2 f(\omega)[\sigma_1, \sigma_2] = q(q-1)A[\sigma_1, \sigma_2, \boldsymbol{\omega}']$$

where we abbreviate $\boldsymbol{\omega}' = (\omega, \dots, \omega) \in \underbrace{\wedge^k \times \cdots \times \wedge^k}_{q-2 \text{ times}}$. Assuming that the increments take the forms $\sigma_1 = \lambda_1 \wedge v_k$, $\sigma_2 = \lambda_2 \wedge v_{2k}$, where $\lambda_1, \lambda_2 \in \wedge^k$, and $v_k, v_{2k} \in \mathbb{R}^n$, and employing antisymmetry condition (4.2), we obtain

$$A[\eta_1 \wedge v_k, \eta_2 \wedge v_{2k}, \boldsymbol{\omega}'] = -A[\eta_1 \wedge v_{2k}, \eta_2 \wedge v_k, \boldsymbol{\omega}']. \tag{4.9}$$

This implies the same equation with $\boldsymbol{\omega}' = (\omega, \dots, \omega)$ replaced by $\boldsymbol{\omega}' = (\omega_3, \dots, \omega_p)$ where $\omega_3, \dots, \omega_p \in \wedge^k$ by polarization. Putting $\eta_1 = v_1 \wedge \cdots \wedge v_{k-1}$, $\eta_2 = v_{k+1} \wedge \cdots \wedge v_{2k-1}$, where v_i are vectors from $\mathbb{R}^n$, we deduce that the $2k$-linear form carrying $(v_1, \dots, v_{2k})$ into

$$A[v_1 \wedge \cdots \wedge v_k, v_{k+1} \wedge \cdots \wedge v_{2k}, \boldsymbol{\omega}'] \tag{4.10}$$

is alternating. Indeed, the alternating property under permutations of $v_1, \dots, v_k$ follows from the alternating property of the wedge product $v_1 \wedge \cdots \wedge v_k$; the alternating property under permutations of $v_{k+1}, \dots, v_{2k}$ follows similarly. Equation (4.9) provides the antisymmetry under the pairwise exchange $v_k \leftrightarrow v_{2k}$. Since the permutations just described generate the group of all permutations of $v_1, \dots, v_{2k}$, the alternating property of (4.10) follows. One obtains in the same way that the $2k$-linear form carrying $(v_{k+1}, \dots, v_{3k})$ into

$$A[\omega_1, v_{k+1} \wedge \cdots \wedge v_{2k}, v_{2k+1} \wedge \cdots \wedge v_{3k}, \omega_4, \dots, \omega_p]$$

is alternating, etc. Hence the kq-form carrying $v_1, \dots, v_{kq}$ into

$$A[v_1 \wedge \cdots \wedge v_k, v_{k+1} \wedge \cdots \wedge v_{2k}, \dots, v_{k(q-1)} \wedge \cdots \wedge v_{kq}] \tag{4.11}$$

is alternating. The representation theorem (2.3) provides the existence of a covector α of degree kq such that the value in (4.11) is given by $\alpha \cdot (v_1 \wedge \cdots \wedge v_{kq})$. In particular, taking

$$(v_1, \dots, v_k) = (v_{k+1}, \dots, v_{2k}) = \dots = (v_{k(q-1)}, \dots, v_{kq}),$$

and setting $\omega = v_1 \wedge \cdots \wedge v_k$, we obtain

$$f(\omega) \equiv A[\omega, \omega, \dots, \omega] = \alpha \cdot \omega^q.$$

This establishes the representation (4.8) for each simple k-vector ω. The extension to general arguments ω follows by multilinearity. □

Next we consider a homogeneous polynomial in two variables.

4.6 Lemma *Let $f : \wedge^{k_1} \times \wedge^{k_2} \to \mathbb{R}$ be a mult. ext. one affine function given by*

$$f(\boldsymbol{\omega}) = A[\underbrace{\omega_1, \dots, \omega_1}_{q_1 \text{ times}} \mid \underbrace{\omega_2, \dots, \omega_2}_{q_2 \text{ times}}] \tag{4.12}$$

for each $\boldsymbol{\omega} = (\omega_1, \omega_2) \in \wedge^{k_1} \times \wedge^{k_2}$, with A a $q_1 + q_2$-linear form on

$$\underbrace{\wedge^{k_1} \times \cdots \times \wedge^{k_1}}_{q_1 \text{ times}} \times \underbrace{\wedge^{k_2} \times \cdots \times \wedge^{k_2}}_{q_2 \text{ times}}.$$

Then there exists an $\alpha \in \wedge^{q_1 k_1 + q_2 k_2}$ such that

$$f(\omega_1, \omega_2) = \alpha \cdot (\omega_1^{q_1} \wedge \omega^{q_2}) \tag{4.13}$$

for every $(\omega_1, \omega_2) \in \wedge^{k_1} \times \wedge^{k_2}$.

In (4.12) we separate the two groups of arguments by a vertical bar to ease the localization of the arguments in the proof.

