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Work conditions and energy functions for ideal elastic–plastic materials

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Abstract The paper deals with the cyclic second law and Il'yushin's condition for isothermal, ideal, isotropic, elastic–plastic materials at large deformations. The second law is equivalent to the existence of the elastic potential and the nonnegativity of plastic power. The material admits infinitely many free energies: the set of all energy functions is described in terms of a dissipation function. Its convexification provides the optimal lower bound for plastic work; it also figures in the maximal and minimal energies. Il'yushin's condition is equivalent to the existence of the elastic potential and a new condition that is stronger than the normality of the plastic stretching and the convexity of the stress range. Il'yushin's condition is also equivalent to the existence of a new kind of energy functions, 'the initial and final extended energy functions. Materials of type C are introduced for which the initial extended energy function has additional convexity properties. It can be viewed as a stored energy of a Hencky hyperelastic material associated with the elastic–plastic material.

Contents

1 Introduction

This paper deals with the cyclic second law and Il'yushin's condition for isothermal, ideal, isotropic elastic–plastic materials at large deformations. The ideal nature of the material excludes hardening. Both the isothermal cyclic second law and Il'yushin's condition assert the nonnegativity of the work of external forces on for certain classes of cyclic processes. The former speaks of cycles in the state space; the latter of cycles in the space of total deformations.

Sections 3–5 analyze the material from the point of view of the general theory of actions and potentials on thermodynamical systems by COLEMAN & OWEN [5]. That theory provides an unbiased derivation of potentials like the free energy; moreover, it enables one to discuss their uniqueness/nonuniqueness and to determine the maximal and minimal potentials. For materials treated here, the plastic flow rule can be ”nonassociated, and the restrictions equivalent to the second law are the existence of the elastic potential for the elastic stress function and the nonnegativity of the plastic power in every process. Using that, and defining the free energy as any state function satisfying the dissipation inequality, one finds that the elastic potential is a possible free energy. However, the free energies are highly nonuinque here: infinitely many of them differ mutually by a nonconstant state function. A complete description of the set of all free energy functions is given using a dissipation function which is closely related to the plastic power. The central result of this part says that the optimal lower bound for the plastic work is the convexification of the dissipation function. The maximal/minimal free energies are the elastic potential plus/minus the convexification of the dissipation function at the logarithmic plastic stretching. Analyses of elastic–plastic materials from the standpoint of [5] have been given previously by COLEMAN & OWEN [6–7] for unidimensional infinitesimal ideal materials, by LUCCHESI [15] for three-dimensional infinitesimal materials with hardening, and by LUCCHESI & ÁILHAVŮ [19] for large-deformations three-dimensional infinitesimal materials with hardening.

In the context of infinitesimal deformations theory, Il'yushin's condition is classically known to imply the convexity of the stress range (CSR) (the stress range is the region below the yield surface) and the normality rule (NR) asserting that during loading, the plastic stretching is proportional to the exterior normal to the yield surface (e.g., [27]). The analysis has been extended to large deformations by LUCCHESI & PODIO-GUIDUGLI [16–17] (see also [18] and [11]). Their analysis showed that Il'yushin's condition still implies the CSR and NR provided that the stress range is interpreted as the set of elastically reachable Kirchhoff stresses (as opposed to Cauchy stresses; see Section 2 for definition); in addition, it implies the existence of the elastic potential (EP). However, their analysis shows that conversely EP, CSR and NR lead to Il'yushin's condition only for deformation cycles which are small in a precisely defined sense. The main feature of the analysis in Sections 6–7, and what distinguishes it from the previous work, is the possibility of treating arbitrarily large deformation cycles. Namely, Il'yushin's condition is shown to be equivalent to EP and a new condition E (Condition (ii) of Theorem 2). Condition E is stronger than NR and CSR. On the basis of Il'yushin's condition for large cycles, new energy functions are constructed satisfying dissipation inequalities stronger than those based on the second law. Their main feature is that they are nonlocal in time (have no localized counterparts). There are two types of the new energy functions; I call these the initial and final (extended) energy functions.

In Sections 8–9 materials called materials of type C are studied which permit a more explicit analysis. The main requirement is a restricted convexity in the logarithmic deformation on symmetric deformation gradients of fixed determinant. Materials of type C satisfy Il'yushin's condition and the initial extended energy admits a description in terms of convex–conjugated functions in the logarithmic deformation. The initial extended energy can be interpreted as a stored energy of some hyperelastic material; this material is the large–deformations analog of the hyperelastic material in the infinitesimal Hencky theory of plasticity. Its stress relation coincides with the nonlinear elastic response of the plastic material for deformation gradients from the original elastic range; if the deformation is outside the elastic range, then the stress is determined by some nonlinear projection onto the elastic range (in the space of the logarithmic deformations, see Section 8 for a precise description). Unexpectedly, the extended energy grows logarithmically at large deformations, in contrast to the infinitesimal Hencky theory, where the growth is linear, cf. TŽMAM [33]. As an illustration, an elastic–plastic material of type C is considered for which the extended energy and all other derived objects can be calculated explicitly.

The above description makes it clear that the logarithmic deformation and logarithmic convexity/concavity emerge naturally from the analysis. The reason is that for large deformations theory, the processes of minimal dissipation are those for which the logarithm of plastic deformation, and not the plastic deformation itself, is linear in time.

2 Ideal elastic–plastic materials

Let Lin denote the set of all second–order tensors on a three-dimensional space; we use the scalar product $\boldsymbol{A}\cdot\boldsymbol{B} = \operatorname{tr}(\boldsymbol{A}\boldsymbol{B}^{\mathrm{T}})$ on Lin and the euclidean norm $|\boldsymbol{A}| := \sqrt{\boldsymbol{A}\cdot\boldsymbol{A}}$. The deformation gradients are interpreted as the elements of the set Lin^+ of all second–order tensors with positive determinant. Sym^+ is the set of all positive definite symmetric tensors.

For the considerations of the paper, it is convenient to describe the response in terms of states and processes (cf. [22, 5–7, 30, 18, 31, 2–3, 26]). The *state space* Σ is

$$\Sigma = \{\sigma = (\boldsymbol{E},\boldsymbol{P}) : \boldsymbol{E} \in \mathscr{E}, \boldsymbol{P} \in \mathrm{Unim}\} = \mathscr{E} \times \mathrm{Unim}$$

where $\mathscr{E} \subset \mathrm{Lin}^+$ and Unim is the set of all tensors with determinant 1. The set $\mathscr{E}$ is closely related to the elastic range, see below. The *states* σ are pairs $(\boldsymbol{E},\boldsymbol{P})$ where $\boldsymbol{E} \in \mathscr{E}$ is the *elastic deformation* and $\boldsymbol{P}$ is an unimodular *plastic deformation.* The *total deformation* and the *Kirchhoff stress* of $\sigma = (\boldsymbol{E},\boldsymbol{P})$ are

$$\hat{\boldsymbol{F}}(\sigma) := \boldsymbol{E}\boldsymbol{P}, \quad \hat{\boldsymbol{T}}(\sigma) := \bar{\boldsymbol{T}}(\boldsymbol{E}) \tag{1}$$

where $\bar{\boldsymbol{T}} : \mathscr{E} \to \mathrm{Sym}$ is a given function and Sym is the set of all symmetric tensors. Equation (1) expresses the multiplicative decomposition of the deformation gradient into the elastic and plastic parts [14]. The Kirchhoff stress $\boldsymbol{T} := (\det \boldsymbol{F})\boldsymbol{T}_C$ is more convenient than the Cauchy stress $\boldsymbol{T}_C$ here. Let $\bar{\boldsymbol{S}} : \mathscr{E} \to \mathrm{Lin}_0$ denote the mapping defined by

$$\bar{\boldsymbol{S}}(\boldsymbol{E}) = \boldsymbol{E}^{\mathrm{T}}\bar{\boldsymbol{T}}(\boldsymbol{E})\boldsymbol{E}^{-\mathrm{T}} - \tfrac{1}{3}(\operatorname{tr}\bar{\boldsymbol{T}}(\boldsymbol{E}))\mathbf{1},$$

where Lin_0 is the set of all traceless tensors. The definition is motivated by the expression for the plastic work (4), below. The *stress range* $\mathscr{S}$ is defined by

$$\mathscr{S} := \bar{\boldsymbol{S}}(\mathscr{E}).$$

To specify the *class of processes* Π, we assume that the plastic deformation changes only when $\boldsymbol{E}$ is on the boundary $\partial\mathscr{E}$ of $\mathscr{E}$, and that the direction of the plastic stretching is determined by a prescribed function $\bar{\boldsymbol{M}}$ mapping $\partial\mathscr{E}$ into the set of all traceless symmetric tensors Sym_0, normalized by $|\bar{\boldsymbol{M}}(\boldsymbol{E})| = 1$, $\boldsymbol{E} \in \partial\mathscr{E}$. Thus we assume the flow rule of the form

$$\boldsymbol{D}^p(t) = \alpha(t)\bar{\boldsymbol{M}}(\boldsymbol{E}(t)), \tag{2}$$

where the *plastic stretching* $\boldsymbol{D}^p(t)$ is the symmetric part of $\boldsymbol{L}^p(t) = \dot{\boldsymbol{P}}(t)\boldsymbol{P}(t)^{-1}$, and $\alpha(t) > 0$ is a coefficient of proportionality. Formally, Π consists of all functions $\pi = (\boldsymbol{E}(\cdot), \boldsymbol{P}(\cdot))$ mapping closed intervals of the type $[0, d_\pi], d_\pi > 0$, into Σ, which are continuous and piecewise continuously differentiable and which satisfy the following condition: for every $t \in [0, d_\pi]$ for which $\boldsymbol{D}^p(t) \neq \boldsymbol{0}$, one has $\boldsymbol{E}(t) \in \partial\mathscr{E}$ and (2) holds for some $\alpha(t) > 0$. For convenience it is assumed that every process starts at time $t = 0$. The number d_π can be different for different processes and it is called the *duration of the process.* The states $\pi^{\mathrm{i}} = \pi(0)$ and $\pi^{\mathrm{f}} = \pi(d_\pi)$ are called the *initial* and *final states* of the process π. We also write $\pi^{\mathrm{i}} = (\boldsymbol{E}^{\mathrm{i}}, \boldsymbol{P}^{\mathrm{i}}), \pi^{\mathrm{f}} = (\boldsymbol{E}^{\mathrm{f}}, \boldsymbol{P}^{\mathrm{f}})$ and $\boldsymbol{F}^{\mathrm{i}} = \hat{\boldsymbol{F}}(\pi^{\mathrm{i}}), \boldsymbol{F}^{\mathrm{f}} = \hat{\boldsymbol{F}}(\pi^{\mathrm{f}})$. The considerations in the paper require to construct new processes from given ones via the operation of continuation. If $\pi_1, \pi_2 \in \Pi$ are two processes with $\pi_1^{\mathrm{f}} = \pi_2^{\mathrm{i}}$ then the *continuation* $\pi_1 * \pi_2$ *of* π_1 *with* π_2, is

$$\pi_1 * \pi_2(t) = \begin{cases} \pi_1(t) & \text{if } \ t \in [0, d_{\pi_1}], \\ \pi_2(t - d_{\pi_1}) & \text{if } \ t \in [d_{\pi_1}, d_{\pi_1} + d_{\pi_2}]. \end{cases}$$

For every $\delta > 0$ let

$$\mathscr{E}_\delta = \{\boldsymbol{A} \in \mathscr{E} : \det \boldsymbol{A} = \delta\}. \tag{3}$$

A set $M \subset \mathrm{Lin}$ is said to be objective or isotropic if

$$\boldsymbol{RA} \in M, \quad \boldsymbol{RAR}^{\mathrm{T}} \in M,$$

respectively, for every $\boldsymbol{A} \in M$ and $\boldsymbol{R} \in \mathrm{Rot}$. Here Rot is the proper orthogonal group. A function $f : M \to \mathbb{R}, M \subset \mathrm{Lin}$, is said to be objective or isotropic if M is objective or isotropic, respectively, and

$$f(\boldsymbol{RA}) = f(\boldsymbol{A}), \quad f(\boldsymbol{RAR}^{\mathrm{T}}) = f(\boldsymbol{A}),$$

respectively, for every $\boldsymbol{A} \in M$ and $\boldsymbol{R} \in \mathrm{Rot}$. A function $\boldsymbol{G} : M \to \mathrm{Lin}$ is said to be objective or isotropic if M is objective or isotropic, respectively, and

$$\boldsymbol{G}(\boldsymbol{RA}) = \boldsymbol{RG}(\boldsymbol{A})\boldsymbol{R}^{\mathrm{T}}, \quad \boldsymbol{G}(\boldsymbol{RAR}^{\mathrm{T}}) = \boldsymbol{RG}(\boldsymbol{A})\boldsymbol{R}^{\mathrm{T}},$$

respectively, for every $\boldsymbol{A} \in M$ and $\boldsymbol{R} \in \mathrm{Rot}$; $\boldsymbol{G}$ is said to be scalar–objective if M is objective and

$$\boldsymbol{G}(\boldsymbol{RA}) = \boldsymbol{G}(\boldsymbol{A})$$

for every $\boldsymbol{A} \in M$ and $\boldsymbol{R} \in \mathrm{Rot}$.

Definition 1 The objects $\mathscr{E}, \bar{\boldsymbol{T}}, \bar{\boldsymbol{M}}$ are said to determine an ideal elastic–plastic material if they satisfy the following conditions:

(i) $\mathscr{E}$ is a closure of its interior, any two points of $\mathscr{E}$ can be connected by a piecewise smooth curve in $\mathscr{E}$, and $\bar{\boldsymbol{T}}$ is continuous.

(ii) For every $\boldsymbol{C} \in \mathrm{Sym}_0$, $|\boldsymbol{C}| = 1$, we have $\bar{\boldsymbol{M}}(\boldsymbol{E}) = \boldsymbol{C}$ for at least one $\boldsymbol{E} \in \partial\mathscr{E}$.
(iii) $\mathscr{S}$ is bounded and $\bar{\boldsymbol{S}}$ maps the interior and the boundary of $\mathscr{E}$ onto the interior and boundary of $\mathscr{S}$, respectively.
(iv) For each $\delta > 0$, the set $\mathscr{E}_\delta$ is nonempty.
(v) The set $\mathscr{E}$ and the function $\bar{\boldsymbol{T}}$ are objective and isotropic and the function $\bar{\boldsymbol{M}}$ is scalar–objective and isotropic.

The quadruple $\mathscr{M} = (\Sigma, \Pi, \hat{\boldsymbol{F}}, \hat{\boldsymbol{T}})$ is called the *ideal elastic–plastic material,* or briefly the *material.* The material just constructed is a special case of a material with elastic range, cf. [27, 24–25, 29, 8, 18]. For processes starting at $\sigma = (\boldsymbol{E}, \boldsymbol{P})$ and remaining in the elastic range $\mathscr{E}(\sigma) := \mathscr{E}\boldsymbol{P}$ the material behaves like an elastic material with the elastic response

$$\hat{\boldsymbol{T}}^*(\sigma, \boldsymbol{F}) = \bar{\boldsymbol{T}}(\boldsymbol{F}\boldsymbol{P}^{-1}), \quad \boldsymbol{F} \in \mathscr{E}(\sigma).$$

Item (v) of definition 1 express the objectivity and isotropy of the material. The objectivity and isotropy of $\mathscr{E}$ and $\bar{\boldsymbol{T}}$ need not be commented; however, note that they imply that $\bar{\boldsymbol{S}}$ is scalar–objective and isotropic; moreover, its values are symmetric tensors. The scalar–objective nature of $\bar{\boldsymbol{M}}$ is consistent with the following transformation rules under a change of frame and change of reference configuration: if $\pi = (\boldsymbol{E}(\cdot), \boldsymbol{P}(\cdot))$ is a process, $\boldsymbol{Q}$ a piecewise continuously differentiable function on $[0, d_\pi]$ with values in Rot, and $\boldsymbol{R}$ a fixed element of Rot, then

$$\boldsymbol{F} \mapsto \boldsymbol{Q}\boldsymbol{F} \Rightarrow \boldsymbol{E} \mapsto \boldsymbol{Q}\boldsymbol{E}, \quad \boldsymbol{P} \mapsto \boldsymbol{P},$$

$$\boldsymbol{F} \mapsto \boldsymbol{F}\boldsymbol{R} \Rightarrow \boldsymbol{E} \mapsto \boldsymbol{E}, \quad \boldsymbol{P} \mapsto \boldsymbol{P}\boldsymbol{R}.$$

Other transformation laws are possible since the plastic deformation is determined only to within a rotation in isotropic materials [29].

