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Mathematics of the Masonry–Like model and Limit Analysis

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Introduction

These notes present a brief introduction to the mathematics of equilibrium of no–tension (masonry–like) materials. We first review the constitutive equations using the idea that the stress of the no–tension material must be always negative semidefinite. The strain tensor is naturally split into the sum of the elastic strain and fracture strain. The stress depends linearly on the elastic strain via the fourth–order tensor of elasticities. Then we consider a body made of a no–tension material, introduce the loads and the total energy of the deformation with is the sum of the internal energy and the energy of the loads. Then we examine the question whether the total energy is bounded from below. That brings us to the important notion of the strong compatibility of loads. The loads are strongly compatible if they can be equilibrated (in the sense of the principle of virtual work) by a square integrable negative semidefinite stress field. The total energy is bounded from below if and only if the loads are strongly compatible. The notion of strong compatibility of loads is central in the limit analysis and in a strengthened form in the theory of existence of equilibrium states. Roughly speaking, if the loads are strongly compatible, then the body is safe, while otherwise strongly incompatible loads lead to the collapse of the body. To determine whether the loads are strongly compatible, it is not necessary to solve the full displacement problem, it suffices to find the negative semidefinite square integrable stress field, which is independent of the constitutive theory.

The considerations concerning the limit analysis and strong compatibility of loads are based on the displacements that belong to the Sobolev space of square integrable maps with square integrable gradients. Roughly speaking, this means that the fracture part of the strain is always without macroscopic cracks. To obtain the existence theory of equilibrium states, it is necessary to enlarge the class of deformations in which the fracture part of the deformation is a measure in the mathematical sense of that word. This introduces discontinuous displacements with macroscopic cracks. We give a brief introduction to the mathematical theory of such displacements, called displacements with bounded deformation. The full theory of equilibrium states is nontrivial and many assertions are presented without proofs. However, the basic line of thought is preserved.

Then the limit analysis for no–tension materials is presented. The loads of the limit analysis are assumed to be linearly dependent on the loading parameter. The ideal goal of the limit analysis is to determine the largest value of the loading parameter for which the loads are strongly compatible. This value is called the collapse multiplier; the loads corresponding to the loading multiplier bigger than the collapse multiplier lead to the collapse of the body. The loading parameters for which the loads are strongly compatible are called statically admissible loading parameters. By the above, they are characterized by the existence of a square integrable negative semidefinite stress field equilibrating the corresponding loads. For concrete loads, it is often easier to find a stress field represented by a negative semidefinite tensor valued measure equilibrating the loads. We call such loads weakly compatible. The difference between the square integrable stress fields and the stress fields represented by measures is that the latter can contain singular part which is concentrated on surfaces and curves in the body. Of course the strong compatibility implies the weak compatibility but not conversely: there are weakly compatible loads that are not strongly compatible. If the loads happen to be weakly compatible on some interval of the loading parameters, then the averaging procedure to be described in Section 6 may lead to square integrable equilibrating stress fields and hence to the strong compatibility. The last section presents an example of the averaging procedure which leads to an explicit determination of the square integrable averaged stress field.

The mechanical tools to be employed include the notion of stress, the virtual power principle, the notion of weak solution, and the notion of the total energy of the body. These notions are established in detail in the treatment below.

The mathematical tools necessary for the understanding include in particular the notions of the convex cones and orthogonal projection upon them, some elements of the convex analysis, vector valued measures, Sobolev spaces, families of measures, and the basic notions associated with the space of displacements of bounded deformation. The basic definitions of the mathematical notions are given in the text below and the basic properties are stated without proof.

1 Constitutive equations

Throughout we use the conventions for vectors and second order tensors identical with those in [15]. Thus Lin denotes the set of all second order tensors on R^n, i.e., linear transformations from R^n into itself, Sym is the subspace of symmetric tensors, Skw is the subspace of skew (antisymmetric) tensors, Sym^+ the set of all positive semidefinite elements of Sym; additionally, Sym^- is the set of all negative semidefinite elements of Sym. The scalar product of $\mathbf{A}, \mathbf{B} \in \mathrm{Lin}$ is defined by $\mathbf{A} \cdot \mathbf{B} = \mathrm{tr}(\mathbf{A}\mathbf{B}^{\mathrm{T}})$ and $|\cdot|$ denotes the associated euclidean norm on Lin. We denote by $\mathbf{1} \in \mathrm{Lin}$ the unit tensor. If $\mathbf{A}, \mathbf{B} \in \mathrm{Sym}$, we write $\mathbf{A} \le \mathbf{B}$ if $\mathbf{B} - \mathbf{A} \in \mathrm{Sym}^+$.

To describe the stress, we assume that $\mathsf{C} : \mathrm{Sym} \to \mathrm{Sym}$ is a given linear transformation, the fourth order tensor of ***elastic constants,*** such that

$$\left.\begin{aligned} \mathbf{E} \cdot \mathsf{C}\mathbf{E} > 0 \quad & \text{for all } \mathbf{E} \in \mathrm{Sym}, \mathbf{E} \neq \mathbf{0}, \\ \mathbf{E}_1 \cdot \mathsf{C}\mathbf{E}_2 = \mathbf{E}_2 \cdot \mathsf{C}\mathbf{E}_1 \quad & \text{for all } \mathbf{E}_1, \mathbf{E}_2 \in \mathrm{Sym}. \end{aligned}\right\} \quad (1.1)$$

For example, the tensor of elastic constants of an isotropic material is of the form

$$\mathsf{C}\mathbf{E} = \lambda(\mathrm{tr}\,\mathbf{E})\mathbf{1} + 2\mu\mathbf{E}$$

for each $\mathbf{E} \in \mathrm{Sym}$ where λ and μ are constants, Lamé moduli, satisfying

$$\mu > 0, \quad n\lambda + 2\mu > 0$$

where $n = 2, 3$ is the dimension of the underlying space.

In the case of a general C, we introduce the ***energetic scalar product*** $(\cdot,\cdot)_E$ and the ***energetic norm*** $|\cdot|_E$ on Sym by setting

$$(\mathbf{A}, \mathbf{B})_E = \mathbf{A} \cdot \mathsf{C}\mathbf{B}, \quad |\mathbf{A}|_E = \sqrt{(\mathbf{A}, \mathbf{A})_E}$$

for each $\mathbf{A}, \mathbf{B} \in \mathrm{Sym}$.

We are now going to review briefly some concepts of convex analysis. We refer to [27] and [11] for more details.

A nonempty subset K of a vector space V is called a ***convex cone*** if $tx + sy \in K$ for each $x, y \in K$ and each $t \ge 0$, $s \ge 0$.

Let K be a closed convex cone in a vector space V with scalar product $(\cdot,\cdot)$ and norm $|\cdot|$. We say that a point $y \in K$ is the ***(orthogonal) projection*** of a point $x \in V$ if y makes the distance $|z - x|$ minimal among all $z \in K$, i.e., if

$$|y - x| \le |z - x|$$

for all $z \in K$. The projection onto a closed convex cone exists and is uniquely determined.

Let K be a nonempty set in a vector space V with scalar product and $x \in V$. We define the ***normal cone*** $\mathrm{Norm}(K, x)$ to K at x by

$$\mathrm{Norm}(K, x) = \{y \in V : (z - x, y) \le 0 \ \text{ for all } \ z \in K\}.$$

1.1 Remark. *Let K be a closed convex cone in a vector space V with scalar product $(\cdot,\cdot)$ and norm $|\cdot|$ and let $x \in K$. Then*

$$\mathrm{Norm}(K, x) = \{y \in V : (y, z) \le 0 \ \text{ for each } \ z \in K \ \text{ and } \ (x, y) = 0\}.$$

Proof If $y \in \mathrm{Norm}(K,x)$ then $(z-x,y) \le 0$ for all $z \in K$. Replacing z by tz where $t > 0$ in the last inequality, dividing the resulting inequality by t and letting $t \to \infty$ we obtain $(z,y) \le 0$. Similarly, taking $z = 0$ in $(z-x,y) \le 0$ we obtain $(x,y) \ge 0$ and as $x \in K$ we have also $(x,y) \le 0$ by the preceding part of the proof. Thus we have $(x,y) = 0$. Thus $y \in \{y \in V : (y,z) \le 0 \text{ for each } z \in K \text{ and } (x,y) = 0\}$. Conversely, if $y \in \{y \in V : (y,z) \le 0 \text{ for each } z \in K \text{ and } (x,y) = 0\}$ then the inequality $(y,z) \le 0$ and the equality $(x,y) = 0$ provide $(z-x,y) \le 0$ for all $z \in K$. □

1.2 Remark. *Let K be a closed convex cone in a vector space V with scalar product $(\cdot,\cdot)$ and norm $|\cdot|$ and let $x \in V$. Then a point $y \in K$ is the projection of x onto K if and only if the following two conditions are satisfied:*
(i) $(w, x-y) \le 0$ *for all* $w \in K$;
(ii) $(x-y,y) = 0$.
Equivalently, a point $y \in K$ is the projection of x onto K if and only if

$$x - y \in \mathrm{Norm}(K,y).$$

Proof Assume that y is the projection of x onto K. Then for every $z \in K$ we have

$$|z-x| \ge |y-x| \tag{1.2}$$

which can be rewritten as

$$|z-y+y-x|^2 \ge |y-x|^2$$

which in turn implies

$$|z-y|^2 + 2(z-y,y-x) \ge 0. \tag{1.3}$$

We now put $z = y + tw$ where $w \in K$ and $t > 0$ to obtain

$$t^2|w|^2 + 2t(w,y-x) \ge 0.$$

Dividing by t and letting $t \to 0$ we obtain (i). Next we put $z = (1+t)y$ where $t \ge -1$ to obtain

$$t^2|y|^2 + 2t(y,y-x) \ge 0.$$

Dividing by $t > 0$ and letting $t \downarrow 0$ we obtain $(y,y-x) \ge 0$; dividing by $t < 0$ and letting $t \uparrow 0$ we obtain $(y,y-x) \le 0$ and hence we have (ii).

Conversely, let (i) and (ii) hold. If $z \in K$ then summing the relations $2(z,y-x) \ge 0$, $-2(y,y-x) = 0$ and $|z-y|^2 \ge 0$ we obtain (1.3) which in turn implies (1.2) and hence y is the projection of x onto K.

This completes the proof of the characterization by (i) and (ii). The equivalent characterization in terms of the normal cone follows from (i) and (ii) via Remark 1.1. □

1.3 Proposition. *Assume* (1.1). *If* $\mathbf{E} \in \mathrm{Sym}$, *there exists a unique triplet* $(\mathbf{T}, \mathbf{E}^{\mathrm{e}}, \mathbf{E}^{\mathrm{f}})$ *of elements of* Sym *such that the following three equivalent characterizations hold:*
(i) *we have*

$$\left.\begin{aligned} \mathbf{E} &= \mathbf{E}^{\mathrm{e}} + \mathbf{E}^{\mathrm{f}}, \\ \mathbf{T} &= \mathsf{C}\mathbf{E}^{\mathrm{e}}, \\ \mathbf{T} \in \mathrm{Sym}^-, &\quad \mathbf{E}^{\mathrm{f}} \in \mathrm{Sym}^+, \\ \mathbf{T} \cdot \mathbf{E}^{\mathrm{f}} &= 0; \end{aligned}\right\} \tag{1.4}$$

(ii) *we have equations* $(1.4)_{1,2}$ *and*

$$\left.\begin{aligned} \mathbf{T} &\in \mathrm{Sym}^-, \\ (\mathbf{T}-\mathbf{T}^*)\cdot\mathbf{E}^{\mathrm f} \geq 0 \quad &\text{for each } \mathbf{T}^* \in \mathrm{Sym}^-; \end{aligned}\right\} \tag{1.5}$$

(iii) *we have equations* $(1.4)_{1,2}$ *and*

$$\mathbf{E}^{\mathrm e} \textit{ is the projection of } \mathbf{E} \textit{ onto } \mathsf{C}^{-1}\mathrm{Sym}^- \textit{ with respect to } (\cdot,\cdot)_E, \tag{1.6}$$

Proof Let us first show that the three characterizations of the triplet $(\mathbf{T}, \mathbf{E}^{\mathrm e}, \mathbf{E}^{\mathrm f})$ are equivalent.

Proof of (iii) $\Rightarrow$ (i). Assume that Characterization (iii) holds. Let $\mathbf{E}^{\mathrm e}$ be the projection of $\mathbf{E}$ onto the convex cone $\mathsf{C}^{-1}\mathrm{Sym}^-$ with respect to the energetic scalar product and let $\mathbf{E}^{\mathrm f}$ and $\mathbf{T}$ be as in $(1.4)_{1,2}$. Since $\mathbf{E}^{\mathrm e} \in \mathsf{C}^{-1}\mathrm{Sym}^-$, we have $\mathbf{T} = \mathsf{C}\mathbf{E}^{\mathrm e} \in \mathrm{Sym}^-$ and by Remark 1.2(i) $(\mathbf{E}^{\mathrm f}, \mathsf{C}^{-1}\mathbf{T}^*)_E = (\mathbf{E}-\mathbf{E}^{\mathrm e}, \mathsf{C}^{-1}\mathbf{T}^*)_E \leq 0$ for all $\mathbf{T}^* \in \mathrm{Sym}^-$ which can be rewritten as $\mathbf{E}^{\mathrm f}\cdot\mathbf{T}^* \leq 0$, which in turn implies that $\mathbf{E}^{\mathrm f} \in \mathrm{Sym}^+$. Thus we have $(1.4)_3$. Finally, by Remark 1.2(ii) we have $(\mathbf{E}^{\mathrm f}, \mathsf{C}^{-1}\mathbf{E}^{\mathrm e})_E = (\mathbf{E}-\mathbf{E}^{\mathrm e}, \mathsf{C}^{-1}\mathbf{T})_E = 0$ which can be rewritten as $\mathbf{E}^{\mathrm f}\cdot\mathbf{T} = 0$. Thus we have $(1.4)_4$. This proves that (iii) $\Rightarrow$ (i).

Proof of (i) $\Rightarrow$ (ii). Assume that Characterization (i) holds. Then we have $(1.5)_1$. Furthermore, if $\mathbf{T}^* \in \mathrm{Sym}^-$ then $\mathbf{T}^*\cdot\mathbf{E}^{\mathrm f} \leq 0$ since $\mathbf{E}^{\mathrm f} \in \mathrm{Sym}^+$ by $(1.4)_3$. Combining with $(1.4)_4$ we obtain $(1.5)_2$. Thus (i) $\Rightarrow$ (ii).

Proof of (ii) $\Rightarrow$ (iii). Assume that Characterization (ii) holds. Using $(1.4)_{1,2}$ we can rewrite (1.5) as

$$\begin{aligned} \mathbf{E}^{\mathrm e} &\in \mathsf{C}^{-1}\mathrm{Sym}^- \\ (\mathbf{E}^*-\mathbf{E}^{\mathrm e}, \mathbf{E}^{\mathrm f}) \leq 0 \quad &\text{for each } \mathbf{E}^* \in \mathsf{C}^{-1}\mathrm{Sym}^-;\ . \end{aligned}$$

Thus

$$\mathbf{E}^{\mathrm f} \equiv \mathbf{E}-\mathbf{E}^{\mathrm e} \in \mathrm{Norm}(\mathsf{C}^{-1}\mathrm{Sym}^-, \mathbf{E}^{\mathrm e}),$$

and Remark 1.2 asserts (1.6). Thus we have shown that (ii) $\Rightarrow$ (iii).

Summarizing, we have shown that the three characterizations of the triplet $(\mathbf{T}, \mathbf{E}^{\mathrm e}, \mathbf{E}^{\mathrm f})$ are equivalent. Characterization (iii) shows that the triplet exists and is unique, because $\mathbf{E}^{\mathrm e}$, being the projection of $\mathbf{E}$ on $\mathsf{C}^{-1}\mathrm{Sym}^-$ is unique, and the uniqueness of $\mathbf{E}^{\mathrm f}$ and $\mathbf{T}$ follows from $(1.4)_{1,2}$. □

We define the ***elastic stress*** $\hat{\mathbf{T}} : \mathrm{Sym} \to \mathrm{Sym}$ and ***stored energy*** $\hat{w} : \mathrm{Sym} \to \mathrm{R}$ of a masonry material by

$$\hat{\mathbf{T}}(\mathbf{E}) = \mathbf{T}, \quad \hat{w}(\mathbf{E}) = \tfrac{1}{2}\hat{\mathbf{T}}(\mathbf{E})\cdot\mathbf{E} = \tfrac{1}{2}|\mathsf{P}\mathbf{E}|_E^2 \tag{1.7}$$

for any $\mathbf{E} \in \mathrm{Sym}$ where $(\mathbf{T}, \mathbf{E}^{\mathrm e}, \mathbf{E}^{\mathrm f})$ is the triplet associated with $\mathbf{E}$ as in Proposition 1.3 and where $\mathsf{P} : \mathrm{Sym} \to \mathsf{C}^{-1}\mathrm{Sym}^-$ denotes the projection from Sym onto $\mathsf{C}^{-1}\mathrm{Sym}^-$ with respect to the energetic scalar product $(\cdot,\cdot)_E$. The tensors $\mathbf{E}^{\mathrm e}$ and $\mathbf{E}^{\mathrm f}$ are called the ***elastic*** and ***fracture*** parts of the deformation $\mathbf{E}$.

