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ON GRAND AND SMALL LEBESGUE AND SOBOLEV SPACES AND SOME APPLICATIONS TO PDE'S

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Abstract. This paper is essentially a survey on grand and small Lebesgue spaces, which are rearrangement-invariant Banach function spaces of interest not only from the point of view of Function Spaces theory, but also from the point of view of their applications: the corresponding Sobolev spaces are of interest, for instance, in the theory of PDEs. We discuss results of existence, uniqueness and regularity of certain Dirichlet problems, where the knowledge of these spaces plays a central role. The novelty of this paper relies in an unified treatment containing a number of equivalent quasinorms, all written making explicit the dependence of $|\Omega|$, in the discussion of the sharpness of Hölder's inequality, and in the connection of the results in PDEs with some existing literature.

1. Grand and small Lebesgue spaces: a short overview

1.1. The original motivation

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $f : \Omega \to \mathbb{R}^n$, $f = (f^1, ..., f^n)$ be a mapping of Sobolev class $W^{1,n}_{loc}(\Omega, \mathbb{R}^n)$. Let us denote by $Df(x) : \mathbb{R}^n \to \mathbb{R}^n$ the differential and by $J(x, f) = detDf(x)$ the Jacobian of f. After the elementary remark that by Hölder's inequality the Jacobian $J(x, f)$ is in $L^1_{loc}(\Omega)$, the first fundamental result on the integrability of the Jacobian was due to Müller ([135]):

$$f \in W^{1,n}(\Omega, \mathbb{R}^n),\ J(x, f) \geqslant 0\ a.e. \quad \Rightarrow \quad J(x, f) \in L\log L_{loc}(\Omega).$$

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Paper DEA-10-03

As a dual result, Iwaniec and Sbordone in [96] introduced the grand Lebesgue space $L^{n)}$, and proved that under the assumption $|Df| \in L^{n)}$ the Jacobian of f is locally integrable. Such assumption, weaker than $|Df| \in L^n$, is shown to be minimal:

$$|Df| \in L^{n)}(\Omega),\ J(x,f) \geqslant 0\ a.e. \quad \Rightarrow \quad J(x,f) \in L^1_{loc}(\Omega).$$

The space $L^{n)}(\Omega)$ has been defined through

$$u \in L^{n)}(\Omega) \quad \Leftrightarrow \quad \|u\|_{L^{n)}(\Omega)} = \sup_{0<\varepsilon<n-1} \left(\varepsilon \fint_\Omega |u(x)|^{n-\varepsilon} dx \right)^{\frac{1}{n-\varepsilon}} < \infty$$

where the symbol $\fint_\Omega$ denotes the integral mean: $\fint_\Omega = \frac{1}{|\Omega|}\int_\Omega$.

Of course, from the point of view of function spaces, it is natural to release the exponent n from the dimension, and therefore to consider, in its place, a generic exponent p, $1 < p < \infty$.

1.2. Background

Let $\Omega \subset \mathbb{R}^n$, $n \geqslant 1$, be (Lebesgue) measurable, $0 < |\Omega| < \infty$, and let $f : \Omega \to \mathbb{R}$ be measurable. Let us recall the familiar notation for the standard, classical Lebesgue spaces (see e.g. the recent monograph [29] on this topic): for $1 \leqslant p < \infty$,

$$f \in L^p(\Omega) \quad \Leftrightarrow \quad \|f\|_{L^p(\Omega)} = \left(\int_\Omega |f(x)|^p dx \right)^{\frac{1}{p}} < \infty$$

while, for $p = \infty$,

$$f \in L^\infty(\Omega) \quad \Leftrightarrow \quad \|f\|_{L^\infty(\Omega)} = \operatorname*{ess\,sup}_\Omega |f| < \infty.$$

A generalization, and at the same time an extraordinary refinement, of the theory of Lebesgue spaces is given by the theory of Orlicz spaces. It is a theory fruitful in applications because the role of the composition by a power is played by a more generic function (see e.g. [2]; for a more general notion of Orlicz space see also, for instance, the paper [97] written for the Orlicz centenary volume; see also the references therein) $\Phi : [0,\infty[\to [0,\infty[$ continuous, strictly increasing, convex, $\Phi(0) = 0$, $\Phi'(0) = 0$, $\Phi'(\infty) = \infty$. The norm of the corresponding space $L^\Phi(\Omega)$ is given by

$$f \in L^\Phi(\Omega) \quad \Leftrightarrow \quad \|f\|_{L^\Phi(\Omega)} = \inf\left\{ \lambda > 0 : \int_\Omega \Phi\left(\frac{|f(x)|}{\lambda} \right) dx \leqslant 1 \right\} < \infty$$

where, by convention, $\inf\emptyset = \infty$. The notation changes for specific choices of the function Φ: even if the norm of a function in $L^\Phi(\Omega)$ depends heavily on Φ, when $0 < |\Omega| < \infty$ the symbol $L^\Phi(\Omega)$, interpreted as set of functions, is not affected when Φ is replaced by an equivalent function at infinity (Φ_1, Φ_2 are equivalent at infinity if for some $c_1, c_2, t_0 > 0$ it is $c_1\Phi_1(t) \leqslant \Phi_2 \leqslant c_2\Phi_2(t)$, $t > t_0$). For instance, here there are examples of growths and symbols for the corresponding sets of functions:

$$\Phi(t) \approx \exp(t^{\beta}), \qquad t \text{ large} \qquad ---\!> \qquad EXP^{\beta}(\Omega) \qquad (\beta > 0)$$
$$\Phi(t) \approx t^p \log^{\alpha p} t, \qquad t \text{ large} \qquad ---\!> \qquad L^p(\log L)^{\alpha p}(\Omega)$$
$$(p > 1, \alpha \in \mathbb{R}; p = 1, \alpha > 0)$$
$$\Phi(t) \approx \frac{t^p}{\log t}, \qquad t \text{ large} \qquad ---\!> \qquad \frac{L^p}{\log L}(\Omega).$$

Orlicz spaces (and therefore, in particular, Lebesgue spaces) enjoy an important property, which involves the notion of decreasing rearrangement f_* of a measurable function $f : \Omega \to \mathbb{R}$, defined as function $f_* : [0, |\Omega|] \to [0, \infty]$ through

$$f_*(t) = \inf\{\lambda \geqslant 0 : \mu_f(\lambda) \leqslant t\} \qquad (\inf \emptyset = \infty),$$

where $\mu_f : [0, \infty[\to [0, |\Omega|]$ is the distribution function of $|f|$, defined by

$$\mu_f(\lambda) = |\{x \in \Omega : |f(x)| > \lambda\}|.$$

The property we highlight is to be *rearrangement-invariant:* two equimeasurable functions f, g (i.e. such that $\mu_f = \mu_g$) have the same norm: this fact is obviously a feature of all norms which can be expressed in terms of the distribution function of $|f|$. Moreover, since f and f_* have the same distribution function, it is $\|f\|_{L^p(\Omega)} = \|f_*\|_{L^p(0,|\Omega|)}$ $(1 \leqslant p \leqslant \infty)$ and more generally $\|f\|_{L^{\Phi}(\Omega)} = \|f_*\|_{L^{\Phi}(0,|\Omega|)}$.

Orlicz spaces are special “Banach function spaces”, a terminology which sometimes is origin of a small confusion: the term “Banach function spaces” does not denote just Banach spaces whose elements are measurable functions (as, for instance, Sobolev spaces), but more frequently denotes Banach spaces whose elements are measurable functions satisfying some additional properties. In the theory developed in [15] a “Banach function space” is a Banach space $(X, \|\cdot\|_X)$ such that the elements of X are measurable functions $f, g : \Omega \to \mathbb{R}$ satisfying

- $|f| \leqslant |g|$ *a.e.* $\Rightarrow \quad \|f\|_X \leqslant \|g\|_X$
- $|f_k| \nearrow |f|$ *a.e.* $\Rightarrow \quad \|f_k\|_X \nearrow \|f\|_X$
- $\|1\|_X < \infty$
- $\int_E |f| dx \leqslant c(E)\|f\|_X \qquad (E \subset \Omega).$

This notion opens the way to a general theory, where many “standard” functions spaces in Analysis represent just examples. Interested readers may consult also other classical monographs, like, for instance, [124, 120, 162].