3 The cyclic second law; its first consequences

The *work* of external forces on the material in the process $\pi = (\boldsymbol{E}(\cdot), \boldsymbol{P}(\cdot))$ is

$$w(\pi) = \int_0^d \boldsymbol{T} \cdot \boldsymbol{L}\, dt,$$

where $\boldsymbol{L} = \dot{\boldsymbol{F}}\boldsymbol{F}^{-1}, \boldsymbol{F} = \boldsymbol{E}\boldsymbol{P}, d = d_\pi$, and $\boldsymbol{T} = \bar{\boldsymbol{T}}(\boldsymbol{E})$ is the time evolution of the Kirchhoff stress in the process. Using the isotropy and symmetry of $\bar{\boldsymbol{T}}$ we find that

$$w(\pi) = w^E(\pi) + w^P(\pi),$$

where

$$w^E(\pi) = \int_0^d \boldsymbol{T} \cdot \dot{\boldsymbol{E}}\boldsymbol{E}^{-1}\, dt, \quad w^P(\pi) = \int_0^d \boldsymbol{S} \cdot \boldsymbol{D}^p\, dt \tag{4}$$

are the *elastic* and *plastic works* in π, respectively, where $\boldsymbol{S} = \bar{\boldsymbol{S}}(\boldsymbol{E})$ is the time-evolution of the traceless part of the Kirchhoff stress. A process π is said to be a σ-cycle if $\pi^{\mathrm{i}} = \pi^{\mathrm{f}}$. For isothermal materials the second law reduces to the following assertion [5].

3.1 The cyclic second law *For every σ-cycle π, $w(\pi) \geq 0$.*

A function $p : \mathscr{E} \to \mathbb{R}$ is said to be an *elastic potential* for $\bar{\boldsymbol{T}}$ if

$$p(\boldsymbol{E}^{\mathrm{f}}) - p(\boldsymbol{E}^{\mathrm{i}}) = w^{E}(\pi) \tag{5}$$

for every process. It follows that if p exists, it is objective and isotropic, and continuously differentiable in the interior of $\mathscr{E}$ with the *stress relation* $\bar{\boldsymbol{T}}(\boldsymbol{E}) = Dp(\boldsymbol{E})\boldsymbol{E}^{\mathrm{T}}$ prevailing there.

Proposition 1 *The material satisfies the cyclic second law if and only if $\bar{\boldsymbol{T}}$ has an elastic potential and*

$$\bar{\boldsymbol{S}}(\boldsymbol{E}) \cdot \bar{\boldsymbol{M}}(\boldsymbol{E}) \geq 0 \tag{6}$$

for every $\boldsymbol{E} \in \partial\mathscr{E}$.

Inequality (6) says that the plastic power, and hence also the plastic work, is nonnegative: $\boldsymbol{S} \cdot \boldsymbol{D}^{p} = \alpha(t)\boldsymbol{S} \cdot \boldsymbol{M} \geq 0$. Let us emphasize that this consequence does not hold for general, nonideal, elastic–plastic materials.

Proof Suppose that the material satisfies the cyclic second law. To prove the existence of the potential, consider the elastic process $\pi = (\boldsymbol{E}(\cdot), \mathbf{1})$ where $\boldsymbol{E}(\cdot) : [0,1] \to \mathscr{E}$ is a path with values in $\mathscr{E}$. If $\boldsymbol{E}(0) = \boldsymbol{E}(1)$ then the cyclic second law asserts $w(\pi) \geq 0$. Applying the same to the time reversal $\bar{\pi}$ we obtain $w(\bar{\pi}) = -w(\pi) \geq 0$ and thus $w(\pi) = 0$. Since $w(\pi) = w^{E}(\pi)$, the vector field $\bar{\boldsymbol{T}}$ is path independent on $\mathscr{E}$; hence it has an elastic potential. Proof of (6): Let $\boldsymbol{E} \in \partial\mathscr{E}$ and set $\boldsymbol{D} := \bar{\boldsymbol{M}}(\boldsymbol{E})$. Consider an orthonormal basis of eigenvectors of $\boldsymbol{D}$ so that $\boldsymbol{D} = \operatorname{diag}(d_1, d_2, d_3)$. Let $\mathbb{P}^3$ be the group of all 3 by 3 permutation matrices Z which we identify with orthogonal tensors $\boldsymbol{Z}$ in our basis. Enumerate the elements of $\mathbb{P}^3$ arbitrarily to obtain a sequence $\boldsymbol{Z}_\alpha$, $\alpha = 1, \ldots, n \equiv 3!$. Since

$$\boldsymbol{Z}_\alpha \boldsymbol{D} \boldsymbol{Z}_\alpha^{\mathrm{T}} = \operatorname{diag}(d_{\sigma_\alpha(1)}, d_{\sigma_\alpha(2)}, d_{\sigma_\alpha(3)})$$

where σ_α is the permutation corresponding to $\boldsymbol{Z}_\alpha$, the family $\{\boldsymbol{Z}_\alpha \boldsymbol{D} \boldsymbol{Z}_\alpha^{\mathrm{T}},\ \alpha = 1, \ldots, n\}$ is commutative. Using $\operatorname{tr} \boldsymbol{D} = 0$, one finds that

$$\sum_{\alpha=1}^{n} \boldsymbol{Z}_\alpha \boldsymbol{D} \boldsymbol{Z}_\alpha^{\mathrm{T}} = \mathbf{0}. \tag{7}$$

Define inductively the processes $\pi_\alpha = (\boldsymbol{E}_\alpha(\cdot), \boldsymbol{P}_\alpha(\cdot))$, of duration 1, by

$$\boldsymbol{E}_1(t) = \boldsymbol{Z}_1 \boldsymbol{E} \boldsymbol{Z}_1^{\mathrm{T}}, \quad \boldsymbol{P}_1(t) = \boldsymbol{Z}_1 e^{\boldsymbol{D}t} \boldsymbol{Z}_1^{\mathrm{T}},$$

and for $\alpha > 1$,

$$\boldsymbol{E}_\alpha(t) = \boldsymbol{Z}_\alpha \boldsymbol{E} \boldsymbol{Z}_\alpha^{\mathrm{T}}, \quad \boldsymbol{P}_\alpha(t) = \boldsymbol{Z}_\alpha e^{\boldsymbol{D}t} \boldsymbol{Z}_\alpha^{\mathrm{T}} \boldsymbol{P}_{\alpha-1}^{\mathrm{f}},$$

$t \in [0,1]$. These are really processes and one finds that $\boldsymbol{D}_\alpha^{p}(t) = \boldsymbol{Z}_\alpha \boldsymbol{D} \boldsymbol{Z}_\alpha^{\mathrm{T}}$, and

$$w^{P}(\pi_\alpha) = \bar{\boldsymbol{S}}(\boldsymbol{E}) \cdot \bar{\boldsymbol{M}}(\boldsymbol{E}). \tag{8}$$

Let $\rho_\alpha = (\boldsymbol{U}_\alpha(\cdot), \boldsymbol{P}_\alpha^{\mathrm{f}})$, $\alpha = 1, \ldots, n$, be elastic processes such that

$$\boldsymbol{U}_\alpha^{\mathrm{i}} = \boldsymbol{Z}_\alpha \boldsymbol{E} \boldsymbol{Z}_\alpha^{\mathrm{T}}, \quad \boldsymbol{U}_\alpha^{\mathrm{f}} = \boldsymbol{Z}_{\alpha+1} \boldsymbol{E} \boldsymbol{Z}_{\alpha+1}^{\mathrm{T}}$$

if $i < n$, and

$$\boldsymbol{U}_n^{\mathrm{i}} = \boldsymbol{Z}_n \boldsymbol{E} \boldsymbol{Z}_n^{\mathrm{T}}, \quad \boldsymbol{U}_n^{\mathrm{f}} = \boldsymbol{Z}_1 \boldsymbol{E} \boldsymbol{Z}_1^{\mathrm{T}}.$$

One finds that the process $\pi = (\boldsymbol{E}(\cdot), \boldsymbol{P}(\cdot)) := \pi_1 * \rho_1 * \pi_2 * \rho_2 \cdots \pi_n * \rho_n$ can be constructed,

$$\boldsymbol{E}^{\mathrm{i}} = \boldsymbol{Z}_1 \boldsymbol{E} \boldsymbol{Z}_1, \quad \boldsymbol{E}^{\mathrm{f}} = \boldsymbol{Z}_1 \boldsymbol{E} \boldsymbol{Z}_1,$$

and

$$\boldsymbol{P}^{\mathrm{i}} = \mathbf{1}, \quad \boldsymbol{P}^{\mathrm{f}} = \exp[\sum_{\alpha=1}^{n} \boldsymbol{Z}_\alpha \boldsymbol{D} \boldsymbol{Z}_\alpha^{\mathrm{T}}] = \mathbf{1}.$$

Here we use (7) and the commutativity of the family $\{\boldsymbol{Z}_\alpha \boldsymbol{D} \boldsymbol{Z}_\alpha^{\mathrm{T}}, \alpha = 1, \ldots, n\}$. As also $\boldsymbol{E}^{\mathrm{i}} = \boldsymbol{E}^{\mathrm{f}}$, the process π is cyclic. Since $w^E(\pi) = 0$ by (5), $w(\pi) \geq 0$ reduces, by (8), to

$$w^P(\pi) = n \bar{\boldsymbol{S}}(\boldsymbol{E}) \cdot \bar{\boldsymbol{M}}(\boldsymbol{E}) \geq 0$$

and (6) follows. Conversely, if the two conditions of the theorem hold and π is a cyclic process then $w^E(\pi) = 0$ by the existence of the potential and $w^P(\pi) \geq 0$ by (6). Hence the cyclic second law holds. □

4 The free energy functions

Any function $\psi : \Sigma \to \mathbb{R}$ satisfying

$$\psi(\pi^{\mathrm{f}}) - \psi(\pi^{\mathrm{i}}) \leq w(\pi) \tag{9}$$

for every process π is referred to as the *free energy function.* Using the nonnegativity of the plastic power, it is shown below that the elastic potential is one example of the free energy function, and a description is given of all free energies. The specific features of the material imply that any free energy function splits into a sum of the reversible elastic potential and a residual energy function that depends only on the plastic deformation.

Proposition 2 *Let the material satisfy the cyclic second law and let p be its elastic potential. A function* $\psi : \Sigma \to \mathbb{R}$ *is a free energy function if and only if*

$$\psi(\sigma) = p(\boldsymbol{E}) + r(\boldsymbol{P}) \tag{10}$$

for every $\sigma = (\boldsymbol{E}, \boldsymbol{P}) \in \Sigma$, *where* $r : \mathrm{Unim} \to \mathbb{R}$ *satisfies*

$$r(\boldsymbol{P}^{\mathrm{f}}) - r(\boldsymbol{P}^{\mathrm{i}}) \leq w^P(\pi) \tag{11}$$

for every process $\pi = (\boldsymbol{E}(\cdot), \boldsymbol{P}(\cdot)) \in \Pi$. *In particular, the function* $\psi : \Sigma \to \mathbb{R}$ *given by*

$$\psi(\sigma) = p(\boldsymbol{E}), \tag{12}$$

$\sigma = (\boldsymbol{E}, \boldsymbol{P}) \in \Sigma$, *is a free energy function.*

Let us emphasize that a general, (nonideal) elastic–plastic material need not have a free energy independent of plastic deformation. Every function r satisfying (11) in every process is called the *residual energy function.* The problem of describing all free energy functions reduces to that of describing all residual energy functions.

Proof By Proposition 1, $\bar{\boldsymbol{T}}$ has an elastic potential p and the plastic work is nonnegative in every process. If ψ is a free energy function and $\boldsymbol{P} \in \mathrm{Unim}$, then for every path

$\boldsymbol{E}(\cdot) : [0,1] \to \mathscr{E}$ the process $\pi = (\boldsymbol{E}(\cdot), \boldsymbol{P})$ is elastic and as $w(\pi) = w^E(\pi)$ is given by (5), the dissipation inequality (9) reads

$$\psi(\boldsymbol{E}^{\mathrm{f}}, \boldsymbol{P}) - \psi(\boldsymbol{E}^{\mathrm{i}}, \boldsymbol{P}) \geq p(\boldsymbol{E}^{\mathrm{f}}) - p(\boldsymbol{E}^{\mathrm{i}}). \tag{13}$$

Replacing π by its time-reversal $\bar{\pi}$ we obtain the opposite inequality and hence the equality must hold in (13). It follows that for each $\boldsymbol{P} \in \mathrm{Unim}$, $\psi(\cdot, \boldsymbol{P})$ differs by a constant $r(\boldsymbol{P})$ from p which leads to (10). Using (10), one sees that (9) reduces to (11). Reversing the direction of the arguments, one finds that every function ψ of the form (10) is a free energy function. Since the plastic work is nonnegative, the function $r \equiv 0$ satisfies (11) and hence the ψ given by (12) is a free energy function. □

To simplify notation, if $\boldsymbol{A}, \boldsymbol{B} \in \mathrm{Lin}$, we write

$$\boldsymbol{A} \sim \boldsymbol{B}$$

to mean that $|\boldsymbol{A}| = 1$, $\boldsymbol{B} \neq \boldsymbol{0}$ and

$$\boldsymbol{A} = \boldsymbol{B}/|\boldsymbol{B}|.$$

Define the *dissipation function* $m : \mathrm{Sym}_0 \to \mathbb{R}$ by

$$m(\boldsymbol{D}) := \inf\{\bar{\boldsymbol{S}}(\boldsymbol{E}) \cdot \boldsymbol{D} : \boldsymbol{E} \in \partial\mathscr{E}, \bar{\boldsymbol{M}}(\boldsymbol{E}) \sim \boldsymbol{D}\}$$

if $\boldsymbol{D} \neq \boldsymbol{0}$ and $m(\boldsymbol{0}) := 0$. Note that the infimum is taken over a nonempty set by Definition 1(ii). If the material satisfies the cyclic second law, then m is nonegative by Proposition 1.

If r is a function on an open subset of Unim and $\boldsymbol{P}$ is in the domain of r, then r is said to have a total differential at $\boldsymbol{P}$ if the function $\boldsymbol{A} \mapsto r(e^{\boldsymbol{A}}\boldsymbol{P})$, $\boldsymbol{A} \in \mathrm{Lin}_0$, has a FrŽchet derivative at $\boldsymbol{0}$ in the sense of [9; Part I, Chapter VIII]. The differential (derivative) $Dr(\boldsymbol{P})$ is then defined as the unique element satisfying $\mathrm{tr}(Dr(\boldsymbol{P})\boldsymbol{P}^{\mathrm{T}}) = 0$ such that

$$\frac{d}{dt} r(\gamma(t))\bigg|_{t=0} = Dr(\boldsymbol{P}) \cdot \dot{\gamma}(0)$$

for every smooth curve in Unim with $\gamma(0) = \boldsymbol{P}$. Note that Unim is a regular 8-dimensional surface in the 9-dimensional space Lin and that the unit normal and the tangent space to Unim at $\boldsymbol{A}$ are $\boldsymbol{A}^{-\mathrm{T}}/|\boldsymbol{A}^{-\mathrm{T}}|$ and $\{\boldsymbol{M} \in \mathrm{Lin} : \mathrm{tr}(\boldsymbol{M}\boldsymbol{A}^{-1}) = 0\}$. The Haar measure ([9; Part II, Chapter XIV and Part IV, Chapter XIX]) and the surface measures on Unim are mutually absolutely continuous and their images in any local coordinate chart on Unim are absolutely continuous with respect to the 8-dimensional Lebesgue measure in that coordinate chart. Hence there is a well defined notion of almost everywhere. If r is a locally Lipschitz continuous function on Unim, Rademachers theorem ([21]) asserts the existence of the total differential for almost every $\boldsymbol{P} \in \mathrm{Unim}$. Finally note that if $\boldsymbol{P}_1, \boldsymbol{P}_2 \in \mathrm{Unim}$ then there exist a unique pair $\boldsymbol{R} \in \mathrm{Rot}$, $\boldsymbol{D} \in \mathrm{Sym}_0$ such that $\boldsymbol{P}_2 = \boldsymbol{R}e^{\boldsymbol{D}}\boldsymbol{P}_1$. This follows from the polar decomposition $\boldsymbol{P}_2\boldsymbol{P}_1^{-1} = \boldsymbol{R}\boldsymbol{U}$ by writing $\boldsymbol{U} = e^{\boldsymbol{D}}$, $\boldsymbol{D} \in \mathrm{Sym}_0$.