Proof As in the preceding lemma, we prove this by evaluating the second derivative of f and employing the antisymmetry condition (4.2) with $j = 2$. For our purpose it suffices to evaluate the second derivative

$$\mathrm{D}^2 f(\boldsymbol{\omega})[(\sigma, 0), (0, \tau)]$$

corresponding to the increments $(\sigma, 0), (0, \tau) \in \wedge^{k_1} \times \wedge^{k_2}$. Then

$$\mathrm{D}^2 f(\boldsymbol{\omega})[(\sigma, 0), (0, \tau)] = q_1 q_2 A[\sigma, \boldsymbol{\omega}_1' \mid \tau, \boldsymbol{\omega}_2']$$

where we abbreviate $\boldsymbol{\omega}_i' = (\omega_i, \dots, \omega_i)$, $i = 1, 2$. Still more specially, we assume that the increments take the forms

$$\sigma = \eta \wedge v_{k_1}, \quad \tau = \theta \wedge v_{k_1+k_2},$$

where $\eta \in \wedge^{k_1-1}$, $\theta \in \wedge^{k_2-1}$, and $v_{k_1}, v_{k_1+k_2} \in \mathbb{R}^n$. The antisymmetry condition (4.2) under the exchange $v_{k_1} \leftrightarrow v_{k_1+k_2}$ gives

$$A[\eta \wedge v_{k_1}, \boldsymbol{\omega}_1' \,|\, \theta \wedge v_{k_1+k_2}, \boldsymbol{\omega}_2'] = -A[\eta \wedge v_{k_1+k_2}, \boldsymbol{\omega}_1' \,|\, \theta \wedge v_{k_1}, \boldsymbol{\omega}_2'].$$

The polarization provides the same equation with $\boldsymbol{\omega}_1' = (\rho_2, \dots, \rho_{k_1})$, $\boldsymbol{\omega}_2' = (\sigma_2, \dots, \sigma_{k_2})$. We now write $\eta = v_1 \wedge \cdots \wedge v_{k_1-1}$, $\theta = v_{k_1+1} \wedge \cdots \wedge v_{k_1+k_2-1}$, and employ the same argument as that in the analogous part of the proof of Lemma 4.5. In this way we see that the $k_1 + k_2$ form carrying $(v_1, \dots, v_{k_1+k_2})$ into

$$A[v_1 \wedge \cdots \wedge v_{k_1}, \boldsymbol{\omega}_1' \,|\, v_{k_1+1} \wedge \cdots \wedge v_{k_1+k_2}, \boldsymbol{\omega}_2']$$

is alternating. Consequently, putting

$$\rho_\alpha = v_{k_1(\alpha-1)+1} \wedge \cdots \wedge v_{k_1\alpha}, \quad \alpha = 1, \dots, q_1,$$

$$\sigma_\beta = v_{k_1q_1+k_2(\beta-1)+1} \wedge \cdots \wedge v_{k_1q_1+k_2\beta}, \quad \beta = 1, \dots, q_2,$$

we deduce that the expression

$$A[\rho_1, \dots, \rho_{k_1} \,|\, \sigma_1, \dots, \sigma_{k_2}]$$

is alternating under all permutations of the totality of vector arguments

$$v_1, \dots, v_{k_1q_1+k_2q_2}.$$

Using the representation theorem (2.3) in a way similar to that in the proof of Lemma 4.5, we obtain the representation (4.13) if the arguments ω_1 and ω_2 are simple. Hence for general arguments by multilinearity. □

4.7 Lemma *Let* $f_{\boldsymbol{q}} : \wedge^k \to \mathbb{R}$ *be a mult. ext. one affine function given by* (4.5) *with* g *identified with* $f_{\boldsymbol{q}}$. *Then*

$$f_{\boldsymbol{q}}(\boldsymbol{\omega}) = \boldsymbol{\alpha}_{\boldsymbol{q}} \cdot \boldsymbol{\omega}^{\boldsymbol{q}}$$

for all $\boldsymbol{\omega} \in \wedge^k$ *and some* $\boldsymbol{\alpha}_{\boldsymbol{q}} \in \wedge^{\deg(\boldsymbol{q})}$.

Proof By applying Lemma 4.6 to every pair of variables $\omega_\alpha, \omega_\beta$ of $f(\boldsymbol{\omega}) = f(\omega_1, \dots, \omega_\alpha, \dots, \omega_\beta, \dots, \omega_s)$, one obtains that the form

$$(v_1, \dots, v_{\deg(\boldsymbol{q})}) \mapsto A[v_1 \wedge \cdots \wedge v_{k_1}, \dots, v_{\deg(\boldsymbol{q})-k_s+1} \wedge \cdots \wedge v_{\deg(\boldsymbol{q})}]$$

is alternating. The representation follows in the same way as in the preceding two lemmas. □

4.8 ***Proof of the implication* (ii) ⇒ (iii) *in Theorem* 4.1** This is a combination of Equation (4.7) and Lemma 4.7. □

4.9 ***Proof of the implication* (iii) ⇒ (i) *in Theorem* 4.1** Our goal is to prove that if f is a polynomial of the form (1.5), then we have

$$\int_Q f(\boldsymbol{\omega} + \boldsymbol{\psi}(x))\,dx = f(\boldsymbol{\omega}) \tag{4.14}$$

for every $\boldsymbol{\omega} \in \boldsymbol{\Lambda}^{\boldsymbol{k}}$ and every $\boldsymbol{\psi} \in C^\infty_{\mathrm{per}}(\mathbb{R}^n, \boldsymbol{\Lambda}^{\boldsymbol{k}})$ such that

$$d\psi_\alpha = 0 \quad \text{on} \quad \mathbb{R}^n \quad \text{and} \quad \int_Q \psi_\alpha(x)\,dx = 0. \tag{4.15}$$