1.3. A first bundle of norms and spaces

Fix $1 < p < \infty$ and consider a measurable function $f : \Omega \to \mathbb{R}$ such that $f \in \bigcap_{0<\varepsilon<p-1} L^{p-\varepsilon}(\Omega)$. If $f \notin L^p(\Omega)$, then

$$\left(\fint_{\Omega} |f|^{p-\varepsilon} dx\right)^{\frac{1}{p-\varepsilon}} \uparrow \infty \quad \text{as} \quad \varepsilon \to 0, \tag{1}$$

while, on the other hand, if $f \in L^p(\Omega)$, then by Hölder's inequality

$$\left(\fint_\Omega |f|^{p-\varepsilon} dx\right)^{\frac{1}{p-\varepsilon}} \leqslant \left(\fint_\Omega |f|^p dx\right)^{\frac{1}{p}} < \infty \quad \forall \varepsilon \in (0, p-1). \tag{2}$$

The functions f such that the blowup in (1) is controlled by a factor ε:

$$\|f\|_{L^{p)}(\Omega)} := \sup_{0<\varepsilon<p-1} \left(\varepsilon \fint_\Omega |f|^{p-\varepsilon} dx\right)^{\frac{1}{p-\varepsilon}} < \infty \tag{3}$$

constitute the *grand L^p space,* denoted by $L^{p)}(\Omega)$. Of course, if there is no blowup as in (2), then (3) holds as well, so that

$$L^{p)}(\Omega) = \{f \,:\, \|f\|_{L^{p)}(\Omega)} < \infty\} \supset L^p(\Omega).$$

The heart of the matter is the balance of the blowup of a norm (depending on a parameter, say ε) and the parameter ε itself, whose main role here is to be infinitesimal as $\varepsilon \to 0$. It is therefore natural to consider also the analogous norm, where ε is substituted by a power ε^θ, where $\theta > 0$. This has been done in [88], where the norm

$$f \in L^{p),\theta}(\Omega) \quad \Leftrightarrow \quad \|f\|_{L^{p),\theta}(\Omega)} = \sup_{0<\varepsilon<p-1} \left(\varepsilon^\theta \fint_\Omega |f|^{p-\varepsilon} dx\right)^{\frac{1}{p-\varepsilon}} < \infty \tag{4}$$

had a role in an existence and uniqueness result for the p-harmonic equation

$$\operatorname{div} |\nabla u|^{p-2} \nabla u = \operatorname{div} f.$$

The space $L^{p),\theta}(\Omega)$ (which gives back $L^{p)}(\Omega)$ when $\theta = 1$) is a rearrangement-invariant Banach function space and, by the same argument as before, it contains $L^p(\Omega)$. Moreover,

$$L^p(\Omega) \subsetneq \frac{L^p}{\log^\theta L}(\Omega) \subsetneq L^{p),\theta}(\Omega) \subsetneq \bigcap_{\alpha>1} \frac{L^p}{\log^{\alpha\theta} L(\Omega)} \subsetneq \bigcap_{0<\varepsilon<p-1} L^{p-\varepsilon}(\Omega). \tag{5}$$

Embeddings between grand Lebesgue spaces and Orlicz spaces have been intensively studied and fully characterized. The interested reader can see the papers [18, 87, 94, 23].

The generalization (4) can be pushed on, substituting ε^θ with a general measurable function $\delta(\varepsilon)$ which is positive a.e. ([26]). In such case, the product of the norm in $L^{p-\varepsilon}$ by $\delta(\varepsilon)$ is a measurable function, and the supremum must be changed into an essential supremum: the norm in $L^{p),\delta(\cdot)}$ is defined by

$$\|f\|_{L^{p),\delta(\cdot)}}(\Omega) = \operatorname*{ess\,sup}_{0<\varepsilon<p-1} \left(\delta(\varepsilon) \fint_\Omega |f|^{p-\varepsilon} dx\right)^{\frac{1}{p-\varepsilon}} < \infty.$$

Of course different δ's may generate the same space $L^{p),\delta(\cdot)}$, and in [26] it has been shown that the same class of spaces can be defined starting from the class of the increasing δ's (hence functions defined pointwise) so that the essential supremum can be equivalently replaced by a simple supremum. In [9] (see also [80]) also the exponent $p-\varepsilon$ has been generalized into a measurable function of ε (in the paper it is shown that this operation makes sense, and that, for instance, this way one can catch certain Orlicz spaces of exponential type for unbonded exponents; see also [12] and references therein), and recently in [8] it has been shown that again such full general definition gives back the spaces in [26] (we mention here the computation of their Boyd indices in [74] and their weighted version in [10]). Moreover, in [8] it has been shown that the same class of spaces is characterized through the norms of the type

$$\rho_\varphi(f) = \sup_{p\in[p_-,p_+)} \varphi(p)\left(\fint_\Omega |f|^p dx\right)^{1/p},$$

where $\varphi=\varphi(\cdot)$ on $[p_-,p_+)$ is positive and decreasing (i.e. $p_-\leqslant p_1<p_2<p_+ \Rightarrow \varphi(p_1)\geqslant\varphi(p_2)>0$). At last, it should be observed that such class of spaces, obtained in literature as endpoint of a sequence of generalizations, appeared maybe for the first time in the final remark in [23].

The *small Lebesgue space* $L^{(p',\theta}(\Omega)$ (here, as usual, $p'=p/(p-1)$) is defined as the associate space of the grand Lebesgue space $L^{p),\theta}(\Omega)$, i.e. $\|\cdot\|_{L^{(p',\theta}(\Omega)}$ is the smallest functional defined on the measurable functions such that a kind of Hölder type inequality holds:

$$\fint_\Omega u(x)v(x)dx \leqslant \|u\|_{L^{(p',\theta}(\Omega)}\|v\|_{L^{p),\theta}(\Omega)}. \tag{6}$$

As a byproduct of the general theory of Banach function spaces, $L^{(p',\theta}(\Omega)$ is a rearrangement-invariant Banach function space. The inequality (6) gives automatically an implicit definition of the norm of $L^{(p',\theta}(\Omega)$ (it suffices to divide by $\|u\|_{L^{(p',\theta}(\Omega)}$ and pass to the supremum, in the left hand side, over $u\not\equiv 0$), however, the explicit (i.e. not depending on the notion of grand Lebesgue space) characterization has been found in [57] (here we exchange p and p' for the sake of simplicity of notation):

$$\|u\|_{L^{(p,\theta}(\Omega)} = \inf_{u=\sum_k u_k}\left\{\sum_{k=1}^{\infty}\inf_{0<\varepsilon<p'-1}\varepsilon^{-\frac{\theta}{p'-\varepsilon}}\left(\fint_\Omega |u_k|^{(p'-\varepsilon)'}dx\right)^{\frac{1}{(p'-\varepsilon)'}}\right\}, \tag{7}$$

see also [67, 68, 25]. Incidentally, let us remark that in (6) equality may hold for every $v\in L^\infty(\Omega)$, as stated in [57, Lemma 2.9]: in fact, if the supremum defining $\|v\|_{L^{p),\theta}(\Omega)}$ is assumed in $\varepsilon=p-1$, then it suffices to set $u=\chi_{\{v>0\}}-\chi_{\{v<0\}}$: hence $|u|\leqslant 1$ and

$$\fint_\Omega uvdx \leqslant \|u\|_{L^{(p',\theta}(\Omega)}\|v\|_{L^{p),\theta}(\Omega)} \leqslant \|v\|_{L^{p),\theta}(\Omega)}\inf_{0<\varepsilon<p-1}\varepsilon^{-\frac{\theta}{p-\varepsilon}}\left(\fint_\Omega |u|^{(p-\varepsilon)'}dx\right)^{\frac{1}{(p-\varepsilon)'}}$$

$$\leqslant (p-1)^\theta\fint_\Omega |v|dx\cdot\frac{1}{(p-1)^\theta} = \fint_\Omega uvdx;$$

therefore the conclusion is that the inequalities are in fact equalities (see [30] for details on this remark).