Proposition 3 *If the material satisfies the cyclic second law and* $r : \mathrm{Unim} \to \mathbb{R}$ *is a function then the following three conditions are equivalent:*
(i) *r is a residual energy;*
(ii) *for every* $\boldsymbol{P}_1, \boldsymbol{P}_2 \in \mathrm{Unim}$ *one has*

$$r(\boldsymbol{P}_2) - r(\boldsymbol{P}_1) \le m(\boldsymbol{D}) \tag{14}$$

where $\boldsymbol{D} \in \mathrm{Sym}_0$ *is determined by the condition that* $\boldsymbol{P}_2 = \boldsymbol{R}e^{\boldsymbol{D}}\boldsymbol{P}_1$ *for some* $\boldsymbol{R} \in \mathrm{Rot}$;

(iii) *r is locally Lipschitz continuous, objective, and for almost every* $\boldsymbol{P} \in \mathrm{Unim}$,

$$Dr(\boldsymbol{P})\boldsymbol{P}^{\mathrm{T}} \cdot \boldsymbol{D} \le m(\boldsymbol{D}) \qquad \text{for all} \qquad \boldsymbol{D} \in \mathrm{Sym}_0. \tag{15}$$

Proof (i) $\Rightarrow$ (ii) : Let (i) hold and let $\boldsymbol{P}_1, \boldsymbol{P}_1, \boldsymbol{D}, \boldsymbol{R}$ be as in (ii). Let us prove that

$$r(\boldsymbol{P}_2) - r(\boldsymbol{P}_1) \le \begin{cases} 0 & \text{if} \quad \boldsymbol{D} = \boldsymbol{0}, \\ \bar{\boldsymbol{S}}(\boldsymbol{E}) \cdot \boldsymbol{D} & \text{if} \quad \boldsymbol{D} \neq \boldsymbol{0}, \end{cases} \tag{16}$$

where $\boldsymbol{E} \in \partial\mathscr{E}$ is any element such that

$$\bar{\boldsymbol{M}}(\boldsymbol{E}) \sim \boldsymbol{D}. \tag{17}$$

Let $\boldsymbol{E}$ be arbitrary if $\boldsymbol{D} = \boldsymbol{0}$ or such that $\boldsymbol{E} \in \partial\mathscr{E}$ and (17) holds if $\boldsymbol{D} \neq 0$. Let $\boldsymbol{Q} : [0,1] \to \mathrm{Rot}$ be any continuously differentiable function such that $\boldsymbol{Q}(0) = \boldsymbol{1}$ and $\boldsymbol{Q}(1) = \boldsymbol{R}$. Define $\pi = (\boldsymbol{E}(\cdot), \boldsymbol{P}(\cdot))$ by

$$\boldsymbol{E}(t) = \boldsymbol{Q}(t)\boldsymbol{E}\boldsymbol{Q}(t)^{\mathrm{T}}, \quad \boldsymbol{P}(t) = \boldsymbol{Q}(t)e^{\boldsymbol{D}t}\boldsymbol{P}_1, \quad t \in [0,1].$$

This is a process, $\boldsymbol{D}^p(t) = \boldsymbol{Q}(t)\boldsymbol{D}\boldsymbol{Q}(t)^{\mathrm{T}}$, $\boldsymbol{P}^{\mathrm{f}} = \boldsymbol{P}_2$, $\boldsymbol{P}^{\mathrm{i}} = \boldsymbol{P}_1$, and $w^P(\pi) = \bar{\boldsymbol{S}}(\boldsymbol{E}) \cdot \boldsymbol{D}$. Thus the residual dissipation inequality gives (16). The definition of $m(\boldsymbol{D})$ then gives (14).

(ii) $\Rightarrow$ (iii) : Assume that (ii) holds. Let us first prove that r is locally Lipschitz continuous. Recall that it is assumed (Definition 1(iii)) $\mathscr{S}$ is bounded. Set

$$c_1 = \max\{|\boldsymbol{S}| : \boldsymbol{S} \in \mathscr{S}\}.$$

By Definition 1(ii), for every $\boldsymbol{D} \in \mathrm{Sym}_0$, $\boldsymbol{D} \neq \boldsymbol{0}$, there exists an $\boldsymbol{E} \in \partial\mathscr{E}$ such that (17) holds. We then have

$$\bar{\boldsymbol{S}}(\boldsymbol{E}) \cdot \boldsymbol{D} \le c_1|\boldsymbol{D}|.$$

Thus for any $\boldsymbol{P}_1, \boldsymbol{P}_2 \in \mathrm{Unim}$ we have

$$r(\boldsymbol{P}_2) - r(\boldsymbol{P}_1) \le c_1|\boldsymbol{D}| \tag{18}$$

where $\boldsymbol{D}$ satisfies $\boldsymbol{P}_2 = \boldsymbol{R}e^{\boldsymbol{D}}\boldsymbol{P}_1$. Since this holds for any $\boldsymbol{P}_1, \boldsymbol{P}_2 \in \mathrm{Unim}$, we can interchange the roles of $\boldsymbol{P}_1, \boldsymbol{P}_2$; one finds that $\boldsymbol{D}$ changes to $\bar{\boldsymbol{D}} = -\boldsymbol{R}\boldsymbol{D}\boldsymbol{R}^{\mathrm{T}}$ and (18) for the interchanged pair provides $r(\boldsymbol{P}_1) - r(\boldsymbol{P}_2) \le c_1|\boldsymbol{D}|$; hence

$$|r(\boldsymbol{P}_2) - r(\boldsymbol{P}_1)| \le c_1|\boldsymbol{D}|. \tag{19}$$

This inequality obviously implies that r is locally Lipschitz continuous. To prove that r is objective, it suffices to note that if $\boldsymbol{P} \in \mathrm{Unim}$, $\boldsymbol{Q} \in \mathrm{Rot}$, then for $\boldsymbol{P}_1 := \boldsymbol{P}$, $\boldsymbol{P}_2 = \boldsymbol{Q}\boldsymbol{P}$, (19) reduces to $r(\boldsymbol{Q}\boldsymbol{P}) = r(\boldsymbol{P})$. We finally prove (15). Let $\boldsymbol{P} \in \mathrm{Unim}$ be such that Dr exists. The application of (14) to $\boldsymbol{P}_2 = e^{\boldsymbol{D}t}\boldsymbol{P}$, $\boldsymbol{P}_1 = \boldsymbol{P}$, $t > 0$, provides

$$r(e^{\boldsymbol{D}t}\boldsymbol{P}) - r(\boldsymbol{P}) \le m(\boldsymbol{D}t) = tm(\boldsymbol{D}).$$

Dividing by t, letting t tend to 0 and using the assumed existence of the total differential of r at $\boldsymbol{P}$ gives (15).

(iii) $\Rightarrow$ (i) : Let r satisfy Condition (iii). Note first that a standard consequence of the objectivity asserts that $Dr(\boldsymbol{P})\boldsymbol{P}^{\mathrm{T}}$ is symmetric for every $\boldsymbol{P} \in \mathrm{Unim}$ for which the total differential exists. Let us now prove (i), i.e., let us prove that the residual dissipation inequality holds for each process $\pi = (\boldsymbol{E}(\cdot), \boldsymbol{P}(\cdot))$. We may assume that $\boldsymbol{P}(\cdot)$ is continuously differentiable; otherwise we divide the process into subintervals where $\boldsymbol{P}(\cdot)$ is continuously differentiable and apply the forthcoming considerations to each such a piece of the process. We want to integrate (15) along the path $\boldsymbol{P}(\cdot)$. Since the differential Dr exists and satisfies (15) only for a.e. $\boldsymbol{P} \in$ Unim, it may happen that $Dr(\boldsymbol{P}(t))$ exists for no $t \in [0, d_\pi]$. Nevertheless, assume first that $Dr(\boldsymbol{P}(t))$ exists for a.e. time t and complete the proof in that case first. At the end of the proof we shall employ some limiting procedure which reduces the general case to the above one. The function $s := r(\boldsymbol{P}(\cdot))$ is Lipschitz continuous and since $\dot{s}(t)$ is given by the chain rule for a.e. time by our assumption and $Dr(\boldsymbol{P}(t))\boldsymbol{P}(t)^{\mathrm{T}}$ is symmetric,

$$\dot{s}(t) = Dr(\boldsymbol{P}(t))\boldsymbol{P}(t)^{\mathrm{T}} \cdot \boldsymbol{D}^p(t) \le m(\boldsymbol{D}^p(t)) \le \bar{\boldsymbol{S}}(\boldsymbol{E}(t)) \cdot \boldsymbol{D}^p(t) \tag{20}$$

for almost every $t \in [0, d_\pi]$; here $\boldsymbol{D}^p(\cdot)$ is the plastic stretching of π. We have also used equation (15) and the definition of $m(\boldsymbol{D})$. The integration of (20) gives the residual dissipation inequality.

To complete the proof in the general case, show that we may perturb $\boldsymbol{P}(\cdot)$ so that Dr exists for L-a.e. $t \in [0, d_\pi]$ during the perturbed process. Here L is the Lebesgue measure on $\mathbb{R}$. We seek the perturbed path in the form $\boldsymbol{P}(\cdot)\boldsymbol{O}$ where $\boldsymbol{O} \in$ Unim is sufficiently close to $\mathbf{1}$. Using Fubinis theorem on the product space Unim $\times\, \mathbb{R}$ with the measure $\nu := \mu \otimes L$, where μ is the Haar measure on Unim, one can prove that for μ-a.e. $\boldsymbol{O} \in$ Unim the differential $Dr(\boldsymbol{P}(t)\boldsymbol{O})$ exists and satisfies (15) for L-a.e. $t \in [0, d_\pi]$. Thus the above part of the proof can be applied to μ-a.e. process $\boldsymbol{P}(\cdot)\boldsymbol{O}$ which gives

$$r(\boldsymbol{P}^{\mathrm{f}}\boldsymbol{O}) - r(\boldsymbol{P}^{\mathrm{i}}\boldsymbol{O}) \le w^P(\pi).$$

Letting $\boldsymbol{O} \to \mathbf{1}$ and using the continuity of r proves the residual dissipation inequality in the general case. □

5 A lower bound for plastic work and the extremal residual energies

In this section we calculate a lower bound for the plastic work in processes of fixed initial and final plastic deformation and determine the maximal and minimal residual energies vanishing at a given point. Both these results are stated in terms of the convex hull m^{**} of the dissipation function m, i.e., the largest convex function on Sym_0 not exceeding m.

The following fact will be useful (see [31; Propositions 18.2.4 and 18.2.5]):

Proposition 4 *Let $f : U \to \mathbb{R}, U = \mathrm{Sym}, \mathrm{Sym}_0$, be an isotropic function. Then f is convex if and only if its restriction f_Δ to diagonal arguments (relative to some fixed orthonormal basis) is convex.*

Theorem 1 *Let the material satisfy the second law, let $\boldsymbol{D} \in \mathrm{Sym}_0$, and let $\mathscr{C}(\boldsymbol{D})$ be the set of all processes $\pi = (\boldsymbol{E}(\cdot), \boldsymbol{P}(\cdot))$ satisfying $\boldsymbol{P}^{\mathrm{f}} = \boldsymbol{R}e^{\boldsymbol{D}}\boldsymbol{P}^{\mathrm{i}}$ for some $\boldsymbol{R} \in \mathrm{Rot}$. Then*

$$m^{**}(\boldsymbol{D}) = \inf\{w^P(\pi) : \pi \in \mathscr{C}(\boldsymbol{D})\}. \tag{21}$$

Proof Define $s : \mathrm{Unim} \to \mathbb{R}$ by

$$s(\boldsymbol{P}) = m^{**}(\boldsymbol{D}), \quad \boldsymbol{P} \in \mathrm{Unim},$$

where we write $\boldsymbol{P} = \boldsymbol{R}e^{\boldsymbol{D}}$, $\boldsymbol{R} \in \mathrm{Rot}$, $\boldsymbol{D} \in \mathrm{Sym}_0$ and prove that s is a residual energy function. By the construction, s is objective. The convex function m^{**} is locally Lipschitz continuous and hence it has the total differential $Dm^{**}(\boldsymbol{A})$ for a.e. $\boldsymbol{A} \in \mathrm{Sym}_0$ with respect to the Lebesgue measure on Sym_0. If $\boldsymbol{A} \in \mathrm{Sym}_0$ and $\boldsymbol{P} \in \mathrm{Unim}$ are related by $\boldsymbol{P} = \boldsymbol{R}e^{\boldsymbol{A}}$ for some $\boldsymbol{R} \in \mathrm{Rot}$, one finds that m^{**} has a differential at $\boldsymbol{A}$ if and only if s has a differential at $\boldsymbol{P}$. Since m is isotropic, nonnegative, and positively homogeneous of degree 1, so also is m^{**}. Using the isotropy, and working in the basis of eigenvectors of $\boldsymbol{A}$ (see, e.g., the corresponding considerations in [1] or [31]), one derives the formula

$$Ds(\boldsymbol{P})\boldsymbol{P}^{\mathrm{T}} \cdot \boldsymbol{D} = Dm^{**}(\boldsymbol{A}) \cdot \boldsymbol{D} \tag{22}$$

for each $\boldsymbol{D} \in \mathrm{Sym}_0$. By the convexity and homogeneity of m^{**},

$$Dm^{**}(\boldsymbol{A}) \cdot \boldsymbol{D} \le m^{**}(\boldsymbol{D}) \le m(\boldsymbol{D})$$

for every $\boldsymbol{A} \in \mathrm{Sym}_0$, $\boldsymbol{A} \ne \boldsymbol{0}$ for which the total differential of m^{**} exists. A combination with (22) provides (15). This proves that s is a residual energy function. Let us now prove that for every process $\pi = (\boldsymbol{E}(\cdot), \boldsymbol{P}(\cdot)) \in \mathscr{C}(\boldsymbol{D})$,

$$w^P(\pi) \ge m^{**}(\boldsymbol{D}). \tag{23}$$

Since $\pi\kappa := (\boldsymbol{E}(\cdot), \boldsymbol{P}(\cdot)(\boldsymbol{P}^{\mathrm{i}})^{-1})$ is also a process, $w^P(\pi) = w^P(\pi\kappa)$, and s is a residual energy,

$$w^P(\pi\kappa) \ge s(\boldsymbol{P}^{\mathrm{f}}(\boldsymbol{P}^{\mathrm{i}})^{-1}) - s(\mathbf{1}) = m^{**}(\boldsymbol{D})$$

which implies (23). This shows that we have the inequality sign $\le$ in (21).

The rest of the proof is devoted to showing the opposite inequality in (21). Thus we seek to prove that for every $\varepsilon > 0$ there exists a process $\pi \in \mathscr{C}(\boldsymbol{D})$ such that

$$w^P(\pi) \le m^{**}(\boldsymbol{D}) + \varepsilon. \tag{24}$$

Let $\{\boldsymbol{e}_i\}$ be any orthonormal basis and let $\Delta \equiv \Delta(\boldsymbol{e}_i)$ be the space of all $\boldsymbol{A} \in \mathrm{Sym}_0$ represented by diagonal matrices in $\{\boldsymbol{e}_i\}$ so that all elements of Δ commute. Let m_Δ and m_Δ^{**} be the restrictions of m and m^{**} to Δ. Let us show that $m_\Delta^{**} = (m_\Delta)^{**}$ where the last symbol denotes the convex hull of m_Δ on Δ. Since m_Δ^{**} is convex, we have

$$m_\Delta^{**} \le (m_\Delta)^{**} \quad \text{on} \quad \Delta. \tag{25}$$

Let us extend the function $(m_\Delta)^{**}$ from Δ to a function $g : \mathrm{Sym}_0 \to \mathbb{R}$ by isotropy, i.e., by

$$g(\boldsymbol{B}) = (m_\Delta)^{**}(\boldsymbol{Q}\boldsymbol{B}\boldsymbol{Q}^{\mathrm{T}}), \quad \boldsymbol{B} \in \mathrm{Sym}_0,$$

where $\boldsymbol{Q} = \boldsymbol{Q}(\boldsymbol{B}) \in \mathrm{Rot}$ is chosen so as to satisfy $\boldsymbol{Q}\boldsymbol{B}\boldsymbol{Q}^{\mathrm{T}} \in \Delta$. Such a $\boldsymbol{Q}$ exists by the spectral decomposition theorem. Then, first, it is easily seen that g is well defined, i.e., independent of the choice of $\boldsymbol{Q}$, and second, by Proposition 4, g is convex as a consequence of the convexity of $(m_\Delta)^{**}$. The construction gives that $g \le m$ on Sym_0

and thus since m^{**} is the maximal convex function not exceeding m, we have $g \le m^{**}$ on Sym_0 and in particular $(m_\Delta)^{**} \le m_\Delta^{**}$ on Δ. Thus combining with (25) we have $(m_\Delta)^{**} = m_\Delta^{**}$. Let now $\boldsymbol{D} \in \mathrm{Sym}_0$ and let $\{\boldsymbol{e}_i\}$ be any basis in which $\boldsymbol{D}$ is diagonal. The above considerations show that

$$m^{**}(\boldsymbol{D}) = m_\Delta^{**}(\boldsymbol{D}) = (m_\Delta)^{**}(\boldsymbol{D}) \tag{26}$$

and applying the familiar construction of the convex hull [28] to m_Δ, we obtain that

$$(m_\Delta)^{**}(\boldsymbol{D}) = \inf\{\sum_{i=1}^{3} \lambda_i m(\boldsymbol{A}_i) : \boldsymbol{A}_i \in \Delta, \lambda_i > 0, \sum_{i=1}^{3} \lambda_i = 1, \sum_{i=1}^{3} \lambda_i \boldsymbol{A}_i = \boldsymbol{D}\}, \tag{27}$$

which also gives the value of $m^{**}(\boldsymbol{D})$ by (26). Here the limit 3 in the sums is related to the dimension 2 of $\Delta(\boldsymbol{e}_i)$ through the CarathŽodory theorem. It is noted, and that is the main conclusion of the above considerations, that all the elements $\boldsymbol{A}_i$ as in (27) mutually commute and commute also with $\boldsymbol{D}$. Let $\varepsilon > 0$ be given. By (27), there exist sequences $\lambda_i > 0, \boldsymbol{A}_i \in \mathrm{Sym}_0$, $i = 1, 2, 3$, such that

$$m^{**}(\boldsymbol{D}) + \frac{\varepsilon}{2} \ge \sum_{i=1}^{3} \lambda_i m(\boldsymbol{A}_i), \quad \sum_{i=1}^{3} \lambda_i = 1, \quad \sum_{i=1}^{3} \lambda_i \boldsymbol{A}_i = \boldsymbol{D}, \tag{28}$$

and $\boldsymbol{A}_i, \boldsymbol{D}$ mutually commute. We can also assume that $\boldsymbol{A}_i \neq \boldsymbol{0}$ for $i = 1, 2, 3$. By the definition of m, for each $i = 1, 2, 3$ there exists a $\boldsymbol{E}_i \in \partial\mathscr{E}$ such that

$$\bar{\boldsymbol{M}}(\boldsymbol{E}_i) \sim \boldsymbol{A}_i \quad \text{and} \quad m(\boldsymbol{A}_i) + \frac{\varepsilon}{2} \ge \bar{\boldsymbol{S}}(\boldsymbol{E}_i) \cdot \boldsymbol{A}_i, \quad i = 1, 2, 3. \tag{29}$$