We shall first prove the particular case $\boldsymbol{\omega} = 0$ in (4.14), i.e., that

$$\int_Q \boldsymbol{\psi}^{\boldsymbol{q}}(x)\,dx = 0$$

for every $\boldsymbol{\psi} \in C^\infty_{\mathrm{per}}(\mathbb{R}^n, \boldsymbol{\Lambda}^{\boldsymbol{k}})$ satisfying (4.15) and every $\boldsymbol{q} \in \mathrm{Adm}(\boldsymbol{k})$. The case $\boldsymbol{q} = \mathbf{0}$ being trivial, we assume $\boldsymbol{q} \neq 0$, i.e., at least one component of $\boldsymbol{q}$ is nonzero. Suppose $q_1 \geq 1$ for definiteness. Then

$$\boldsymbol{\psi}^{\boldsymbol{q}}(x) = \psi_1(x) \wedge \boldsymbol{\psi}^{\boldsymbol{q}'}(x)$$

where $\boldsymbol{q}' = (q_1 - 1, q_2, \dots, q_s)$. By (4.15) with $\alpha = 1$, there exists a $\varphi \in C^\infty_{\mathrm{per}}(\mathbb{R}^n, \Lambda^{k_1})$ such that $\psi_1 = d\varphi$. Then $d\big(\varphi \wedge \boldsymbol{\psi}^{\boldsymbol{q}}(x)\big) = \boldsymbol{\psi}^{\boldsymbol{q}}(x)$ and hence the Stokes theorem gives

$$\int_Q \boldsymbol{\psi}^{\boldsymbol{q}}(x)\,dx = \int_Q d\big(\varphi(x) \wedge \boldsymbol{\psi}^{\boldsymbol{q}}(x)\big)dx = \int_{\partial Q} \nu \wedge \big(\varphi(x) \wedge \boldsymbol{\psi}^{\boldsymbol{q}}(x)\big)\,dx$$

where ν is the outer normal to the boundary ∂Q. Noting that the boundary integral vanishes since $\varphi(x) \wedge \boldsymbol{\psi}^{\boldsymbol{q}}(x)$ is periodic, we have (4.14) in the case $\boldsymbol{\omega} = 0$.

We now prove (4.14) generally. Any function given by (1.5) is a polynomial in $\boldsymbol{\omega}$ in the sense of (4.1). We shall give a proof by induction on the power m of f in (4.1). The case $m = 0$ is immediate. Let $m \geq 1$. Fixing $\boldsymbol{\psi}$, let $F : \boldsymbol{\Lambda}^{\boldsymbol{k}} \to \mathbb{R}$ be defined by

$$F(\boldsymbol{\omega}) = \int_Q f(\boldsymbol{\omega} + \boldsymbol{\psi}(x))\,dx,$$

$\boldsymbol{\omega} \in \boldsymbol{\Lambda}^{\boldsymbol{k}}$. Then

$$\mathrm{D}F(\boldsymbol{\omega})[\boldsymbol{\sigma}] = \int_Q \mathrm{D}f(\boldsymbol{\omega} + \boldsymbol{\psi}(x))[\boldsymbol{\sigma}]\,dx = \mathrm{D}f(\boldsymbol{\omega})[\boldsymbol{\sigma}] \tag{4.16}$$

for each $\boldsymbol{\sigma} \in \boldsymbol{\Lambda}^{\boldsymbol{k}}$ by induction hypothesis since $\mathrm{D}f(\cdot)[\boldsymbol{\sigma}]$ is a polynomial of degree $m-1$. The integration of the equality of the two derivatives in (4.16) provides $F(\boldsymbol{\omega}) = f(\boldsymbol{\omega}) + C$ for all $\boldsymbol{\omega} \in \boldsymbol{\Lambda}^{\boldsymbol{k}}$ and some $C \in \mathbb{R}$; in particular $F(0) = f(0) + C$. The proof in the special case $\boldsymbol{\omega} = 0$ shows that $F(0) = f(0)$ and hence $F(\boldsymbol{\omega}) = f(\boldsymbol{\omega})$ which is (4.14). □

5 Existence of minimizers of energy under mult. ext. polyconvexity

In this section we present a simple theorem on the existence of a minimizer for an integral functional (5.1) with a mult. ext. polyconvex integrand. The only goal is to show that the standard arguments of the direct method of the calculus of variations, well known in the case of the standard polyconvexity, can be modified to work also if mul. ext. polyconvexity is used. A more detailed treatment, with special cases, will be given elsewhere.

