BOREL REDUCIBILITY AND CARDINAL ARITHMETIC

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Subject A.

Borel reducibility of analytic equivalence relations.

- most proofs in ZF+DC;
- Borel, analytic sets, actions of Polish groups;
- motivation from mathematical analysis.

Subject B.

Combinatorial set theory with choice.

- proofs with large cardinals, independence results;
- transfinite induction, forcing, pcf;
- set theoretic motivation.

The two subjects are connected.

With a pipe of large diameter

Example.

$$E \not\leq F$$

- E, F are both Borel equivalence relations classifiable by countable structures;
- the natural proof of $E \not\leq F$ uses the fact that the Singular Cardinal Hypothesis can fail at $\aleph_\omega$.

The pinned cardinal–purpose.

- $\kappa(E)$ is a cardinal invariant respecting the Borel reducibility: $E \le F \to \kappa(E) \le \kappa(F)$;

- I produce E such that (provably) $\kappa(E) = (\aleph_\omega^{\aleph_0})^+$;

- . . . and also F such that (provably) $\kappa(F) = \max\{\mathfrak{c}, \aleph_{\omega+1}\}^+$;

- the pinned cardinal can reflect many other combinatorial issues.

The pinned cardinal–definition.

Let E be an analytic equivalence relation on Polish X, let τ be a P-name for an element of X.

- The name τ is *pinned* if $P \times P \Vdash \tau_{\text{left}}\ E\ \tau_{\text{right}}$;
- $\langle P, \tau \rangle\ \bar{E}\ \langle Q, \sigma \rangle$ if $P \times Q \Vdash \tau\ E\ \sigma$;
- $\kappa(E)$ is the smallest cardinal such that every pinned name has an $\bar{E}$-equivalent on a poset of size $< \kappa$.

The pinned cardinal–features.

- $\kappa(E) = \aleph_1$ for E pinned, as a definitory matter;

- $\kappa(E) \le \beth_{\omega_1}$ for Borel E;

- $\kappa(E)$ stays below the first measurable cardinal if not ∞;

- $\kappa(E) = \infty$ iff $E_{\omega_1} \le E$;

- natural behavior vis-a-vis usual operations.

Evaluation I.

Definition. An $L_{\omega_1\omega}$ sentence is *set-like* if it has an extensional relation $\in$ about which it proves that $\in$ is wellfounded.

Theorem. If ϕ is set-like and E_ϕ is the isomorphism of models of ϕ, then $\kappa(E_\phi)$ =supremum of possible sizes of models of ϕ.

Proof. Each pinned name in this case corresponds to a collapse of a (uncountable) model of ϕ.

Evaluation II.

Theorem. For every countable ordinal $\alpha > 0$ there is a set-like sentence ϕ_α which has models of exactly all sizes $< \aleph_\alpha$. Thus $\kappa(E_\alpha) = \aleph_\alpha$.

Proof. Induce on α. At limit stage, take disjunction of previous sentences. At successor stage $\alpha + 1$, let $\phi_{\alpha+1}$ be a sentence whose model consists of

- one model M_β for each ϕ_β for all $\beta \leq \alpha$;
- a separate linear ordering $\prec$;
- for each $p \in \text{dom}(\prec)$, a single bijection between the set $\{q \in \text{dom}(\prec) : q \prec p\}$ and one of the models M_β for $\beta \leq \alpha$.

Evaluation III.

Theorem. There is a set-like sentence ϕ which has models of size $\aleph_\omega^{\aleph_0}$ but no larger.

Proof. Models of ϕ are sets of maps from a model of ϕ_1 to a model of $\phi_{\omega+1}$.

Theorem. There is a set-like sentence ϕ which has models of size $\max\{\mathfrak{c}, \aleph_{\omega+1}\}$ but no larger.

Proof. The disjunction of $\phi_{\omega+2}$ and $\phi_{\mathfrak{c}}$.