The desired process is constructed in the form $\pi = \pi_1 * \rho_1 * \pi_2 * \rho_2 * \pi_3$ as follows. We take $\pi_i, i = 1, 2, 3$, as processes of duration λ_i, where $\pi_i = (\boldsymbol{E}_i, \boldsymbol{P}_i(\cdot))$ and

$$\begin{aligned} \boldsymbol{P}_1(t) &= e^{\boldsymbol{A}_1 t}, & t &\in [0, \lambda_1], \\ \boldsymbol{P}_2(t) &= e^{\boldsymbol{A}_2 t} e^{\lambda_1 \boldsymbol{A}_1}, & t &\in [0, \lambda_2], \\ \boldsymbol{P}_3(t) &= e^{\boldsymbol{A}_3 t} e^{\lambda_2 \boldsymbol{A}_2} e^{\lambda_1 \boldsymbol{A}_1}, & t &\in [0, \lambda_3]. \end{aligned}$$

We further take ρ_1, ρ_2 as elastic processes of the form $\rho_1 = (\tilde{\boldsymbol{E}}_1(\cdot), e^{\lambda_1 \boldsymbol{A}_1}), \rho_2 = (\tilde{\boldsymbol{E}}_1(\cdot), e^{\lambda_1 \boldsymbol{A}_1 + \lambda_2 \boldsymbol{A}_2})$ such that $\boldsymbol{E}_1 = \tilde{\boldsymbol{E}}_1^{\mathrm{i}}, \boldsymbol{E}_2 = \tilde{\boldsymbol{E}}_1^{\mathrm{f}}, \boldsymbol{E}_2 = \tilde{\boldsymbol{E}}_2^{\mathrm{i}}, \boldsymbol{E}_3 = \tilde{\boldsymbol{E}}_2^{\mathrm{f}}$. Then the process $\pi = \pi_1 * \rho_1 * \pi_2 * \rho_2 * \pi_3$ can be constructed and

$$\boldsymbol{P}^{\mathrm{i}} = \boldsymbol{1}, \quad \boldsymbol{P}^{\mathrm{f}} = e^{\lambda_1 \boldsymbol{A}_1 + \lambda_2 \boldsymbol{A}_2 + \lambda_3 \boldsymbol{A}_3} = e^{\boldsymbol{D}}$$

where we have used the commutativity of the $\boldsymbol{A}_i$ and $(28)_3$. Thus $\pi \in \mathscr{C}(\boldsymbol{D})$. Furthermore,

$$w^P(\pi) = \sum_{i=1}^{3} w^P(\pi_i) = \sum_{i=1}^{3} \lambda_i \bar{\boldsymbol{S}}(\boldsymbol{E}_i) \cdot \boldsymbol{A}_i \le \sum_{i=1}^{3} \lambda_i m(\boldsymbol{A}_i) + \frac{\varepsilon}{2} \le m^{**}(\boldsymbol{D}) + \varepsilon$$

by $(29)_2$ and (24). □

A residual energy function r is said to be maximal or minimal at $\boldsymbol{H} \in \mathrm{Unim}$ if $r(\boldsymbol{H}) = 0$ and $r \ge \bar{r}$ or $r \le \bar{r}$, respectively, for any residual energy function $\bar{r}$ such that $\bar{r}(\boldsymbol{H}) = 0$. Coleman & Owen [5] show that the set of all free energy functions that vanish at a given state is convex and has the largest and the smallest elements. By Proposition 2(ii) the problem of describing the extremal free energy functions is equivalent to that of describing the extremal residual energy functions.

Proposition 5 *Let the material satisfy the second law and define $s, t : \mathrm{Unim} \to \mathbb{R}$ by*

$$s(\boldsymbol{P}) = m^{**}(\boldsymbol{D}), \quad t(\boldsymbol{P}) = -m^{**}(-\boldsymbol{D}),$$

$\boldsymbol{P} \in \mathrm{Unim}$, *where we write $\boldsymbol{P} = \boldsymbol{R}e^{\boldsymbol{D}}$, $\boldsymbol{R} \in \mathrm{Rot}$, $\boldsymbol{D} \in \mathrm{Sym}_0$. Then s, t are the maximal and minimal residual energies at* $\mathbf{1}$. *Moreover they satisfy*

$$s(\boldsymbol{OP}) \le s(\boldsymbol{O}) + s(\boldsymbol{P}), \quad t(\boldsymbol{OP}) \ge t(\boldsymbol{O}) + t(\boldsymbol{P}) \tag{30}$$

for any $\boldsymbol{O}, \boldsymbol{P} \in \mathrm{Unim}$.

The extremal residual energies at a general $\boldsymbol{H} \in \mathrm{Unim}$ are

$$s_{\boldsymbol{H}}(\boldsymbol{P}) = s(\boldsymbol{PH}), \quad t_{\boldsymbol{H}}(\boldsymbol{P}) = t(\boldsymbol{PH}),$$

$\boldsymbol{P} \in \mathrm{Unim}$.

Proof By the proof of Theorem 1, s is a residual energy; the proof that t is a residual energy is similar. To prove that s is maximal at $\mathbf{1}$, let r be any residual energy vanishing at $\mathbf{1}$, let $\pi = (\boldsymbol{E}(\cdot), \boldsymbol{P}(\cdot))$ be any process in $\mathscr{C}(\boldsymbol{D})$, and let $\pi\kappa = (\boldsymbol{E}(\cdot), \boldsymbol{P}(\cdot)(\boldsymbol{P}^{\mathrm{i}})^{-1})$. The residual dissipation inequality for $\pi\kappa$ reads

$$r(\boldsymbol{P}^{\mathrm{f}}(\boldsymbol{P}^{\mathrm{i}})^{-1}) = r(e^{\boldsymbol{D}}) \le w^P(\pi\kappa) = w^P(\pi);$$

taking the infimum over all processes $\pi \in \mathscr{C}(\boldsymbol{D})$ and using Theorem 1 we obtain

$$r(e^{\boldsymbol{D}}) \le s(e^{\boldsymbol{D}});$$

combining with the objectivity of both r, s this gives $r \le s$. To prove (30), consider only s. For any function r on Unim and any $\boldsymbol{H} \in \mathrm{Unim}$ let $r_{\boldsymbol{H}}$ denote the shifted function given by $r_{\boldsymbol{H}}(\boldsymbol{P}) = r(\boldsymbol{PH})$. It follows immediately from, e.g., Condition (iii) of Proposition 3 that r is a residual energy function if and only if $r_{\boldsymbol{H}}$ is a residual energy function. Note that for any $\boldsymbol{P} \in \mathrm{Unim}$, the function $s_{\boldsymbol{P}}(\cdot) - s(\boldsymbol{P})$ is a residual energy vanishing at $\mathbf{1}$. Hence $s_{\boldsymbol{P}}(\cdot) - s(\boldsymbol{P}) \le s(\cdot)$ and $(30)_1$ follows by inserting $\boldsymbol{O}$. The rest is immediate. □

6 Il'yushin's condition; its first consequences

A process $\pi = (\boldsymbol{V}(\cdot), \boldsymbol{P}(\cdot))$ is said to be an $\boldsymbol{F}$-cycle if $\hat{\boldsymbol{F}}(\pi^{\mathrm{f}}) = \hat{\boldsymbol{F}}(\pi^{\mathrm{i}})$. Every σ-cycle (see Section 3) is an $\boldsymbol{F}$-cycle.

6.1 Il'yushin's condition *For every $\boldsymbol{F}$-cycle π, $w(\pi) \ge 0$.*

Theorem 2, below, describes the consequences of Il'yushin's condition on the constitutive objects, which include the normality rule.

Definition 2 The material is said to obey the normality rule if for every $\boldsymbol{E} \in \partial\mathscr{E}$,

$$\bar{\boldsymbol{M}}(\boldsymbol{E}) \in \mathrm{N}_{\bar{\boldsymbol{S}}(\boldsymbol{E})}\mathscr{S}, \tag{31}$$

where $\mathrm{N}_{\bar{\boldsymbol{S}}(\boldsymbol{E})}\mathscr{S}$ denotes the normal cone to $\mathscr{S}$ at $\bar{\boldsymbol{S}}(\boldsymbol{E})$, defined by

$$\mathrm{N}_{\boldsymbol{S}}\mathscr{S} := \{\boldsymbol{D} \in \mathrm{Sym}_0 : (\boldsymbol{Z} - \boldsymbol{S}) \cdot \boldsymbol{D} \le 0 \ \text{ for every } \ \boldsymbol{Z} \in \mathscr{S}\},$$

$\boldsymbol{S} \in \mathrm{Sym}_0$.

The following lemma gathers some consequences of the normality rule.

Lemma 1 *If the material obeys the normality rule then*
(i) *the stress range* $\mathscr{S}$ *is convex;*
(ii) *for every* $\boldsymbol{D} \in \mathrm{Sym}_0$,

$$m(\boldsymbol{D}) = \sup \{\boldsymbol{S} \cdot \boldsymbol{D} : \boldsymbol{S} \in \mathscr{S}\} \tag{32}$$

and in particular,

$$m(\boldsymbol{D}) = \boldsymbol{S} \cdot \boldsymbol{D} \tag{33}$$

for any $\boldsymbol{S} \in \mathscr{S}$ *such that* $\boldsymbol{D} \in \mathrm{N}_{\boldsymbol{S}}\mathscr{S}$;
(iii) *m is isotropic, convex, positively homogeneous of degree* 1;
(iv) *the subdifferential of m is given by*

$$\partial m(\boldsymbol{D}) = \begin{cases} \{\boldsymbol{S} \in \mathrm{Sym}_0 : \boldsymbol{D} \in \mathrm{N}_{\boldsymbol{S}}\mathscr{S}\} & \text{if} \quad \boldsymbol{D} \neq \boldsymbol{0}, \\ \mathscr{S} & \text{if} \quad \boldsymbol{D} = \boldsymbol{0}; \end{cases} \tag{34}$$

(v) *the convex conjugate* m^* *of m is given by*

$$m^*(\boldsymbol{S}) = \begin{cases} 0 & \text{if} \quad \boldsymbol{S} \in \mathscr{S}, \\ \infty & \text{otherwise.} \end{cases} \tag{35}$$

Proof (i): Let us derive (i) from the following assertion, which is easy to prove: *If* $M \subset \mathbb{R}^d$ *is a closed set with nonempty interior such that for each* $x \in \partial M$ *we have* $\mathrm{N}_x M \neq \{0\}$, *then M is convex.* Let us verify that $\mathscr{S}$ satisfies the hypotheses of the assertion. Clearly, $\mathscr{S}$ is closed since it is an image of the closed set $\mathscr{E}$ under the continuous mapping $\bar{\boldsymbol{S}}$; further, $\mathscr{S}$ has nonempty interior since $\mathscr{E}$ has nonempty interior by Definition 1(i) and (iv) and $\bar{\boldsymbol{S}}$ maps the interior of $\mathscr{E}$ onto the interior of $\mathscr{S}$ by Definition 1(iii). Finally, for each $\boldsymbol{S} \in \partial\mathscr{S}$ we have $\mathrm{N}_{\boldsymbol{S}}\mathscr{S} \neq \{\boldsymbol{0}\}$ since $\bar{\boldsymbol{S}}$ maps $\partial\mathscr{E}$ onto $\partial\mathscr{S}$ by Definition 1(iii) and for each $\boldsymbol{E} \in \partial\mathscr{E}$ we have (31). (ii): Let $\boldsymbol{D} \neq \boldsymbol{0}$ and $\boldsymbol{E} \in \partial\mathscr{E}$ be any point such that $\bar{\boldsymbol{M}}(\boldsymbol{E}) \sim \boldsymbol{D}$ so that $\boldsymbol{D} \in \mathrm{N}_{\bar{\boldsymbol{S}}(\boldsymbol{E})}\mathscr{S}$ by the normality rule which means that $(\boldsymbol{S} - \bar{\boldsymbol{S}}(\boldsymbol{E})) \cdot \boldsymbol{D} \le 0$, i.e.,

$$\boldsymbol{S} \cdot \boldsymbol{D} \le \bar{\boldsymbol{S}}(\boldsymbol{E}) \cdot \boldsymbol{D}$$

for any $\boldsymbol{S} \in \mathscr{E}$. Fixing $\boldsymbol{S}$ and taking the infimum of the right-hand side over all $\boldsymbol{E} \in \partial\mathscr{E}$ such that $\bar{\boldsymbol{M}}(\boldsymbol{E}) \sim \boldsymbol{D}$ we obtain

$$\boldsymbol{S} \cdot \boldsymbol{D} \le m(\boldsymbol{D})$$

with the equality holding if $\boldsymbol{S} = \bar{\boldsymbol{S}}(\boldsymbol{E})$ where $\boldsymbol{E}$ is any element as above. Once (ii) has been established, (iii)–(v) follow from the standard duality theory for homogeneous degree 1 convex functions, [10, 28]. □

Theorem 2 *The material satisfies Il'yushin's condition if and only if the following two conditions are satisfied:*
(i) $\bar{\boldsymbol{T}}$ *has an elastic potential p;*
(ii) *if* $\boldsymbol{D} \in \mathrm{Sym}_0$ *and* $\boldsymbol{E}, \boldsymbol{E}e^{-\boldsymbol{D}} \in \mathscr{E}$ *then*

$$p(\boldsymbol{E}e^{-\boldsymbol{D}}) \ge p(\boldsymbol{E}) - m(\boldsymbol{D}). \tag{36}$$

Moreover, if (i) *and* (ii) *hold then the material obeys the normality rule.*

Item (ii) is called Condition E in the subsequent discussion.

Proof Assume that the material satisfies Il'yushin's condition. Condition (i) follows from the reversibility of elastic processes via the path-independence argument as in the proof of Proposition 2(i). (ii): Let $\boldsymbol{E}_0 \in \partial\mathscr{E}$ be such that

$$\bar{\boldsymbol{M}}(\boldsymbol{E}_0) \sim \boldsymbol{D}. \tag{37}$$

Consider a process $\pi = \pi_1 * \pi_0 * \pi_2$, where π_1 is any elastic process connecting $(\boldsymbol{E}, \mathbf{1})$ with $(\boldsymbol{E}_0, \mathbf{1})$, π_2 any elastic process connecting $(\boldsymbol{E}_0, e^{\boldsymbol{D}})$ with $(\boldsymbol{E}e^{-\boldsymbol{D}}, \boldsymbol{R}e^{\boldsymbol{D}})$, and π_0 is a process of duration 1 with

$$\boldsymbol{E}_0(t) = \boldsymbol{E}_0, \quad \boldsymbol{P}_0(t) = e^{\boldsymbol{D}t}, \quad t \in [0,1]. \tag{38}$$

Then π_0 is really a process by (37) and the process $\pi_1 * \pi_0 * \pi_2$ is an $\boldsymbol{F}$-cycle. Il'yushin's condition says

$$w(\pi) = w^E(\pi) + w^P(\pi) \geq 0$$

where $w^E(\pi) = p(\boldsymbol{E}e^{-\boldsymbol{D}}) - p(\boldsymbol{E})$, $w^P(\pi) = w^P(\pi_0) = \bar{\boldsymbol{S}}(\boldsymbol{E}_0) \cdot \boldsymbol{D}$. The last three relations yield $p(\boldsymbol{E}e^{-\boldsymbol{D}}) \geq p(\boldsymbol{E}) - \bar{\boldsymbol{S}}(\boldsymbol{E}_0) \cdot \boldsymbol{D}$. Since $\boldsymbol{E}_0$ is arbitrary subject to condition (37), the definition of m gives (36). This completes the proof of (ii). Assume that (i), (ii) hold and prove the normality rule. Let $\boldsymbol{G} \in \partial\mathscr{E}$. The goal is to prove that

$$(\boldsymbol{S} - \bar{\boldsymbol{S}}(\boldsymbol{G})) \cdot \bar{\boldsymbol{M}}(\boldsymbol{G}) \leq 0 \tag{39}$$

for every $\boldsymbol{S} \in \mathscr{S}$. Assume first that $\boldsymbol{S}$ is in the interior of $\mathscr{S}$. Use Definition 1(iii) to find that there exists an interior point $\boldsymbol{E}$ of $\mathscr{E}$ such that $\boldsymbol{S} = \bar{\boldsymbol{S}}(\boldsymbol{E})$. Set $\boldsymbol{D} = \bar{\boldsymbol{M}}(\boldsymbol{G})$ and note that for all $t > 0$ sufficiently small we have $\boldsymbol{E}e^{-t\boldsymbol{D}} \in \mathscr{E}$ since $\boldsymbol{E}$ is in the iterior of $\mathscr{E}$. The application of (36) and the use of $m(t\boldsymbol{D}) \leq t\bar{\boldsymbol{S}}(\boldsymbol{E}_0) \cdot \boldsymbol{D}$, which follows from the definition of m, provide

$$p(\boldsymbol{E}e^{-t\boldsymbol{D}}) \geq p(\boldsymbol{E}) - t\bar{\boldsymbol{S}}(\boldsymbol{G}) \cdot \boldsymbol{D}.$$