The notion of associate space has a nice behavior with respect to inclusion (note that the *inclusion* is equivalent to the *continuous embedding* in the case of Banach function spaces, see [15, Thm 1.8 p. 7]), namely, for associate spaces the inverted inclusions hold (note also that, on the contrary, this property does not hold in general for the notion of dual space). Hence ([23, 36])

$$\bigcup_{\varepsilon>0} L^{p+\varepsilon} \subsetneq \bigcup_{\beta>1} L^p(\log L)^{\frac{\beta\theta}{p'-1}} \subsetneq L^{(p,\theta}(\Omega) \subsetneq L^p(\log L)^{\frac{\theta}{p'-1}} \subsetneq L^p(\Omega).$$

We conclude this subsection mentioning that comparisons with another class of Banach function spaces of recent interest, namely, the variable Lebesgue spaces (see e.g. the books [149, 129, 48, 39, 139, 51, 112, 113] and [98]) appear in [71, 38].

1.4. The point of view of interpolation-extrapolation theory: the first results

The structure of the norm of grand and small Lebesgue spaces is familiar to researchers in interpolation theory, from which it is possible to get the expression of a quasinorm, equivalent to that one in (4) and (7), respectively. In the case of small Lebesgue spaces, the new expression is also "nicer", because one can get rid of decompositions and can "play" just with integrals. We are going to write the expression of such quasinorms, trying to "read" them in terms of the language of interpolation theory. We begin by recalling the classical notion of Lorentz space (see e.g. the recent monograph [139] on function spaces), starting from a classical result by Peetre in 1966 (for the sake of history, without claiming to be complete on interpolation theory, we mention the precedent papers [78, 123, 122, 21]; see also [37, 40] and references therein).

Let $1 < p < \infty$. We recall that from the equality (see e.g. [138]; [160, 5.10 p.213]; [15, Thm 6.2 p. 74])

$$\int_0^t f_*(s)ds = \inf_{f=g+h}\left\{\|g\|_{L^1(\Omega)} + t\|h\|_{L^\infty(\Omega)}\right\} := K(f,t,L^1(\Omega),L^\infty(\Omega)),$$

with the help of the classical Hardy's inequality, one has

$$\|f\|_{L^p(\Omega)} = \|f_*\|_{L^p(0,|\Omega|)} \approx \left\|t^{-1}\int_0^t f_*\right\|_{L^p(0,|\Omega|)} = \left\|t^{-1}K(f,t,L^1(\Omega),L^\infty(\Omega))\right\|_{L^p(0,|\Omega|)}.$$

The expression on the right hand side is a special case of the quasinorm of the Lorentz spaces, which represent a generalization of the Lebesgue spaces in a different "direction" with respect to the Orlicz spaces; for our goals we recall here few standard facts. Fix a parameter q, $1 < q < \infty$, and set $\|f\|_{L^{p,q}(\Omega)} = \|t^{\frac{1}{p}-\frac{1}{q}} f_*(t)\|_{L^q(0,|\Omega|)}$. The parameter q, roughly speaking, gives a "tuning" of the Lebesgue space $L^p(\Omega)$: notice that, for any $1 < q < \infty$, if for instance $\Omega = (0,1) \subset \mathbb{R}$, the "boundary" power function $t^{-\frac{1}{p}}$ remains unchanged. Proceeding as above,

$$\|f\|_{L^{p,q}(\Omega)} = \|t^{\frac{1}{p}-\frac{1}{q}} f_*(t)\|_{L^q(0,|\Omega|)} \approx \left\|t^{\frac{1}{p}-\frac{1}{q}}\cdot t^{-1}\int_0^t f_*\right\|_{L^q(0,|\Omega|)}$$

$$= \left\| t^{\frac{1}{p}-\frac{1}{q}} \cdot t^{-1} K(f,t,L^1(\Omega),L^\infty(\Omega)) \right\|_{L^q(0,|\Omega|)}$$

$$= \left\| t^{-\left(1-\frac{1}{p}\right)} \cdot t^{-\frac{1}{q}} K(f,t,L^1(\Omega),L^\infty(\Omega)) \right\|_{L^q(0,|\Omega|)}$$

$$:= \|f\|_{(L^1,L^\infty)_{1-\frac{1}{p},q}} . \tag{8}$$

Note that $\|f\|_{L^p(\Omega)} = \|f\|_{L^{p,p}(\Omega)} = \|f\|_{(L^1,L^\infty)_{1-\frac{1}{p},p}}$. For any $1 < q < \infty$, the norm in (8) gives a space "close" to the Lebesgue space $L^p(\Omega)$. The parameter p appears in the first index, namely in the expression $1-1/p$ which has the advantage to be always a number between 0 and 1: when it is 0, the space is close to L^1, the first space in the pair (L^1,L^∞); when it is 1, the space is close to L^∞, the second space in the pair (L^1,L^∞). About the parameter q, it can be shown that when q increases, the space becomes larger (with respect to inclusion).

The whole machinery can be made again considering one more parameter $\alpha \in \mathbb{R}$, which appears as power of a logarithm, put as factor of $K(f,t,L^1(\Omega),L^\infty(\Omega))$: a further refinement, again classical, is that one of the Lorentz-Zygmund spaces. For any $1 < q < \infty$, $\alpha \in \mathbb{R}$, set

$$\|f\|_{L^{p,q;\alpha}(\Omega)} = \|t^{\frac{1}{p}-\frac{1}{q}}(1-\log t)^\alpha f_*(t)\|_{L^q(0,|\Omega|)} .$$

Proceeding again as above,

$$\|f\|_{L^{p,q;\alpha}(\Omega)} = \left\| t^{-\left(1-\frac{1}{p}\right)} (1-\log t)^\alpha \cdot t^{-\frac{1}{q}} K(f,t,L^1(\Omega),L^\infty(\Omega)) \right\|_{L^q(0,|\Omega|)}$$

$$:= \|f\|_{(L^1,L^\infty)_{1-\frac{1}{p},q;\alpha}} .$$

At last, we may state the characterization of grand and small Lebesgue spaces given in [61]:

$$L^{p)} = (L^1,L^p)_{1,\infty;-\frac{1}{p}} \tag{9}$$

$$L^{(p} = (L^p,L^\infty)_{0,1;-\frac{1}{p}} . \tag{10}$$

From the point of view of the digression above, in (9) one can see that the first index 1 tells that the space $L^{p)}(\Omega)$ is "close" to $L^p(\Omega)$, the second space in the pair (L^1,L^p); the second index ∞ tells that the space is larger (with respect to inclusion) than $L^p(\Omega)$, and, finally, the third parameter $-\frac{1}{p}$ gives the correct logarithmic tuning of the space. In an analogous way one can read the relation in (10). After the computations of $K(f,t,L^1(\Omega),L^p(\Omega))$, $K(f,t,L^p(\Omega),L^\infty(\Omega))$ which are implicit in the symbols in (9), (10), respectively, one gets the equivalent quasinorms

$$\|u\|_{L^{p),\theta}(\Omega)} \approx \sup_{0<t<|\Omega|} \left(1-\log\left(\frac{t}{|\Omega|}\right)\right)^{-\frac{\theta}{p}} \left(\int_t^{|\Omega|} u_*(s)^p ds\right)^{\frac{1}{p}} \tag{11}$$

$$\|u\|_{L^{(p,\theta}(\Omega)} \approx \int_0^{|\Omega|} \left(1-\log\left(\frac{t}{|\Omega|}\right)\right)^{\frac{\theta}{p'}-1} \left(\int_0^t u_*(s)^p ds\right)^{\frac{1}{p}} \frac{dt}{t}.$$