We introduce the following terminology and notation. We define the scalar product on $\boldsymbol{\Lambda}^{\boldsymbol{k}}$ by putting

$$\boldsymbol{\psi} \cdot \boldsymbol{\omega} = \sum_{\alpha=1}^{s} \psi_\alpha \cdot \omega_\alpha$$

for every $\boldsymbol{\psi} = (\psi_1, \dots, \psi_s)$, $\boldsymbol{\omega} = (\omega_1, \dots, \omega_s) \in \boldsymbol{\Lambda}^{k}$. Furthermore, if $\boldsymbol{p} = (p_1, \dots, p_s)$ is a collection of numbers satisfying $1 \le p_\alpha \le \infty$, we introduce the Banach space

$$L^{\boldsymbol{p}}(\Omega, \boldsymbol{\Lambda}^k) = L^{p_1}(\Omega, \wedge^{k_1}) \times \dots \times L^{p_s}(\Omega, \wedge^{k_s})$$

with the norm

$$|\boldsymbol{\omega}|_{L^{\boldsymbol{p}}(\Omega, \boldsymbol{\Lambda}^k)} = \sum_{\alpha=1}^{s} |\omega_\alpha|_{L^{p_\alpha}(\Omega, \wedge^{k_\alpha})}.$$

We write

$$\mathbf{d}\,\boldsymbol{\omega} := (d\omega_1, \dots, d\omega_s), \qquad \mathbf{div}\,\boldsymbol{\omega} := (\operatorname{div}\omega_1, \dots, \operatorname{div}\omega_s)$$

for any $\boldsymbol{\omega} : \Omega \to \boldsymbol{\Lambda}^{\boldsymbol{k}}$.

We shall minimize the integral functional

$$I(\boldsymbol{\omega}) = \int_\Omega (f(\boldsymbol{\omega}) - \boldsymbol{\phi} \cdot \boldsymbol{\omega})\, dx \tag{5.1}$$

depending on the collection $\boldsymbol{\omega} = (\omega_1, \dots, \omega_s)$ of closed differential forms on an open bounded region $\Omega \subset \mathbb{R}^n$. The external influences, such as the body forces and boundary tractions in elasticity, are modeled by the second term in the integral in (5.1). In that term, $\boldsymbol{\phi} = (\phi_1, \dots, \phi_s) : \Omega \to \boldsymbol{\Lambda}^{\boldsymbol{k}}$ is a prescribed function.

To formulate the assumptions, we let let $p_1, \dots, p_s$ be numbers in $(1, \infty)$, and consider the following conditions:

$\mathbf{H}_1$ f is a mult. ext. polyconvex (continuous) integrand;

$\mathbf{H}_2$ f satisfies

$$f(\boldsymbol{\omega}) \ge c(|\omega_1|^{p_1} + \dots + |\omega_s|^{p_s} - 1)$$

for some $c > 0$ and all $(\omega_1, \dots, \omega_s) \in \boldsymbol{\Lambda}^{\boldsymbol{k}}$ where the numbers $p_1, \dots, p_s$ satisfy

$$q_1/p_1 + \dots + q_s/p_s \le 1 \quad \text{for all } (q_1, \dots, q_s) \in \operatorname{Adm}(\boldsymbol{k});$$

$\mathbf{H}_3$ $\boldsymbol{\phi} \in L^{\boldsymbol{p}'}(\Omega, \boldsymbol{\Lambda}^{\boldsymbol{k}})$, where

$$\boldsymbol{p}' = (p_1', \dots, p_s'), \quad p_\alpha' = p_\alpha/(p_\alpha - 1).$$

For the purpose of the treatment below, we define the domain $\mathscr{D}$ of the functional I to be the set of all $\boldsymbol{\omega} \in L^{\boldsymbol{p}}(\Omega, \boldsymbol{\Lambda}^{\boldsymbol{k}})$ which satisfy

$$\mathbf{d}\,\boldsymbol{\omega} = \mathbf{0} \quad \text{on } \Omega \tag{5.2}$$

in the weak sense, see Definition 2.6.

It is not assumed that the integrand f is finite. Condition $\mathbf{H}_2$ implies that f is bounded from below and thus the integral in (5.1) is well defined as a finite number or ∞. We denote by

$$\operatorname{dom} f = \{\boldsymbol{\sigma} \in \Lambda^k : f(\boldsymbol{\sigma}) < \infty\}$$

the effective domain of f. The assumed continuity of f (which is a part of the definition of an integrand) implies that $\operatorname{dom} f$ is an open subset of Λ^k.

The following theorem presents an existence result for a minimizer of I under inhomogeneous Neumann's boundary conditions.

5.1 Theorem *Suppose that Hypotheses* $\mathbf{H}_1$–$\mathbf{H}_3$ *hold. Then*

(i) *if I is not identically equal to ∞ on $\mathscr{D}$ then I has a minimizer $\boldsymbol{\omega}$ in $\mathscr{D}$, i.e., an element such that*

$$I(\boldsymbol{\omega}) \le I(\boldsymbol{\sigma})$$

for all $\boldsymbol{\sigma} \in \mathscr{D}$;

(ii) *each minimizer $\boldsymbol{\omega}$ satisfies $f(\boldsymbol{\omega}(x)) < \infty$ for almost every $x \in \Omega$;*

(iii) *if f is differentiable on* $\operatorname{dom} f$ *and $\boldsymbol{\omega}$ is a minimizer whose range is contained in a compact subset of* $\operatorname{dom} f$ *then we have the weak form of the Euler-Lagrange equations*

$$\mathbf{div}(\mathrm{D}_{\boldsymbol{\omega}} f - \boldsymbol{\phi}) = 0 \quad \text{on} \quad \mathbb{R}^n, \tag{5.3}$$

where **div** *denotes the weak divergence* (*Definition* 2.6).