Dividing by t, letting t tend to 0 and using the stress relation, we obtain $\bar{\boldsymbol{S}}(\boldsymbol{E}) \cdot \boldsymbol{D} \leq \bar{\boldsymbol{S}}(\boldsymbol{G}) \cdot \boldsymbol{D}$ and thus eventually (39). Since $\mathscr{E}$ is the closure of its interior, the limit gives (39) for each $\boldsymbol{S} = \bar{\boldsymbol{S}}(\boldsymbol{E})$ where $\boldsymbol{E}$ is a boundary point of $\mathscr{E}$ and since every boundary point of $\mathscr{S}$ is of the form $\boldsymbol{S} = \bar{\boldsymbol{S}}(\boldsymbol{E})$ where $\boldsymbol{E}$ is a boundary point of $\mathscr{E}$ (see Definition 1) inequality (39) holds for all $\boldsymbol{S} \in \mathscr{S}$. Assume conversely that (i) and (ii) hold and prove Il'yushin's condition. Let $\pi = (\boldsymbol{E}(\cdot), \boldsymbol{P}(\cdot))$ be an $\boldsymbol{F}$-cycle. Since (i), (ii) imply the normality rule, Lemma 1(iii) sayst that m is convex and thus $m = m^{**}$. Then by Theorem 1,

$$w(\pi) = w^E(\pi) + w^P(\pi) \geq p(\boldsymbol{E}^{\mathrm{f}}) - p(\boldsymbol{E}^{\mathrm{i}}) + m(\boldsymbol{D}) \tag{40}$$

where $\boldsymbol{D}$ is such that $\boldsymbol{P}^{\mathrm{f}} = \boldsymbol{R}e^{\boldsymbol{D}}\boldsymbol{P}^{\mathrm{i}}$ for some $\boldsymbol{R} \in \mathrm{Rot}$. Combining with $\boldsymbol{E}^{\mathrm{f}}\boldsymbol{P}^{\mathrm{f}} = \boldsymbol{E}^{\mathrm{i}}\boldsymbol{P}^{\mathrm{i}}$ we obtain $\boldsymbol{E}^{\mathrm{f}} = \boldsymbol{E}^{\mathrm{i}}e^{-\boldsymbol{D}}\boldsymbol{R}^{\mathrm{T}}$ and (40) reads $w(\pi) \geq p(\boldsymbol{E}^{\mathrm{i}}e^{-\boldsymbol{D}}) - p(\boldsymbol{E}^{\mathrm{i}}) + m(\boldsymbol{D}) \geq 0$ where the last inequality is Condition (ii). □

Remark 1 *Suppose that the material obeys the normality rule and that* $\bar{\boldsymbol{T}}$ *has an elastic potential* p. *Then:*

(i) *Condition* E *holds for all pairs* $\boldsymbol{E}, \boldsymbol{D}$ *as in that condition with* $|\boldsymbol{D}|$ *sufficiently small and* $\boldsymbol{E}$ *in the interior of* $\mathscr{E}$.

(ii) *If* $\mathscr{E}$ *is logarithmically convex in the sense that for every pair of* $\boldsymbol{E}, \boldsymbol{D}$ *of tensors with* $\boldsymbol{D} \in \mathrm{Sym}_0, \boldsymbol{E}, \boldsymbol{E}e^{-\boldsymbol{D}} \in \mathscr{E}$ *one also has* $\boldsymbol{E}e^{-\boldsymbol{D}t} \in \mathscr{E}$ *for every* $t \in [0,1]$, *then* E *holds.*

The logarithmic convexity of Item (ii) seems to be hard to verify. The following section gives other sufficient conditions to guarantee E.

Proof (i): Since $\boldsymbol{E}$ is an interior point of $\mathscr{E}$, if $|\boldsymbol{D}|$ is small enough, also $\boldsymbol{E}e^{-\boldsymbol{D}t} \in \mathscr{E}$ for all $t \in [0,1]$. Writing $\boldsymbol{H}(t) := \boldsymbol{E}e^{-\boldsymbol{D}t}$, one obtains $dp(\boldsymbol{H}(t))/dt = -\bar{\boldsymbol{S}}(\boldsymbol{H}(t)) \cdot \boldsymbol{D}$. Lemma 1(ii) gives $\bar{\boldsymbol{S}}(\boldsymbol{H}(t)) \cdot \boldsymbol{D} \le m(\boldsymbol{D})$; thus $dp(\boldsymbol{H}(t))/dt \ge -m(\boldsymbol{D})$ and the integration over $[0,1]$ yields (36). (ii): If $\mathscr{E}$ is logarithmically convex, then in the notation of the proof of (i), one has $\boldsymbol{H}(t) \in \mathscr{E}$ for all $t \in [0,1]$. The proof is then identical with that of (i). □

7 The extended energy functions

This section shows that Il'yushin's condition leads to energy functions that satisfy the dissipation inequalities stronger than those arising from the second law.

Theorem 3 *Suppose that the material satisfies Il'yushin's condition. Let $\boldsymbol{F} \in \mathrm{Lin}^+$ and define*

$$e(\boldsymbol{F}) = \inf\{p(\boldsymbol{F}e^{-\boldsymbol{D}}) + m(\boldsymbol{D}) : \boldsymbol{D} \in \mathrm{Sym}_0, \boldsymbol{F}e^{-\boldsymbol{D}} \in \mathscr{E}\}, \tag{41}$$

$$f(\boldsymbol{F}) = \sup\{p(\boldsymbol{F}e^{-\boldsymbol{D}}) - m(-\boldsymbol{D}) : \boldsymbol{D} \in \mathrm{Sym}_0, \boldsymbol{F}e^{-\boldsymbol{D}} \in \mathscr{E}\}. \tag{42}$$

Then

(i) $$-\infty < f \le e < \infty;$$

(ii) *for every $\boldsymbol{F} \in \mathscr{E}$ the infima and suprema in* (41) *and* (42) *are attained for $\boldsymbol{D} = \boldsymbol{0}$ and hence*

$$e(\boldsymbol{F}) = f(\boldsymbol{F}) = p(\boldsymbol{F});$$

(iii) *if $\boldsymbol{F} \notin \mathscr{E}$ then* (41) *and* (42) *hold with the condition $\boldsymbol{F}e^{-\boldsymbol{D}} \in \mathscr{E}$ replaced by $\boldsymbol{F}e^{-\boldsymbol{D}} \in \partial\mathscr{E}$;*

(iv) *for every process $\pi = (\boldsymbol{E}(\cdot), \boldsymbol{P}(\cdot))$,*

$$e(\boldsymbol{F}^{\mathrm{f}}(\boldsymbol{P}^{\mathrm{i}})^{-1}) - e(\boldsymbol{F}^{\mathrm{i}}(\boldsymbol{P}^{\mathrm{i}})^{-1}) \le w(\pi), \tag{43}$$

$$f(\boldsymbol{F}^{\mathrm{f}}(\boldsymbol{P}^{\mathrm{f}})^{-1}) - f(\boldsymbol{F}^{\mathrm{i}}(\boldsymbol{P}^{\mathrm{f}})^{-1}) \le w(\pi). \tag{44}$$

The proof will show that all the infima and suprema in the above theorem are taken over nonempty sets but it is not apriori clear that they are finite. The function e is called the *initial extended energy* or briefly *initial energy* and f the *final extended energy* or *final energy.* For a concrete material, the function e is calculated in Section 9. Since for $\boldsymbol{F}$-cycles the left–hand side s of (43) and (44) vanish and thus $w(\pi) \ge 0$, we see that the existence of a function $e : \mathrm{Lin}^+ \to \mathbb{R}$ satisfying (43) or similarly the existence a function $f : \mathrm{Lin}^+ \to \mathbb{R}$ satisfying (44) implies that the material satisfies Il'yushin's condition; thus the existence of such functions is equivalent to Il'yushin's condition. Since Il'yushin's condition is strictly stronger than the cyclic second law, the dissipation inequalities (43) and (44) are strictly stronger than the dissipation inequality (9) stemming from the second law. In Remark 2 and the subsequent discussion we shall see that there are qualitative differences between the inequalities (43) and (44). It is also noted that similar potentials have been introduced by Lucchesi

& ÁILHAVŮ [18], but the dissipation inequalities have been proved only for a restricted class of processes. Recently, a potential similar to e has been introduced by ORTIZ & REPETTO [23] and CARSTENSEN, HACKL & MIELKE [4] and MIELKE [20] to treat plastic materials from the variational point of view.

Proof (i): If we set $\delta = \det \boldsymbol{F}$ and take any element $\boldsymbol{E}$ of $\mathscr{E}_\delta$ (see Definition 1(iv)) then by the polar decomposition theorem there exists an $\boldsymbol{R} \in \mathrm{Rot}$ and $\boldsymbol{D} \in \mathrm{Sym}_0$ such that $\boldsymbol{F}e^{-\boldsymbol{D}} = \boldsymbol{E}\boldsymbol{R}$ and hence $\boldsymbol{F}e^{-\boldsymbol{D}} \in \mathscr{E}$ for some $\boldsymbol{D} \in \mathrm{Sym}_0$. Thus the suprema and infima in (41) and (42) are taken over nonempty sets and hence $e(\boldsymbol{F}) < \infty, f(\boldsymbol{F}) > -\infty$. It remains to be proved that $f(\boldsymbol{F}) \le e(\boldsymbol{F})$. Note first that if $\tilde{\boldsymbol{D}}_1, \tilde{\boldsymbol{D}}_2, \tilde{\boldsymbol{D}} \in \mathrm{Sym}_0$ are such that $e^{\tilde{\boldsymbol{D}}_1}e^{\tilde{\boldsymbol{D}}_2} = \boldsymbol{R}e^{\tilde{\boldsymbol{D}}}$ for some $\boldsymbol{R} \in \mathrm{Rot}$, then we have the triangle inequality

$$m(\tilde{\boldsymbol{D}}) \le m(\tilde{\boldsymbol{D}}_1) + m(\tilde{\boldsymbol{D}}_2). \tag{45}$$

To establish (45), let $\boldsymbol{E}_\alpha \in \partial\mathscr{E}$, $\alpha = 1, 2$, be such that

$$\bar{\boldsymbol{M}}(\boldsymbol{E}_\alpha) \sim \tilde{\boldsymbol{D}}_\alpha, \tag{46}$$

see Definition 1(ii). Let π_α be processes of duration 1 of the form $\pi_\alpha = (\boldsymbol{E}_\alpha, \boldsymbol{P}_\alpha(\cdot))$ where

$$\boldsymbol{P}_2(t) = e^{t\tilde{\boldsymbol{D}}_2}, \quad \boldsymbol{P}_1(t) = e^{t\tilde{\boldsymbol{D}}_1}e^{\tilde{\boldsymbol{D}}_2}, \quad t \in [0,1].$$

Let finally ρ be an elastic process connecting π_2^{f} with π_1^{i}. Then $\pi = \pi_2 * \rho * \pi_1$ is a process in which

$$w^P(\pi) = \bar{\boldsymbol{S}}(\boldsymbol{E}_1) \cdot \tilde{\boldsymbol{D}}_1 + \bar{\boldsymbol{S}}(\boldsymbol{E}_2) \cdot \tilde{\boldsymbol{D}}_2$$

and we have $\boldsymbol{P}^{\mathrm{f}} = e^{\tilde{\boldsymbol{D}}_1}e^{\tilde{\boldsymbol{D}}_2} = \boldsymbol{R}e^{\tilde{\boldsymbol{D}}}$. Thus the lower bound for the plastic work (Theorem 1; recall $m = m^{**}$) gives

$$\bar{\boldsymbol{S}}(\boldsymbol{E}_1) \cdot \tilde{\boldsymbol{D}}_1 + \bar{\boldsymbol{S}}(\boldsymbol{E}_2) \cdot \tilde{\boldsymbol{D}}_2 \ge m(\tilde{\boldsymbol{D}}).$$

Using (33) we obtain (45). Next use (45) to show that if $\boldsymbol{F} \in \mathrm{Lin}^+$ and $\boldsymbol{D}_1, \boldsymbol{D}_2 \in \mathrm{Sym}_0$ then

$$p(\boldsymbol{F}e^{-\boldsymbol{D}_1}) - m(-\boldsymbol{D}_1) \le p(\boldsymbol{F}e^{-\boldsymbol{D}_2}) + m(\boldsymbol{D}_2).$$

Indeed, let $\boldsymbol{D} \in \mathrm{Sym}_0$ be such that $\boldsymbol{R}e^{\boldsymbol{D}} = e^{\boldsymbol{D}_2}e^{-\boldsymbol{D}_1}$ for some $\boldsymbol{R} \in \mathrm{Rot}$ so that, by (45),

$$m(-\boldsymbol{D}_1) + m(\boldsymbol{D}_2) \ge m(\boldsymbol{D}).$$

Condition E says

$$p(\boldsymbol{F}e^{-\boldsymbol{D}_1}e^{-\boldsymbol{D}}) \ge p(\boldsymbol{F}e^{-\boldsymbol{D}_1}) - m(\boldsymbol{D})$$

and hence

$$p(\boldsymbol{F}e^{-\boldsymbol{D}_2}) \ge p(\boldsymbol{F}e^{-\boldsymbol{D}_1}) - m(-\boldsymbol{D}_1) - m(\boldsymbol{D}_2)$$

Taking the supremum over all $\boldsymbol{D}_1$ such that $\boldsymbol{F}e^{-\boldsymbol{D}_1} \in \mathscr{E}$ and the infimum over all $\boldsymbol{D}_2$ such that $\boldsymbol{F}e^{-\boldsymbol{D}_2} \in \mathscr{E}$ completes the proof of (i). (ii): This is trivial by (36). (iii): Let

$$\bar{e}(\boldsymbol{F}) = \inf\{p(\boldsymbol{F}e^{-\boldsymbol{D}}) + m(\boldsymbol{D}) : \boldsymbol{D} \in \mathrm{Sym}_0, \boldsymbol{F}e^{-\boldsymbol{D}} \in \partial\mathscr{E}\}. \tag{47}$$

This is an infimum over a smaller set than in (41) and therefore,

$$e(\boldsymbol{F}) \le \bar{e}(\boldsymbol{F}). \tag{48}$$

To prove the opposite inequality, let $\boldsymbol{D} \in \mathrm{Sym}_0$ be such that $\boldsymbol{F}e^{-\boldsymbol{D}} \in \mathscr{E}$. As $\boldsymbol{F} \notin \mathscr{E}$, there exists a $\tau \in [0,1]$ such that $\boldsymbol{F}e^{-\boldsymbol{D}\tau} \in \partial\mathscr{E}$. Inequality (36) gives

$$p(\boldsymbol{F}e^{-\boldsymbol{D}}) = p(\boldsymbol{F}e^{-\boldsymbol{D}\tau}e^{-(1-\tau)\boldsymbol{D}}) \geq p(\boldsymbol{F}e^{-\boldsymbol{D}\tau}) - m(\boldsymbol{D}) + m(\boldsymbol{D}\tau),$$

where we have used the homogeneity of m. Thus

$$p(\boldsymbol{F}e^{-\boldsymbol{D}}) + m(\boldsymbol{D}) \geq p(\boldsymbol{F}e^{-\boldsymbol{D}\kappa}) + m(\boldsymbol{D}\kappa)$$

where $\boldsymbol{D}\kappa := \boldsymbol{D}\tau \in \partial\mathscr{E}$. Thus for each $\boldsymbol{D}$ as in the infimum (41) there exists a $\boldsymbol{D}\kappa$ as in the infimum (47) with a value of $p(\boldsymbol{F}e^{-\boldsymbol{D}\kappa}) + m(\boldsymbol{D}\kappa)$ that does not exceed $p(\boldsymbol{F}e^{-\boldsymbol{D}}) + m(\boldsymbol{D})$. This proves the opposite inequality in (48). The assertion about $f(\boldsymbol{F})$ is proved similarly. (iv): If $\pi = (\boldsymbol{E}(\cdot), \boldsymbol{P}(\cdot))$ is a process and if we write $\boldsymbol{P}^{\mathrm{f}} = \boldsymbol{R}e^{\boldsymbol{D}}\boldsymbol{P}^{\mathrm{i}}$ for some $\boldsymbol{R} \in \mathrm{Rot}, \boldsymbol{D} \in \mathrm{Sym}_0$ then Theorem 1 (recall $m = m^{**} \geq 0$) gives

$$w(\pi) \geq p(\boldsymbol{E}^{\mathrm{f}}) - p(\boldsymbol{E}^{\mathrm{i}}) + m(\boldsymbol{D}). \tag{49}$$

We have

$$e(\boldsymbol{F}^{\mathrm{f}}(\boldsymbol{P}^{\mathrm{i}})^{-1}) \leq p(\boldsymbol{F}^{\mathrm{f}}(\boldsymbol{P}^{\mathrm{i}})^{-1}e^{-\boldsymbol{D}}) + m(\boldsymbol{D}) = p(\boldsymbol{E}^{\mathrm{f}}) + m(\boldsymbol{D})$$

by the definition of e and

$$e(\boldsymbol{F}^{\mathrm{i}}(\boldsymbol{P}^{\mathrm{i}})^{-1}) = e(\boldsymbol{E}^{\mathrm{i}}) = p(\boldsymbol{E}^{\mathrm{i}})$$

by (ii); hence (49) leads to (43). Equation (44) is proved similarly. □

Let us estimate the extended energy functions at large values of $|\boldsymbol{F}|$.