A different proof of these relations, for researchers not expert in interpolation theory, appears in [47], where sharp estimates are proved by using the Euler's Gamma function. We stress that the relations above, differently from [47], are written for domains Ω having finite measure, not necessarily equal to 1. Finally, we highlight that the right hand side of (11), by a simple change of variable (see [47, Proposition 3.1]), can be written also in a different, expressive form, so that

$$\|u\|_{L^{p),\theta}(\Omega)} \approx \sup_{0<\varepsilon<p-1} \left[\left(\frac{\varepsilon}{p-1}\right)^{\theta} \int_{|\Omega|e^{1-(p-1)/\varepsilon}}^{|\Omega|} u_*(s)^p ds\right]^{\frac{1}{p}}.$$

1.5. The development of grand and small spaces

Starting from the norms listed in subsection 1.3, in literature several authors devoted their attention either to the spaces themselves (see e.g. [105]), either to a number of generalizations and variants, applied to questions related e.g. to the boundedness of operators, to integral inequalities, to interpolation theory. Applications to certain optimization problems of the type considered in [13] will appear in a forthcoming paper.

Here we give just an idea of the large literature appeared in recent years, where generalizations and variants have been considered.

First of all, for the reader's convenience, we list some quasinorms of some spaces already mentioned above, so that it is possible to make evident the differences and the analogies between them. Assume now, for simplicity, $|\Omega| = 1$, and let $1 < p, q < \infty$, $\alpha \in \mathbb{R}$, $\theta > 0$.

We recall the quasinorms for Lorentz spaces, Zygmund spaces (which can be seen as special Orlicz spaces) and small Lebesgue spaces, respectively:

$$\|u\|_{L^{p,q}(\Omega)} \approx \left(\int_0^1 t^{\frac{q}{p}-1} u_*(t)^q \, dt\right)^{\frac{1}{q}}$$

$$\|u\|_{L^p(\log L)^\alpha(\Omega)} \approx \left(\int_0^1 (1-\log t)^\alpha u_*(t)^p \, dt\right)^{\frac{1}{p}}$$

$$\|u\|_{L^{(p,\theta}(\Omega)} \approx \int_0^1 (1-\log t)^{\frac{\theta}{p'}-1} \left(\int_0^t u_*(s)^p ds\right)^{\frac{1}{p}} \frac{dt}{t}.$$

This last norm can be considered as special case of the so-called $G\Gamma(p,m,w)$ spaces, introduced in [70] and considered e.g. in [72, 86, 147, 58] (see these references for values and digressions on the parameters p and m, and the weight w):

$$\|u\|_{G\Gamma(p,m,w)} \approx \left[\int_0^1 w(t) \left(\int_0^t u_*(s)^p ds\right)^{\frac{m}{p}} \frac{dt}{t}\right]^{1/m}. \tag{12}$$

We mention also the following variant, appeared in [132]:

$$\|u\|_{\mathscr{H}_{x_0}^{p,m,w}(\Omega)} \approx \left[\int_0^{2diam(\Omega)} w(x_0,t)^m \left(\int_{B(x_0,t)} u(s)^p ds\right)^{\frac{m}{p}} \frac{dt}{t}\right]^{1/m};$$

moreover, the variant of $G\Gamma(p,m,w)$ with double weights appears in [59] and the $G\Gamma(p,\infty;w,v)$ spaces appeared in [83]:

$$\|u\|_{G\Gamma(p,\infty;w,v)} \approx \operatorname*{ess\,sup}_{t>0} w(t) \left(\int_0^t v(s)u_*(s)^p ds\right)^{\frac{1}{p}}.$$

Iteration type results, where the Lebesgue norm in the expression of the norm of grand Lebesgue spaces is replaced by the norm of grand Lebesgue spaces, are in [7] (see also [56]).

Generalizations of small Lebesgue spaces appear in two papers (see [103, 11]), where some common results have been obtained independently (see also [28]).

As to variants of grand Lebesgue spaces, we mention the bilateral grand Lebesgue spaces considered in [119], the grand Orlicz spaces in [115], [24], [52], the grand Lorentz spaces in [99], the weighted grand Lebesgue spaces in [60, 102, 101, 108], the weighted fully measurable grand Lebesgue spaces in [104], the grand Morrey spaces in [130, 164, 109, 108, 110], the grand Morrey spaces of variable exponent in [77, 136], the grand Musielak-Orlicz-Morrey spaces in [133, 125], the grand grand Morrey spaces in [143], the grand Bochner-Lebesgue spaces in [111, 100] and, finally, the grand variable exponent Lebesgue spaces in [107, 62].

Finally, we mention that abstract notions of grand spaces appear in [61, 112, 113], and a way to extend the notion of grand Lebesgue spaces when the underlying measure space has infinity measure appears in [150, 151, 163].

2. Applications to Sobolev spaces theory and to PDEs

The grand and small Lebesgue spaces generate the corresponding Sobolev analogs: for a given p, $1<p<\infty$, the grand Sobolev space $W^{1,p)}(\Omega)$ consists of all functions $u \in W^{1,1}(\Omega)$ such that $|\nabla u| \in L^{p)}(\Omega)$; similarly, the small Sobolev space $W^{1,(p}(\Omega)$ consists of all functions $u \in W^{1,1}(\Omega)$ such that $|\nabla u| \in L^{(p}(\Omega)$. The definitions of the corresponding spaces $W_0^{1,p)}(\Omega)$, $W_0^{1,p)}(\Omega)$ constituted by the functions which, roughly speaking, assume value zero on the boundary, follow the same lines of the standard definitions given in the Sobolev spaces theory; the same words can be spent for grand and small Sobolev spaces of higher order. This whole class of spaces has been intensively studied and their properties found several applications in Geometric function theory and PDEs. For a review, the interested reader may consult [42] (see also [152]) and the rich bibliography therein. In the following we will confine ourselves to some results involving Sobolev embeddings and to some existence, uniqueness, regularity results for certain elliptic PDEs.

2.1. The role of grand Sobolev spaces in embedding theorems

Let us begin by recalling some classical embedding results. For $\Omega \subset \mathbb{R}^n$ open, bounded, connected, $\partial\Omega$ Lipschitz, $n \geqslant 2$, we have

$$W^{1,p}(\Omega) \hookrightarrow L^{\frac{np}{n-p},p}(\Omega) \hookrightarrow L^{\frac{np}{n-p}}(\Omega) \qquad 1 \leqslant p < n$$

$$W^{1,n}(\Omega) \hookrightarrow L^q(\Omega) \qquad (p = n) \qquad p \leqslant q < \infty$$

$$W^{1,p}(\Omega) \hookrightarrow \mathscr{C}^{0,1-\frac{n}{p}}(\Omega) \qquad (p > n)$$

$$W^{1,n}(\Omega) \hookrightarrow EXP^{\frac{n}{n-1}}(\Omega) \qquad W^{1,n}(\Omega) \not\hookrightarrow L^\infty(\Omega) \qquad (p = n).$$