5.2 Remarks

- In contrast to Theorem 5.1, the existence theorem under the weaker condition of mult. ext. quasiconvexity requires finitely-valued integrand since the prerequisite lower semicontinuity theorem [19; Theorem 3.7] requires conditions of the type

$$c_1(|\boldsymbol{\omega}|^p - 1) \le f(\boldsymbol{\omega}) \le c_2(|\boldsymbol{\omega}|^p - 1)$$

  for some $p > 1$, $c_1 > 0$, $c_2 > 0$ and all $\boldsymbol{\omega}$.
- Theorem 5.1 or its minor modifications contains the existence theorems in nonlinear elasticity [1] and in nonlinear electro-magneto-elasticity [34].
- The coercivity hypothesis $\mathbf{H}_2$ may be unnecessarily strong in concrete cases. For example, in nonlinear elasticity (see Section 6, below), the unknown $\boldsymbol{\omega}$ is a triplet 1-forms in $n = 3$ and thus $\mathbf{H}_3$ requires

$$p_1 \ge 3, \quad p_2 \ge 3, \quad p_3 \ge 3,$$

  i.e., condituion of type

$$W(F) \ge c(|F|^p - 1), \quad p \ge 3,$$

  where F is the deformation gradient. However, weaker coercivity conditions suffice to establish existence theorems in nonlinear elasticity [1, 27].
- Since the weak form of the Euler–Lagrange equation (5.3) holds on $\mathbb{R}^n$, it includes the inhomogeneous Neumann-type boundary conditions.

To prove the existence of the solution, we need the compensated compactness and the lower semicontinuity results in Theorems 5.3 and Theorem 5.5.

5.3 Theorem *Let* $\boldsymbol{q} = (q_1, \dots, q_s) \in \mathrm{Adm}(\boldsymbol{k})$ *and let* $\boldsymbol{p} = (p_1, \dots, p_s)$, $\boldsymbol{r} = (r_1, \dots, r_s)$ *be s-tuples that satisfy*

$$1 < p_\alpha \le \infty, \quad r_\alpha \ge 1, \quad np_\alpha/(n + p_\alpha) < r_\alpha \le p_\alpha,$$

and

$$q_1/p_1 + \dots + q_s/p_s \le 1. \tag{5.4}$$

If ω_j, $j = 1, \dots,$ *is a sequence in* $L^{\boldsymbol{p}}(\Omega, \Lambda^{\boldsymbol{k}})$ *and* $\boldsymbol{\omega}$ *an element of* $L^{\boldsymbol{p}}(\Omega, \Lambda^{\boldsymbol{k}})$ *such that*★

$$\boldsymbol{\omega}_j \rightharpoonup \boldsymbol{\omega} \quad \text{as } j \to \infty \text{ in } L^{\boldsymbol{p}}(\Omega, \Lambda^{\boldsymbol{k}}), \tag{5.5}$$

and

$$\sup \{ |\mathbf{d}\,\omega_j|_{L^{\boldsymbol{r}}(\Omega, \Lambda^{\boldsymbol{k}+\mathbf{1}})} : j = 1, \dots \} < \infty$$

then

$$\boldsymbol{\omega}_j^{\boldsymbol{q}} \overset{*}{\rightharpoonup} \boldsymbol{\omega} \quad \text{in } \mathscr{M}(\Omega, \wedge^{\deg(\boldsymbol{q})}). \tag{5.6}$$

If (5.4) *holds with the strict inequality sign, then* (5.6) *can be strengthened to assert*

$$\boldsymbol{\omega}_j^{\boldsymbol{q}} \rightharpoonup \boldsymbol{\omega} \quad \text{in } L^p(\Omega, \wedge^{\deg(\boldsymbol{q})})$$

where p is defined by

$$1/p + q_1/p_1 + \dots + q_s/p_s = 1.$$

5.4 Remarks

- Theorem 5.3 is due to Robbin, Rogers & Temple [31; Theorem 1.1] and Iwaniec & Lutoborski [21; Theorem 5.1].

The weak continuity of the exterior product of differential forms has been considered previously:

- In a forgotten result, Whitney [36; Chapter IX, Theorem 17A] establishes, in 1957, the case $s = 2$, $\boldsymbol{k} = (k_1, k_2)$ arbitrary, $\boldsymbol{q} = (1, 1)$ and $\boldsymbol{p} = \boldsymbol{r} = (\infty, \infty)$ of Theorem 5.3. *This seems to be the first compensated compactness result ever proved.* This, however, went unnoticed, as emphasized in my earlier paper [33]. Since the product of any elements of L^∞ is in L^∞, an obvious iteration of Whitney's result provides the case of a general $s \ge 2$, $\boldsymbol{k} = (k_1, \dots, k_s)$ and $\boldsymbol{p} = \boldsymbol{r} = (\infty, \dots, \infty)$.
- The div-curl lemma by Murat [28] and Tartar [35] is $s = 2$, $\boldsymbol{k} = (1, n-1)$, $\boldsymbol{p} = \boldsymbol{r} = (t, t')$, $1 < t < \infty$.