Lemma 2 *For every* $\boldsymbol{U} \in \mathrm{Sym}^+$ *with* $\det \boldsymbol{U} = 1$ *we have*

$$\frac{1}{2\sqrt{3}}|\ln \boldsymbol{U}| \leq \ln|\boldsymbol{U}| \leq \ln\sqrt{3} + |\ln \boldsymbol{U}|. \tag{50}$$

Proof Denote by $|\cdot|_\infty$ the maximum norm of Sym, i.e.,

$$|\boldsymbol{D}|_\infty := \max\{|d_1|, |d_2|, |d_3|\},$$

where $d_1 \geq d_2 \geq d_3$ are the eigenvalues of $\boldsymbol{D} \in \mathrm{Sym}$. Write $\boldsymbol{U} = e^{\boldsymbol{D}}$ with $\boldsymbol{D} \in \mathrm{Sym}_0$ since $\det \boldsymbol{U} = 1$. We have $d_1 + d_2 + d_3 = 0$; hence $d_3 \leq 0$ and $|\boldsymbol{D}|_\infty = \max\{d_1, -d_3\}$. If $|\boldsymbol{D}|_\infty = d_1$ then trivially $|\boldsymbol{D}|_\infty \leq 2d_1$. If $|\boldsymbol{D}|_\infty = -d_3$, then $d_1 + d_2 - |\boldsymbol{D}|_\infty = 0$, i.e., $|\boldsymbol{D}|_\infty = d_1 + d_2 \leq 2d_1$. Hence in every case $|\boldsymbol{D}|_\infty \leq 2d_1$ and so

$$|e^{\boldsymbol{D}}|^2 \geq e^{2d_1} \geq e^{|\boldsymbol{D}|_\infty}$$

which implies $\frac{1}{2}|\boldsymbol{D}|_\infty \leq \ln|e^{\boldsymbol{D}}|$. Combining this with $|\boldsymbol{D}| \leq \sqrt{3}|\boldsymbol{D}|_\infty$ we obtain $(50)_1$. To prove $(50)_2$, it suffices to note that

$$|e^{\boldsymbol{D}}|^2 \leq 3e^{2d_1} \leq 3e^{2|\boldsymbol{D}|_\infty},$$

i.e., $\ln|e^{\boldsymbol{D}}| \leq \ln\sqrt{3} + |\boldsymbol{D}|_\infty$. Combining this with $|\boldsymbol{D}|_\infty \leq |\boldsymbol{D}|$ we obtain the desired result. □

Remark 2 *Suppose that the material satisfies Il'yushin's condition and assume that for each fixed* $\delta > 0$, *the set* $\mathscr{E}_\delta$, *given by* (3), *is bounded and that* $\delta^{1/3}\mathbf{1} \in \mathscr{E}_\delta$; *moreover, assume that*

$$m(\boldsymbol{D}) \geq \bar{c}_0|\boldsymbol{D}|, \quad \boldsymbol{D} \in \mathrm{Sym}_0 \tag{51}$$

for some $\bar{c}_0 > 0$. *Then there exist positive constants* $\beta_e, \bar{\beta}_e, \beta_f, \bar{\beta}_f$ *and functions* $\gamma_e, \bar{\gamma}_e, \gamma_f, \bar{\gamma}_f : (0,\infty) \to \mathbb{R}$ *such that*

$$\bar{\gamma}_e(\delta) + \bar{\beta}_e \ln |\boldsymbol{F}| \le e(\boldsymbol{F}) \le \gamma_e(\delta) + \beta_e \ln^+ |\boldsymbol{F}|, \tag{52}$$

$$\gamma_f(\delta) - \beta_f \ln^+ |\boldsymbol{F}| \le f(\boldsymbol{F}) \le \bar{\gamma}_f(\delta) - \bar{\beta}_f \ln |\boldsymbol{F}| \tag{53}$$

for every $\boldsymbol{F} \in \mathrm{Lin}^+$ *where* $\delta = \det \boldsymbol{F}$ *and* $\ln^+$ *is the positive part of* ln.

Proof Let us first prove the upper bound $(52)_2$. Since $\mathscr{S}$ is bounded, there exists a $c_0 > 0$ such that

$$m(\boldsymbol{D}) \le c_0 |\boldsymbol{D}| \tag{54}$$

for each $\boldsymbol{D} \in \mathrm{Sym}_0$. Let $\gamma_e^0 : (0,\infty) \to \mathbb{R}$ be defined by

$$\gamma_e^0(\delta) = \max\{p(\boldsymbol{E}) : \boldsymbol{E} \in \mathscr{E}_\delta\}, \tag{55}$$

for each $\delta > 0$. The maximum exist and is finite since $\mathscr{E}_\delta$ is bounded p is continuous. Let

$$\beta_e = 2\sqrt{3}c_0, \quad \gamma_e(\delta) := \gamma_e^0(\delta) + (2c_0/\sqrt{3})|\ln \delta| \tag{56}$$

for each $\delta > 0$. Since the functions on both sides of $(52)_2$ are objective and isotropic, it suffices to prove $(52)_2$ only if $\boldsymbol{F} \in \mathrm{Sym}^+$. If $\boldsymbol{F} \in \mathscr{E}$ then the proof is immediate. Let $\boldsymbol{F} \in \mathrm{Sym}^+ \sim \mathscr{E}$. Set $\delta = \det \boldsymbol{F}$ and

$$\boldsymbol{D} := \ln(\boldsymbol{F}/\delta^{1/3}), \qquad \text{i.e.,} \qquad \boldsymbol{F}e^{-\boldsymbol{D}} = \delta^{1/3}\mathbf{1} \tag{57}$$

Then by (41) we have $e(\boldsymbol{F}) \le p(\delta^{1/3}\mathbf{1}) + m(\boldsymbol{D})$ and using successively (57), (55), (54), (57), $(50)_1$ and (56), we obtain $(52)_2$. Let us prove $(52)_2$. We have $|\boldsymbol{AB}| \le m_0|\boldsymbol{A}||\boldsymbol{B}|$ for some $m_0 > 0$ and all second-order tensors $\boldsymbol{A}, \boldsymbol{B}$. Let $\boldsymbol{H} \in \partial\mathscr{E}$ so that $\boldsymbol{H}$ is invertible and $|\boldsymbol{H}| \ne 0$. Then $|\boldsymbol{F}| = |\boldsymbol{H}\boldsymbol{H}^{-1}\boldsymbol{F}| \le m_0|\boldsymbol{H}||\boldsymbol{H}^{-1}\boldsymbol{F}|$ for each $\boldsymbol{F} \in \mathrm{Lin}^+$ and hence

$$|\boldsymbol{H}^{-1}\boldsymbol{F}| \ge c_3|\boldsymbol{F}|$$

for all $\boldsymbol{H} \in \partial\mathscr{E}$ and $\boldsymbol{F} \in \mathrm{Lin}^+$ where $c_3 = \min\{m_0^{-1}|\boldsymbol{H}|^{-1} : \boldsymbol{H} \in \partial\mathscr{E}\} > 0$. The compactness assumption of $\mathscr{E}_\delta$ implies that the minimum exists and is positive. For the same reason,

$$\bar{\gamma}_e^0(\delta) := \min\{p(\boldsymbol{H}) : \boldsymbol{H} \in \partial\mathscr{E}_\delta\} \tag{58}$$

is finite. We now prove that $(52)_2$ holds with

$$\bar{\beta}_e = \bar{c}_0 > 0, \quad \bar{\gamma}_e(\delta) = \bar{\gamma}_e^0(\delta) - \bar{\beta}_e \ln\sqrt{3} + \bar{\beta}_e \ln c_3. \tag{59}$$

We can again assume that $\boldsymbol{F} \in \mathrm{Sym}^+$. Let $\boldsymbol{D} \in \mathrm{Sym}_0$ be such that $\boldsymbol{H} := \boldsymbol{F}e^{-\boldsymbol{D}} \in \partial\mathscr{E}$. Then using successively (58), (51), (50) and (59) we obtain

$$p(\boldsymbol{F}e^{-\boldsymbol{D}}) + m(\boldsymbol{D}) \ge \bar{\gamma}_e(\delta) + \bar{\beta}^{\mathrm{i}} \ln |\boldsymbol{F}|.$$

Equation (47) implies $(52)_2$. Inequality (53) is proved similarly. □

Remark 2 shows that there are strong differences in the behavior of e, f for large values of $|\boldsymbol{F}|$ on the surfaces $\det \boldsymbol{F} = \delta$. We see that f is not even bounded from below. For this reason, we restrict ourselves to e. The growth of e is sublinear on surfaces of constant determinant since the set $\mathscr{S}$ of all values of the stress $\boldsymbol{S}$, as opposed to the Piola Kirchhoff stress, is bounded: the assumption that

$$|\boldsymbol{S}| \le c < \infty \tag{60}$$

leads to the logarithmic growth due to the extra factor $\boldsymbol{F}^{\mathrm{T}}$ in $\boldsymbol{S} = \boldsymbol{F}^{\mathrm{T}} Dp(\boldsymbol{F})$. The sublinear growth is at variance with the linear growth of the energy in the infinitesimal deformations Hencky plasticity theory (see TŽMAM [33]). There the stress relation reads $\boldsymbol{S} = Dp(\boldsymbol{E})$ with $\boldsymbol{S}$ satisfying (60), which excludes the superlinear growth.

7.1 Appendix: a strange conjugation Next let us show that Condition E has little or nothing to do with the convexity properties of p. To this end, we introduce a transformation on the set of materials which does not change the work in appropriately changed processes but changes the signs of potentials like the elastic potential, free energy, extended energies, etc.

Consider a material $\mathscr{M} = (\Sigma, \Pi, \hat{\boldsymbol{F}}, \hat{\boldsymbol{T}})$ determined by the constitutive objects $\mathscr{E}, \bar{\boldsymbol{T}}, \bar{\boldsymbol{M}}$ and define ${}^{*}\bar{\boldsymbol{T}} : \mathscr{E} \to \mathrm{Sym}$, ${}^{*}\bar{\boldsymbol{M}} : \partial\mathscr{E} \to \mathrm{Sym}_0$ by

$$ {}^{*}\bar{\boldsymbol{T}}(\boldsymbol{E}) = -\bar{\boldsymbol{T}}(\boldsymbol{E}), \quad \boldsymbol{E} \in \mathscr{E}, \quad {}^{*}\bar{\boldsymbol{M}}(\boldsymbol{E}) = -\bar{\boldsymbol{M}}(\boldsymbol{E}), \quad \boldsymbol{E} \in \partial\mathscr{E}. $$

It is easily seen that $\mathscr{E}, {}^{*}\bar{\boldsymbol{T}}, {}^{*}\bar{\boldsymbol{M}}$ satisfy Conditions (i)–(v) in Definition 1 and thus $\mathscr{E}, {}^{*}\bar{\boldsymbol{T}}, {}^{*}\bar{\boldsymbol{M}}$ determine a material which we denote ${}^{*}\mathscr{M} = ({}^{*}\Sigma, {}^{*}\Pi, {}^{*}\hat{\boldsymbol{F}}, {}^{*}\hat{\boldsymbol{T}})$. Clearly,

$$ {}^{*}\Sigma = \Sigma, \quad {}^{*}\hat{\boldsymbol{F}} = \hat{\boldsymbol{F}}, \quad {}^{*}\hat{\boldsymbol{T}} = -\hat{\boldsymbol{T}}. $$

To determine the relationship between ${}^{*}\Pi$ and Π, define, for every $\pi = (\boldsymbol{E}(\cdot), \boldsymbol{P}(\cdot)) \in \Pi$, a pair ${}^{*}\pi = ({}^{*}\boldsymbol{E}(\cdot), {}^{*}\boldsymbol{P}(\cdot))$ by

$$ {}^{*}\boldsymbol{E}(t) = \boldsymbol{E}(d_\pi - t), \quad {}^{*}\boldsymbol{P}(t) = \boldsymbol{P}(d_\pi - t), \quad t \in [0, d_\pi]. $$

It is easily seen that for every process $\pi \in \Pi$ we have ${}^{*}\pi \in {}^{*}\Pi$ and the operation ${}^{*}(\cdot)$ establishes a one-to-one correspondence between Π and ${}^{*}\Pi$. One finds that

$$ w^E({}^{*}\pi) = w^E(\pi), \quad w^P({}^{*}\pi) = w^P(\pi), $$

$$ {}^{*}m(\boldsymbol{D}) = m(-\boldsymbol{D}), \quad \boldsymbol{D} \in \mathrm{Sym}_0, $$

and:

$\mathscr{M}$ satisfies the cyclic second law	$\Leftrightarrow$	${}^{*}\mathscr{M}$ satisfies the cyclic second law,
p is an elastic potential for $\mathscr{M}$	$\Leftrightarrow$	${}^{*}p := -p$ is an elastic potential for ${}^{*}\mathscr{M}$,
ψ is a free energy for $\mathscr{M}$	$\Leftrightarrow$	${}^{*}\psi := -\psi$ is a free energy for ${}^{*}\mathscr{M}$,
r is a residual energy for $\mathscr{M}$	$\Leftrightarrow$	${}^{*}r := -r$ is a residual energy for ${}^{*}\mathscr{M}$,
$\mathscr{M}$ satisfies Il'yushin's condition	$\Leftrightarrow$	${}^{*}\mathscr{M}$ satisfies Il'yushin's condition,
$\mathscr{S}$ is convex	$\Leftrightarrow$	${}^{*}\mathscr{S}$ is convex,
$\mathscr{M}$ satisfies the normality rule	$\Leftrightarrow$	${}^{*}\mathscr{M}$ satisfies the normality rule,
e is an initial energy for $\mathscr{M}$	$\Leftrightarrow$	${}^{*}f := -e$ is a final energy for ${}^{*}\mathscr{M}$,
f is a final energy for $\mathscr{M}$	$\Leftrightarrow$	${}^{*}e := -f$ is an initial energy for ${}^{*}\mathscr{M}$.

8 Materials of type C

The rest of the paper is devoted to a class of materials for which the elastic potential is logarithmically convex on deformation gradients of fixed determinant. They are shown to satisfy Il'yushin's condition and the extended energy e can be calculated using the double convex conjugation with respect to the logarithmic deformation. An example of a material of type C is in Section 9.

If f is any mapping defined on Lin^+ and $\delta > 0$, define the mapping f_δ on Sym_0 by

$$ f_\delta(\boldsymbol{A}) = f(\delta^{1/3} e^{\boldsymbol{A}}), \quad \boldsymbol{A} \in \mathrm{Sym}_0. $$

If $\tilde{p} : \mathrm{Lin}^+ \to \mathbb{R}$ is a continuously differentiable objective isotropic function then $\tilde{p}_\delta$ is isotropic and continuously differentiable and

$$D\tilde{p}_\delta = \tilde{\boldsymbol{S}}_\delta, \tag{61}$$

where $\tilde{\boldsymbol{S}}$ is the traceless part of $\tilde{\boldsymbol{T}}$ defined by

$$\tilde{\boldsymbol{T}}(\boldsymbol{F}) := D\tilde{p}(\boldsymbol{F})\boldsymbol{F}^{\mathrm{T}}. \tag{62}$$

To establish (61), note that in view of the isotropy of both sides of (61) it suffices to verify (61) only on diagonal elements (relative to some orthonormal basis), which is trivial.

The function $\tilde{p}$ is said to be (strictly) logarithmically convex if for every $\delta > 0$, the function $\tilde{p}_\delta$ is (strictly) convex. It must be emphasized that this definition involves only symmetric tensors of constant determinant. By (61) the convexity inequality reads

$$\tilde{p}_\delta(\boldsymbol{B}) \geq \tilde{p}_\delta(\boldsymbol{A}) + \tilde{\boldsymbol{S}}_\delta(\boldsymbol{A}) \cdot (\boldsymbol{B} - \boldsymbol{A}) \tag{63}$$

for any $\boldsymbol{A}, \boldsymbol{B} \in \mathrm{Sym}_0$, with the strict inequality sign if $\tilde{p}$ is strictly logarithmically convex and $\boldsymbol{A} \neq \boldsymbol{B}$. The convexity in the logarithmic deformation has been examined by Hill [12–13] for elastic materials, and was shown to be free from undesirable consequences (in contrast to the convexity in the principal stretches); our assumption is actually weaker because of the determinant restriction. Note also that the logarithmic convexity is consistent with the polyconvexity [32]. Let us show that the logarithmic convexity implies the Baker–Ericksen inequalities.