When derivatives are in Lorentz spaces (see [159, 46, 6, 127]), recalling that

$$L^{n,1}(\Omega) \hookrightarrow L^n(\Omega) \hookrightarrow L^{n,\infty}(\Omega),$$

we have

$$WL^{n,1}(\Omega) \hookrightarrow L^\infty(\Omega)\,,\ WL^{n,1}(\Omega) \hookrightarrow \mathscr{C}^0(\Omega) \tag{13}$$

and

$$WL^{n,p}(\Omega) \hookrightarrow EXP^{\frac{p}{p-1}}(\Omega)\,,\ WL^{n,p}(\Omega) \not\hookrightarrow L^\infty(\Omega) \qquad 1 < p \leqslant \infty.$$

When derivatives are in Orlicz spaces $L^A(\Omega)$ (see [49, 22, 34, 32, 33]), a notion of A^* corresponding to the Sobolev conjugate exponent can be given in a sharp way, and we have, roughly speaking,

$$W^{1,A}(\Omega) \hookrightarrow L^{A^*}(\Omega) \qquad A(t) \lesssim t^n(\log t)^{n-1}(\log\log t)^{n-1}\cdots, \quad t \to \infty.$$

The role of grand Lebesgue spaces becomes significant if we give a look at the special case $L^A(\Omega) = \frac{L^n}{\log^\theta L}(\Omega)$: from the computation of A^* we get, in this case,

$$W_0\frac{L^n}{\log^\theta L}(\Omega) \hookrightarrow EXP^{n/(n-1+\theta)}(\Omega) \qquad \forall\theta > 0 \tag{14}$$

and now, in spite of the embedding (see (5))

$$\frac{L^n}{\log^\theta L}(\Omega) \subsetneq L^{n),\theta}(\Omega), \tag{15}$$

we have ([76, 85])

$$u \in W_0^{1,1}(\Omega), |\nabla u| \in L^{n),\theta}(\Omega) \Rightarrow u \in EXP^{n/(n-1+\theta)}(\Omega). \tag{16}$$

The imbedding (15) and the fact that on the right hand side of (14) and (16) the exponential type spaces are the same, tell that, in this critical setting, the fact to consider the gradient in grand Lebesgue spaces gives no loss of integrability.

In this framework we mention that if we write (14) in the form

$$\sup_{0<t<1} (1-\log t)^{-\frac{\theta}{n}} \left(\int_t^1 |\nabla u|_*(s)^n ds \right)^{\frac{1}{n}} < \infty \Rightarrow \sup_{0<t<1} \frac{|u|_{**}(t)}{(1-\log t)^{\frac{n-1+\theta}{n}}} < \infty,$$

the following generalized form holds ([58]):

$$\sup_{0<t<1} w(t) \left(\int_t^1 |\nabla u|_*(s)^n ds \right)^{\frac{1}{n}} < \infty \Rightarrow \sup_{0<t<1} \frac{w(t)|u|_{**}(t)}{(1-\log t)^{\frac{n-1}{n}}} < \infty.$$

It should be noted, here, that the norm on the right hand side is that one of a Marcinkiewicz space (see [15]).

Let us examine now embedding theorems for small Sobolev spaces, beginning with the critical situation of derivatives in the small $L^n(\Omega)$ spaces. Recalling that

$$L^{n,\infty}(\Omega) \hookrightarrow L^{n)}(\Omega)$$

(for results about embeddings between Lorentz and grand Lebesgue spaces see [96, 87, 3]), by associativity one gets immediately

$$L^{(n}(\Omega) \hookrightarrow L^{n,1}(\Omega)$$

and therefore, since by (13) it is $WL^{n,1}(\Omega) \hookrightarrow \mathscr{C}^0(\Omega)$, the same holds for $W^{1,(n}(\Omega)$ and the oscillation of these functions can be estimated as follows ([67, 68, 145]):

$$u \in W^{1,1}(\Omega), |\nabla u| \in L^{(n}(\Omega) \Rightarrow osc_{B(x,t)} u \leqslant c_n |\Omega|^{1/n} \| |\nabla u| \chi_{B(x,t)} \|_{L^{(n}(\Omega)}.$$

We conclude this subsection by discussing some compactness results and a dimension-free Sobolev estimate.

The well known classical results about compactness, for higher order Sobolev spaces, are the following: if $1 \leqslant p < \frac{n}{k}$, $k \in \mathbb{N}$, then (the historical references are [148] for the case $k = 1$, [114] for the case $k \geqslant 1$)

$$W^{k,p}(\Omega) \hookrightarrow L^q(\Omega) \qquad 1 \leqslant q < p^* = \frac{np}{n-kp}$$

is compact, while $W^{k,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$ is not compact. With the help of the notions of grand and small Lebesgue and Sobolev spaces, the loss of compactness in the extreme case $q = p^*$ can be replaced by the following gains of compactness (see [69] for the case $k = 1$, [36] for the case $k \geqslant 1$): the embeddings

$$W^{k,p}(\Omega) \hookrightarrow L^{p^*)}(\Omega) \qquad W^{k,(p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$$

are compact. Hence, for instance, every sequence of functions equibounded in $W^{k,(p}(\Omega)$ has a subsequence strongly convergent in $L^{p^*}(\Omega)$.

Let us spend now few words on a so-called dimension-free Sobolev estimate (for classical results on this topic, for instance the logarithmic Sobolev inequality by Gross and its equivalence to the hypercontractive inequality by Nelson, see e.g. [1] and references therein). Let $1 < p < \infty$ be fixed, and, for any $n > p$, let B_n be a given open ball in $\mathbb{R}^n$ of measure 1. The following version of the Sobolev inequality is well known (see e.g. [95]):

$$\|f\|_{L^{np/(n-p)}(B_n)} \leqslant \frac{p(n-1)}{n-p}\|\nabla f\|_{L^p(B_n)} \quad \forall f \in W_0^{1,p}(B_n), \quad \forall n > p$$

and, noticing the boundedness, with respect to the dimension, of the constant in the right hand side, we can write simply

$$\|f\|_{L^{np/(n-p)}(B_n)} \leqslant c_p\|\nabla f\|_{L^p(B_n)} \quad \forall f \in W_0^{1,p}(B_n), \quad \forall n > p$$

or, equivalently, taking into account of the rearrangement-invariance of the Lebesgue spaces (see subsection 1.2),

$$\|f_*\|_{L^{np/(n-p)}(0,1)} \leqslant c_p\|\nabla f\|_{L^p(B_n)}.$$

This inequality makes visible the gain of summability given by the Sobolev inequality, such gain being dependent of the dimension n, and being smaller and smaller as $n \to \infty$: in fact we have $\frac{np}{n-p} \searrow p$ and $\|f_*\|_{L^p(0,1)} \leqslant \|f_*\|_{L^{np/(n-p)}(0,1)}$ $\forall n > p$. If, on the left hand side, one wishes to see a Lebesgue space not depending on n, the best inequality one can write is

$$\|f_*\|_{L^p(0,1)} \leqslant c_p\|\nabla f\|_{L^p(B_n)} \ \forall n > p$$

which means, in fact, that there is no gain of summability (with respect to the gradient) in the scale of Lebesgue spaces. However, in the scale of rearrangement-invariant Banach function spaces, a gain of summability independent of the dimension exists: if one looks for the best (=”smallest”) space $Y_p(0,1)$ such that

$$\|f_*\|_{Y_p(0,1)} \leqslant c_p\|\nabla f\|_{L^p(B_n)} \quad \forall f \in W_0^{1,p}(B_n), \quad \forall n > p,$$

one has that $Y_p(0,1) \subset L^p(\log L)^{p/2}(0,1)$, the exponent $p/2$ being optimal (see [126], where the authors answered positively a conjecture by Hans Triebel). The point here is that a further refinement holds: one has $Y_p(0,1) \subset L^{(p,p'/2}(0,1)$ and also that the inclusion $Y_p(0,1) \subset L^p(\log L)^{p/2}(0,1)$ is optimal in the scale of Orlicz spaces ([64], see also [128]).