5.5 Theorem (Reshetnyak [30], Ball & Murat [3]) *Let* $\Phi : \mathbb{R}^h \to \bar{\mathbb{R}}$ *be convex, lower semicontinuous and bounded below. If* θ, $\theta_k \in L^1(\Omega, \mathbb{R}^h)$ *satisfy*

$$\theta_k \overset{*}{\rightharpoonup} \theta \quad \text{in } \mathscr{M}(\Omega, \mathbb{R}^h)$$

then

$$\liminf_{j \to \infty} \int_\Omega \Phi(\theta_k)\, dx \ge \int_\Omega \Phi(\theta)\, dx.$$

★ Equation (5.5) denotes the weak (weak*) convergence of the components of ω_j, i.e., $\int_\Omega \boldsymbol{\psi} \cdot \boldsymbol{\omega}_j\, dx \to \int_\Omega \boldsymbol{\psi} \cdot \boldsymbol{\omega}\, dx$ for every $\boldsymbol{\psi} \in L^{\boldsymbol{p}'}(\Omega, \Lambda^{\boldsymbol{k}})$. Similarly, (5.6) (below) denotes the weak convergence in the sense of measures, i.e., $\int_\Omega \gamma \cdot \boldsymbol{\omega}_j^{\boldsymbol{q}}\, dx \to \int_\Omega \gamma \cdot \boldsymbol{\omega}^{\boldsymbol{q}}\, dx$ for every $\gamma \in C_0(\Omega, \wedge^{\deg(\boldsymbol{q})})$.

***Proof of Theorem* 5.1** (i): Let $\boldsymbol{\omega}_j \in \mathscr{D}$ be a minimization sequence, which is bounded in $L^p(\Omega, \Lambda^k)$ by $\mathbf{H}_2$. By reflexivity, there exists a subsequence, again denoted by $\boldsymbol{\omega}_j$, such that we have the weak convergence (5.5). Then

$$\boldsymbol{\omega}_j^{\boldsymbol{q}} \xrightarrow{*} \boldsymbol{\omega}^{\boldsymbol{q}} \quad \text{in} \quad \mathscr{M}(\Omega, \wedge^{\dim(r)}) \tag{5.7}$$

for every $\boldsymbol{q} \in \operatorname{Adm}(\boldsymbol{k})$ by Theorem 5.3. By $\mathbf{H}_1$ and Theorem 1.3 there exists a convex lower semicontinuous function $\Phi : \wedge^{\deg(\boldsymbol{q}_1)} \times \cdots \times \wedge^{\deg(\boldsymbol{q}_r)} \to \bar{\mathbb{R}}$ such that

$$f(\boldsymbol{\omega}) = \Phi(\boldsymbol{\omega}^{\boldsymbol{q}_1}, \ldots, \boldsymbol{\omega}^{\boldsymbol{q}_r})$$

or each $\boldsymbol{\omega} \in \Lambda^k$, where $\boldsymbol{q}_1, \ldots, \boldsymbol{q}_r$ is a fixed collection of distinct, nonzero elements of $\operatorname{Adm}(\boldsymbol{k})$. By Theorem 5.5 and (5.7),

$$\liminf_{j\to\infty} \int_\Omega \Phi(\boldsymbol{\omega}_j^{\boldsymbol{q}_1}, \ldots, \boldsymbol{\omega}_j^{\boldsymbol{q}_r})\, dx \ge \int_\Omega \Phi(\boldsymbol{\omega}^{\boldsymbol{q}_1}, \ldots, \boldsymbol{\omega}^{\boldsymbol{q}_r})\, dx,$$

i.e.,

$$\liminf_{j\to\infty} \int_\Omega f(\boldsymbol{\omega}_j)\, dx \ge \int_\Omega f(\boldsymbol{\omega})\, dx.$$

As also

$$\lim_{j\to\infty} \int_\Omega \boldsymbol{\phi} \cdot \boldsymbol{\omega}_j\, dx = \int_\Omega \boldsymbol{\phi} \cdot \boldsymbol{\omega}\, dx,$$

we have

$$\liminf_{j\to\infty} I(\boldsymbol{\omega}_j) \ge I(\boldsymbol{\omega}).$$

Since the condition (5.2) survives the limit, we see that $\boldsymbol{\omega}$ is in $\mathscr{D}$ and thus it minimzes I on $\mathscr{D}$. The proof of (i) is complete.

(ii): Follows immediately form $I(\boldsymbol{\omega}) < \infty$.

(iii): If $\boldsymbol{\psi} \in C_0^\infty(\mathbb{R}^n, \Lambda^k)$ then $\boldsymbol{\omega} + t\boldsymbol{\psi} \in \operatorname{dom} f$ for all sufficiently small $|t|$ by the hypothesis of (iii). Then

$$\int_\Omega \boldsymbol{\psi} \cdot (\mathrm{D}_\omega f - \boldsymbol{\phi})\, dx = 0$$

by a standard argument. Taking $\boldsymbol{\psi} = \mathbf{d}\,\boldsymbol{\chi}$ and invoking Definition 2.6 we obtain (5.3). □

6 Example A: Classical calculus of variations and nonlinear elasticity

The goal of this section is to discuss the relationship of the weakened convexity and affinity notions of this paper to their original counterparts from the calculus of variations.