The no–tension materials have been introduced in the eighties [9, 5, 14, 7, 4]. The explicit form of the response function $\hat{\mathbf{T}}$ and its further analysis have been given in the case of C isotropic in [5, 14] in dimension 2 and in [16–17] in dimension 3; see also [18]. We also note for the reader's curiosity that the membranes with continuously distributed wrinkles differ from the no–tension materials by the exchange of the roles of the cones Sym^- and Sym^+ [12, 6].

If $F : V \to \bar{\mathrm{R}} := \mathrm{R} \cup \{\infty, -\infty\}$ is a function on an inner product space V then $F^* : V \to \bar{\mathrm{R}}$ is the ***convex conjugate function*** defined by

$$F^*(y) = \sup \{(x, y) - F(x) : x \in V\},$$

$y \in V$, [10; Part One].

1.4 Proposition. *The map* $\hat{\mathbf{T}}$ *is monotone and Lipschitz continuous and the function* $\hat{w}$ *is continuously differentiable, convex and* $D\hat{w} = \hat{\mathbf{T}}$; *in fact, we have the following inequalities:*

$$(\hat{\mathbf{T}}(\mathbf{F}) - \hat{\mathbf{T}}(\mathbf{E})) \cdot (\mathbf{F} - \mathbf{E}) \geq k|\hat{\mathbf{T}}(\mathbf{F}) - \hat{\mathbf{T}}(\mathbf{E})|^2, \tag{1.8}$$

$$|\hat{\mathbf{T}}(\mathbf{F}) - \hat{\mathbf{T}}(\mathbf{E})| \leq k^{-1}|\mathbf{F} - \mathbf{E}|, \tag{1.9}$$

$$\hat{w}(\mathbf{F}) \geq \hat{w}(\mathbf{E}) + \hat{\mathbf{T}}(\mathbf{E}) \cdot (\mathbf{F} - \mathbf{E}) + \tfrac{1}{2}k|\hat{\mathbf{T}}(\mathbf{F}) - \hat{\mathbf{T}}(\mathbf{E})|^2 \tag{1.10}$$

for any $\mathbf{E}, \mathbf{F} \in \mathrm{Sym}$ *where*

$$k := \inf \{\mathbf{A} \cdot \mathsf{C}^{-1}\mathbf{A} : \mathbf{A} \in \mathrm{Sym}, |\mathbf{A}| = 1\} > 0. \tag{1.11}$$

We have

$$\hat{w}^*(\mathbf{T}) = \begin{cases} \tfrac{1}{2}\mathbf{T} \cdot \mathsf{C}^{-1}\mathbf{T} & \text{if} \quad \mathbf{T} \in \mathrm{Sym}^-, \\ \infty & \text{if} \quad \mathbf{T} \in \mathrm{Sym} \sim \mathrm{Sym}^-. \end{cases} \tag{1.12}$$

Cf. Del Piero [7; Proposition 4.4 and Lemma 5.1] for (1.8)–(1.10).

Proof Let $\mathbf{E}, \mathbf{F} \in \mathrm{Sym}$ and put $\mathbf{T} = \hat{\mathbf{T}}(\mathbf{E}), \mathbf{U} = \hat{\mathbf{T}}(\mathbf{F})$. From $(1.5)_2$ we obtain

$$(\mathbf{T} - \mathbf{U}) \cdot (\mathbf{E} - \mathsf{C}^{-1}\mathbf{T}) \geq 0, \quad (\mathbf{U} - \mathbf{T}) \cdot (\mathbf{F} - \mathsf{C}^{-1}\mathbf{U}) \geq 0; \tag{1.13}$$

summing these two inequalities and rearranging we obtain

$$(\mathbf{T} - \mathbf{U}) \cdot \mathsf{C}^{-1}(\mathbf{T} - \mathbf{U}) \leq (\mathbf{T} - \mathbf{U}) \cdot (\mathbf{E} - \mathbf{F});$$

using (1.11) we obtain (1.8). Using the Schwarz inequality on the left hand side of (1.8) we obtain (1.9). To prove (1.10), one finds that

$$\hat{w}(\mathbf{F}) - \hat{w}(\mathbf{E}) - \mathbf{T} \cdot (\mathbf{F} - \mathbf{E}) - \tfrac{1}{2}(\mathbf{T} - \mathbf{U}) \cdot \mathsf{C}^{-1}(\mathbf{T} - \mathbf{U}) = (\mathbf{U} - \mathbf{T}) \cdot (\mathbf{F} - \mathsf{C}^{-1}\mathbf{U});$$

the last expression is nonnegative by $(1.13)_2$ and hence

$$\hat{w}(\mathbf{F}) - \hat{w}(\mathbf{E}) - \mathbf{T} \cdot (\mathbf{F} - \mathbf{E}) - \tfrac{1}{2}(\mathbf{T} - \mathbf{U}) \cdot \mathsf{C}^{-1}(\mathbf{T} - \mathbf{U}) \geq 0;$$

a reference to (1.11) then yields (1.10) and hence also the convexity of $\hat{w}$. To prove that $\hat{w}$ is continuously differentiable and $\hat{\mathbf{T}}$ is its derivative, we note that using (1.10) twice we obtain

$$\hat{\mathbf{T}}(\mathbf{F}) \cdot (\mathbf{F} - \mathbf{E}) \geq \hat{w}(\mathbf{F}) - \hat{w}(\mathbf{E}) \geq \hat{\mathbf{T}}(\mathbf{E}) \cdot (\mathbf{F} - \mathbf{E})$$

for any $\mathbf{E}, \mathbf{F} \in \mathrm{Sym}$; dividing by $|\mathbf{E} - \mathbf{F}|$, letting $\mathbf{F} \to \mathbf{E}$, using $\hat{\mathbf{T}}(\mathbf{F}) \to \hat{\mathbf{T}}(\mathbf{E})$ and invoking the definition of the Fréchet derivative we obtain $D\hat{w}(\mathbf{E}) = \hat{\mathbf{T}}(\mathbf{E})$. To prove (1.12), let $\hat{h} : \mathrm{Sym} \to \mathrm{R} \cup \{\infty\}$ be the function defined by the right hand side of (1.12). We calculate the convex conjugate $\hat{h}^*(\mathbf{E})$ of h at $\mathbf{E} \in \mathrm{Sym}$. We note that if $(\mathbf{T}, \mathbf{E}^{\mathrm{e}}, \mathbf{E}^{\mathrm{f}})$ is the triplet associated with $\mathbf{E}$ as in Proposition 1.3, then algebraic manipulations show that $(1.5)_2$ can be rewritten as

$$\mathbf{T} \cdot \mathbf{E} - \hat{h}(\mathbf{T}) \geq \mathbf{S} \cdot \mathbf{E} - \hat{h}(\mathbf{S}) + \tfrac{1}{2}(\mathbf{T} - \mathbf{S}) \cdot \mathsf{C}^{-1}(\mathbf{T} - \mathbf{S}) \tag{1.14}$$

for every $\mathbf{S} \in \mathrm{Sym}^-$ with the equality if $\mathbf{S} = \mathbf{T}$. Since (1.14) holds also if $\mathbf{S} \notin \mathrm{Sym}^-$ as the right hand side is $-\infty$ in that case, we have

$$\mathbf{T} \cdot \mathbf{E} - \hat{h}(\mathbf{T}) \geq \mathbf{S} \cdot \mathbf{E} - \hat{h}(\mathbf{S})$$

for all $\mathbf{S} \in \mathrm{Sym}$ and thus the definition gives $\hat{h}^*(\mathbf{E}) = \mathbf{T} \cdot \mathbf{E} - \hat{h}(\mathbf{T}) \equiv \hat{w}(\mathbf{E})$. Then $\hat{w}^* = \hat{h}^{**} = \hat{h}$ by [13; Theorem 4.92(iii)] since $\hat{h}$ is lowersemicontinuous, convex and bounded from below by an affine (continuous) function. The proof of (1.12) is complete. □

1.5 Proposition. *The stored energy $\hat{w}$ is decreasing in the sense that*

$$\hat{w}(\mathbf{E} + \mathbf{P}) \leq \hat{w}(\mathbf{E}) \tag{1.15}$$

for any $\mathbf{E} \in \mathrm{Sym}$ and any $\mathbf{P} \in \mathrm{Sym}^+$. Moreover, the function $\hat{w}$ is completely characterized by the following two equivalent requirements:

(i) *$\hat{w}$ is the largest decreasing function such that*

$$\hat{w}(\mathbf{E}) \leq \tfrac{1}{2}\mathbf{E} \cdot \mathbf{C}\mathbf{E} \tag{1.16}$$

for every $\mathbf{E} \in \mathrm{Sym}$;

(ii) *we have*

$$\hat{w}(\mathbf{E}) = \inf\{\tfrac{1}{2}(\mathbf{E} - \mathbf{P}) \cdot \mathbf{C}(\mathbf{E} - \mathbf{P}) : \mathbf{P} \in \mathrm{Sym}^+\}. \tag{1.17}$$

Proof To prove (i), we invoke (1.10) in which we omit the last term on the right hand side to obtain

$$\hat{w}(\mathbf{E} + \mathbf{P}) - \hat{\mathbf{T}}(\mathbf{E} + \mathbf{P}) \cdot \mathbf{P} \leq \hat{w}(\mathbf{E}).$$

Noting that $\hat{\mathbf{T}}(\mathbf{E} + \mathbf{P}) \cdot \mathbf{P} \leq 0$ since $\hat{\mathbf{T}}(\mathbf{E} + \mathbf{P}) \in \mathrm{Sym}^-$ and $\mathbf{P} \in \mathrm{Sym}^+$ completes the proof of (1.15).

Next let $\hat{w} : \mathrm{Sym} \to \mathrm{R}$ be any function and let us prove that if it is given by (1.17) then it is the largest decreasing function satisfying (1.16). Clearly, $\hat{w}$ satisfies (1.16). Let $\mathbf{E} \in \mathrm{Sym}$ and $\mathbf{Q} \in \mathrm{Sym}^+$. Then

$$\hat{w}(\mathbf{E} - \mathbf{Q}) = \inf\{\tfrac{1}{2}(\mathbf{E} - \mathbf{Q} - \mathbf{P}) \cdot \mathbf{C}(\mathbf{E} - \mathbf{Q} - \mathbf{P}) : \mathbf{P} \in \mathrm{Sym}^+\}$$

and noting that $\mathbf{Q} + \mathbf{P} \in \mathrm{Sym}$, we see that $S' := \{\mathbf{Q} + \mathbf{P} : \mathbf{P} \in \mathrm{Sym}^+\} \subset \{\mathbf{P} \in \mathrm{Sym}^+\}$ thus

$$\begin{aligned} \hat{w}(\mathbf{E} - \mathbf{Q}) &= \{\tfrac{1}{2}(\mathbf{E} - \mathbf{P}') \cdot \mathbf{C}(\mathbf{E} - \mathbf{P}') : \mathbf{P} \in S'\} \\ &\geq \{\tfrac{1}{2}(\mathbf{E} - \mathbf{P}) \cdot \mathbf{C}(\mathbf{E} - \mathbf{P}) : \mathbf{P} \in \mathrm{Sym}^+\} \\ &= \hat{w}(\mathbf{E}). \end{aligned}$$

Thus $\hat{w}$ is decreasing. Next assume that $\hat{w}'$ is a decreasing function satisfying (1.16) and prove that $\hat{w}'(\mathbf{E}) \leq \hat{w}(\mathbf{E})$ for every $\mathbf{E} \in \mathrm{Sym}$. Let $\mathbf{P} \in \mathrm{Sym}^+$. We have

$$\hat{w}'(\mathbf{E}) \leq \hat{w}'(\mathbf{E} - \mathbf{P}) \leq \tfrac{1}{2}(\mathbf{E} - \mathbf{P}) \cdot \mathbf{C}(\mathbf{E} - \mathbf{P}).$$

Taking the infimum over all $\mathbf{P}$ we obtain

$$\hat{w}'(\mathbf{E}) \leq \inf\{\tfrac{1}{2}(\mathbf{E} - \mathbf{P}) \cdot \mathbf{C}(\mathbf{E} - \mathbf{P})\} = \hat{w}(\mathbf{E}).$$

This proves the equivalence of (i) and (ii).

Let now $\hat{w}$ be the stored energy of a no–tension material and prove that it satisfies (i). We have already seen that $\hat{w}$ is decreasing. To prove (1.16), we note that

$$\begin{aligned}\mathbf{E}\cdot\mathsf{C}\mathbf{E} &= \mathbf{E}\cdot\mathsf{C}\mathbf{E}^{\mathrm{e}} + \mathbf{E}\cdot\mathsf{C}\mathbf{E}^{\mathrm{f}}\\ &= \mathbf{E}^{\mathrm{e}}\cdot\mathsf{C}\mathbf{E}^{\mathrm{e}} + (\mathbf{E}^{\mathrm{e}}+\mathbf{E}^{\mathrm{f}})\cdot\mathsf{C}\mathbf{E}^{\mathrm{f}}\\ &= 2\hat{w}(\mathbf{E}) + \mathbf{E}^{\mathrm{e}}\cdot\mathsf{C}\mathbf{E}^{\mathrm{f}} + \mathbf{E}^{\mathrm{f}}\cdot\mathsf{C}\mathbf{E}^{\mathrm{f}}\\ &\ge 2\hat{w}(\mathbf{E}).\end{aligned}$$

Thus $\hat{w}$ satisfies (1.16). Furthermore, we have

$$\hat{w}(\mathbf{E}) = \tfrac{1}{2}(\mathbf{E}-\mathbf{E}^{\mathrm{f}})\cdot\mathsf{C}(\mathbf{E}-\mathbf{E}^{\mathrm{f}})$$

and noting that $\mathbf{E}^{\mathrm{f}} \in \mathrm{Sym}^{+}$ we see that

$$\hat{w}(\mathbf{E}) \ge \inf\{\tfrac{1}{2}(\mathbf{E}-\mathbf{P})\cdot\mathsf{C}(\mathbf{E}-\mathbf{P}) : \mathbf{P}\in\mathrm{Sym}^{+}\}.$$

Thus $\hat{w}$ is a decreasing function satisfying (1.16) that is larger than the function defined by the right hand side of (1.17). However, since this right hand side defines the largest function with this property, we have the equality (1.17). □

2 Vector valued measures

In this section we introduce the main object which will represent stresses in masonry bodies and loads applied to the bodies. These two classes of objects will be represented by tensor valued measures and vector valued measures, respectively. Among measures there are ordinary functions (up to an identification), i.e., stresses and loads in the usual sense, but most importantly, measures can represent stresses or loads concentrated on lower–dimensional objects–surfaces or curves. The main goal in this section is to introduce the terminology and notation for measures with values in a finite dimensional vector space. We refer to [3; Chapter 1] for further details.

2.1 Definition. Let V be a finite-dimensional vector space. By a V ***valued measure*** in R^n we mean a map m from a system of all Borel sets in R^n to V which is countably additive in the sense that if $B_1, B_2, \ldots$ is a disjoint family of Borel sets in R^n then

$$\mathsf{m}\Big(\bigcup_{i=1}^{\infty} B_i\Big) = \sum_{i=1}^{\infty} \mathsf{m}(B_i).$$

Below we need the choices $V = \mathrm{Sym}$ and $V = \mathrm{R}^n$. We call the Sym valued measures tensor valued measures; this particular case is used to model the stress fields over the body. We call the R^n valued measures vector valued measures. These are used to model the loads applied to the body.

We say that a function ϕ defined on the system of all Borel sets in R^n is a ***nonnegative measure*** if it takes the values from the set $[0,\infty]$ of nonnegative numbers or ∞ which is countably additive in the sense that if $B_1, B_2, \ldots$ is a disjoint family of Borel sets in R^n then

$$\phi\Big(\bigcup_{i=1}^{\infty} B_i\Big) = \sum_{i=1}^{\infty} \phi(B_i)$$

and

$$\phi(\emptyset) = 0.$$

If Ω is a Borel subset of $\mathbb{R}^n$ and m a V valued measure or a nonnegative measure, we say that m is supported by Ω if $\mathsf{m}(A) = \mathbf{0}$ for any Borel set A such that $A \cap \Omega = \emptyset$. We denote by $\mathscr{M}(\Omega, V)$ the set of all V valued measures supported by Ω.