3. Some applications to PDEs

In the previous section we already mentioned the paper [42] and references therein. Without claiming to be complete, we point out the presence of the notion of grand and small Lebesgue and Sobolev spaces also in several other papers, see e.g. [27, 45, 41, 53, 54, 55, 82, 89, 90, 91, 106, 134, 141, 142, 144, 147, 152, 153, 166].

In this section we will be not interested in applications to parabolic problems (however, the interested reader may consult the papers [65, 66]) and we will deal with three particular cases of the following problem: let $\Omega \subset \mathbb{R}^n$ $(n \geqslant 2)$ be open, bounded, connected, $\partial\Omega$ smooth, and let $a : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ be such that

$$\begin{cases} x \to a(x,\xi) & measurable \quad \forall \xi \in \mathbb{R}^n \\ \xi \to a(x,\xi) & continuous \quad \forall x \in \Omega \text{ a.e.} \end{cases}$$

$$\begin{cases} a(x,\xi)\xi \geqslant \alpha|\xi|^2 \\ |a(x,\xi)| \leqslant \beta|\xi| \end{cases} \qquad \forall x \in \Omega \text{ a.e.}, \quad \forall \xi \in \mathbb{R}^n$$

$$\langle a(x,\xi) - a(x,\eta), \xi - \eta \rangle > 0, \quad \xi \neq \eta \quad \text{(strict monotonicity)}.$$

We consider the Dirichlet problem:

$$\begin{cases} -\text{div} a(x,\nabla u) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{17}$$

for data f lying in some Banach spaces. We recall that a function $u \in W_0^{1,1}(\Omega)$ is said to be a distributional solution if

$$\begin{cases} |a(x,\nabla u)| \in L^1(\Omega) \\ \displaystyle\int_\Omega a(x,\nabla u)\nabla\varphi\, dx = \int_\Omega f\varphi\, dx \quad \forall \varphi \in \mathscr{C}_0^\infty(\Omega) \end{cases}.$$

The typical result of the classical variational method (originated in [117, 157, 158] in the linear case, and extended to the nonlinear case in [118], [121, Thm 2.8 p.183], see e.g. also [75, Thm 26.14 p.205], [155, Prop. 5.1 p. 60; Thm 6.1 p.76]) is the following:

$$f \in W^{-1,2}(\Omega) = (W_0^{1,2}(\Omega))' \qquad \Rightarrow \quad \exists\, u \quad distributional\ solution\ in\ W_0^{1,2}(\Omega).$$

In order to distinguish the numbers "2" above, let us mention that in the case of $a = a(x,\xi)$ of "growth p" (our assumptions fit into the case $p = 2$), the above statement reads as

$$f \in W^{-1,p'}(\Omega) = (W_0^{1,p}(\Omega))' \qquad \Rightarrow \quad \exists\, u \quad distributional\ solution\ in\ W_0^{1,p}(\Omega);$$

moreover, solutions assumed in $W_0^{1,p}(\Omega)$ are unique by the strict monotonicity.

Let us begin with the case $n = 2$ and the datum f in an Orlicz space. It is easy to find the spaces for f such that the variational method can be applied: in fact, from the Sobolev embedding

$$W_0^{1,2}(\Omega) \hookrightarrow \exp_2(\Omega) := \overline{L^\infty(\Omega)}^{EXP^2(\Omega)},$$

by duality,

$$L(\log L)^{\frac{1}{2}}(\Omega) \hookrightarrow W^{-1,2}(\Omega),$$

hence the variational method works when the datum f is in a Lebesgue or Orlicz space contained in $L(\log L)^{\frac{1}{2}}$, namely, all the spaces $L^m(\Omega)$, $m > 1$, and all the spaces $L(\log L)^\delta(\Omega)$, $\delta \geqslant \frac{1}{2}$. The following estimate appears in [3]:

$$\|\nabla u\|_{L^2(\log L)^{2\delta-1}(\Omega)} \leqslant c\|f\|_{L(\log L)^{\delta}(\Omega)} \qquad \delta \in \left[\frac{1}{2},1\right]. \tag{18}$$

For completeness, we recall that in the linear case, the datum f in $L\log L(\Omega)$ was studied in [5] where sharp L^∞ estimates have been derived. Moreover, the case $f \in L(\log L)^{\frac{1}{2}}(\log\log L)^{\frac{1}{2}}(\Omega)$ has been treated in [165], the case $f \in L(\log L)^{\frac{1}{2}}(\log\log L)^{\frac{\beta}{2}}(\Omega)$ in [45], the case $f \in L(\log L)^{\frac{1}{2}}(\log\log\log L)^{\frac{\beta}{2}}(\Omega)$ in [43], the case $f \in L(\log L)^{\delta}(\log\log L)^{\frac{\beta}{2}}(\Omega)$ in [44], and, finally, some optimal results appear in [4].

The same procedure can be carried on in the case $n > 2$, and it may be easily seen that the variational method fails for f "free to move" in a space larger than

$$\begin{cases} L(\log L)^{\frac{1}{2}}(\Omega) & (n=2) \\ L^{\frac{2n}{n+2}}(\Omega) & (n>2) \end{cases}.$$

After these preliminary facts, we may therefore assert that, speaking about the spaces for the datum f, the space $L^1(\Omega)$ is never "variational", while the space $L(\log L)(\Omega)$ is "variational" in the case $n=2$, and it is not in the case $n>2$. For each of these two spaces we are going to mention a result, where through the notion of grand Lebesgue space one can state an existence and uniqueness result, and a regularity result, respectively. Let us begin with the following result related to the non variational case $f \in L^1(\Omega)$:

THEOREM 1. *([73], see also [88, 131]) If $\Omega \subset \mathbb{R}^2$ is open, bounded, connected, $\partial\Omega$ smooth, and if $a : \Omega \times \mathbb{R}^2 \to \mathbb{R}^2$ is such that*

$$\begin{cases} x \to a(x,\xi) \text{ measurable } \forall \xi \in \mathbb{R}^2 \\ \xi \to a(x,\xi) \text{ continuous } \forall x \in \Omega \text{ a.e.} \end{cases}$$

$$\begin{cases} |a(x,\xi)-a(x,\eta)| \leqslant m|\xi-\eta| & \text{(Lipschitz continuity)} \\ \frac{1}{m}|\xi-\eta|^2 \leqslant \langle a(x,\xi)-a(x,\eta),\xi-\eta\rangle & \text{(strong monotonicity)} \\ a(x,0)=0 \end{cases},$$

then the Dirichlet problem

$$\begin{cases} -\operatorname{div} a(x,\nabla u) = f \in L^1(\Omega) \\ u=0 \quad \text{on } \partial\Omega \end{cases}$$

admits one and only one distributional solution $u \in W_0^{1,2)}(\Omega)$.

As a consequence, from the Sobolev embedding (16), one gets $u \in EXP(\Omega)$, already known from [19, 31] in the linear case). Theorem 1 is in agreement also with [79], where existence and uniqueness of a distributional solution $u \in \bigcap_{q<2} W_0^{1,q}(\Omega)$ has been proved in the linear case, for data f bounded Radon measure, that is, f in the dual of $\mathscr{C}(\overline{\Omega})$.