6.1 The integrand and its variables The classical calculus of variations deals with the integral functionals of the form

$$I(u_1, \ldots, u_s) = \int_\Omega f(\nabla u_1, \ldots, \nabla u_s)\, dx$$

where Ω is a bounded open subset of $\mathbb{R}^n$ as always and $u = (u_1, \dots, u_s)$ is an s-tuple of scalar functions on Ω. We obtain the format (1.1)–(1.3) with

$$\omega_\alpha = u_{\alpha,i}\, dx_i. \tag{6.1}$$

The forms ω_α are closed since the combination of (2.6) with $dx_i \wedge dx_j = -dx_j \wedge dx_i$ provides

$$d\omega_\alpha = \frac{1}{2}(\omega_{\alpha j,i} - \omega_{\alpha i,j})\, dx_i \wedge dx_j = \frac{1}{2}(u_{\alpha,ji} - u_{\alpha,ij})\, dx_i \wedge dx_j = 0.$$

The domain of f is $\boldsymbol{\wedge}^k := \underbrace{\wedge^1 \times \dots \times \wedge^1}_{s \text{ times}}$.

6.2 Quasiconvexity and rank one convexity The reader is referred to [24–25, 1, 26] for the weakened notions of convexity in the calculus of variations. It is immediate that in the present case the mult. ext. quasiconvexity coincides with the quasiconvexity, mult. ext. quasiaffinity with the quasiaffinity derived from the quasiconvexity. Also the mult. ext. one convexity and mult. ext. one affinity coincide with the rank one convexity and rank one affinity, respectively. Indeed, the discussion in Subsection 3.4 shows that in the present case, the components of $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_s)$ are real numbers, $\lambda_\alpha \in \mathbb{R}$. Thus $\lambda_\alpha \wedge \xi \equiv \lambda_\alpha \xi \in \mathbb{R}^n$ and $\boldsymbol{\lambda} \wedge \xi = (\lambda_1 \xi, \dots, \lambda_s \xi) = \boldsymbol{\lambda} \otimes \xi$. Thus Formulas in (3.3) and (3.4) from the definition of mult. ext. one convexity and mult. ext. one affinity coincide with those in the definition of the rank one convexity and rank one affinity, respectively.

We now turn to the central notion of this paper, polyconvexity.

6.3 Mult. ext. polyconvexity and Ball's polyconvexity According to the construction described after the statement of Theorem 1.2, the set of all nonconstant and nonzero mult. ext. quasiaffine integrands is the span of the list

$$\omega_{\alpha_1} \wedge \dots \wedge \omega_{\alpha_k}, \quad 1 \le k \le r, \quad 1 \le \alpha_1 < \dots < \alpha_k \le m, \tag{6.2}$$

where $r := \min\{m, n\}$. An integrand $f : \boldsymbol{\wedge}^k \to \bar{\mathbb{R}}$ is mult. ext. polyconvex if it is a convex function of the forms occurring in (6.2). The standard rules for the exterior product and (6.1) show that

$$\omega_{\alpha_1} \wedge \dots \wedge \omega_{\alpha_k} = \sum_{1 \le i_1 < \dots < i_k \le n} M_{\alpha_1, \dots, \alpha_k; i_1, \dots, i_k}\, dx_{i_1} \wedge \dots \wedge dx_{i_k}$$

where for each $\alpha_1, \dots, \alpha_k$ and $i_1, \dots, i_k$ we denote by

$$M_{\alpha_1, \dots, \alpha_k; i_1, \dots, i_k} = \det\, [u_{\alpha_\mathfrak{a}, i_\mathfrak{b}}]_{1 \le \mathfrak{a}, \mathfrak{b} \le k}$$

the minor of the $n \times m$ matrix

$$F := [u_{\alpha,i}]_{1 \le \alpha \le m,\ 1 \le i \le n}$$

corresponding to the rows and columns labeled by $\alpha_1, \dots, \alpha_k$ and $i_1, \dots, i_k$, respectively. This shows that the list of exterior products (6.2) is linearly isomorphic with the collection of all minors of the matrix F. Consequently, the mult. ext. polyconvexity coincides with the standard polyconvexity, defined as a convex function of minors of $F = \nabla u$; [2, 26].

Nonlinear elasticity [1] is a particular case $m = n = 3$; then we have a convex function of F, $\operatorname{cof} F$ and $\det F$. Recalling that

$$(\operatorname{cof} F)_{pq} = \frac{1}{2}\,\varepsilon_{pij}\varepsilon_{qkl}F_{ik}F_{jl}, \qquad \det F = \frac{1}{6}\,\varepsilon_{pij}\varepsilon_{qkl}F_{pq}F_{ik}F_{jl}$$

we obtain the following explicit relationship between the lists in (1.8) and (1.9):

$$\omega_j \wedge \omega_k = \frac{1}{2}\,\varepsilon_{jkq}\varepsilon_{qrs}(\operatorname{cof} F)_{pq}\,dx_r \wedge dx_s, \tag{6.3}$$

$$\omega_1 \wedge \omega_2 \wedge \omega_3 = (\det F)\,dx_1 \wedge dx_2 \wedge dx_3. \tag{6.4}$$

7 Example B: Electro-magneto-elastostatics

As mentioned in the introduction, the statics of electro- and magneto-sensitive elastomers has received considerable attention in recent years [8, 14–17, 9–10, 13, 20]. The main point in modeling these materials is the coupling of the nonlinear mechanical response with the electric or magnetic response. The goal of this section is to determine the mult. ext. polyconvexity corresponding to this case; the reader is referred to [34] for more details. Let $\Omega \subset \mathbb{R}^n$ where $n = 2$ or 3.

The basic electromagnetic variables are the referential (lagrangean) electric displacement D, magnetic induction B satisfying the static Maxwell's equations

$$\operatorname{div} D = 0, \qquad \operatorname{div} B = 0.$$

The mechanical variables are the deformation $u : \Omega \to \mathbb{R}^n$ and the deformation gradient $F = \nabla u$.