If $\mathsf{m} \in \mathscr{M}(\Omega, V)$ and if $\alpha : \Omega \to V$ is a bounded Borel function then we have a well defined integral

$$\int_\Omega \alpha \cdot d\mathsf{m},$$

which is a real number.

We denote by $\mathscr{L}^n$ the Lebesgue measure in $\mathbb{R}^n$ [3; Definition 1.52] and if k is an integer, $0 \le k \le n$, we denote by $\mathscr{H}^k$ the k-dimensional Hausdorff measure ("k dimensional area") in $\mathbb{R}^n$ [3; Section 2.8]. If ϕ is a nonnegative measure or a V valued measure, we denote by $\phi \llcorner A$ the ***restriction*** of ϕ to a Borel set $A \subset \mathbb{R}^n$ defined by

$$\phi \llcorner A(B) = \phi(A \cap B)$$

for any Borel subset B of $\mathbb{R}^n$. Thus if $\mathscr{N}$ is an $n-1$ dimensional surface in $\mathbb{R}^n$ then $\mathscr{H}^{n-1} \llcorner \mathscr{N}$ is the area measure on $\mathscr{N}$.

If ϕ is a nonnegative measure, we denote by $f\phi$ the ***product of the measure ϕ*** by a ϕ integrable V valued function f on $\mathbb{R}^n$; one has

$$(f\phi)(A) = \int_A f \, d\phi$$

for any Borel subset A of $\mathbb{R}^n$.

The operations of restriction and multiplication of measures are employed to construct tensor valued measures concentrated on surfaces as follows:

2.2 Examples. Consider a body $\Omega \subset \mathbb{R}^n$ and an $n-1$ dimensional surface $\mathscr{N} \subset \Omega$ and let $\mathbf{E} : \Omega \to \mathbb{R}^n$ be a bounded continuous function, interpreted as a field of strain over Ω.

(i) The measure $\mathscr{H}^{n-1} \llcorner \mathscr{N}$ is supported by $\mathscr{N}$ and thus if $\mathbf{T}_s : \mathscr{N} \to \mathrm{Sym}$ is an $\mathscr{H}^{n-1}$ integrable tensor field on $\mathscr{N}$, then the measure

$$\boldsymbol{\mathsf{T}}_s := \mathbf{T}_s \mathscr{H}^{n-1} \llcorner \mathscr{N}$$

is a tensor valued measure in $\mathscr{M}(\Omega, \mathrm{Sym})$ concentrated on $\mathscr{N}$. One has

$$\int_\Omega \mathbf{E} \cdot d\boldsymbol{\mathsf{T}}_s = \int_{\mathscr{N}} \mathbf{E} \cdot \mathbf{T}_s \, d\mathscr{H}^{n-1}.$$

(ii) If $\mathbf{T}_r : \Omega \to \mathrm{Sym}$ is an $\mathscr{L}^n$ integrable tensor field, then the tensor valued measure

$$\boldsymbol{\mathsf{T}}_r := \mathbf{T}_r \mathscr{L}^n \llcorner \Omega$$

belongs to $\mathscr{M}(\Omega, \mathrm{Sym})$ and faithfully represents $\mathbf{T}_r$; the measure is distributed over Ω. One has

$$\int_\Omega \mathbf{E} \cdot d\boldsymbol{\mathsf{T}}_r = \int_\Omega \mathbf{E} \cdot \mathbf{T}_r \, d\mathscr{L}^n.$$

The measures of the type $\boldsymbol{\mathsf{T}}_s$ and $\boldsymbol{\mathsf{T}}_r$ and their combinations $\boldsymbol{\mathsf{T}} = \boldsymbol{\mathsf{T}}_r + \boldsymbol{\mathsf{T}}_s$ will be employed in Sections 6–7 where we deal with weakly compatible loads. Vector valued measures will be employed in the following section to define the loads of the body.

The ***polar decomposition of measures*** (cf. [28; Theorem 6.12]) says that if $\mathsf{m} \in \mathscr{M}(\Omega, V)$, there exists a pair $(r, |\mathsf{m}|)$ consisting of a Borel function $r : \Omega \to V$ and of a nonnegative measure $|\mathsf{m}|$ on Ω such that

$$\mathsf{m} = r|\mathsf{m}|$$

and

$$|r(\mathbf{x})| = 1 \quad \text{for } |\mathsf{m}| \text{ almost every } \mathbf{x} \in \Omega.$$

The measure $|\mathsf{m}|$ is unique and the function r is unique up to a change on a $|\mathsf{m}|$ null set. The measure $|\mathsf{m}|$ is called the total variation measure of m, and r the amplitude. We denote by $\mathrm{M}(\mathsf{m})$ the ***mass*** of m, defined by $\mathrm{M}(\mathsf{m}) = |\mathsf{m}|(\mathrm{R}^n)$.

If Ω is an open subset of R^n, we denote by $C_0(\Omega, V)$ the space of all continuous V valued functions on R^n with compact support that is contained in Ω, and denote by $|\cdot|_{C_0}$ the maximum norm on $C_0(\mathrm{R}^n, V)$.

3 Loads

We consider a continuous body represented by a Lipschitz domain [1] $\Omega \subset \mathrm{R}^n$ and assume that $\mathscr{D}, \mathscr{S}$ are two disjoint subsets of $\partial\Omega$ such that $\mathscr{D} \cup \mathscr{S} = \partial\Omega$, to be identified below as the set of prescribed boundary displacement and prescribed boundary force. We assume that $\mathscr{D}$ is a closed set.

We put

$$V_0 = \{\mathbf{v} \in C^1(\mathrm{cl}\,\Omega, \mathrm{R}^n) : \mathbf{v} = \mathbf{0} \text{ on } \mathscr{D}\}$$

and

$$V = \{\mathbf{v} \in W^{1,2}(\Omega, \mathrm{R}^n) : \mathbf{v} = \mathbf{0} \text{ almost everywhere on } \mathscr{D}\};$$

here $C^1(\mathrm{cl}\,\Omega, \mathrm{R}^n)$ is the set of all continuously differentiable mappings $\mathbf{v} : \Omega \to \mathrm{R}^n$ such that $\mathbf{v}$ and its derivative $\nabla\mathbf{v}$ have a continuous extension to the closure $\mathrm{cl}\,\Omega$ of Ω and $W^{1,2}(\Omega, \mathrm{R}^n)$ is the Sobolev space of all R^n valued maps such that $\mathbf{v}$ and the weak gradient $\nabla\mathbf{v}$ of $\mathbf{v}$ are square integrable on Ω, i.e.,

$$\int_\Omega |\mathbf{v}|^2 \, d\mathscr{L}^n < \infty, \quad \int_\Omega |\nabla\mathbf{v}|^2 \, d\mathscr{L}^n < \infty,$$

[1]. We assume that V_0 is a dense subset of V. For any $\mathbf{v} \in V$ we define the infinitesimal strain tensor $\hat{\mathbf{E}}(\mathbf{v})$ of $\mathbf{v}$ by

$$\hat{\mathbf{E}}(\mathbf{v}) = \tfrac{1}{2}(\nabla\mathbf{v} + \nabla\mathbf{v}^{\mathrm{T}}).$$

We assume that the body is subjected to loads which consist of a body force acting in the interior of Ω and of the surface force acting on $\mathscr{S}$. We represent both the body and surface forces as vector valued measures supported by Ω and $\mathscr{S}$, respectively. Thus we assume that we are given $\mathbf{b} \in \mathscr{M}(\Omega, \mathrm{R}^n)$ and $\mathbf{s} \in \mathscr{M}(\mathscr{S}, \mathrm{R}^n)$ with the following meaning. For any Borel subset P of Ω the value $\mathbf{b}(P) \in \mathrm{R}^n$ is the body force acting on P from the exterior and for any Borel $S \subset \mathscr{S}$ the value $\mathbf{s}(S) \in \mathrm{R}^n$ is the force acting on the surface S from the exterior of the body. We call the pair $(\mathbf{s}, \mathbf{b})$ the ***loads*** acting on the body. We note that we define the loads as measures, which allows for the concentration of the body force and more importantly surface tractions.

We define for each $\mathbf{u} \in V$ the ***internal energy*** of the body by

$$\mathsf{E}(\mathbf{u}) = \int_\Omega \hat{w}(\hat{\mathbf{E}}(\mathbf{u}))\, d\mathscr{L}^n$$

and for each $\mathbf{u} \in V_0$ the ***energy of the loads*** by

$$\langle \mathbf{l}, \mathbf{u} \rangle = \int_\Omega \mathbf{u} \cdot d\mathbf{b} + \int_{\mathscr{S}} \mathbf{u} \cdot d\mathbf{s}.$$

The ***total energy*** of the deformation $\mathbf{u} \in V_0$ is defined by

$$\mathsf{F}(\mathbf{u}) = \mathsf{E}(\mathbf{u}) - \langle \mathbf{l}, \mathbf{u} \rangle.$$

An important special case arises when there are square integrable functions $\mathbf{b} \in L^2(\Omega, \mathrm{R}^n)$, $\mathbf{s} \in L^2(\mathscr{S}, \mathrm{R}^n)$ such that

$$\mathbf{b} = \mathbf{b}\mathscr{L}^n \llcorner \Omega, \quad \mathbf{s} = \mathbf{s}\mathscr{H}^{n-1} \llcorner \mathscr{S}. \tag{3.1}$$

Here $L^2(\Omega, \mathrm{R}^n)$ is the set of all $\mathscr{L}^n$ measurable maps $\mathbf{b} : \Omega \to \mathrm{R}^n$ such that

$$\int_\Omega |\mathbf{b}|^2\, d\mathscr{L}^n < \infty$$

and $L^2(\mathscr{S}, \mathrm{R}^n)$ is the set of all $\mathscr{H}^{n-1}$ measurable maps $\mathbf{s} : \mathscr{S} \to \mathrm{R}^n$ such that

$$\int_{\mathscr{S}} |\mathbf{s}|^2\, d\mathscr{H}^{n-1} < \infty.$$

In the case (3.1) one can define the potential energy of the loads $\langle \mathbf{l}, \mathbf{u} \rangle$ for each $\mathbf{u}$ from the larger space V by

$$\langle \mathbf{l}, \mathbf{u} \rangle = \int_\Omega \mathbf{u} \cdot \mathbf{b}\, d\mathscr{L}^n + \int_{\mathscr{S}} \mathbf{u} \cdot \mathbf{s}\, d\mathscr{H}^{n-1}$$

$\mathbf{u} \in V$.

Given the loads $(\mathbf{s}, \mathbf{b})$ and $\mathbf{u} \in V$, we say that $\mathbf{u}$ is an ***equilibrium state*** of Ω under the given loads if

$$\int_\Omega \hat{\mathbf{T}}(\hat{\mathbf{E}}(\mathbf{u})) \cdot \hat{\mathbf{E}}(\mathbf{v})\, d\mathscr{L}^n = \langle \mathbf{l}, \mathbf{v} \rangle \tag{3.2}$$

for each $\mathbf{v} \in V_0$.

We note that if the loads are of the special form (3.1) where $\mathbf{b} \in L^2(\Omega, \mathrm{R}^n)$, $\mathbf{s} \in L^2(\mathscr{S}, \mathrm{R}^n)$ and if $\hat{\mathbf{T}}(\hat{\mathbf{E}}(\mathbf{u})) \in C^1(\mathrm{cl}\,\Omega, \mathrm{Sym})$ then the variational equation (3.2) is equivalent to the strong form

$$\operatorname{div} \mathbf{T} + \mathbf{b} = \mathbf{0} \ \text{ in } \ \Omega \qquad \text{and} \qquad \mathbf{T}\mathbf{n} = \mathbf{s} \ \text{ on } \ \mathscr{S}$$

where $\mathbf{n}$ is the outer normal to $\partial\Omega$. We note that in general the existence of the equilibrium state is not guaranteed. The existence theory of equilibrium state requires the extension of the states to admit fracture. See Section 4, below. On the other hand, the given loads may admit more than one equilibrium state $\mathbf{u}$.

3.1 Remark. *Assume the loads of the special form* (3.1) *where* $\mathbf{b} \in L^2(\Omega, \mathrm{R}^n)$, $\mathbf{s} \in L^2(\mathscr{S}, \mathrm{R}^n)$. *Then* $\mathbf{u} \in V$ *is an equilibrium state under the given loads if and only if* $\mathbf{u}$ *is a minimizer of the total energy under the given loads.*

Proof Let $\mathbf{u}$ be an equilibrium state under the given loads. Let $\mathbf{v} \in V_0$. Then

$$\begin{aligned}\mathsf{F}(\mathbf{u}+\mathbf{v}) &= \int_\Omega \hat{w}(\hat{\mathbf{E}}(\mathbf{u})+\hat{\mathbf{E}}(\mathbf{v}))\,d\mathscr{L}^n - \langle \mathbf{l},\mathbf{u}\rangle - \langle \mathbf{l},\mathbf{v}\rangle\\ &\geq \int_\Omega [\hat{w}(\hat{\mathbf{E}}(\mathbf{u})) + \hat{\mathbf{T}}(\hat{\mathbf{E}}(\mathbf{u})\cdot\hat{\mathbf{E}}(\mathbf{v})]\,d\mathscr{L}^n - \langle \mathbf{l},\mathbf{u}\rangle - \langle \mathbf{l},\mathbf{v}\rangle\\ &= \mathsf{F}(\mathbf{u}) + [\int_\Omega \hat{\mathbf{T}}(\hat{\mathbf{E}}(\mathbf{u})\cdot\hat{\mathbf{E}}(\mathbf{v})\,d\mathscr{L}^n - \langle \mathbf{l},\mathbf{v}\rangle]\end{aligned}$$

by (1.10). Since $\mathbf{u}$ is an equilibrium state, the square bracket on the last line vanishes and we obtain

$$\mathsf{F}(\mathbf{u}+\mathbf{v}) \geq \mathsf{F}(\mathbf{u}) \tag{3.3}$$

for each $\mathbf{v}\in V_0$. Since V_0 is dense in V and the energy a continuous functional, we have (3.3) for each $\mathbf{v}\in V$. Thus $\mathbf{u}$ is a point of minimum energy.

Conversely, assume that $\mathbf{u}$ is a point of minimum energy. Let $\mathbf{v}\in V_0$ and $t\in \mathrm{R}$. We have

$$\mathsf{F}(\mathbf{u}+t\mathbf{v}) \geq \mathsf{F}(\mathbf{u})$$

for all $t\in\mathrm{R}$ with the equality sign for $t=0$ and thus the derivative of the function $t\mapsto \mathsf{F}(\mathbf{u}+t\mathbf{v})$ at $t=0$ vanishes. One has

$$\mathsf{F}(\mathbf{u}+t\mathbf{v}) = \int_\Omega \hat{w}(\hat{\mathbf{E}}(\mathbf{u})+t\hat{\mathbf{E}}(\mathbf{v}))\,d\mathscr{L}^n - \langle \mathbf{l},\mathbf{u}\rangle - t\langle \mathbf{l},\mathbf{v}\rangle$$

and differentiating under the integral sign we obtain

$$\frac{d}{dt}\mathsf{F}(\mathbf{u}+t\mathbf{v})\Big|_{t=0} = \int_\Omega \hat{\mathbf{T}}(\hat{\mathbf{E}}(\mathbf{u}))\cdot\hat{\mathbf{E}}(\mathbf{v}))\,d\mathscr{L}^n - \langle \mathbf{l},\mathbf{v}\rangle = 0.$$

Hence $\mathbf{u}$ is an equilibrium state of Ω under the given loads. □

Let $(\mathbf{s},\mathbf{b})$ be given loads of Ω and let $\mathbf{T}\in L^2(\Omega,\mathrm{Sym})$ where $L^2(\Omega,\mathrm{Sym})$ is the set of all $\mathscr{L}^n$ measurable maps $\mathbf{T}:\Omega\to\mathrm{Sym}$ such that $\int_\Omega |\mathbf{T}|^2\,d\mathscr{L}^n<\infty$. We say that $\mathbf{T}$ ***equilibrates the loads*** $(\mathbf{s},\mathbf{b})$ if

$$\int_\Omega \mathbf{T}\cdot\hat{\mathbf{E}}(\mathbf{v})\,d\mathscr{L}^n = \langle \mathbf{l},\mathbf{v}\rangle$$

for all $\mathbf{v}\in V_0$. We say that $\mathbf{T}$ is ***admissible*** if $\mathbf{T}(\mathbf{x})\leq \mathbf{0}$ for $\mathscr{L}^n$ almost every $\mathbf{x}\in\Omega$. We say that the lods $(\mathbf{s},\mathbf{b})$ are ***strongly compatible*** if there exists an admissible stress field equilibrating the loads. Using this terminology we can say that if $\mathbf{u}$ is an equilibrium state of Ω under the given loads then the stress field corresponding to $\mathbf{u}$ is admissible and equilibrates the loads. Thus the loads must be strongly compatible for an equilibrium state to exist.