Let us consider, now, the case of the right hand side $f \in L(\log L)$ for the problem (17) in the linear case. We already stated that when $n = 2$ there is existence and uniqueness of a distributional solution; otherwise, when $n > 2$, in general, uniqueness fails if one looks for distributional solutions in spaces "much" larger than $W_0^{\frac{n}{n-1}}(\Omega)$ ([154], see e.g. also [140]). By considering appropriate notions of solutions, for several ranges for the parameter $\delta \in]0, +\infty[$, unifying results in [156, 16, 137, 81, 63, 3, 116], we can state the following

THEOREM 2. *If $\Omega \subset \mathbb{R}^n$ is open, bounded, connected, $\partial\Omega$ smooth, and if $A = (a_{ij})_{i,j=1,\dots,n}$, $a_{ij} = a_{ji}$ is sufficiently regular, then the problem*

$$\begin{cases} \operatorname{div} A(x)\nabla u = f \in L(\log L)^{\delta}(\Omega) \\ u = 0 \qquad\qquad\qquad\qquad \text{on } \partial\Omega \end{cases} \qquad \delta > 0$$

admits the existence of a solution in $W_0^{1,\frac{n}{n-1}}(\Omega)$, and the following estimate holds:

$$\|\nabla u\|_{L^{\frac{n}{n-1}}(\log L)^{\frac{n\delta}{n-1}-1}(\Omega)} \leqslant c\|f\|_{L(\log L)^{\delta}(\Omega)} \qquad \delta > 0\,.$$

Note that (18) corresponds to the case $n = 2$, $\delta \in [1/2, 1]$, of Theorem 2. The knowledge of the small Sobolev spaces allows us to state the following improvement of the estimate in Theorem 2 (see [58]):

$$\|\nabla u\|_{L^{(n',n\delta-n+1}(\Omega)} \leqslant c\|f\|_{L(\log L)^{\delta}(\Omega)}, \qquad \delta \in \left[\frac{n-1}{n}, +\infty\right[.$$

We conclude with the following regularity result (see [58]):

THEOREM 3. *If $\Omega \subset \mathbb{R}^n$ is open, bounded, connected, $\partial\Omega$ smooth, then there exists one and only one very weak solution u of the problem*

$$\begin{cases} -\Delta u = f \in L^1(\Omega; \delta(1 + |\log\delta|)^{\frac{n-1+\theta}{n}}) \\ u = 0 \qquad\qquad\qquad\qquad\qquad\qquad \text{on } \partial\Omega \end{cases},$$

namely, one and only one $u \in L^{(\frac{n}{n-1},\theta}(\Omega)$, $\theta > 0$, such that

$$-\int_{\Omega} u\Delta\varphi dx = \int_{\Omega} f\varphi dx \qquad \forall\varphi \in C^2(\overline{\Omega}), \quad \varphi = 0 \text{ on } \partial\Omega.$$

Here $\delta(x) = \operatorname{dist}(x, \partial\Omega)$.

Observe that when $\theta = 1$ we have $L^{(\frac{n}{n-1}}(\Omega) \subset L^{\frac{n}{n-1},1}(\Omega)$, hence Theorem 3 improves the corresponding results appeared in [17, 50, 146].

4. The point of view of interpolation theory: some recent results

Using the celebrated Holmstedt's formula ([93]) and the computation of the K-functional, in the paper [59] (see also [35]) a new, recent characterization of grand and small Lebesgue spaces as interpolation spaces in the sense of Peetre, has been obtained. Namely, the following extension of formulas (9), (10) from [61], recalled in subsection 1.4, hold (here and in the remaining part of this section, for simplicity of notation, we omit to write "(Ω)" in the symbols denoting the function spaces and we assume $|\Omega| = 1$):

THEOREM 4. ([59], Theorem 3.1) *If* $1 \leqslant p < q$, $\alpha > 0$, *then*

$$L^{q),\alpha} = \left(L^p, L^q\right)_{1,\infty;-\frac{\alpha}{q}}.$$

THEOREM 5. ([59], Proposition 4.2) *If* $1 < p < q$, $\alpha > 0$, *then*

$$L^{(p,\alpha} = \left(L^p, L^q\right)_{0,1;-1+\alpha-\frac{\alpha}{p}}.$$

The main approach to get such results is based on the computation of the K-functional based on results from [14]; the following explicit equivalent expression of the K-functional between the grand Lebesgue spaces and Lebesgue spaces was found.

THEOREM 6. ([59], Theorem 3.2) *If* $1 < p < q$, $\alpha > 0$, *then*

$$K(f,t;L^{p),\alpha},L^q) \approx \sup_{0<s<\varphi(t)} (1-\log s)^{-\frac{\alpha}{p}} \left(\int_s^{\varphi(t)} f_*^p(x)dx\right)^{\frac{1}{p}} + t\left(\int_{\varphi(t)}^1 f_*^q(s)ds\right)^{\frac{1}{q}},$$

where φ *is the inverse of the increasing function* $\psi(t) = t^{\frac{1}{p}-\frac{1}{q}}(1-\log t)^{-\frac{\alpha}{p}}$, $t \in (0,1)$. *Thus*

$$t = \varphi(t)^{\frac{1}{p}-\frac{1}{q}}(1-\log\varphi(t))^{-\frac{\alpha}{p}}.$$

As a consequence, the following new characterization of grand Lebesgue spaces holds:

THEOREM 7. ([59], Theorem 1.1) *If* $1 < p < q$, $\alpha > 0$, *then*

$$L^{q),\alpha} = (L^{p),\alpha}, L^q)_{1,\infty;-\frac{\alpha}{q}}.$$

The same paper [59] contains also the computation of the K-functional between two Grand Lebesgue spaces.

THEOREM 8. ([59], Theorem 3.3) *If* $1 < p < q$, $\alpha > 0$, *then*

$$K(f,t;L^{p),\alpha},L^{q),\alpha}) \approx \sup_{0<s<\varphi(t)} (1-\log s)^{-\frac{\alpha}{p}} \left(\int_s^{\varphi(t)} f_*^p(x)dx\right)^{\frac{1}{p}}$$

$$+t \sup_{\varphi(t)<s<1} (1-\log s)^{-\frac{\alpha}{q}} \left(\int_s^1 f_*^q(x)dx \right)^{\frac{1}{q}},$$

where φ *is the inverse of the increasing function* $\psi(t) = t^{\frac{1}{p}-\frac{1}{q}}(1-\log t)^{-\frac{\alpha}{p}+\frac{\alpha}{q}}$, $t \in (0,1)$. *Thus*

$$t = \varphi(t)^{\frac{1}{p}-\frac{1}{q}}(1-\log\varphi(t))^{-\frac{\alpha}{p}+\frac{\alpha}{q}}.$$

Again, after the computation of the K-functionals above, the following characterization of the interpolation spaces between two Grand Lebesgue spaces holds:

THEOREM 9. ([59], Theorem 1.2) *If* $0<\theta<1$, $1 \leqslant r < +\infty$, $\alpha > 0$, $1<p<q$, *then*

$$\left(L^{p),\alpha}, L^{q),\alpha}\right)_{\theta,r} = L^{p_\theta,r}\left(\log L\right)^{-\frac{\alpha}{p_\theta}} \qquad \textit{where } \frac{1}{p_\theta} = \frac{1-\theta}{p} + \frac{\theta}{q}. \tag{19}$$

Readers are warned to beware of the different symbol for Lorentz-Zygmund spaces on the right hand side of (19): in this section we follow the notation commonly used in interpolation theory and adopted in [59], which is different from that one used in previous sections, which is, on the contrary, used in the theory of Orlicz spaces. According to this latter notation (introduced in subsection 1.2), the space on the right hand side of (19) should be written $L^{p_\theta,r}\left(\log L\right)^{-\frac{\alpha r}{p_\theta}}$.

We mention that also in [92, Theorem 1.1] the interpolation between two grand Lebesgue spaces appear, but the methods used therein are different. They employed the complex interpolation functors defined by Calderón in 1964 ([21]), obtaining that the first complex interpolation spaces between two grand Lebesgue spaces coincides with the closure of L^∞ in intermediate grand Lebesgue spaces, while the second complex interpolation spaces between two grand Lebesgue spaces coincide with the intermediate grand Lebesgue spaces.

Let us now go back to the interpolation in the sense of Peetre, analyzing the corresponding results for small Lebesgue spaces. Using the duality result on interpolation spaces, (see [20, 161]), the following characterization of the interpolation spaces of small Lebesgue spaces hold.