Lemma 3 *Let $\tilde{p}$ be objective and isotropic, let $\tilde{\boldsymbol{T}}$ be given by* (62)*, let $\boldsymbol{E} = \mathrm{diag}(e_1, e_2, e_3) \in \mathrm{Sym}^+$ be diagonal so that also $\boldsymbol{T} := \tilde{\boldsymbol{T}}(\boldsymbol{E}) = \mathrm{diag}(t_1, t_2, t_3)$ is diagonal. If $\tilde{p}$ is logarithmically convex then we have the Baker–Ericksen inequalities*

$$t_i \geq t_j \quad \text{if} \quad e_i \geq e_j \tag{64}$$

with the strict inequality sign if $\tilde{p}$ is strictly logarithmically convex and $e_i > e_j$.

Proof Write $\boldsymbol{E} = \delta^{1/3} e^{\boldsymbol{A}}$ where $\boldsymbol{A} = \mathrm{diag}(a_1, a_2, a_3) \in \mathrm{Sym}_0$, and fix the pair i,j. Apply the monotonicity $(\tilde{\boldsymbol{S}}_\delta(\boldsymbol{B}) - \tilde{\boldsymbol{S}}_\delta(\boldsymbol{A})) \cdot (\boldsymbol{B} - \boldsymbol{A}) \geq 0$ to $\boldsymbol{B} = \mathrm{diag}(b_1, b_2, b_3)$ where the triple (b_1, b_2, b_3) is obtained from (a_1, a_2, a_3) by interchanging the i,j-components. Then $\tilde{\boldsymbol{S}}_\delta(\boldsymbol{B})$ is a diagonal tensor which differs from $\tilde{\boldsymbol{S}}_\delta(\boldsymbol{A})$ by interchanging the i,j-components and one obtains

$$(t_i - t_j)(e_i - e_j) \geq 0 \tag{65}$$

with the strict inequality sign if $e_i \neq e_j$ and $\tilde{p}_\delta$ is strictly convex. Thus if $e_i > e_j$, we have $t_i \geq t_j$. Since the derivative of $\tilde{p}$ is continuous, a limit provides that $t_i \geq t_j$ also if $e_i = e_j$. □

Lemma 4 (i) *If $\tilde{p}$ is objective, isotropic, of class C^1, and strictly logarithmically convex and $\boldsymbol{E}, \boldsymbol{M} \in \mathrm{Sym}^+$ satisfy*

$$\tilde{p}(\boldsymbol{ERMR}^{\mathrm{T}}) \geq \tilde{p}(\boldsymbol{EM})$$

for each $\boldsymbol{R} \in \mathrm{Rot}$ then $\boldsymbol{E}$ and $\boldsymbol{M}$ commute. (ii) *Similarly, if $f : \mathrm{Sym}_0 \to \mathbb{R}$ is strictly convex and continuously differentiable and $\boldsymbol{A}, \boldsymbol{B} \in \mathrm{Sym}_0$ satisfy*

$$f(A + RBR^{\mathrm{T}}) \geq f(A + B)$$

for each $R \in$ Rot *then* A, B *commute.*

Proof (i): If W is any skew tensor, then the differentiation of

$$\tilde{p}(Ee^{W} Me^{-W}) = \tilde{p}(Ee^{W} M) \geq \tilde{p}(EM)$$

with respect to W at $W = \mathbf{0}$ gives $ETE^{-1} \cdot W = 0$ where $T := \tilde{T}(F)$ and $F := EM$. Hence ETE^{-1} is symmetric which means that $E^2 T = TE^2$ and hence

$$ET = TE. \tag{66}$$

Since $\tilde{T}(\cdot)$ is objective isotropic, we have $T = \tilde{T}(C)$ where $C = (FF^{\mathrm{T}})^{1/2}$. Let us show that (66) implies that

$$EC = CE. \tag{67}$$

In a suitable orthonormal basis we have $C = \operatorname{diag}(c_1, c_2, c_3)$, $T = \operatorname{diag}(t_1, t_2, t_3)$ and (67) reads (no summation)

$$(c_i - c_j)E_{ij} = 0, \quad 1 \leq i,j \leq 3, \tag{68}$$

where E_{ij} are the components of E in our basis. If $c_i = c_j$ then (68) holds trivially. If $c_i \neq c_j$ then the strict convexity of $\tilde{p}_\delta$ implies $t_i \neq t_j$ by the strict version of the Baker–Ericksen inequalities (65) (Remark 3), and as (66) reads $(t_i - t_j)E_{ij} = 0$ we have $V_{ij} = 0$ and (68) holds again. Hence also E and FF^{T} commute. This leads to $EM^2 = M^2 E$. That is, E commutes with M^2 and hence also with $(M^2)^{1/2} = M$. (ii) is proved similarly. □

Definition 3 An ideal elastic–plastic material is said to be of type C if it satisfies the following conditions:

(i) the material obeys the normality rule (see Definition 2);

(ii) $\bar{T}$ has an elastic potential p which admits an objective, isotropic, class C^1 extension $\tilde{p} : \mathrm{Lin}^+ \to \mathbb{R}$ which is strictly logarithmically convex and bounded from below;

(iii) for every $\delta > 0$, $\mathscr{S} = \bar{S}(\mathscr{E}_\delta)$ (see (3) for the definition of $\mathscr{E}_\delta$);

(iv) $\mathbf{0}$ is in the interior of $\mathscr{S}$.

Note that (iv) and the boundedness of $\mathscr{S}$ imply that

$$c_1|D| \leq m(D) \leq c_2|D| \tag{69}$$

for some positive constants c_1, c_2 and all $D \in \mathrm{Sym}_0$.

Proposition 6 *Each material of type* C *satisfies Il'yushin's condition.*

Proof Verify Condition (ii) of Theorem 2. It suffices to establish (36) for pairs E and D as in that condition and satisfying additionally $E \in \mathrm{Sym}^+$. Let Q be a point of minimum of the function $R \mapsto \tilde{p}(ERe^{-D}R^{\mathrm{T}}) = \tilde{p}(ERe^{-D})$, $R \in$ Rot, so that

$$\tilde{p}(Ee^{-D}) \geq \tilde{p}(EQe^{-D}Q^{\mathrm{T}}). \tag{70}$$

By Lemma 4(i), E and $Qe^{-D}Q^{\mathrm{T}}$ commute and hence so also do E and QDQ^{T}. Let $\delta = \det E$ and $A = \ln(E/\delta^{1/3})$. By the commutativity, $Ee^{-QDQ^{\mathrm{T}}} = \delta^{1/3} e^{A - QDQ^{\mathrm{T}}}$ and thus $\tilde{p}(Ee^{-QDQ^{\mathrm{T}}}) = p_\delta(A - QDQ^{\mathrm{T}})$; the convexity of p_δ gives

$$p_\delta(\boldsymbol{A} - \boldsymbol{QDQ}^{\mathrm{T}}) \geq p_\delta(\boldsymbol{A}) - \tilde{\boldsymbol{S}}_\delta(\boldsymbol{A}) \cdot (\boldsymbol{QDQ}^{\mathrm{T}}).$$

Since $\boldsymbol{E} \in \mathscr{E}$, this reads

$$\tilde{p}(\boldsymbol{EQ}e^{-\boldsymbol{D}}\boldsymbol{Q}^{\mathrm{T}}) \geq p(\boldsymbol{E}) - \bar{\boldsymbol{S}}(\boldsymbol{E}) \cdot (\boldsymbol{QDQ}^{\mathrm{T}}) = p(\boldsymbol{E}) - \bar{\boldsymbol{S}}(\boldsymbol{Q}^{\mathrm{T}}\boldsymbol{EQ}) \cdot \boldsymbol{D}.$$

By Lemma 1(ii), $\bar{\boldsymbol{S}}(\boldsymbol{Q}^{\mathrm{T}}\boldsymbol{EQ}) \cdot \boldsymbol{D} \leq m(\boldsymbol{D})$ which in conjunction with (70) leads to (36). □

We now give simplified constructions of the initial energy for materials of type C.

Theorem 4 *Consider a material of type* C *and let* $\boldsymbol{F} \in \mathrm{Lin}^+$. *Then*
(i) *there exists a unique* $\boldsymbol{D} \in \mathrm{Sym}_0$ *such that* $\boldsymbol{E} := \boldsymbol{F}e^{-\boldsymbol{D}} \in \mathscr{E}$ *and*

$$e(\boldsymbol{F}) = p(\boldsymbol{E}) + m(\boldsymbol{D}); \tag{71}$$

(ii) *if* $\boldsymbol{F} \in \mathscr{E}$ *then* $\boldsymbol{E} = \boldsymbol{F}, \boldsymbol{D} = \boldsymbol{0}$;
(iii) *if* $\boldsymbol{F} \notin \mathscr{E}$ *then* $\boldsymbol{E} \in \partial\mathscr{E}$ *and*

$$\boldsymbol{D} \in \mathrm{N}_{\bar{\boldsymbol{S}}(\boldsymbol{E})}\mathscr{S}; \tag{72}$$

if additionally $\boldsymbol{F} \in \mathrm{Sym}^+$, *then* $\boldsymbol{F}, \boldsymbol{E}, \boldsymbol{D}$ *commute.*

Hence, setting

$$\boldsymbol{P} = e^{\boldsymbol{D}},$$

we have the elastic–plastic decomposition

$$\boldsymbol{F} = \boldsymbol{EP} \tag{73}$$

with $\boldsymbol{E} \in \mathscr{E}$, $\boldsymbol{D} \in \mathrm{N}_{\bar{\boldsymbol{S}}(\boldsymbol{E})}\mathscr{S}$ and

$$e(\boldsymbol{F}) = p(\boldsymbol{E}) + m(\boldsymbol{D}).$$

For $\boldsymbol{F} \in \mathrm{Sym}^+$ the logarithm of (73) and the commutativity give

$$\ln \boldsymbol{F} = \ln \boldsymbol{E} + \boldsymbol{D}$$

which decomposes $\ln \boldsymbol{F}$ into its projection $\ln \boldsymbol{E}$ onto $\ln(\mathscr{E} \cap \mathrm{Sym}^+)$ and the complement $\boldsymbol{D}$ in the normal direction to $\mathscr{S}$ at the corresponding stress. If we interpret the energy e as a stored energy of the associated nonlinear Hencky material, the above shows that the constuction of e involves the same pojections as in the small deformation theory, but in the space of logarithmic deformations.

Proof It is enough to give the proof in the case $\boldsymbol{F} \in \mathrm{Sym}^+$. Write $\boldsymbol{F} = \delta^{1/3}e^{\boldsymbol{A}}$ where $\delta = \det \boldsymbol{F}$, $\boldsymbol{A} \in \mathrm{Sym}_0$, and consider an auxilliary minimum problem

$$\bar{e} = \min\{\tilde{p}_\delta(\boldsymbol{A} - \boldsymbol{D}) + m(\boldsymbol{D}) : \boldsymbol{D} \in \mathrm{Sym}_0\}. \tag{74}$$

The minimum exists since $\tilde{p}$ is bounded from below and m coercive (see (69)). Let $\boldsymbol{D}$ be a point of minimum. The optimality conditions say that

$$\tilde{\boldsymbol{S}}_\delta(\boldsymbol{A} - \boldsymbol{D}) \in \partial m(\boldsymbol{D}). \tag{75}$$

Furthermore, we have in particular

$$\tilde{p}_\delta(\boldsymbol{A} - \boldsymbol{D}) \leq \tilde{p}_\delta(\boldsymbol{A} - \boldsymbol{RDR}^{\mathrm{T}})$$

for each $\boldsymbol{R} \in \mathrm{Rot}$ and hence $\boldsymbol{A}, \boldsymbol{D}$ commute by Lemma 4(ii). We conclude from (75) and $\partial m(\boldsymbol{D}) \subset \mathscr{S}$ (see (34)) that $\tilde{\boldsymbol{S}}_\delta(\boldsymbol{A} - \boldsymbol{D}) \in \mathscr{S}$. From Definition 3(iii) we have $\bar{\boldsymbol{S}}(\mathscr{E}_\delta) = \tilde{\boldsymbol{S}}(\mathscr{E}_\delta) = \mathscr{S}$ and thus $\tilde{\boldsymbol{S}}_\delta(\boldsymbol{M}) = \tilde{\boldsymbol{S}}_\delta(\boldsymbol{A} - \boldsymbol{D})$ for some $\boldsymbol{M} \in \mathrm{Sym}_0$ such that $\delta^{1/3} e^{\boldsymbol{M}} \in \mathscr{E}_\delta$. The strict monotonicity of $\tilde{\boldsymbol{S}}_\delta$, which is a consequence od the strict convexity of $\tilde{p}_\delta$, implies that $\boldsymbol{M} = \boldsymbol{A} - \boldsymbol{D}$ and as $\boldsymbol{A}, \boldsymbol{D}$ commute, we obtain $\boldsymbol{F} e^{-\boldsymbol{D}} \in \mathscr{E}$. Hence

$$\bar{e} = p(\boldsymbol{F} e^{-\boldsymbol{D}}) + m(\boldsymbol{D})$$

with $\boldsymbol{F} e^{-\boldsymbol{D}} \in \mathscr{E}$ which implies that

$$e(\boldsymbol{F}) \le \bar{e}. \tag{76}$$

Let us further prove that the opposite inequality holds in (76). Note that the infimum in (41) exists as a minimum in the present case. Let $\boldsymbol{D}$ be the point of minimum in (41). Let further $\boldsymbol{Q}$ be a point of minimum of the function $\boldsymbol{R} \mapsto \tilde{p}(\boldsymbol{F} e^{-\boldsymbol{R}\boldsymbol{D}\boldsymbol{R}^{\mathrm{T}}})$, $\boldsymbol{R} \in \mathrm{Rot}$. Lemma 4(i) tells us that $\boldsymbol{F}, \boldsymbol{Q}\boldsymbol{D}\boldsymbol{Q}^{\mathrm{T}}$ commute and hence

$$\begin{aligned} p(\boldsymbol{F} e^{-\boldsymbol{D}}) + m(\boldsymbol{D}) &\ge \tilde{p}(\boldsymbol{F} e^{-\boldsymbol{Q}\boldsymbol{D}\boldsymbol{Q}^{\mathrm{T}}}) + m(\boldsymbol{Q}\boldsymbol{D}\boldsymbol{Q}^{\mathrm{T}}) \\ &= \tilde{p}_\delta(\boldsymbol{A} - \boldsymbol{Q}\boldsymbol{D}\boldsymbol{Q}^{\mathrm{T}}) + m(\boldsymbol{Q}\boldsymbol{D}\boldsymbol{Q}^{\mathrm{T}}) \\ &\ge \bar{e}, \end{aligned}$$

which proves the opposite inequality in (76). Moreover, the argument shows that $\boldsymbol{D}$ is a point of minimum in (41) if and only if $\boldsymbol{D}$ is a point of minimum in (74) and that such a point commutes with $\boldsymbol{F}$.

(i): To prove the uniqueness of the point of minimum in (41), it suffices to prove the uniqueness of the point of minimum in (74). Let $\boldsymbol{D}_\alpha$, $\alpha = 1, 2$, be two distinct points of minimum in (74), so that, in particular, $\boldsymbol{F}, \boldsymbol{A}, \boldsymbol{D}_\alpha$ commute,

$$\tilde{\boldsymbol{S}}_\delta(\boldsymbol{A} - \boldsymbol{D}_\alpha) \in \partial m(\boldsymbol{D}_\alpha), \tag{77}$$

and

$$\tilde{p}_\delta(\boldsymbol{A} - \boldsymbol{D}_1) + m(\boldsymbol{D}_1) = \tilde{p}_\delta(\boldsymbol{A} - \boldsymbol{D}_2) + m(\boldsymbol{D}_2). \tag{78}$$

The inclusions (77) imply

$$\tilde{\boldsymbol{S}}_\delta(\boldsymbol{A} - \boldsymbol{D}_\alpha) \cdot \boldsymbol{D}_\alpha = m(\boldsymbol{D}_\alpha), \tag{79}$$

and the strict convexity provides

$$\tilde{p}_\delta(\boldsymbol{A} - \boldsymbol{D}_1) > \tilde{p}_\delta(\boldsymbol{A} - \boldsymbol{D}_2) + \tilde{\boldsymbol{S}}_\delta(\boldsymbol{A} - \boldsymbol{D}_2) \cdot (\boldsymbol{D}_2 - \boldsymbol{D}_1)$$

which in combination with (78) and (79) leads to

$$\tilde{\boldsymbol{S}}_\delta(\boldsymbol{A} - \boldsymbol{D}_2) \cdot \boldsymbol{D}_1 > m(\boldsymbol{D}_1)$$

in conttradiction with (32). Thus the minimizer is unique. (ii): This follows from the uniqueness and Theorem 3(ii). (iii): The inclusion $\boldsymbol{E} \in \partial\mathscr{E}$ follows from the uniqueness and Theorem 3(iii). The inclusion (72) follows from the above proof. □

Remark 3 If we consider the unique minimizer $\boldsymbol{D}$ and the $\boldsymbol{E}$ from Theorem 4 as a function of $\boldsymbol{F} \in \mathrm{Lin}^+$, written $\boldsymbol{D} = \hat{\boldsymbol{D}}(\boldsymbol{F})$, $\boldsymbol{E} = \hat{\boldsymbol{E}}(\boldsymbol{F})$, then the form of (41) implies that the functions have the following transformation properties:

$$\hat{\boldsymbol{E}}(\boldsymbol{QF}) = \boldsymbol{Q}\hat{\boldsymbol{E}}(\boldsymbol{F}), \quad \hat{\boldsymbol{P}}(\boldsymbol{QF}) = \hat{\boldsymbol{P}}(\boldsymbol{F}),$$
$$\hat{\boldsymbol{E}}(\boldsymbol{FQ}^{\mathrm{T}}) = \hat{\boldsymbol{E}}(\boldsymbol{F})\boldsymbol{Q}^{\mathrm{T}}, \quad \hat{\boldsymbol{P}}(\boldsymbol{FQ}^{\mathrm{T}}) = \boldsymbol{Q}\hat{\boldsymbol{P}}(\boldsymbol{F})\boldsymbol{Q}^{\mathrm{T}},$$

for every $\boldsymbol{F} \in \mathrm{Lin}^+$, $\boldsymbol{Q} \in \mathrm{Rot}$. We define $\hat{\boldsymbol{Z}} : \mathrm{Lin}^+ \to \mathrm{Sym}$ by

$$\hat{\boldsymbol{Z}}(\boldsymbol{F}) = \bar{\boldsymbol{T}}_0(\hat{\boldsymbol{E}}(\boldsymbol{F})).$$

where $\bar{\boldsymbol{T}}_0(\boldsymbol{E}) = \bar{\boldsymbol{T}}(\boldsymbol{E}) - \frac{1}{3}(\mathrm{tr}\,\bar{\boldsymbol{T}}(\boldsymbol{E}))\mathbf{1}$ is the traceless part of the Kirchhoff stresss. Then $\hat{\boldsymbol{Z}}$ is objective and isotropic and for every $\delta > 0$ we have the strerss relation

$$\hat{\boldsymbol{Z}}_\delta(\boldsymbol{A}) = De_\delta(\boldsymbol{A})$$

for a.e. $\boldsymbol{A} \in \mathrm{Sym}_0$. More precisely, e_δ is convex and $\hat{\boldsymbol{Z}}_\delta(\boldsymbol{A})$ is a subgradient of e_δ at $\boldsymbol{A}$ for every $\boldsymbol{A} \in \mathrm{Sym}_0$. Thus the Hencky material with the stored energy e is such that traceless part of the Kirschhoff stress always belong to the stress range $\mathscr{S}$.