3.2 Theorem. *Let* $\mathbf{u}\in V_0$ *be an equilibrium state under the loads* $(\mathbf{s},\mathbf{b})$. *Then the stress field* $\mathbf{S}:=\hat{\mathbf{T}}(\hat{\mathbf{E}}(\mathbf{u}))$ *is a minimum point of the complementary energy functional*

$$\mathsf{G}(\mathbf{T}) = \tfrac{1}{2}\int_\Omega \mathbf{T}\cdot\mathsf{C}^{-1}\mathbf{T}\,d\mathscr{L}^n$$

among all admissible stress fields $\mathbf{T}$ *equilibrating the loads.*

We call G the ***complementary energy.***

Proof Let $\mathbf{T}$ be an admissible stress field equilibrating the loads and let $\mathbf{E}^{\mathrm{e}}$ and $\mathbf{E}^{\mathrm{f}}$ be the elastic and fracture parts of the strain corresponding to $\mathbf{u}$. Using the convexity of the function $\mathbf{U}\mapsto\frac{1}{2}(\mathsf{C}^{-1}\mathbf{U}\cdot\mathbf{U})$ we find

$$\begin{aligned}\mathsf{G}(\mathbf{T})-\mathsf{G}(\mathbf{S}) &\geq \int_\Omega \mathsf{C}^{-1}\mathbf{S}\cdot(\mathbf{T}-\mathbf{S})\,d\mathscr{L}^n\\ &= \int_\Omega \mathbf{E}^{\mathrm{e}}\cdot(\mathbf{T}-\mathbf{S})\,d\mathscr{L}^n\\ &= \int_\Omega \mathbf{E}^{\mathrm{e}}\cdot\mathbf{T}\,d\mathscr{L}^n-\langle\mathbf{l},\mathbf{u}\rangle\\ &= \Big[\int_\Omega \hat{\mathbf{E}}(\mathbf{u})\cdot\mathbf{T}\,d\mathscr{L}^n-\langle\mathbf{l},\mathbf{u}\rangle\Big]-\int_\Omega \mathbf{E}^{\mathrm{f}}\cdot\mathbf{T}\,d\mathscr{L}^n\\ &= -\int_\Omega \mathbf{E}^{\mathrm{f}}\cdot\mathbf{T}\,d\mathscr{L}^n \geq 0\end{aligned}$$

since $\mathbf{u}$ is an equilibrium state and because the square bracket vanishes as $\mathbf{T}$ is a stress field equilibrating the loads. □

As a corollary to Theorem 3.2 we have that while the displacement corresponding to equilibrium may be nonunique, the equilibrium stress is unique as the complementary energy has a unique minimum point. We note also that the complementary energy may admit a minimum among the admissible stress fields equilibrating the given loads, and yet the equilibrium state need not exist.

Let $(\mathbf{s},\mathbf{b})$ be the loads of Ω. We put

$$I_0=\inf\{\mathsf{F}(\mathbf{u}):\mathbf{u}\in V_0\}.$$

In general,

$$-\infty\leq I_0<\infty$$

and we have $I_0>-\infty$ if and only if the total energy F is bounded from below. Let $H:L^2(\Omega,\mathrm{Sym})\to\mathrm{R}$ be defined by

$$H(\mathbf{A})=\int_\Omega \hat{w}(\mathbf{A})\,d\mathscr{L}^n$$

for each $\mathbf{A}\in L^2(\Omega,\mathrm{Sym})$. Let

$$H^*(\mathbf{T})=\sup\{\mathbf{A}\cdot\mathbf{T}-H(\mathbf{A}):\mathbf{A}\in L^2(\Omega,\mathrm{Sym})\}$$

for any $\mathbf{T}\in L^2(\Omega,\mathrm{Sym})$. Then [10]

$$H^*(\mathbf{T})=\int_\Omega \hat{w}^*(\mathbf{T})\,d\mathscr{L}^n$$

and hence

$$H^*(\mathbf{T})=\begin{cases}\frac12\int_\Omega \mathbf{T}\cdot\mathsf{C}^{-1}\mathbf{T}\,d\mathscr{L}^n & \text{if}\quad \mathbf{T}\quad\text{is negative semidefinite},\\ \infty & \text{otherwise},\end{cases}$$

$\mathbf{T}\in L^2(\Omega,\mathrm{Sym})$, by (1.12).

3.3 Proposition. *Consider the general loads* $(\mathbf{s},\mathbf{b})$. *Then the loads are strongly compatible if and only if*

$$I_0>-\infty.$$

Proof Let $Y:=L^2(\Omega,\mathrm{Sym})$ and $X_0:=\{\hat{\mathbf{E}}(\mathbf{v}):\mathbf{v}\in V_0\}$ so that $X_0\subset Y$. Assume that $c=I_0\in\mathrm{R}$. Prove a preliminary result: if $\mathbf{v}_1\ \mathbf{v}_2\in V_0$ satisfy $\hat{\mathbf{E}}(\mathbf{v}_1)=\hat{\mathbf{E}}(\mathbf{v}_2)$ then $\langle\mathbf{l},\mathbf{v}_1\rangle=\langle\mathbf{l},\mathbf{v}_2\rangle$. Indeed, let $t\in\mathrm{R}$ and put $\mathbf{v}=(1-t)\mathbf{v}_1+t\mathbf{v}_2$. Then $\mathbf{v}\in V_0$, $\hat{\mathbf{E}}(\mathbf{v})=\mathbf{E}:=\hat{\mathbf{E}}(\mathbf{v}_1)=\hat{\mathbf{E}}(\mathbf{v}_2)$ and thus

$$\mathsf{F}(\mathbf{v}) = \int_\Omega \hat{w}(\mathbf{E})\, d\mathscr{L}^n - (1-t)\langle \mathbf{l}, \mathbf{v}_1 \rangle - t\langle \mathbf{l}, \mathbf{v}_2 \rangle \geq c.$$

Assuming $t > 0$, dividing the inequality by t and letting $t \to \infty$ we obtain $\langle \mathbf{l}, \mathbf{v}_1 \rangle - \langle \mathbf{l}, \mathbf{v}_2 \rangle \geq 0$; similarly, assuming $t < 0$, dividing by t and letting $t \to -\infty$ we obtain $\langle \mathbf{l}, \mathbf{v}_1 \rangle - \langle \mathbf{l}, \mathbf{v}_2 \rangle \leq 0$ and thus $\langle \mathbf{l}, \mathbf{v}_1 \rangle = \langle \mathbf{l}, \mathbf{v}_2 \rangle$ which completes the proof of the preliminary result. Let $L_0 : X_0 \to \mathrm{R}$ be defined by

$$L_0(\hat{\mathbf{E}}(\mathbf{v})) = \langle \mathbf{l}, \mathbf{v} \rangle \tag{3.4}$$

for each $\mathbf{v} \in V_0$, where we use the preliminary result to see that the right hand side of (3.4) depends only on $\hat{\mathbf{E}}(\mathbf{v})$. Then

$$L_0(\mathbf{A}) \leq H(\mathbf{A}) - c \quad \text{for all} \quad \mathbf{A} \in X_0.$$

The convexity of $\hat{w}$ implies the convexity of H and hence by the version of the Hahn Banach theorem [13; Theorem A.35] there exists a linear extension $L : Y \to \mathrm{R}$ of L_0 such that

$$L(\mathbf{A}) \leq H(\mathbf{A}) - c \quad \text{for all} \quad \mathbf{A} \in Y. \tag{3.5}$$

The continuity of H on Y, which follows from the properties of $\hat{w}$, implies that H is bounded on the unit ball in Y and hence L is bounded on the unit ball and hence continuous. Thus it can be represented by an element $\mathbf{T} \in Y$ as a scalar product in Y, i.e., there exists a $\mathbf{T} \in Y$ such that

$$L(\mathbf{A}) = \int_\Omega \mathbf{T} \cdot \mathbf{A}\, d\mathscr{L}^n$$

for each $\mathbf{A} \in Y$. Taking in particular $\mathbf{A} \in L^2(\Omega, \mathrm{Sym}^+)$ and noting that then $\hat{w}(\mathbf{A}) = 0$, we find from (3.5) that

$$L(\mathbf{A}) \leq -c.$$

Repalcing $\mathbf{A}$ by $t\mathbf{A}$ where $t > 0$, dividing by t and letting $t \to \infty$ we obtain $L(\mathbf{A}) \leq 0$ which implies that $\mathbf{T} \leq \mathbf{0}$ for almost all points of Ω. Further, relation (3.4) gives

$$\int_\Omega \mathbf{T} \cdot \hat{\mathbf{E}}(\mathbf{v})\, d\mathscr{L}^n = \langle \mathbf{l}, \mathbf{v} \rangle$$

for each $\mathbf{v} \in V_0$ and thus $\mathbf{T}$ strongly equilibrates the loads (s, b).

To prove the converse part of the statement, we let $\mathbf{T}$ be a stress field strongly equilibrating the loads (s, b). Since $\mathbf{T}$ is negative semidefinite and square integrable, we have

$$H^*(\mathbf{T}) = \tfrac{1}{2} \int_\Omega \mathbf{T} \cdot \mathsf{C}^{-1} \mathbf{T}\, d\mathscr{L}^n < \infty$$

and hence

$$\infty > H^*(\mathbf{T}) := \sup \{ \int_\Omega \mathbf{T} \cdot \mathbf{A}\, d\mathscr{L}^n - H(\mathbf{A}) : \mathbf{A} \in Y \}$$

from which

$$H(\mathbf{A}) - \int_\Omega \mathbf{T} \cdot \mathbf{A}\, d\mathscr{L}^n \geq -H^*(\mathbf{T}) \qquad \text{for all} \quad \mathbf{A} \in Y;$$

taking $\mathbf{A} = \hat{\mathbf{E}}(\mathbf{v})$ where $\mathbf{v} \in V_0$, this is rewritten as

$$\mathsf{F}(\mathbf{v}) \geq c$$

for all $\mathbf{v} \in V_0$ [with $c = -H^*(\mathbf{T})$]. □

4 The existence of equilibrium states

In this section we outline the theory of existence of equilibrium for masonry materials. This theory is due to G. Anzellotti [5] and M. Giaquinta & E. Giusti [14]. The presentation below follows [5]. The reader is referred to the cited paper for further details and proofs. The theories of Anzellotti and Giaquinta & Giusti are based on the existence of uniformly negative definite stress field, see Definition 4.10, below. A theory based on an alternative assumption of the strong absence of collapse mechanism is presented in [30].

4.1 Definition. Let $\Omega \subset \mathrm{R}^n$ be an open set. We denote by $BD(\Omega)$ the set of all $\mathbf{u} \in L^1(\Omega, \mathrm{R}^n)$ such that there exist a measure $\hat{\mathbf{E}}(\mathbf{u}) \in \mathscr{M}(\Omega, \mathrm{Sym})$ such that

$$\int_\Omega \mathbf{u} \cdot \operatorname{div} \mathbf{T}\, d\mathscr{L}^n = -\int_\Omega \mathbf{T} \cdot d\hat{\mathbf{E}}(\mathbf{u}) \tag{4.1}$$

for all $\mathbf{T} \in C_0^\infty(\Omega, \mathrm{Sym})$. Here $C_0^\infty(\Omega, \mathrm{Sym})$ is the set of all $\mathbf{T} : \mathrm{R}^n \to \mathrm{Sym}$ such that the support

$$\operatorname{spt} \mathbf{T} = \operatorname{cl}\{\mathbf{x} \in \mathrm{R}^n : \mathbf{T}(\mathbf{x}) \neq \mathbf{0}\}$$

is contained in Ω and is compact. We denote by $|\mathbf{u}|_{BD(\Omega)}$ the BD norm defined by

$$|\mathbf{u}|_{BD(\Omega)} = |\mathbf{u}|_{L^1(\Omega, \mathrm{R}^n)} + \mathrm{M}(\hat{\mathbf{E}}(\mathbf{u}))$$

where we recall that $\mathrm{M}(\hat{\mathbf{E}}(\mathbf{u}))$ is the mass of $\hat{\mathbf{E}}(\mathbf{u})$ defined in Section 2. We call the elements of $BD(\Omega)$ ***displacements of bounded deformation.***

In other words, the strain tensor, being generally a distribution defined by

$$\hat{\mathbf{E}}(\mathbf{u}) = \tfrac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^{\mathrm{T}}),$$

is a Sym valued measure if $\mathbf{u} \in BD(\Omega)$. If $\varphi \in C_0^\infty(\Omega, \mathrm{R})$ and if we apply (4.1) with $T_{ij} = T_{ji} = \frac{1}{2}\varphi$ for some pair (i,j) of indices and $T_{kl} = 0$ otherwise we obtain

$$\tfrac{1}{2}\int_\Omega (\varphi_{,i} u_j + \varphi_{,j} u_i)\, d\mathscr{L}^n = -\int_\Omega \varphi\, d\hat{E}_{ij}(\mathbf{u}),$$

which is the ‘index form’ of the definition of the space $BD(\Omega)$.

The space $BD(\Omega)$ endowed with the norm $|\cdot|_{BD(\Omega)}$ is a Banach space.

We refer to [32, 31] and [2] and the references therein for further details and proofs of displacements with bounded deformation.

4.2 Example (Fracture in $BD(\Omega)$). Let $\Sigma \subset \Omega$ be a surface of dimension $n-1$ which divides Ω into two open sets Ω_1 and Ω_2. Let $\mathbf{u}$ be a function such that its restriction $\mathbf{u}_k$, $k = 1, 2$, onto Ω_k belongs to $C^1(\operatorname{cl}\Omega_k, \mathrm{R}^n)$. Then $\mathbf{u} \in BD(\Omega)$ and

$$\hat{\mathbf{E}}(\mathbf{u}) = \{\hat{\mathbf{E}}(\mathbf{u})\}\mathscr{L}^n \llcorner \Omega + [\mathbf{u}] \odot \mathbf{m}\mathscr{H}^{n-1} \llcorner \Sigma \tag{4.2}$$

where $\{\hat{\mathbf{E}}(\mathbf{u})\}$ is the function equal to $\hat{\mathbf{E}}(\mathbf{u}_k)$ on Ω_k $(k = 1, 2)$, $[\mathbf{u}](\mathbf{x}) = \mathbf{u}_2(\mathbf{x}) - \mathbf{u}_1(\mathbf{x})$ $(\mathbf{x} \in \Sigma)$, $\mathbf{m}$ is the normal to Σ pointing from Ω_1 to Ω_2, and

$$\mathbf{a} \odot \mathbf{b} = \tfrac{1}{2}(\mathbf{a} \otimes \mathbf{b} + \mathbf{b} \otimes \mathbf{a})$$

for every $\mathbf{a}, \mathbf{b} \in \mathrm{R}^n$.

Proof Let $\mathbf{T} \in C_0^\infty(\Omega, \mathrm{R}^n)$. Applying the divergence theorem to Ω_1 and Ω_2, we obtain

$$\begin{aligned}\int_{\Omega} \mathbf{u}\cdot \operatorname{div}\mathbf{T}\, d\mathscr{L}^n &= \int_{\Omega_1} \mathbf{u}_1\cdot \operatorname{div}\mathbf{T}\, d\mathscr{L}^n + \int_{\Omega_2} \mathbf{u}_2\cdot \operatorname{div}\mathbf{T}\, d\mathscr{L}^n\\ &= -\int_{\Omega_1} \hat{\mathbf{E}}(\mathbf{u}_1)\cdot \mathbf{T}\, d\mathscr{L}^n + \int_{\Sigma} \mathbf{Tm}\cdot \mathbf{u}_1\, d\mathscr{H}^{n-1}\\ &\quad - \int_{\Omega_2} \hat{\mathbf{E}}(\mathbf{u}_2)\cdot \mathbf{T}\, d\mathscr{L}^n - \int_{\Sigma} \mathbf{Tm}\cdot \mathbf{u}_2\, d\mathscr{H}^{n-1}\\ &= -\int_{\Omega} \{\hat{\mathbf{E}}(\mathbf{u})\}\cdot \mathbf{T}\, d\mathscr{L}^n - \int_{\Sigma} \mathbf{T}\cdot ([\mathbf{u}]\odot \mathbf{m})\, d\mathscr{H}^{n-1}. \qquad \square\end{aligned}$$

4.3 Theorem. *Let $\Omega \subset \mathrm{R}^n$ be an open set with Lipschitz boundary. There exists a linear map $\gamma_0 : BD(\Omega) \to L^1(\partial\Omega, \mathrm{R}^n)$ such that we have*

$$\int_{\Omega} \mathbf{u}\cdot \operatorname{div}\mathbf{T}\, d\mathscr{L}^n + \int_{\Omega} \mathbf{T}\cdot d\hat{\mathbf{E}}(\mathbf{u}) = \int_{\partial\Omega} \mathbf{Tn}\cdot \gamma_0(\mathbf{u})\, d\mathscr{H}^{n-1}$$

for each $\mathbf{u} \in BD(\Omega)$ and $\mathbf{T} \in C^1(\mathrm{cl}\,\Omega, \mathrm{Sym})$. One has

$$|\gamma_0(\mathbf{u})|_{L^1(\partial\Omega, \mathrm{R}^n)} \le c|\mathbf{u}|_{BD(\Omega)}$$

for each $\mathbf{u} \in BD(\Omega)$ and some $c \in \mathrm{R}$.