THEOREM 10. ([59], Theorem 3.4) *If* $0<\theta<1$, $1<r<+\infty$, $1<p<q$, $\alpha>0$, *then*

$$\left(L^{(p,\alpha}, L^{(q,\alpha}\right)_{\theta,r} = L^{p_\theta,r}(\log L)^{\frac{\alpha}{p'_\theta}},$$

where $\dfrac{1}{p_\theta} = \dfrac{1-\theta}{p} + \dfrac{\theta}{q}$, $\dfrac{1}{p_\theta} + \dfrac{1}{p'_\theta} = 1$.

We highlight the following consequence of Theorem 9 and Theorem 10:

THEOREM 11. ([59], Theorem 3.5) *Let* $1<a<+\infty$, $\beta \in \mathbb{R}$, $\beta \neq 0$, $1<r<+\infty$. *The Lorentz-Zygmund space* $L^{a,r}(\log L)^\beta$ *is an interpolation space in the sense of Peetre of two Grand Lebesgue spaces if* $\beta < 0$ *and of two small Lebesgue spaces if* $\beta > 0$.

The results above concern interpolation between two grand or two small Lebesgue spaces. It may be interesting to consider also the off-diagonal case: this has been obtained in the following result, where the interpolation between small Lebesgue spaces $L^{(p}$ and grand Lebesgue spaces $L^{q)}$ when $p<q$ is considered.

THEOREM 12. ([59], Theorem 5.1) *If* $0<\theta<1,\ 1\leqslant r<+\infty,\ p<q$, *then*

$$\left(L^{(p},L^{q)}\right)_{\theta,r}=L^{p_\theta,r}\left(\log L\right)^{\alpha_\theta},$$

where $\dfrac{1}{p_\theta}=\dfrac{1-\theta}{p}+\dfrac{\theta}{q},\ \ \alpha_\theta=1-\theta-\dfrac{1}{p_\theta}$.

The proof of Theorem 12 is based of the following estimate of the K-functional.

THEOREM 13. ([59], Theorem 5.2) *If* $1<p<q<+\infty$, *then for all* $t>0,\ f\in L^{(p}+L^{q)}$

$$\begin{aligned}K(f,t;L^{(p},L^{q)})\approx&\int_0^{\varphi(t)}(1-\log s)^{\frac{-1}{p}}\left(\int_0^s f_*^p(\tau)d\tau\right)^{\frac{1}{p}}\frac{ds}{s}\\&+(1-\log t)^{\frac{p-1}{p}}\left(\int_0^{\varphi(t)} f_*^p(\tau)d\tau\right)^{\frac{1}{p}}\\&+t\sup_{\varphi(t)<s<1}(1-\log s)^{\frac{-1}{q}}\left(\int_s^1 f_*^q(\tau)d\tau\right)^{\frac{1}{q}},\end{aligned}$$

where φ *is an invertible function from* $[0,1]$ *into itself satisfying the equivalence*

$$\varphi(t)^{\frac{1}{p}-\frac{1}{q}}(1-\log\varphi(t))^{\frac{p-q+pq}{pq}}\approx t.$$

The preceding study can be extended to the case where $p=q$, where the inverse function of $\psi_1(x)=(1-\log x)^{-1}$, say $\varphi_1(t)=e^{1-\frac{1}{t}}$, will play a fundamental role in order to express the K-functional. Note that in this case the reiteration theorem from [14] cannot be used, and the computation must be made directly.

THEOREM 14. ([59], Theorem 6.1) *If* $1<p<+\infty,\ 0<t<1,\ f\in L^{p)}+L^{(p}$, *then*

$$\begin{aligned}K(f,t;L^{p)},L^{(p})\approx&\sup_{0<s<e^{1-\frac{1}{t}}}(1-\log s)^{\frac{-1}{p}}\left(\int_s^{e^{1-\frac{1}{t}}} f_*^p(x)dx\right)^{\frac{1}{p}}\\&+t\int_{e^{1-\frac{1}{t}}}^1(1-\log s)^{\frac{-1}{p}}\left(\int_{e^{1-\frac{1}{t}}}^s f_*^p(x)dx\right)^{\frac{1}{p}}\frac{ds}{s}.\end{aligned}$$

From Theorem 14 we may characterize the interpolation space $(L^{p)},L^{(p})_{\theta,r}$ as follows.

THEOREM 15. ([59], Theorem 6.2) *If* $1 < p < +\infty$, $0 < \theta < 1$, $1 \leqslant r < +\infty$, *then* $Z_{\theta,r} := (L^{p)}, L^{(p})_{\theta,r}$ *has the following equivalent norm, namely, for* $f \in Z_{\theta,r}$, $\beta_\theta = \theta - \frac{1}{p} - \frac{1}{r}$

- *if* $\theta < \frac{1}{p}$ *then*

$$\|f\|_{Z_{\theta,r}} \approx \left[\int_0^1 \left[(1-\log t)^{\beta_\theta}\left(\int_t^1 f_*^p\right)^{\frac{1}{p}}\right]^r \frac{dt}{t}\right]^{\frac{1}{r}};$$

- *if* $\theta > \frac{1}{p}$ *then* $Z_{\theta,r} = G\Gamma(p,r,w), w(t) = t^{-1}(1-\log t)^{\beta_\theta r}$ *and*

$$\|f\|_{Z_{\theta,r}} \approx \left[\int_0^1 \left[(1-\log t)^{\beta_\theta}\left(\int_0^t f_*^p\right)^{\frac{1}{p}}\right]^r \frac{dt}{t}\right]^{\frac{1}{r}};$$

- *if* $\theta = \frac{1}{p}$

$$\|f\|_{Z_{\theta,r}} \approx \left[\sum_{k=0}^{+\infty}\left(\int_{2^{1-2^{k+1}}}^{2^{1-2^k}} f_*^p\right)^{\frac{r}{p}}\right]^{\frac{1}{r}}.$$

In particular

$$(L^{p)}, L^{(p})_{\frac{1}{p},p} = L^p \qquad \textit{and} \qquad (L^{p)}, L^{(p})_{\theta,p} = L^{p,p}(\log L)^{\theta-\frac{1}{p}}.$$

We conclude with the following two results, where the quasinorm of the interpolation space $(L^{p)}, L^{(p})_{\theta,r}$ is given in a unified form (see next Corollary 1) and in terms of the notion of Generalized Gamma space with double weights ([59, 84]), which is a further evolution of the $G\Gamma(p,m,w)$ spaces mentioned above in (12) (see next Theorem 16).

COROLLARY 1. ([59], Corollary 6.1) *If* $\theta \in (0,1)$, $1 \leqslant r < +\infty$ *and* $f \in Z_{\theta,r}$, *then*

$$\|f\|_{Z_{\theta,r}}^r \approx \sum_{k\in\mathbb{N}} 2^{kr(\theta-\frac{1}{p})}\left(\int_{t_{k+1}}^{t_k} f_*^p(y)\,dy\right)^{\frac{r}{p}},$$

where $t_k = 2^{1-2^k}$, $k \in \mathbb{N}$.

THEOREM 16. ([59], Theorem 6.4) *If* $1 < p < +\infty$, $0 < \theta < 1$, $1 \leqslant r < +\infty$, *then*

$$(L^{p)}, L^{(p})_{\theta,r} = G\Gamma(p,r;w_1,w_2),$$

where $w_1(t) = t^{-1}(1-\log t)^{\theta r-1}$, $w_2(t) = (1-\log t)^{-1}$, $t \in (0,1)$, *i.e.*

$$\|f\|_{Z_{\theta,r}} \approx \left[\int_0^1 (1-\log t)^{\theta r}\left(\int_0^t (1-\log x)^{-1} f_*^p(x)dx\right)^{\frac{r}{p}} \frac{dt}{(1-\log t)t}\right]^{\frac{1}{r}}.$$

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