Proof Let $\boldsymbol{A}$ be fixed and $\boldsymbol{D}$ the corresponding minimizer. If $\boldsymbol{B}, \bar{\boldsymbol{D}} \in \mathrm{Sym}_0$ then the convexity of $\tilde{p}_\delta$ says that

$$\tilde{p}_\delta(\boldsymbol{B} - \bar{\boldsymbol{D}}) \geq \tilde{p}_\delta(\boldsymbol{A} - \boldsymbol{D}) + \tilde{\boldsymbol{S}}_\delta(\boldsymbol{A} - \boldsymbol{D}) \cdot (\boldsymbol{B} - \boldsymbol{A} + \boldsymbol{D} - \bar{\boldsymbol{D}}) \tag{80}$$

and the convexity of m that

$$m(\bar{\boldsymbol{D}}) \geq m(\boldsymbol{D}) + \tilde{\boldsymbol{S}}_\delta(\boldsymbol{A} - \boldsymbol{D}) \cdot (\bar{\boldsymbol{D}} - \boldsymbol{D}) \tag{81}$$

since $\tilde{\boldsymbol{S}}_\delta(\boldsymbol{A} - \boldsymbol{D})$ is a subgradient of m at $\boldsymbol{D}$. A combination of (80), (81) with

$$e_\delta(\boldsymbol{A}) = \tilde{p}_\delta(\boldsymbol{A} - \boldsymbol{D}) + m(\boldsymbol{D})$$

provides

$$\tilde{p}_\delta(\boldsymbol{B} - \bar{\boldsymbol{D}}) + m(\bar{\boldsymbol{D}}) \geq e_\delta(\boldsymbol{A}) + \tilde{\boldsymbol{S}}_\delta(\boldsymbol{A} - \boldsymbol{D}) \cdot (\boldsymbol{B} - \boldsymbol{A})$$

Fixing $\boldsymbol{B}$ and taking the infimum of the left-hand side over all $\bar{\boldsymbol{D}} \in \mathrm{Sym}_0$ we obtain

$$e_\delta(\boldsymbol{B}) \geq e_\delta(\boldsymbol{A}) + \tilde{\boldsymbol{S}}_\delta(\boldsymbol{A} - \boldsymbol{D}) \cdot (\boldsymbol{B} - \boldsymbol{A})$$

which shows that e_δ is convex and $\tilde{\boldsymbol{S}}_\delta(\boldsymbol{A} - \boldsymbol{D})$ is its subgradient at $\boldsymbol{A}$. It follows that e_δ is locally Lischitz continuous and the subgradient coincides with the derivative for a.e. $\boldsymbol{A} \in \mathrm{Sym}_0$. □

9 The duality; example

This section gives a description of the initial energy for materials of type C in terms of the convex conjugation in the logarithmic measure of deformation (Proposition 7). The dual description is the analog, for large deformations, of the duality considerations presented in TŽmam [33; Chapter 1, Section 3] in the context of the small deformation theory of plasticity.

Proposition 7 *Consider a material of type* C. *Then for every* $\delta > 0$ *we have*

$$e_\delta = \varphi_\delta^*$$

where φ_δ^* *is the convex conjugate of* φ_δ *given by*

$$\varphi_\delta(\boldsymbol{S}) = \begin{cases} \tilde{p}_\delta^*(\boldsymbol{S}) & \text{if} \quad \boldsymbol{S} \in \mathscr{S}, \\ \infty & \text{otherwise,} \end{cases}$$

where $\tilde{p}_\delta^*$ *is the convex conjugate of* $\tilde{p}_\delta$ *on* Sym_0.

Proof The proof of Theorem 4 shows that

$$e(\boldsymbol{F}) = \min\{\tilde{p}_\delta(\boldsymbol{A} - \boldsymbol{D}) + m(\boldsymbol{D}) : \boldsymbol{D} \in \mathrm{Sym}_0\} \tag{82}$$

and the proof of Remark 3 that e_δ is convex on Sym_0 for every $\delta > 0$. For a fixed δ, consider the family $\{f_{\boldsymbol{D}} : \boldsymbol{D} \in \mathrm{Sym}_0\}$ of convex functions on Sym_0 given by

$$f_{\boldsymbol{D}}(\boldsymbol{A}) = \tilde{p}_\delta(\boldsymbol{A} - \boldsymbol{D}) + m(\boldsymbol{D}). \tag{83}$$

Equation (82) asserts that

$$e_\delta(\boldsymbol{A}) = \min\{f_{\boldsymbol{D}}(\boldsymbol{A}) : \boldsymbol{D} \in \mathrm{Sym}_0\}, \quad \boldsymbol{A} \in \mathrm{Sym}_0. \tag{84}$$

The application of the general formula for the convex conjugate of a minimum of a family of convex functions [10; Chapter I, Equation (4.6)] in the present case gives

$$e_\delta^*(\boldsymbol{S}) = \big(\min\{f_{\boldsymbol{D}} : \boldsymbol{D} \in \mathrm{Sym}_0\}\big)^*(\boldsymbol{S}) = \sup\{f_{\boldsymbol{D}}^*(\boldsymbol{S}) : \boldsymbol{D} \in \mathrm{Sym}_0\}, \tag{85}$$

$\boldsymbol{S} \in \mathrm{Sym}_0$. We have

$$f_{\boldsymbol{D}}^*(\boldsymbol{S}) = \big(\tilde{p}_\delta(\cdot - \boldsymbol{D}) + m(\boldsymbol{D})\big)^*(\boldsymbol{S}) \tag{86}$$

where $\tilde{p}_\delta(\cdot - \boldsymbol{D})$ denotes the shifted function $\boldsymbol{A} \mapsto \tilde{p}_\delta(\boldsymbol{A} - \boldsymbol{D})$. In (86), $\boldsymbol{D}$ is a parameter and hence $m(\boldsymbol{D})$ an additive constant during the evaluation of the convex conjugate. The rules for the evaluation of the conjugate of a function shifted by an additive constant and shifted in the domain space [10; Chapter I, Equations (4.8) and (4.9)] yield in the present case

$$f_{\boldsymbol{D}}^*(\boldsymbol{S}) = \tilde{p}_\delta^*(\boldsymbol{S}) + \boldsymbol{S} \cdot \boldsymbol{D} - m(\boldsymbol{D}), \tag{87}$$

where $\tilde{p}_\delta^*$ is the conjugate of $\tilde{p}_\delta$. Inserting (87) into (85),

$$e_\delta^*(\boldsymbol{S}) = \tilde{p}_\delta^*(\boldsymbol{S}) + \sup\{\boldsymbol{S} \cdot \boldsymbol{D} - m(\boldsymbol{D}) : \boldsymbol{D} \in \mathrm{Sym}_0\} \tag{88}$$

and we note that

$$m^*(\boldsymbol{S}) := \sup\{\boldsymbol{S} \cdot \boldsymbol{D} - m(\boldsymbol{D}) : \boldsymbol{D} \in \mathrm{Sym}_0\},$$

is the conjugate of the convex dissipation function m. A combination of (35) with (88) then gives $e_\delta^*(\boldsymbol{S}) = \varphi_\delta(\boldsymbol{S})$. Since e_δ is convex, we have $e_\delta = (e_\delta)^{**} = \varphi_\delta^*$. □

9.1 Example Finally, an elastic–plastic material is described for which the initial energy and the Hencky type response are calculated explicitly. Let

$$\mathscr{E} = \{\boldsymbol{E} := \boldsymbol{V}\boldsymbol{R} \in \mathrm{Lin}^+ : \boldsymbol{V} \in \mathrm{Sym}^+, \boldsymbol{R} \in \mathrm{Rot}, |\boldsymbol{L}| \le \tau/\beta\},$$

where τ, β are given positive numbers and $\boldsymbol{L} = \ln \boldsymbol{V} - \frac{1}{3}\operatorname{tr}(\ln \boldsymbol{V})\mathbf{1}$. Define $\bar{\boldsymbol{T}} : \mathrm{Lin}^+ \to \mathrm{Sym}$ by

$$\bar{\boldsymbol{T}}(\boldsymbol{E}) = \alpha\kappa(\delta)\delta\mathbf{1} + \beta\boldsymbol{L}$$

for each $\boldsymbol{E} \in \mathrm{Lin}^+$, where $\delta = \det \boldsymbol{E}$ and $\alpha : (0, \infty) \to \mathbb{R}$ is a continuously differentiable function. We use the same symbol $\bar{\boldsymbol{T}}$ for the restriction of $\bar{\boldsymbol{T}}$ to $\mathscr{E}$. Define $\bar{\boldsymbol{M}} : \partial\mathscr{E} \to \mathrm{Sym}_0$ by

$$\bar{\boldsymbol{M}}(\boldsymbol{E}) = (\beta/\tau)\boldsymbol{R}^{\mathrm{T}}\boldsymbol{L}\boldsymbol{R}, \quad \boldsymbol{E} \in \partial\mathscr{E}.$$

The objects $\mathscr{E}, \bar{\boldsymbol{T}}, \bar{\boldsymbol{M}}$ determine an ideal elastic–plastic material $(\Sigma, \Pi, \hat{\boldsymbol{F}}, \hat{\boldsymbol{T}})$ (see Definition 1) and one finds that

$$\bar{\boldsymbol{S}}(\boldsymbol{E}) = \beta\boldsymbol{R}^{\mathrm{T}}\boldsymbol{L}\boldsymbol{R}, \quad \boldsymbol{E} \in \mathrm{Sym}^+,$$

$$\mathscr{S} = \{\boldsymbol{S} \in \mathrm{Sym}_0 : |\boldsymbol{S}| \le \tau\}.$$

The material is of type C. Indeed, other things being obvious, it suffices to verify Condition (ii) of Definition 3. One finds that if $p : \mathrm{Lin}^+ \to \mathbb{R}$ is defined by

$$p(\boldsymbol{E}) = \alpha(\delta) + \tfrac{1}{2}\beta|\boldsymbol{L}|^2$$

for each $\boldsymbol{E} \in \mathrm{Lin}^+$, where $\boldsymbol{E} = \boldsymbol{V}\boldsymbol{R}$ is the polar decomposition of $\boldsymbol{E}$, then p is an elastic potential for $\bar{\boldsymbol{T}}$ on Lin^+. We identify $\tilde{p}$ with p. One finds that

$$\tilde{p}_\delta(\boldsymbol{A}) = \alpha(\delta) + \tfrac{1}{2}\beta|\boldsymbol{A}|^2, \quad \boldsymbol{A} \in \mathrm{Sym}_0,$$

which is a strictly convex function. Thus the material is of type C. The dissipation function is given by

$$m(\boldsymbol{D}) = m^{**}(\boldsymbol{D}) = \tau|\boldsymbol{D}|, \quad \boldsymbol{D} \in \mathrm{Sym}_0,$$

and since it is convex, by Proposition 5, the maximal and minimal residual energies are

$$s(\boldsymbol{R}e^{\boldsymbol{D}}) = \tau|\ln\boldsymbol{D}|, \quad t(\boldsymbol{R}e^{\boldsymbol{D}}) = -\tau|\ln\boldsymbol{D}|,$$

$\boldsymbol{R} \in \mathrm{Rot}$, $\boldsymbol{D} \in \mathrm{Sym}_0$. The initial energy is given by

$$e(\boldsymbol{F}) = \begin{cases} \alpha(\delta) + \tfrac{1}{2}\beta|\boldsymbol{L}|^2 & \text{if } |\boldsymbol{L}| \le \tau/\beta, \\ \alpha(\delta) - \tfrac{1}{2}(\tau/\beta) + \tau|\boldsymbol{L}| & \text{if } |\boldsymbol{L}| > \tau/\beta, \end{cases} \tag{89}$$

for each $\boldsymbol{F} \in \mathrm{Lin}^+$, where throughout, $\boldsymbol{F} = \boldsymbol{V}\boldsymbol{R}$ is the polar decomposition of $\boldsymbol{F}$ and $\delta = \det\boldsymbol{F}$. The case $|\boldsymbol{L}| \le \tau/\beta$ in (89) follows from Theorem 3. Let us consider the case $|\boldsymbol{L}| > \tau/\beta$. By the objectivity and isotropy we may restrict ourselves to $\boldsymbol{F} = \boldsymbol{V} \in \mathrm{Sym}^+ \sim \mathscr{E}$; in fact it suffices to consider an $\boldsymbol{F}$ that is diagonal in some orthonormal basis. By Theorem 4, it suffices to seek the infimum in (47) only on those $\boldsymbol{D}$ which are diagonal in some basis of eigenvectors of $\boldsymbol{F}$. If $\boldsymbol{F}e^{-\boldsymbol{D}} \in \partial\mathscr{E}$ then

$$p(\boldsymbol{F}e^{-\boldsymbol{D}}) = \alpha(\delta) + \tau^2/2\beta.$$

Consider an orthonormal basis of eigenvectors of $\boldsymbol{F}$ so that $\boldsymbol{F} = \mathrm{diag}(v_1, v_2, v_3)$. By (47),

$$e(\boldsymbol{F}) = \alpha(\delta) + \tau^2/2\beta + \tau M \tag{90}$$

where

$$M := \inf\{|\boldsymbol{D}| : \boldsymbol{D} \in \mathrm{Sym}_0, \boldsymbol{F}e^{-\boldsymbol{D}} \in \partial\mathscr{E},\ \boldsymbol{D} \text{ diagonal}\}. \tag{91}$$

For a $\boldsymbol{D} = \mathrm{diag}(d_1, d_2, d_3)$ as in (91) the condition $\boldsymbol{F}e^{-\boldsymbol{D}} \in \partial\mathscr{E}$ gives $|\ln_0 \boldsymbol{F} - \boldsymbol{D}| = \tau/\beta$ and we are led to

$$M = \inf\{|d| : d \in C\}$$

where $C = \{d \in P, |d - c| = \tau/\beta\}$ is the circle with the center $c, c_i = \ln(v_i/\delta^{1/3}), \delta := v_1 v_2 v_3$, in the plane $P = \{d \in \mathbb{R}^3 : d_1 + d_2 + d_3 = 0\}$. By $|\ln_0 \boldsymbol{F}| > \tau/\beta$ the origin

$(0, 0, 0)$ is not in the interior of C. Thus we are looking for the point of the circle C whose distance from the origin is minimal and hence $M = |\ln_0 \boldsymbol{F}| - \tau/\delta$. for each $\boldsymbol{F} \in \mathrm{Lin}^+$. A combination with (90) gives (89). Note the linear growth of e in $|\boldsymbol{L}|$ outside $\mathscr{E}$. A differentiation of e gives

$$\hat{\boldsymbol{Z}}(\boldsymbol{F}) = \begin{cases} \beta \boldsymbol{L} & \text{if} \quad |\boldsymbol{L}| \leq \tau/\beta, \\ \tau \boldsymbol{L}/|\boldsymbol{L}| & \text{if} \quad |\boldsymbol{L}| > \tau/\beta. \end{cases}$$

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