The function $\gamma_0(\mathbf{u})$ represents the boundary values of $\mathbf{u}$. We often simplify the notation and write $\mathbf{u}$ for $\gamma_0(\mathbf{u})$. With this notation we have

$$\int_{\Omega} \mathbf{u}\cdot \operatorname{div}\mathbf{T}\, d\mathscr{L}^n + \int_{\Omega} \mathbf{T}\cdot d\hat{\mathbf{E}}(\mathbf{u}) = \int_{\partial\Omega} \mathbf{Tn}\cdot \mathbf{u}\, d\mathscr{H}^{n-1}.$$

4.4 Theorem. *Let $\Omega \subset \mathrm{R}^n$ be an open set with Lipschitz boundary. If $\mathbf{u} \in BD(\Omega)$ then one has $\mathbf{u} \in L^{n/(n-1)}(\Omega, \mathrm{R}^n)$ and there exists a $c \in \mathrm{R}$ such that*

$$|\mathbf{u}|_{L^{n/(n-1)}(\Omega, \mathrm{R}^n)} \le c|\mathbf{u}|_{BD(\Omega)}$$

for all $\mathbf{u} \in BD(\Omega)$.

4.5 Theorem. *Let $\Omega \subset \mathrm{R}^n$ be an open set with Lipschitz boundary. The operator imbedding $BD(\Omega)$ into $L^p(\Omega, \mathrm{R}^n)$, $1 \le p < n/(n-1)$, is compact; i.e., if $\mathbf{u}_j \in BD(\Omega)$ is a sequence bounded in the $|\cdot|_{BD(\Omega)}$ norm and $1 \le p < n/(n-1)$ then there exists a subsequence of $\mathbf{u}_j$, still denoted by $\mathbf{u}_j$, such that*

$$\mathbf{u}_j \to \mathbf{u} \quad \text{in} \quad L^p(\Omega, \mathrm{R}^n)$$

for some $\mathbf{u} \in BD(\Omega)$.

4.6 Theorem. *Let $\Omega \subset \mathrm{R}^n$ be an open connected set and $\mathbf{u} \in BD(\Omega)$. Then $\hat{\mathbf{E}}(\mathbf{u}) = \mathbf{0}$ if and only if $\mathbf{u}$ is of the form*

$$\mathbf{u}(\mathbf{x}) = \mathbf{Wx} + \mathbf{a} \tag{4.3}$$

for all $\mathbf{x} \in \Omega$ where $\mathbf{a} \in \mathrm{R}^n$ and $\mathbf{W} \in \mathrm{Skw}$.

If $n = 3$ and $\mathbf{b}$ is the polar vector of $\mathbf{W}$, we can write

$$\mathbf{u}(\mathbf{x}) = \mathbf{b} \times \mathbf{x} + \mathbf{a}.$$

We call any $\mathbf{u}$ of the form (4.3) a rigid body displacement and denote by $\mathscr{R}(\Omega)$ the linear space of all rigid body displacements of Ω.

4.7 Theorem. *Let* $\Omega \subset \mathrm{R}^n$ *be an open connected set. There exists a linear map* $\mathbf{u} \mapsto \mathbf{r}(\mathbf{u})$ *from* $BD(\Omega)$ *to* $\mathscr{R}(\Omega)$ *such that* $\mathbf{r}(\mathbf{u}) = \mathbf{u}$ *for* $\mathbf{u} \in \mathscr{R}(\Omega)$ *and*

$$|\mathbf{u} - \mathbf{r}(\mathbf{u})|_{L^{n/(n-1)}(\Omega, \mathrm{R}^n)} \le c\mathrm{M}(\hat{\mathbf{E}}(\mathbf{u}))$$

for all $\mathbf{u} \in BD(\Omega)$ *and some* $c \in \mathrm{R}$.

Recall from Section 1 that $\mathsf{P} : \mathrm{Sym} \to \mathsf{C}^{-1}\mathrm{Sym}^-$ denotes the orthogonal projection from Sym onto $\mathsf{C}^{-1}\mathrm{Sym}^-$ with respect to the energetic scalar product $(\cdot, \cdot)_E$. Endow Sym with the energetic scalar product. Let $\mathbf{u} \in BD(\Omega)$ and write

$$\hat{\mathbf{E}}(\mathbf{u}) = \mathbf{D}|\hat{\mathbf{E}}(\mathbf{u})|$$

for the polar decomposition of the measure $\hat{\mathbf{E}}(\mathbf{u})$, with $|\hat{\mathbf{E}}(\mathbf{u})|$ a nonnegative measure on Ω and $\mathbf{D} : \Omega \to \mathrm{Sym}$ a function satisfying $|\mathbf{D}|_E = 1$ for $|\hat{\mathbf{E}}(\mathbf{u})|$ almost every point of Ω. We denote by $\mathscr{P}\hat{\mathbf{E}}(\mathbf{u})$ the measure defined by

$$\mathscr{P}\hat{\mathbf{E}}(\mathbf{u}) = (\mathsf{P}\mathbf{D})|\hat{\mathbf{E}}(\mathbf{u})|.$$

4.8 Definition. We denote by $\mathscr{U}(\Omega)$ the set of all $\mathbf{u} \in BD(\Omega)$ such that the measure $\mathscr{P}\hat{\mathbf{E}}(\mathbf{u})$ is absolutely continuous with respect to the Lebesgue measure, with the density $\tilde{\mathscr{P}}\hat{\mathbf{E}}(\mathbf{u})$ such that $\tilde{\mathscr{P}}\hat{\mathbf{E}}(\mathbf{u}) \in L^2(\Omega, \mathrm{Sym})$. We call the elements of $\mathscr{U}(\Omega)$ ***admissible displacements.*** We define the internal energy of the admissible displacement $\mathbf{u}$ by

$$\mathsf{E}(\mathbf{u}) = \int_\Omega |\tilde{\mathscr{P}}\hat{\mathbf{E}}(\mathbf{u})|_E^2 \, d\mathscr{L}^n.$$

4.9 Example (Fracture in $\mathscr{U}(\Omega)$). The set $\mathscr{U}(\Omega)$ is the basic set of competitors for the equilibrium problem. Note that $\mathscr{U}(\Omega)$ is not a linear space since if $\mathbf{u} \in \mathscr{U}(\Omega)$ then it may happen that $-\mathbf{u} \notin \mathscr{U}(\Omega)$. As an example, let $\mathbf{u} \in BD(\Omega)$ be as in Example 4.2. Let us show that $\mathbf{u} \in \mathscr{U}(\Omega)$ if and only if there exists a function $\lambda : \Sigma \to [0, \infty)$ such that

$$[\mathbf{u}] = \lambda \mathbf{m} \quad \text{on } \Sigma, \tag{4.4}$$

i.e., the jump in $\mathbf{u}$ across Σ is positively proportional to the normal to Σ. This seems to be in agreement with the observation of fractured masonry structures.

Proof Assume that there exists a point $\mathbf{x}$ on Σ such that $[\mathbf{u}]$ and $\mathbf{m}$ are not positively proportional. Then by the continuity there exists a neighborhood N of $\mathbf{x}$ in Σ such that $[\mathbf{u}]$ and $\mathbf{m}$ are not positively proportional. One easily finds that in this case the tensor $[\mathbf{u}] \odot \mathbf{m}$ has a nonzero negative definite part, which further implies

$$\mathsf{P}([\mathbf{u}] \odot \mathbf{m}) \neq \mathbf{0} \quad \text{on } N,$$

since otherwise $[\mathbf{u}] \odot \mathbf{m}$ would be positive semidefinite. Equation (4.2) then gives

$$\mathscr{P}\hat{\mathbf{E}}(\mathbf{u}) = \mathsf{P}\{\hat{\mathbf{E}}(\mathbf{u})\}\mathscr{L}^n \llcorner \Omega + \mathsf{P}([\mathbf{u}] \odot \mathbf{m})\mathscr{H}^{n-1} \llcorner \Sigma$$

and thus $\mathscr{P}\hat{\mathbf{E}}(\mathbf{u})$ has a nonzero singular part, in contradiction with the definition of $\mathscr{U}(\Omega)$. Therefore we have (4.4) with a nonnegative λ. The converse implication: under (4.4) we have $\mathsf{P}([\mathbf{u}] \odot \mathbf{m}) = \mathbf{0}$ and thus

$$\mathscr{P}\hat{\mathbf{E}}(\mathbf{u}) = \mathsf{P}\{\hat{\mathbf{E}}(\mathbf{u})\}\mathscr{L}^n \llcorner \Omega$$

and hence $\mathscr{P}\hat{\mathbf{E}}(\mathbf{u})$ is absolutely continuous with respect to $\mathscr{L}^n$ with a square integrable density. □

We shall deal with the existence of solution for the Neumann problem. Thus we assume that $\mathscr{D} = \emptyset$. We call a pair $(\mathbf{s}, \mathbf{b})$ loads for the system if $\mathbf{s} \in L^\infty(\partial\Omega, \mathrm{R}^n)$, $\mathbf{b} \in L^n(\Omega, \mathrm{R}^n)$. In view of the fact that for any $\mathbf{u} \in BD(\Omega)$ we have $\gamma_0(\mathbf{u}) \in L^1(\partial\Omega)$, $\mathbf{u} \in L^{n/(n-1)}(\Omega, \mathrm{R}^n)$, by the Hölder inequality we have a well defined energy of the loads

$$\langle \mathbf{l}, \mathbf{u} \rangle = \int_{\partial\Omega} \mathbf{u} \cdot \mathbf{s} \, d\mathscr{H}^{n-1} + \int_\Omega \mathbf{u} \cdot \mathbf{b} \, d\mathscr{L}^n$$

for any $\mathbf{u} \in BD(\Omega)$ and in particular for any $\mathbf{u} \in \mathscr{U}(\Omega)$. We define the total energy of $\mathbf{u} \in \mathscr{U}(\Omega)$ by

$$\mathsf{F}(\mathbf{u}) = \mathsf{E}(\mathbf{u}) - \langle \mathbf{l}, \mathbf{u} \rangle.$$

We say that $\Omega \subset \mathrm{R}^n$ is an ***admissible domain*** if for any $\mathbf{u} \in \mathscr{U}(\Omega)$ there exists a sequence $\mathbf{u}_j \in C^1(\mathrm{cl}\,\Omega, \mathrm{R}^n)$ such that

$$\begin{gathered} \mathbf{u}_j \to \mathbf{u} \quad \text{in} \quad L^{n/(n-1)}(\Omega, \mathrm{R}^n), \\ \mathrm{M}(\hat{\mathbf{E}}(\mathbf{u}_j)) \to \mathrm{M}(\hat{\mathbf{E}}(\mathbf{u})), \\ \tilde{\mathscr{P}}(\mathbf{u}_j) \to \tilde{\mathscr{P}}(\mathbf{u}) \quad \text{in} \quad L^2(\Omega, \mathrm{Sym}). \end{gathered}$$

It turns out that all Lipschitz domains in R^2 are admissible and that for any n all star shaped Lipschitz domains are admissible.

4.10 Definition. We say that a stress field $\mathbf{T} \in L^2(\Omega, \mathrm{Sym})$ is ***safe*** if there exists an $\alpha > 0$ such that

$$-\mathbf{T}(\mathbf{x}) \cdot \mathbf{A} \geq \alpha |\mathbf{A}|$$

for all $\mathbf{A} \in \mathrm{Sym}^+$ and $\mathscr{L}^n$ almost every $\mathbf{x} \in \Omega$.

In other words, $\mathbf{T}$ is uniformly negative definite over Ω, which in particular implies that $\mathbf{s} \cdot \mathbf{n} \leq -\alpha < 0$ on $\partial\Omega$, i.e., the body must be uniformly compressed on the boundary. As before we say that a stress field $\mathbf{T}$ equilibrates the loads $(\mathbf{s}, \mathbf{b})$ if

$$\langle \mathbf{l}, \mathbf{v} \rangle = \int_\Omega \mathbf{T} \cdot \hat{\mathbf{E}}(\mathbf{v}) \, d\mathscr{L}^n$$

for every $\mathbf{v} \in C^1(\mathrm{cl}\,\Omega, \mathrm{R}^n)$.

4.11 Theorem. *Let Ω be an admissible Lipschitz domain in R^n and consider loads $(\mathbf{s}, \mathbf{b})$. If there exists a safe stress field $\mathbf{T}$ equilibrating the loads then the functional F is coercive on $\mathscr{U}(\Omega)$ in the sense that*

$$\mathsf{F}(\mathbf{u}) \geq c_1 \Big\{ \int_\Omega |\tilde{\mathscr{P}}(\hat{\mathbf{E}}(\mathbf{u}))|_E^2 \, d\mathscr{L}^n + \mathrm{M}(\hat{\mathbf{E}}(\mathbf{u})) \Big\} + c_2$$

for some constants c_1, c_2 with $c_1 > 0$ and all $\mathbf{u} \in \mathscr{U}(\Omega)$.

We say that a sequence $\mathbf{u}_j \in \mathscr{U}(\Omega)$ ***converges weakly*** to $\mathbf{u} \in BD(\Omega)$ if we have the relations

$$\left.\begin{gathered} \mathbf{u}_j \to \mathbf{u} \quad \text{in} \quad L^1(\Omega, \mathrm{R}^n), \\ \mathbf{u}_j \rightharpoonup \mathbf{u} \quad \text{in} \quad L^{n/(n-1)}(\Omega, \mathrm{R}^n), \\ \mathrm{M}(\hat{\mathbf{E}}(\mathbf{u}_j)) \leq M, \\ \int_\Omega |\tilde{\mathscr{P}}(\hat{\mathbf{E}}(\mathbf{u}_j))|_E^2 \, d\mathscr{L}^n \leq M \end{gathered}\right\} \tag{4.5}$$

for some $M \in \mathrm{R}$.

4.12 Theorem. *Let Ω be an admissible Lipschitz domain in* R^n *and consider loads* $(\mathbf{s}, \mathbf{b})$. *Assume that there exists a stress field* $\mathbf{S} \in L^2(\Omega, \mathrm{Sym}^-)$ *and a function* $\mathbf{c} \in L^n(\Omega, \mathrm{R}^n)$ *such that* $\mathbf{S}$ *is bounded and* $\mathbf{S}$ *equilibrates the loads* $(\mathbf{s}, \mathbf{c})$. *Then the functional* F *is weakly sequentially lowersemicontinuous in the sense that for any sequence* $\mathbf{u}_j \in \mathscr{U}(\Omega)$ *which converges weakly to* $\mathbf{u} \in \mathscr{U}(\Omega)$, *we have*

$$\liminf_{j\to\infty} \mathsf{F}(\mathbf{u}_j) \ge \mathsf{F}(\mathbf{u}).$$

4.13 Proposition. *The set* $\mathscr{U}(\Omega)$ *is closed under the weak convergence of sequences, i.e., if* $\mathbf{u}_j \in \mathscr{U}(\Omega)$ *converges weakly to* $\mathbf{u} \in BD(\Omega)$ *then* $\mathbf{u} \in \mathscr{U}(\Omega)$.

4.14 Theorem. *Let Ω be an admissible Lipschitz domain in* R^n *and consider loads* $(\mathbf{s}, \mathbf{b})$. *Assume that there exists a safe stress field equilibrating the loads and moreover there exists a stress field* $\mathbf{S}$ *as in Theorem* 4.12. *Then there exists in* $\mathscr{U}(\Omega)$ *a minimizer of* F *on* $\mathscr{U}(\Omega)$.