To formulate the constitutive equations, we note that many choices of independent/dependent variables are possible. We take the triplet (F, D, B) and start from the free energy function $f : \mathbb{M}^{n\times n} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, i.e., $f = f(F, D, B)$.

The energy of the body is given by

$$I(u, D, B) = \int_\Omega f(F, D, B)\,dx; \tag{7.1}$$

the total energy then consists of this term plus the energy of the vacuum electromagnetic field in the exterior of Ω and the term describing the loads. The integral in (7.1) falls within the format (1.1) under the identifications which we now describe separately for $n = 3$ and 2.

7.1 Dimension three Here $s = 5$ and the the variable $\boldsymbol{\omega} = (\omega_1, \dots, \omega_5)$ is formed by ω_1, ω_2, ω_3 as in (1.7) and by ω_4, ω_5 as in (1.10). That the forms ω_1, ω_2, ω_3 are closed has been explained in Section 6. In the case of ω_4 and ω_5 Formula (2.6) provides

$$d\omega_4 = \frac{1}{2}\,\varepsilon_{ijk}D_{i,l}\,dx_l \wedge dx_j \wedge dx_k = D_{i,i}dx_1 \wedge dx_2 \wedge dx_3$$

because

$$\frac{1}{2}\,\varepsilon_{ijk}\,dx_l \wedge dx_j \wedge dx_k = \delta_{il}\,dx_1 \wedge dx_2 \wedge dx_3$$

and as a consequence of

$$dx_l \wedge dx_j \wedge dx_k = \varepsilon_{ljk}\, dx_1 \wedge dx_2 \wedge dx_3 .$$

Referring to Theorem 4.1 and to the rule that follows it, we form the complete list of mult. ext. quasiaffine functions by taking all mutual products of $\omega_1 \dots , \omega_5$ of degree ≤ 3 (since $n = 3$). This gives the set of differential forms which consists of the constant form of degree 0 equal to 1 identically and of the forms

$$\omega_\alpha, \quad \omega_\beta \wedge \omega_\gamma, \quad \omega_1 \wedge \omega_2 \wedge \omega_3, \quad 1 \le \alpha \le 3, \quad 1 \le \beta < \gamma \le 3, \tag{7.2}$$

$$\omega_4, \quad \omega_5, \quad \omega_\alpha \wedge \omega_4, \quad \omega_\alpha \wedge \omega_5, \quad 1 \le \alpha \le 3. \tag{7.3}$$

This is isomorphic to

$$F, \quad \operatorname{cof} F, \quad \det F, \quad D, \quad B, \quad FD, \quad FB. \tag{7.4}$$

Indeed, it was shown in Section 6 that (7.2) is isomorphic to the first three members of (7.4); the first two members of (7.3) of course correspond to D and B, and for the remaining two members it suffices to note that

$$\omega_\alpha \wedge \omega_4 = (FD)_\alpha\, dx_1 \wedge dx_2 \wedge dx_3, \quad \omega_\alpha \wedge \omega_5 = (FB)_\alpha\, dx_1 \wedge dx_2 \wedge dx_3 .$$

The list (7.4) was determined by a direct calculation in [34]. Thus the free energy $f = f(F, D, B)$ is mult. ext. polyconvex if and only if there exists a convex function Φ such that

$$f(F, D, B) = \Phi(F, \operatorname{cof} F, \det F, D, B, FD, FB)$$

for each $(F, D, B) \in \mathbb{M}^{n\times n} \times \mathbb{R}^n \times \mathbb{R}^n$ [34; Theorem 6.5].

7.2 Dimension two Then (7.1) takes the form (1.1) with $s = 4$ and with the 1-forms $\omega_1, \dots , \omega_4$ given by

$$\omega_\alpha = F_{\alpha i} dx_i, \quad 1 \le \alpha \le 2,$$

$$\omega_3 = D_1 dx_2 - D_2 dx_1, \quad \omega_4 = B_1 dx_2 - B_2 dx_1 .$$

The reader will have no difficulty to check that the list of mult. ext. quasiaffine functions is

$$\omega_1, \quad \omega_2, \quad \omega_3, \quad \omega_4, \quad \omega_{\mathfrak{b}} \wedge \omega_{\mathfrak{c}}, \quad 1 \le \mathfrak{b} < \mathfrak{c} \le 4,$$

which is isomorphic to

$$F, \quad \det F, \quad D, \quad B, \quad FD, \quad FB, \quad D \times B. \tag{7.5}$$

We note that the term $D \times B$ comes from the product $\omega_3 \wedge \omega_4$, which is a 2-form in dimension 2, since ω_3 and ω_4 are 1-forms. This has no analog in dimension $n = 3$ since the corresponding term $\omega_4 \wedge \omega_5$ (where ω_4, ω_5 are as in (1.10)), being a product of two 2-forms, is a 4-form in dimension 3, and hence $\omega_4 \wedge \omega_5$ vanishes. From (7.5) one finds that $f = f(F, D, B)$ is mult. ext. polyconvex if and only if there exists a convex function Φ such that

$$f(F, D, B) = \Phi(F, \det F, D, B, FD, FB, D \times B)$$

[34; Theorem 6.5].

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