Proof There exists a sequence $\mathbf{u}_j \in \mathscr{U}(\Omega)$ such that

$$\lim_{j\to\infty} \mathsf{F}(\mathbf{u}_j) = \inf\{\mathsf{F}(\mathbf{u}) : \mathbf{u} \in \mathscr{U}(\Omega)\}.$$

Since the energy functional F is coercive by Theorem 4.11, the boundedness of the sequence $\mathsf{F}(\mathbf{u}_j)$ implies that there exists a $M \in \mathrm{R}$ such that

$$\mathrm{M}(\hat{\mathbf{E}}(\mathbf{u}_j)) \le M,$$

$$\int_\Omega |\tilde{\mathscr{P}}(\hat{\mathbf{E}}(\mathbf{u}_j))|_E^2 \, d\mathscr{L}^n \le M$$

If $\mathbf{u} \mapsto \mathbf{r}(\mathbf{u})$ is the map from Theorem 4.7, we have

$$|\mathbf{u}_j - \mathbf{r}(\mathbf{u}_j)| \le \mathrm{M}(\hat{\mathbf{E}}(\mathbf{u}_j)) \le M$$

and thus the sequence $\mathbf{v}_j := \mathbf{u}_j - \mathbf{r}(\mathbf{u}_j)$ is bounded in $L^{n/(n-1)}(\Omega, \mathrm{R}^n)$ and hence it contains a subsequence, again denoted by $\mathbf{v}_j$, such that

$$\mathbf{v}_j \rightharpoonup \mathbf{v} \quad \text{in} \quad L^{n/(n-1)}(\Omega, \mathrm{R}^n) \tag{4.6}$$

for some $\mathbf{v} \in L^{n/(n-1)}(\Omega, \mathrm{R}^n)$. Since $\hat{\mathbf{E}}(\mathbf{v}_j) = \hat{\mathbf{E}}(\mathbf{u}_j)$, we also have

$$\mathrm{M}(\hat{\mathbf{E}}(\mathbf{v}_j)) \le M, \tag{4.7}$$

$$\int_\Omega |\tilde{\mathscr{P}}(\hat{\mathbf{E}}(\mathbf{v}_j))|_E^2 \, d\mathscr{L}^n \le M. \tag{4.8}$$

Since the sequence $\mathbf{v}_j$ is bounded in $L^{n/(n-1)}(\Omega, \mathrm{R}^n)$, it is also bounded in $L^1(\Omega, \mathrm{R}^n)$. Thus we conclude that $|\mathbf{v}_j|_{BD(\Omega)}$ is bounded. The compactness of the imbedding of $BD(\Omega)$ into $L^1(\Omega, \mathrm{R}^n)$ by Theorem 4.5 implies that we have

$$\mathbf{v}_j \to \mathbf{v} \quad \text{in} \quad L^1(\Omega, \mathrm{R}^n). \tag{4.9}$$

We thus summarize (4.6)–(4.9) by saying that the sequence $\mathbf{v}_j \in \mathscr{U}(\Omega)$ converges weakly to $\mathbf{v} \in BD(\Omega)$. The weak closedness of $\mathscr{U}(\Omega)$ (Proposition 4.13) then says that $\mathbf{v} \in \mathscr{U}(\Omega)$. Moreover, one easily finds that $\mathsf{F}(\mathbf{v}_j) = \mathsf{F}(\mathbf{u}_j)$ and thus

$$\lim_{j\to\infty} \mathsf{F}(\mathbf{v}_j) = \min\{\mathsf{F}(\mathbf{u}) : \mathbf{u} \in \mathscr{U}(\Omega)\}.$$

On the other hand, the functional F is sequentially weakly lowersemicontinuous (Theorem 4.12) and thus

$$\lim_{j\to\infty} \mathsf{F}(\mathbf{v}_j) \ge \mathsf{F}(\mathbf{v}).$$

This gives

$$\mathsf{F}(\mathbf{v}) \le \inf\{\mathsf{F}(\mathbf{u}) : \mathbf{u} \in \mathscr{U}(\Omega)\}$$

and thus

$$\mathsf{F}(\mathbf{v}) = \min\{\mathsf{F}(\mathbf{u}) : \mathbf{u} \in \mathscr{U}(\Omega)\}. \quad \square$$

5 Limit analysis

The limit analysis deals with the loads that depend linearly (affinely) on a scalar parameter $\lambda \in \mathrm{R}$ [8]. We thus assume [21, 23] that the body and surface forces $\mathbf{b}^\lambda \in \mathscr{M}(\Omega, \mathrm{R}^n)$ and $\mathbf{s}^\lambda \in \mathscr{M}(\mathscr{S}, \mathrm{R}^n)$ corresponding to λ are given by

$$\mathbf{b}^\lambda = \mathbf{b}_0 + \lambda \mathbf{b}_1, \quad \mathbf{s}^\lambda = \mathbf{s}_0 + \lambda \mathbf{s}_1 \tag{5.1}$$

where

$$\mathbf{b}_0, \mathbf{b}_1 \in \mathscr{M}(\Omega, \mathrm{R}^n) \quad \mathbf{s}_0, \mathbf{s}_1 \in \mathscr{M}(\mathscr{S}, \mathrm{R}^n).$$

We call $(\mathbf{s}(\lambda), \mathbf{b}(\lambda)) = (\mathbf{s}^\lambda, \mathbf{b}^\lambda)$ the loads corresponding to λ. If $\mathbf{v} \in V_0$ then the work of the loads $(\mathbf{s}(\lambda), \mathbf{b}(\lambda))$ corresponding to $\mathbf{v}$ is

$$\langle \mathbf{l}(\lambda), \mathbf{v} \rangle = \int_{\mathscr{S}} \mathbf{v} \cdot d\mathbf{s}^\lambda + \int_{\Omega} \mathbf{v} \cdot d\mathbf{b}^\lambda.$$

If the loads have square integrable densities, i.e., if

$$\mathbf{b}_0 = \mathrm{b}_0 \mathscr{L}^n, \quad \mathbf{b}_1 = \mathrm{b}_1 \mathscr{L}^n, \quad \mathbf{s}_0 = \mathrm{s}_0 \mathscr{H}^{n-1}, \quad \mathbf{s}_1 = \mathrm{s}_1 \mathscr{H}^{n-1},$$

where

$$\mathrm{b}_0,\ \mathrm{b}_1 \in L^2(\Omega, \mathrm{R}^n), \quad \mathrm{s}_0,\ \mathrm{s}_1 \in L^2(\mathscr{S}, \mathrm{R}^n),$$

then one can extend the definition of $\mathbf{l}(\lambda)$ to elements $\mathbf{v}$ of V.

In the general context of loads represented by measures we define the total energy $\mathsf{F}(\mathbf{v}, \lambda)$ of the body corresponding to the loads $(\mathbf{s}(\lambda), \mathbf{b}(\lambda))$ and displacement $\mathbf{v} \in V_0$ by

$$\mathsf{F}(\mathbf{v}, \lambda) = \mathsf{E}(\mathbf{v}) - \langle \mathbf{l}(\lambda), \mathbf{v} \rangle$$

so that $\mathsf{F}(\cdot, \lambda) : V_0 \to \mathrm{R}$. Central to our considerations is the ***infimum energy*** $I_0(\lambda) \in \mathrm{R} \cup \{-\infty\}$ of the loads $(\mathbf{s}(\lambda), \mathbf{b}(\lambda))$ defined by

$$I_0(\lambda) = \inf\{\mathsf{F}(\mathbf{v}, \lambda) : \mathbf{v} \in V_0\}.$$

We denote by $\mathscr{A}(\lambda)$ the set of all admissible stress fields equilibrating the loads $(\mathbf{s}(\lambda), \mathbf{b}(\lambda))$. Recal that the loads $(\mathbf{s}(\lambda), \mathbf{b}(\lambda))$ are strongly compatible if $\mathscr{A}(\lambda) \neq \emptyset$.

We now follow [19].

5.1 Proposition.
(i) *The loads* $(\mathbf{s}(\lambda), \mathbf{b}(\lambda))$ *are strongly compatible if and only if* $I_0(\lambda) > -\infty$.
(ii) *The function* $I_0 : \mathrm{R} \to \mathrm{R} \cup \{-\infty\}$ *is concave and uppersemicontinuous, i.e.,*

$$I_0(\alpha\lambda + (1-\alpha)\mu) \geq \alpha I_0(\lambda) + (1-\alpha) I_0(\mu)$$

for every $\lambda, \mu \in \mathrm{R}$ *and* $\alpha \in [0,1]$ *and*

$$I_0(\lambda) \geq \limsup_{k\to\infty} I_0(\lambda_k)$$

for every $\lambda \in \mathrm{R}$ *and every sequence* $\lambda_k \to \lambda$. *Hence the set*

$$\Lambda = \{\lambda \in \mathrm{R} : I_0(\lambda) > -\infty\} \equiv \{\lambda \in \mathrm{R} : \mathscr{A}(\lambda) \neq \emptyset\} \tag{5.2}$$

is an interval.

Since the notion of compatibility of loads is independent of the tensor of elastic constants C, also the finiteness of $I_0(\lambda)$ is independent of C [within the class specified by (1.1)], even though the concrete value of $I_0(\lambda)$ depends on C. We emphasize the role of the square integrability requirement of the stress field in the definition of strongly compatible loads; there are loads $(\mathbf{s}(\lambda), \mathbf{b}(\lambda))$ with $I_0(\lambda) = -\infty$ and yet with $(\mathbf{s}(\lambda), \mathbf{b}(\lambda))$ being weakly equilibrated by a stress field $\mathbf{T} \in L^1(\Omega, \mathrm{Sym}) \sim L^2(\Omega, \mathrm{Sym})$ with values in Sym^-.

Proof (i): This follows from Proposition 3.3.

(ii): The affine dependence of $\mathbf{l}(\lambda)$ on λ implies that the function $\lambda \mapsto \mathsf{F}(\mathbf{u}, \lambda)$ is affine for each $\mathbf{u} \in V_0$; thus the function $\lambda \mapsto I_0(\lambda)$, being the lower envelope of the family of affine continuous functions over the parameter set $\{\mathbf{u} \in V_0\}$, is concave and uppersemicontinuous [10; Chapter I, Section 2]. □

5.2 Definitions. Let Λ be given by (5.2). A loading multiplier $\lambda \in \mathrm{R}$ is said to
(i) be ***statically admissible*** if $\lambda \in \Lambda$; otherwise λ is said to be ***statically inadmissible***;
(ii) be a ***collapse multiplier*** if it is a finite endpoint of Λ;
(iii) be ***kinematically admissible*** if there exists a $\mathbf{v} \in V_0$ such that $\hat{\mathbf{E}}(\mathbf{v}) \geq \mathbf{0}$ $\langle \bar{\mathbf{l}}, \mathbf{v} \rangle = 1$ and

$$\lambda = -\langle \mathbf{l}_0, \mathbf{v} \rangle; \tag{5.3}$$

(iv) ***admit a collapse mechanism*** if λ is kinematically admissible and $\lambda \leq \sup \Lambda$.

5.3 Remarks.

(i) The collapse multiplier can be statically admissible as well as statically inadmissible.

(ii) The notion of collapse multiplier can be given a dynamical meaning. The paper [26] considers no–tension bodies in dynamical situations with a viscous perturbation of the equations of motion. It turns out that if $I_0(\lambda) > -\infty$ then the motion with arbitrary initial data stabilizes in the sense that the kinetic energy satisfies $K(t) \to 0$ as $t \to \infty$ while if $I_0(\lambda) = -\infty$ then the total energy, given by the sum of the total potential energy and the kinetic energy, $T(t) = F(t) + K(t)$, satisfies $T(t) \to -\infty$ as $t \to \infty$.

(iii) If λ admits a collapse mechanism then there exists a $\mathbf{v} \in V$ with $\hat{\mathbf{E}}(\mathbf{v}) \geq \mathbf{0}$, $\langle \bar{\mathbf{l}}, \mathbf{v} \rangle = 1$ and $\langle \mathbf{l}(\lambda), \mathbf{v} \rangle = 0$; each such a $\mathbf{v}$ is said to be a ***collapse mechanism for the loads*** $(\mathbf{s}(\lambda), \mathbf{b}(\lambda))$.

(iv) If λ admits a collapse mechanism and if additionally λ is statically admissible then each admissible equilibrating stress field for $(\mathbf{s}(\lambda), \mathbf{b}(\lambda))$ is called a ***collapse stress field.*** A stronger version of the definition of collapse mechanism $\mathbf{v}$ in [8] requires that $\mathbf{v}$ be as in (iii) and that additionally λ be statically admissible.

5.4 Remark. If $\mathbf{T} \in L^2(\Omega, \mathrm{Sym})$, we denote the normal cone to the set $L^2(\Omega, \mathrm{Sym}^-)$ at $\mathbf{T}$ by $\mathrm{Norm}(L^2(\Omega, \mathrm{Sym}^-), \mathbf{T})$ and applying the definition from Section 1 we obtain

$$\begin{aligned} \mathrm{Norm}\,(L^2(\Omega, \mathrm{Sym}^-), \mathbf{T}) &= \Big\{\mathbf{D} \in L^2(\Omega, \mathrm{Sym}) : \int_\Omega \mathbf{D} \cdot (\mathbf{T} - \mathbf{S})\, d\mathscr{L}^n \geq 0 \\ &\qquad\qquad \text{for each admissible stress field } \mathbf{S}\Big\} \\ &= \Big\{\mathbf{D} \in L^2(\Omega, \mathrm{Sym}) : \mathbf{D} \cdot (\mathbf{T} - \mathbf{U}) \geq 0 \\ &\qquad\qquad \text{for every } \mathbf{U} \in \mathrm{Sym}^- \text{ and } \mathscr{L}^n \text{ almost every point of } \Omega\Big\}. \end{aligned}$$

Let $\lambda \in \mathrm{R}$. If $\mathbf{v} \in V$ satisfies

$$\hat{\mathbf{E}}(\mathbf{v}) \in \mathrm{Norm}(L^2(\Omega, \mathrm{Sym}^-), \mathbf{T}), \quad \langle \bar{\boldsymbol{l}}, \mathbf{v} \rangle = 1 \tag{5.4}$$

and

$$\lambda = -\langle \mathbf{l}_0, \mathbf{v} \rangle \tag{5.5}$$

for some $\mathbf{T} \in \mathscr{A}(\lambda)$ then $\mathbf{v}$ is a collapse mechanism for the loads $(\mathbf{s}(\lambda), \mathbf{b}(\lambda))$. Indeed, the second characterization of $\mathrm{Norm}(L^2(\Omega, \mathrm{Sym}^-), \mathbf{T})$ implies that $\hat{\mathbf{E}}(\mathbf{v}) \geq \mathbf{0}$ almost everywhere on Ω.

The number of collapse multipliers ranges from 0 to 2. In applications, one is interested in the larger of the possibly two collapse multipliers. Motivated by this, we introduce the multiplier

$$\lambda_{\mathrm{c}}^+ := \sup\{\lambda \in \mathrm{R} : \lambda \text{ is statically admissible}\} \tag{5.6}$$

$-\infty \leq \lambda_{\mathrm{c}}^+ \leq \infty$; thus if λ_{c}^+ is finite, then λ_{c}^+ is a collapse multiplier, and if there are two collapse multipliers, then λ_{c}^+ is the larger of these two. Also, we consider the multiplier

$$\bar{\lambda}_{\mathrm{c}}^+ = \inf\{\lambda \in \mathrm{R} : \lambda \text{ is kinematically admissible}\}. \tag{5.7}$$

5.5 Remark. The above definitions of λ_{c}^+ and $\bar{\lambda}_{\mathrm{c}}^+$ are based on the square integrability: in the definition of λ_{c}^+ the admissible equilibrating stresses are square integrable and in the definition of $\bar{\lambda}_{\mathrm{c}}^+$ we consider mechanisms that are square integrable with the square integrable gradients. The definitions of the analogs of λ_{c}^+ and $\bar{\lambda}_{\mathrm{c}}^+$ using different function spaces is treated in detail in [24]. However, it must be emphasized that the definitions based on the square integrability are well motivated by Proposition 3.3.

5.6 Remark. It turns out [24] that the definition of the kinematic multiplier (5.7) can be reformulated to the format of the variational problem by Ekeland & Temam [11] and then the static multiplier (5.6) takes the form of the dual problem in the sense of the cited reference.

Our first result shows that our definition of the collapse multiplier generalizes that based on the collapse mechanism:

5.7 Theorem.
(i) *We have* $\lambda_c^+ \le \bar{\lambda}_c^+$.
(ii) *If* $\lambda \in \mathrm{R}$ *admits a collapse mechanism then* $\lambda = \lambda_c^+ = \bar{\lambda}_c^+$.

Proof (i): Let $\lambda \in \mathrm{R}$ be kinematically admissible, i.e., there exists a $\mathbf{v} \in V_0$ such that $\hat{\mathbf{E}}(\mathbf{v}) \ge \mathbf{0}$ almost everywhere on Ω, $\langle \bar{\boldsymbol{l}}, \mathbf{v} \rangle = 1$, and $\langle \mathbf{l}(\lambda), \mathbf{v} \rangle = 0$. Then $\lambda = -\langle \mathbf{l}_0, \mathbf{v} \rangle$ and thus

$$\bar{\lambda}_c^+ = \inf \{ -\langle \mathbf{l}_0, \mathbf{v} \rangle : \mathbf{v} \in V_0,\ \hat{\mathbf{E}}(\mathbf{v}) \ge \mathbf{0},\ \langle \bar{\boldsymbol{l}}, \mathbf{v} \rangle = 1 \}. \tag{5.8}$$

Let $\lambda \in \mathrm{R}$ be statically admissible with the admissible equilibrating stress field $\mathbf{T}$ and let $\mathbf{v} \in V_0$ be such that $\hat{\mathbf{E}}(\mathbf{v}) \ge \mathbf{0}$ almost everywhere on Ω and $\langle \bar{\boldsymbol{l}}, \mathbf{v} \rangle = 1$. Then we have

$$0 \ge \int_\Omega \mathbf{T} \cdot \hat{\mathbf{E}}(\mathbf{v})\, d\mathscr{L}^n = \langle \mathbf{l}(\lambda), \mathbf{v} \rangle = \lambda + \langle \mathbf{l}_0, \mathbf{v} \rangle,$$

Thus

$$\lambda \le -\langle \mathbf{l}_0, \mathbf{v} \rangle.$$

Taking the infimum over all $\mathbf{v}$ with the indicated properties and using (5.8) we find $\lambda \le \bar{\lambda}_c^+$ and taking the supremum over all $\mathbf{T}$ with the indicated properties, we obtain the inequality in (i).

(ii): Assume that $\lambda \in \mathrm{R}$ admits a collapse mechanism. Prove first that $\lambda = \lambda_c^+$. Since λ admits a collapse mechanism, λ is kinematically admissible and hence there exists a $\mathbf{v} \in V_0$ with

$$\hat{\mathbf{E}}(\mathbf{v}) \ge \mathbf{0}, \quad \langle \bar{\boldsymbol{l}}, \mathbf{v} \rangle = 1 \quad \text{and} \quad \langle \mathbf{l}(\lambda), \mathbf{v} \rangle = 0. \tag{5.9}$$

Prove that $I_0(\mu) = -\infty$ for all $\mu > \lambda$. We have $\mathsf{E}(t\mathbf{v}) = 0$ and hence

$$\mathsf{F}(t\mathbf{v}, \mu) = -\langle \mathbf{l}(\mu), t\mathbf{v} \rangle = -\langle \mathbf{l}(\lambda), t\mathbf{v} \rangle - (\mu - \lambda)\langle \bar{\boldsymbol{l}}, t\mathbf{v} \rangle = -(\mu - \lambda) t \langle \bar{\boldsymbol{l}}, \mathbf{v} \rangle = -(\mu - \lambda) t.$$

Letting $t \to \infty$ we thus obtain $\mathsf{F}(t\mathbf{v}, \mu) \to -\infty$ as $t \to \infty$ for all $\mu > \lambda$. Hence $\lambda_c^+ \le \lambda$; on the other hand, $\lambda \le \lambda_c^+$ as part of the definition of the multiplier admitting a collapse mechanism. This completes the proof of $\lambda = \lambda_c^+$.

On the other hand, λ is kinematically admissible as part of the definition of the property of λ admitting a collapse multiplier. Thus $\bar{\lambda}_c^+ \le \lambda = \lambda_c^+$ and Item (i) completes the proof. □

6 Families of measures and the weak compatibility of loads

In this section we introduce the generalized compatibility of loads called weak compatibility. This involves balancing the loads by a stressfield represented by a measure T from $\mathscr{M}(\Omega, \mathrm{Sym})$. Thus in contrast to the strong compatibility, which is based on balancing by a square integrable function, we here admit concentrations of stress on objects of dimension lower than the dimension n of the physical space R^n, see Example 2.2(i). In many concrete cases, it is easier to prove the weak compatibility than the strong compatibility. However, Proposition 3.3 on the boundedness below of the total energy requires strong compatibility. In the case of the limit analysis, where we deal with the loads depending on the loading parameter λ, we have the balancing measures depending on λ as well. Then we can use a procedure of integrating the balancing stress measures with respect to the loading parameter to smear out the singularities of the stress measure to obtain a square integrable function.

We say that $\mathbf{T} \in \mathscr{M}(\Omega, \mathrm{Sym})$ is admissible if $\mathbf{T}$ takes the values in the set Sym^- of the negative semidefinite symmetric tensors, i.e., if $\mathbf{T}(A)\mathbf{a} \cdot \mathbf{a} \le 0$ for any Borel set $A \subset \Omega$ and for any $\mathbf{a} \in \mathrm{R}^n$. We say that $\mathbf{T}$ ***weakly equilibrates the loads*** $(\mathbf{s}, \mathbf{b})$ if

$$\int_\Omega \hat{\mathrm{E}}(\mathbf{v}) \cdot d\mathbf{T} = \int_\Omega \mathbf{v} \cdot d\mathbf{b} + \int_{\mathscr{N}} \mathbf{v} \cdot d\mathbf{s}$$

for any $\mathbf{v} \in V_0$. We say that the loads $(\mathbf{s}, \mathbf{b})$ are ***weakly compatible*** if there exists an admissible $\mathbf{T} \in \mathscr{M}(\Omega, \mathrm{Sym})$ which weakly equilibrates them. The reader is referred to [29] for the general properties of stresses represented by measures.

If the loads are strongly compatible then they are weakly compatible; however, there are examples of loads that are weakly compatible but not strongly compatible.

6.1 Example. Consider a stress measure of the form

$$\mathbf{T} = \mathbf{T}_r + \mathbf{T}_s, \qquad \text{where} \qquad \mathbf{T}_r = \mathrm{T}_r \mathscr{L}^n \llcorner \Omega, \quad \mathbf{T}_s = \mathrm{T}_s \mathscr{H}^{n-1} \llcorner \mathscr{N}$$

where $\mathrm{T}_r : \mathrm{cl}\,\Omega \to \mathrm{Sym}$ is a smooth $\mathscr{L}^n$ integrable function over the body and $\mathrm{T}_s : \mathrm{cl}\,\mathscr{N} \to \mathrm{Sym}$ is a smooth $\mathscr{H}^{n-1}$ integrable function over a smooth surface $\mathscr{N}$ contained in Ω. Assume furthermore given the loads $(\mathbf{s}, \mathbf{b})$ of the form (3.1) with $\mathbf{b}$ and $\mathbf{s}$ continuous integrable functions over Ω and $\mathscr{S}$, respectively. Then $\mathbf{T}$ weakly equilibrates the loads $(\mathbf{s}, \mathbf{b})$ if and only if the following two conditions hold:

(i) we have

$$\mathrm{div}\,\mathrm{T}_r + \mathbf{b} = \mathbf{0} \quad \text{in} \quad \Omega \sim \mathscr{N}, \tag{6.1}$$

where div is the classical divergence operator;

(ii) the stress field T_s is superficial in the sense that $\mathrm{T}_s\mathbf{m} = \mathbf{0}$ where $\mathbf{m}$ is the normal to $\mathscr{N}$ and we have

$$[\mathrm{T}_r]\mathbf{m} + \mathrm{div}^{\mathscr{N}}\,\mathrm{T}_s = \mathbf{0} \quad \text{on} \quad \mathscr{N} \tag{6.2}$$

where $[\mathrm{T}_r] = \mathrm{T}_r^+ - \mathrm{T}_r^-$ is the jump discontinuity of T_r on $\mathscr{N}$ and $\mathrm{div}^{\mathscr{N}}$ is the superficial divergence [20; Section 4];

(iii) we have

$$\mathrm{T}_r\mathbf{n} = \mathbf{s} \qquad \text{on } \mathscr{S}$$

where $\mathbf{n}$ is the outer normal to $\partial\Omega$.

We now pass to the details of the integration procedure.

6.2 Definition. An ***integrable parametric measure*** [21] is a family $\{\mathsf{m}^\lambda : \lambda \in \Lambda\}$ of V valued measures on R^n where $\Lambda \subset \mathrm{R}$ is a $\mathscr{L}^1$ measurable set of parameters such that

(i) for every continuous V valued function f on R^n with compact support the function $\lambda \mapsto \int_{\mathrm{R}^n} f \cdot d\mathsf{m}^\lambda$ is $\mathscr{L}^1$ measurable on Λ;

(ii) we have

$$c := \int_\Lambda \mathrm{M}(\mathsf{m}^\lambda)\, d\lambda < \infty.$$

We note that parametric measures similar to those defined above occur in the context of disintegration (slicing) of measures [3; Section 2.5] and, what is related, in the context of Young's measures [25; Chapter 5].

6.3 Proposition. *If* $\{\mathsf{m}^\lambda : \lambda \in \Lambda\}$ *is an integrable parametric measure then there exists a unique* V *valued measure* m *on* $\mathbb{R}^n$ *such that*

$$\int_{\mathbb{R}^n} f \cdot d\mathsf{m} = \int_\Lambda \int_{\mathbb{R}^n} f \cdot d\mathsf{m}^\lambda \, d\lambda \tag{6.3}$$

for each continuous V *valued function* f *on* $\mathbb{R}^n$ *with compact support.*

We write

$$\mathsf{m} = \int_\Lambda \mathsf{m}^\lambda \, d\lambda \tag{6.4}$$

and call m the integral of the family $\{\mathsf{m}^\lambda : \lambda \in \Lambda\}$ with respect to λ.

Proof We note that for each continuous V valued function f on $\mathbb{R}^n$ with compact support the right hand side of (6.3) is a well defined real number: indeed

$$\begin{aligned} \Big| \int_\Lambda \int_{\mathbb{R}^n} f \cdot d\mathsf{m}^\lambda \, d\lambda \Big| &\le \int_\Lambda \int_{\mathbb{R}^n} |f| \, d|\mathsf{m}^\lambda| \, d\lambda \\ &\le \max\{|f(\mathbf{x})| : \mathbf{x} \in \mathbb{R}^n\} \int_\Lambda \mathsf{M}(\mathsf{m}^\lambda) \, d\lambda \\ &\le c \max\{|f(\mathbf{x})| : \mathbf{x} \in \mathbb{R}^n\}. \end{aligned}$$

Thus by the Riesz representation theorem [3; Theorem 1.54] there exists a measure m such that (6.3) holds. □

The following two propositions give two important examples of integrable parametric measures. In both cases the corresponding integral (6.4) is absolutely continuous with respect to the Lebesgue measure.

6.4 Proposition. *Let* $\{\mathbf{T}_r^\lambda : \lambda \in \Lambda\}$ *be a family of* Sym *valued functions on* $\Omega \subset \mathbb{R}^n$ *defined for all* λ *from a* $\mathscr{L}^1$ *measurable set* $\Lambda \subset \mathbb{R}$ *such that the mapping* $(\mathbf{x}, \lambda) \mapsto \mathbf{T}_r^\lambda(\mathbf{x})$ *is* $\mathscr{L}^{n+1}$ *integrable on* $\Omega \times \Lambda$, *i.e.,*

$$\int_\Lambda \int_\Omega |\mathbf{T}_r^\lambda(\mathbf{x})| \, d\mathbf{x} d\lambda < \infty. \tag{6.5}$$

If we define a Sym *valued measure* $\boldsymbol{\mathsf{T}}_r^\lambda$ *by*

$$\boldsymbol{\mathsf{T}}_r^\lambda = \mathbf{T}_r^\lambda \mathscr{L}^n \llcorner \Omega$$

then $\{\boldsymbol{\mathsf{T}}_r^\lambda : \lambda \in \Lambda\}$ *is an integrable parametric measure and we have*

$$\int_\Lambda \boldsymbol{\mathsf{T}}_r^\lambda \, d\lambda = \mathbf{T}_r \mathscr{L}^n \llcorner \Omega$$

where

$$\mathbf{T}_r(\mathbf{x}) = \int_\Lambda \mathbf{T}_r^\lambda(\mathbf{x}) \, d\lambda$$

for $\mathscr{L}^n$ *almost every* $\mathbf{x} \in \Omega$.

Proof This follows directly from Fubini's theorem. □

6.5 Proposition. *Let* $\Omega_0 \subset \mathbb{R}^n$ *be open, let* $\varphi : \Omega_0 \to \mathbb{R}$ *be locally Lipschitz continuous and let* $\mathbf{T}_s : \Omega_0 \to$ Sym *be* $\mathscr{L}^n$ *measurable on* Ω_0, *with*

$$\int_{\Omega_0} |\mathbf{T}_s| |\nabla \varphi| \, d\mathscr{L}^n < \infty. \tag{6.6}$$

Then for $\mathscr{L}^1$ *almost every* $\lambda \in \mathrm{R}$ *the function* T_s *is* $\mathscr{H}^{n-1} \llcorner \varphi^{-1}(\lambda)$ *integrable; denoting by* Λ *the set of all such* λ *we define the measure* m^λ *by*

$$\mathbf{T}^\lambda := \mathrm{T}_s \mathscr{H}^{n-1} \llcorner \varphi^{-1}(\lambda)$$

for each $\lambda \in \Lambda$. *Then* $\{\mathbf{T}^\lambda : \lambda \in \Lambda\}$ *is an integrable parametric measure and we have*

$$\int_\Lambda \mathbf{T}^\lambda \, d\lambda = \mathrm{T}_s |\nabla\varphi| \mathscr{L}^n \llcorner \Omega_0. \tag{6.7}$$

Proof Let $\mathbf{T}$ be given by

$$\mathbf{T} = \int_\Lambda \mathbf{T}^\lambda \, d\lambda. \tag{6.8}$$

If $\mathbf{E} \in C_0(\mathrm{R}^n, \mathrm{Sym})$ then by the coarea formula [3; Section 2.12] we have

$$\begin{aligned}\int_{\Omega_0} \mathbf{E} \cdot \mathrm{T}_s |\nabla\varphi| \, d\mathscr{L}^n &= \int_{\mathrm{R}} \int_{\varphi^{-1}(\lambda)} \mathbf{E} \cdot \mathrm{T}_s \, d\mathscr{H}^{n-1} d\lambda \\ &= \int_\Lambda \int_{\mathrm{R}^n} \mathbf{E} \cdot d\mathbf{T}^\lambda d\lambda \\ &= \int_{\mathrm{R}^n} \mathbf{E} \cdot d\mathbf{T}. \quad \square\end{aligned}$$

We now assume that the loads depend affinely on the loading parameter as in (5.1), thus we have loads $(\mathbf{s}^\lambda, \mathbf{b}^\lambda)$ defined for each $\lambda \in \mathrm{R}$. It may happen that there exists an interval $(\lambda - \varepsilon, \lambda + \varepsilon)$ such that each load $(\mathbf{s}^\mu, \mathbf{b}^\mu)$, with μ from this interval, is weakly equilibrated by a stress field $\mathbf{T}^\mu \in \mathscr{M}(\Omega, \mathrm{Sym})$ in such a way that

$$\{\mathbf{T}^\mu, \mu \in \Lambda\} \tag{6.9}$$

is an integrable parametric measure. In this situation, we have

6.6 Remark. The measure $\mathbf{U}$ defined by

$$\mathbf{U} = \frac{1}{2\varepsilon} \int_{\lambda-\varepsilon}^{\lambda+\varepsilon} \mathbf{T}^\mu \, d\mu$$

weakly equilibrates the loads $(\mathbf{s}^\lambda, \mathbf{b}^\lambda)$. However, depending on the nature of the family (6.9), it may happen that the measure $\mathbf{U}$ is such that

$$\mathbf{U} = \mathrm{U} \mathscr{L}^n \llcorner \Omega \quad \mathrm{U} \in L^2(\Omega, \mathrm{Sym}^-),$$

i.e., the loads $(\mathbf{s}^\lambda, \mathbf{b}^\lambda)$ are strongly equilibrated by the stress field U. Such a situation arises when the family (6.9) satisfies the hypothesis of Proposition 6.4 or Proposition 6.5, or is a sum of families such that one satisfies Proposition 6.4 and the other Proposition 6.5. In many concrete cases, it is hard to evalueate $\mathbf{U}$ explicitly but for the analysis it suffices to know that $\mathbf{U}$ is represented by a square integrable function.

7 Integration with gravity

In this section we consider a rectangular panel made of a no-tension material that is fixed at its base and subject to both the weight $\mathbf{b}$ which constitutes the permanent part of the loads, and a horizontal compressive load $\mathbf{s}^\lambda$ of intensity λ, which is uniformly distributed on the right lateral side of the panel and constitutes the variable part of the loads. Then, for every λ in an appropriate interval $(0, \lambda_c)$, we determine a negative semidefinite and square integrable stress field $\mathbf{T}^\lambda$ which is in equilibrium with the given loads and we conclude that every $\lambda \in (0, \lambda_c)$ is statically admissible (see Definition 5.2). Let

$$\Omega = (0, B) \times (0, H) \subset \mathrm{R}^2$$

be the rectangular panel. We introduce the coordinate system x, y in R^2 with the origin in the upper right corner of the panel and with the x axis pointing to the left and the y axis pointing downward. We denote a general point of Ω by $\mathbf{r} = (x, y)$ and the coordinate vectors along the axes x, y by $\mathbf{i}$, $\mathbf{j}$, respectively. We put

$$\mathscr{D} = (0, B) \times \{H\}, \quad \mathscr{S} = \partial\Omega \sim \mathscr{D}$$

and consider the loads $(\mathbf{s}^\lambda, \mathbf{b}^\lambda)$ where, for $b > 0$, $\mathbf{b}^\lambda = b\mathbf{j}$ in Ω, and for $\mathbf{r} = (x, y) \in \mathscr{S}$ and $\lambda > 0$,

$$\mathbf{s}^\lambda(\mathbf{r}) = \begin{cases} \lambda\mathbf{i} & \text{on} \quad \{0\} \times (0, H), \\ \mathbf{0} & \text{elsewhere.} \end{cases} \tag{7.1}$$

The stress field $\mathbf{T}^\lambda$ will be constructed in two steps. Firstly, for every $\lambda \in (0, \lambda_c)$ we determine a measure stress field $\mathsf{T}^\lambda \in \mathscr{M}(\Omega, \mathrm{Sym}^-)$ that is in equilibrium with the loads, i.e., such that

$$\int_\Omega \hat{\mathbf{E}}(\mathbf{v}) \cdot d\mathsf{T}^\lambda = b \int_\Omega \mathbf{v} \cdot \mathbf{j} \, d\mathscr{L}^2 + \lambda \int_{\{0\} \times (0, H)} \mathbf{v} \cdot \mathbf{i} \, d\mathscr{H}^1$$

for every $\mathbf{v} \in V = W^{1,2}(\Omega, \mathrm{R}^2)$ (we recall that $\mathscr{L}^2$ stands for the Lebesgue measure and $\mathscr{H}^1$ for the 1-dimensional Hausdorff measure in R^2). This expression is well defined because the loads are of the special form (3.1). Secondly, we determine a square integrable negative semidefinite and equilibrated stress field $\mathbf{T}^\lambda$ for every $\lambda \in (0, \lambda_c)$ by the integration procedure described in Section 6, i.e. $\mathbf{T}^\lambda \in \mathscr{A}(\lambda)$. This will prove that the loads $(\mathbf{s}^\lambda, \mathbf{b}^\lambda)$ are strongly compatible for every $\lambda \in (0, \lambda_c)$ (see (5.2)).

To determine the stress measure T^λ, we use the results of [22]. We consider a smooth curve $\mathscr{N}^\lambda$ which starts at the upper right corner and ends at the bottom of the panel, and which is the graph of an increasing function $\omega^\lambda : [0, t^\lambda] \to [0, H]$ to be specified below. In this way, Ω is divided into the regions Ω_+^λ (on the left) and Ω_-^λ (on the right) by the curve $\mathscr{N}^\lambda$. We are looking for a weakly equilibrated measure stress field T^λ which is absolutely continuous with respect to the Lebesgue measure in Ω_+^λ and Ω_-^λ with a piecewise continuously differentiable density $\mathbf{T}_r^\lambda$ and has a concentration on $\mathscr{N}^\lambda$ with a continuously differentiable density $\mathbf{T}_s^\lambda$, i.e.,

$$\mathsf{T}^\lambda = \mathbf{T}_r^\lambda \mathscr{L}^2 \llcorner \Omega + \mathbf{T}_s^\lambda \mathscr{H}^1 \llcorner \mathscr{N}^\lambda.$$

The equilibrium condition (6.1) implies that

$$\operatorname{div} \mathbf{T}_r^\lambda + \mathbf{b}^\lambda = \mathbf{0} \quad \text{in} \quad \Omega \sim \mathscr{N}^\lambda. \tag{7.2}$$

Furthermore, $\mathbf{T}_s^\lambda$ is superficial by Item (ii) of Example 6.1, which means here

$$\mathbf{T}_s^\lambda = \sigma^\lambda \mathbf{t}^\lambda \otimes \mathbf{t}^\lambda, \tag{7.3}$$

where $\mathbf{t}^\lambda$ is the unit tangent vector to $\mathscr{N}^\lambda$ and σ^λ is a scalar function on $\mathscr{N}^\lambda$. By (6.2) we have

$$[\mathbf{T}_r^\lambda]\mathbf{m} + \mathrm{div}^{\mathscr{N}}\,\mathbf{T}_s^\lambda = \mathbf{0} \quad \text{on} \quad \mathscr{N}^\lambda \tag{7.4}$$

where $[\mathbf{T}_r^\lambda]$ is the jump discontinuity $\mathbf{T}_r^\lambda$ on $\mathscr{N}^\lambda$, $\mathbf{m}$ is the unit normal to $\mathscr{N}^\lambda$ pointing toward Ω_+^λ.

Following the method presented in [22] we obtain (see [22; Eqs. (4.6), (4.7) and (3.1)])

$$\mathbf{T}_r^\lambda(\mathbf{r}) = \begin{cases} -by\mathbf{j}\otimes\mathbf{j} & \text{if} \quad \mathbf{r}\in\Omega_+^\lambda, \\ -\lambda\mathbf{i}\otimes\mathbf{i} - 2bx\mathbf{i}\odot\mathbf{j} - \dfrac{b^2x^2}{\lambda}\mathbf{j}\otimes\mathbf{j} & \text{if} \quad \mathbf{r}\in\Omega_-^\lambda \end{cases} \tag{7.5}$$

which satisfies the equilibrium equation (7.2) and the boundary conditions (7.1). From (7.4) we deduce the equation of $\mathscr{N}^\lambda$ (see [22; Eq. (4.16)] with $\lambda = \beta H$ and $p_0 = 0$)

$$\omega^\lambda(x) = cbx^2/\lambda, \quad c = 1/2 + \sqrt{3}/6 \tag{7.6}$$

which has the unit tangent vector

$$\mathbf{t}^\lambda(\mathbf{r}) = \frac{x\mathbf{i} + 2y\mathbf{j}}{\sqrt{x^2+4y^2}} \tag{7.7}$$

Moreover, from (7.6) and (7.4) we obtain (see [22; Eq. (2.19)], with $s_0 = 0$, and $f(x,y)$ at the end of page 229)

$$\sigma^\lambda(\mathbf{r}) = -\frac{\sqrt{3}}{6}bx\sqrt{x^2+4y^2} \tag{7.8}$$

$\mathbf{r}\in\mathscr{N}^\lambda$. If $\lambda\in(0,\lambda_c)$, with $\lambda_c = cbB^2/H$, then $\mathscr{N}^\lambda$ is contained in Ω, except for the endpoints and the measure stress field $\mathbf{T}^\lambda$ is well defined by relations (7.5)–(7.8).

The parametric measure $\mathbf{T}_r^\lambda\mathscr{L}^2 \llcorner \Omega$ is of the form considered in Proposition 6.4 and the integrability condition (6.5) is satisfied because we have

$$\int_0^{\lambda_c}\int_\Omega |\mathbf{T}_r^\lambda(\mathbf{r})|\, d\mathbf{r}d\lambda < \infty$$

Hence for $0 < \lambda < \lambda_c$ and $\varepsilon > 0$ such that

$$\Lambda = (\lambda-\varepsilon, \lambda+\varepsilon) \subset (0,\lambda_c) \tag{7.9}$$

the measure

$$\bar{\mathbf{T}}_r^\lambda = \frac{1}{2\varepsilon}\int_\Lambda \mathbf{T}_r^\mu\, d\mu \tag{7.10}$$

is an absolutely continuous measure with respect to $\mathscr{L}^2 \llcorner \Omega$,

$$\bar{\mathbf{T}}_r^\lambda = \mathbf{U}_r^\lambda\mathscr{L}^2 \llcorner \Omega.$$

To compute $\mathbf{U}_r^\lambda$, let us put

$$A = \{\mathbf{r} = (x,y) : bcx^2/y \in \Lambda\}. \tag{7.11}$$

We obtain

$$\mathbf{U}_r^\lambda(\mathbf{r}) = \begin{cases} -by\mathbf{j}\otimes\mathbf{j} & \text{if} \quad \mathbf{r}\in\Omega_+^\lambda \sim A, \\ -\lambda\mathbf{i}\otimes\mathbf{i} - 2bx\mathbf{i}\odot\mathbf{j} - (2\varepsilon)^{-1}b^2x^2\ln\left(\dfrac{\lambda+\varepsilon}{\lambda-\varepsilon}\right)\mathbf{j}\otimes\mathbf{j} & \text{if} \quad \mathbf{r}\in\Omega_-^\lambda \sim A, \\ (2\varepsilon)^{-1}(\xi_1(\mathbf{r})\mathbf{i}\otimes\mathbf{i} + 2\xi_2(\mathbf{r})\mathbf{i}\odot\mathbf{j} + \xi_3(\mathbf{r})\mathbf{j}\otimes\mathbf{j}) & \text{if} \quad \mathbf{r}\in A, \end{cases} \tag{7.12}$$

where

$$\xi_1(\mathbf{r}) = \frac{b^2c^2x^4}{2y^2} - \frac{1}{2}(\lambda+\varepsilon)^2$$

$$\xi_2(\mathbf{r}) = bx(\frac{bcx^2}{y} - \lambda - \varepsilon),$$

$$\xi_3(\mathbf{r}) = by(\lambda-\varepsilon) - b^2x^2\left(c + \ln\frac{y(\lambda+\varepsilon)}{bcx^2}\right).$$

In order to verify the first two regimes in (7.12) we note that if $\mathbf{r}\in\Omega_+^\mu \sim A$ or $\mathbf{r}\in\Omega_-^\mu \sim A$ then, for all values of μ in A, the expression of $\mathbf{T}_r^\mu(\mathbf{r})$ is that given by $(7.5)_1$ and $(7.5)_2$, respectively. Thus, $(7.12)_1$ and $(7.12)_2$ can be immediately obtained from (7.6). For $\mathbf{r}\in A$, we have $\mathbf{r}\in\Omega_+^\mu$ for $\mu\in(\lambda-\varepsilon, bcx^2/y)$ and $\mathbf{r}\in\Omega_-^\mu$ for $\mu\in(bcx^2/y, \lambda+\varepsilon)$. Therefore

$$\begin{aligned}\mathbf{U}_r^\lambda(\mathbf{r}) &= (2\varepsilon)^{-1}\int_{\lambda-\varepsilon}^{\lambda+\varepsilon}\mathbf{T}_r^\mu\,d\mu \\ &= (2\varepsilon)^{-1}\Big\{\int_{\lambda-\varepsilon}^{bcx^2/y} -by\mathbf{j}\otimes\mathbf{j}\,d\mu + \int_{bcx^2/y}^{\lambda+\varepsilon}(-\mu\mathbf{i}\otimes\mathbf{i} - 2bx\mathbf{i}\odot\mathbf{j} - \frac{b^2x^2}{\mu}\mathbf{j}\otimes\mathbf{j})\,d\mu\Big\} \\ &= (2\varepsilon)^{-1}\Big\{\Big(\frac{b^2c^2x^4}{2y^2} - \frac{1}{2}(\lambda+\varepsilon)^2\Big)\mathbf{i}\otimes\mathbf{i} + 2bx\Big(\frac{bcx^2}{y} - \lambda - \varepsilon\Big)\mathbf{i}\odot\mathbf{j} \\ &\qquad + \Big[by(\lambda-\varepsilon) - b^2x^2\Big(c + \ln\frac{y(\lambda+\varepsilon)}{bcx^2}\Big)\Big]\mathbf{j}\otimes\mathbf{j}\Big\}.\end{aligned}$$

The density $\mathbf{U}_r^\lambda$ is bounded in Ω (we note that for $\mathbf{r}\in A$ we have $\lambda-\varepsilon < cbx^2/y < \lambda+\varepsilon$ by (7.7)).

Next we consider the measures $\mathbf{T}_s^\lambda$. Let $\varphi : \Omega \to \mathrm{R}$ be defined by

$$\varphi(\mathbf{r}) = cbx^2/y, \tag{7.13}$$

$\mathbf{r} = (x, y) \in \Omega$. Then, for any $\lambda \in (0, \lambda_c)$, the curve $\mathcal{N}^\lambda$ is the level set of φ corresponding to the value of λ, i.e.

$$\mathcal{N}^\lambda(\mathbf{r}) = \{\mathbf{r}\in\Omega : \varphi(\mathbf{r}) = \lambda\}.$$

Moreover, φ is continuously differentiable and

$$|\nabla\varphi(\mathbf{r})| = \frac{cbx\sqrt{x^2+4y^2}}{y^2} \tag{7.14}$$

Firstly we note that

$$|\nabla\varphi(\mathbf{r})||\mathbf{T}_s^\lambda(\mathbf{r})| = |\nabla\varphi(\mathbf{r})||\sigma^\lambda(\mathbf{r})| = \frac{\sqrt{3}}{6}c^2b^2(4x^2 + \frac{x^4}{y^2})$$

is bounded in

$$\Omega_0 = \{\mathbf{r} = (x,y) : cbx^2/y \in (0,\lambda_c)\} = \varphi^{-1}(0,\lambda_c),$$

in view of (7.14), (7.3) and (7.8). Then, Proposition 6.5 says that for any interval Λ as in (7.9) the measure

$$\bar{\mathbf{T}}_s^\lambda = \frac{1}{2\varepsilon}\int_\Lambda \mathbf{T}_s^\mu \, d\mu$$

is $\mathscr{L}^2$ absolutely continuous over Ω, i.e.,

$$\bar{\mathbf{T}}_s^\lambda = \mathbf{U}_s^\lambda(\mathbf{r})\mathscr{L}^2 \llcorner \Omega_0,$$

with the density given by (6.7), i.e.

$$\mathbf{U}_s^\lambda(\mathbf{r}) = \begin{cases} (2\varepsilon)^{-1}\mathbf{T}_s^\lambda(\mathbf{r})|\nabla\varphi(\mathbf{r})| & \text{if} \quad \mathbf{r} \in A, \\ \mathbf{0} & \text{otherwise.} \end{cases} \tag{7.15}$$

Note that $\varphi(\mathbf{r}) \in \Lambda$ if and only if $\mathbf{r} \in A$, by (7.13) and (7.11). In the present case we have

$$\begin{aligned} \mathbf{T}_s^\lambda(\mathbf{r})\,|\nabla\varphi(\mathbf{r})| &= \sigma^\lambda(\mathbf{r})\mathbf{t}^\lambda(\mathbf{r}) \otimes \mathbf{t}^\lambda(\mathbf{r})\,|\nabla\varphi(\mathbf{r})| \\ &= -\frac{\sqrt{3}cb^2x^2}{6y^2}(x\mathbf{i} + 2y\mathbf{j}) \otimes (x\mathbf{i} + 2y\mathbf{j}) \\ &= -\frac{\sqrt{3}cb^2x^2}{6y^2}(x^2\mathbf{i} \otimes \mathbf{i} + 4xy\mathbf{i} \odot \mathbf{j} + 4y^2\mathbf{j} \otimes \mathbf{j}) \end{aligned}$$

$\mathbf{r} \in A$, by (7.14), (7.3) and (7.8).

Finally, we obtain the negative semidefinite and square integrable (in fact bounded) stress field $\mathbf{U}^\lambda = \mathbf{U}_r^\lambda + \mathbf{U}_s^\lambda$,

$$\mathbf{U}^\lambda(\mathbf{r}) = \begin{cases} -by\mathbf{j} \otimes \mathbf{j} & \text{if} \quad \mathbf{r} \in \Omega_+^\lambda \sim A, \\ -\lambda\mathbf{i} \otimes \mathbf{i} - 2bx\mathbf{i} \odot \mathbf{j} - (2\varepsilon)^{-1}b^2x^2 \ln\left(\dfrac{\lambda+\varepsilon}{\lambda-\varepsilon}\right)\mathbf{j} \otimes \mathbf{j} & \text{if} \quad \mathbf{r} \in \Omega_-^\lambda \sim A, \\ \mathbf{S}(\mathbf{r}) & \text{if} \quad \mathbf{r} \in A, \end{cases}$$

where

$$\begin{aligned} \mathbf{S}(\mathbf{r}) = -(2\varepsilon)^{-1}\Big\{ &\Big[-\frac{b^2x^4}{12y^2} + \frac{1}{2}(\lambda+\varepsilon)^2\Big]\mathbf{i} \otimes \mathbf{i} \\ &+ 2\Big[-\frac{b^2x^3}{3y} + bx(\lambda+\varepsilon)\Big]\mathbf{i} \odot \mathbf{j} \\ &+ \Big[(\frac{\sqrt{3}}{2} + \frac{5}{6} + \ln\frac{y(\lambda+\varepsilon)}{bcx^2})b^2x^2 - by(\lambda-\varepsilon)\Big]\mathbf{j} \otimes \mathbf{j} \end{aligned}$$

by (7.12) and (7.15). It is an easy matter to verify that, for every $\lambda \in (0,\lambda_c)$, $\mathbf{U}^\lambda$ verifies the equilibrium equation $\operatorname{div}\mathbf{U}^\lambda + b\mathbf{j} = \mathbf{0}$ in Ω and the boundary conditions (7.1), so that λ is statically admissible and the loads $(\mathbf{s}^\lambda, \mathbf{b}^\lambda)$ are strongly compatible.

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