Flux reconstructions for lower bounds on eigenvalues

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Menu

1. Soup.
   Guaranteed error bounds for elliptic boundary value problems
2. Main course.
   Lehmann–Goerisch lower bound on eigenvalues
3. Dessert.
   Surprise.

1. Soup.

Guaranteed error bounds for elliptic boundary value problems

Poisson problem

$$-\Delta u = f \quad \text{in } \Omega \subset \mathbb{R}^d$$
$$u = 0 \quad \text{on } \partial\Omega$$

Weak formulation:

$$u \in V: \quad (\nabla u, \nabla v) = (f, v) \quad \forall v \in V$$

Finite element method:

$$u_h \in V_h: \quad (\nabla u_h, \nabla v_h) = (f, v_h) \quad \forall v_h \in V_h$$

Notation:

- $V = H_0^1(\Omega)$
- $(u, v) = \int_\Omega uv \,\mathrm{d}x$
- $V_h = \{v_h \in V : v_h|_K \in P^1(K), K \in \mathcal{T}_h\}$

Poisson problem

$$-\Delta u = f \quad \text{in } \Omega \subset \mathbb{R}^d$$
$$u = 0 \quad \text{on } \partial\Omega$$

Weak formulation:

$$u \in V : \quad (\nabla u, \nabla v) = (f, v) \quad \forall v \in V$$

Finite element method:

$$u_h \in V_h : \quad (\nabla u_h, \nabla v_h) = (f, v_h) \quad \forall v_h \in V_h$$

Error bound:

$$\|\nabla u - \nabla u_h\| \leq \eta$$

Guaranteed error bound

Theorem.

Let $u_h \in V$, $\gamma > 0$, and C_{F} be the Friedrichs constant. Then

$$\|\nabla u - \nabla u_h\| \leq \left(1 + C_{\mathrm{F}}^2 \gamma\right)^{\frac{1}{2}} \left(\|\mathbf{q} - \nabla u_h\|^2 + \frac{1}{\gamma} \|f + \operatorname{div} \mathbf{q}\|^2 \right)^{\frac{1}{2}} \quad \forall \mathbf{q} \in \mathbf{H}(\operatorname{div}, \Omega)$$

Note:

$$C_{\mathrm{F}} \leq \frac{1}{\pi} \left(\sum_{i=1}^{d} \frac{1}{L_i^2} \right)^{-1/2}$$

where L_i are lenghts of sides of a box containing Ω.

Guaranteed error bound

Theorem.

Let $u_h \in V$, $\gamma > 0$, and C_{F} be the Friedrichs constant. Then

$$\|\nabla u - \nabla u_h\| \leq \left(1 + C_{\mathrm{F}}^2 \gamma\right)^{\frac{1}{2}} \left(\|\mathbf{q} - \nabla u_h\|^2 + \frac{1}{\gamma} \|f + \operatorname{div} \mathbf{q}\|^2 \right)^{\frac{1}{2}} \quad \forall \mathbf{q} \in \mathbf{H}(\operatorname{div}, \Omega)$$

Proof. $v \in V$

$$(\nabla u - \nabla u_h, \nabla v) = (f, v) - (\nabla u_h, \nabla v)$$

Guaranteed error bound

Theorem.

Let $u_h \in V$, $\gamma > 0$, and C_{F} be the Friedrichs constant. Then

$$\|\nabla u - \nabla u_h\| \leq \left(1 + C_{\mathrm{F}}^2 \gamma\right)^{\frac{1}{2}} \left(\|\mathbf{q} - \nabla u_h\|^2 + \frac{1}{\gamma} \|f + \operatorname{div} \mathbf{q}\|^2 \right)^{\frac{1}{2}} \quad \forall \mathbf{q} \in \mathbf{H}(\operatorname{div}, \Omega)$$

Proof. $v \in V$

$$(\nabla u - \nabla u_h, \nabla v) = (f, v) - (\nabla u_h, \nabla v) + (\mathbf{q}, \nabla v) + (\operatorname{div} \mathbf{q}, v)$$

Guaranteed error bound

Theorem.

Let $u_h \in V$, $\gamma > 0$, and C_{F} be the Friedrichs constant. Then

$$\|\nabla u - \nabla u_h\| \leq \left(1 + C_{\mathrm{F}}^2 \gamma\right)^{\frac{1}{2}} \left(\|\mathbf{q} - \nabla u_h\|^2 + \frac{1}{\gamma} \|f + \operatorname{div} \mathbf{q}\|^2 \right)^{\frac{1}{2}} \quad \forall \mathbf{q} \in \mathbf{H}(\operatorname{div}, \Omega)$$

Proof. $v \in V$

$$\begin{aligned}(\nabla u - \nabla u_h, \nabla v) &= (f, v) - (\nabla u_h, \nabla v) + (\mathbf{q}, \nabla v) + (\operatorname{div} \mathbf{q}, v) \\ &= (\mathbf{q} - \nabla u_h, \nabla v) + (f + \operatorname{div} \mathbf{q}, v)\end{aligned}$$

Guaranteed error bound

Theorem.
Let $u_h \in V$, $\gamma > 0$, and C_{F} be the Friedrichs constant. Then

$$\|\nabla u - \nabla u_h\| \le \left(1 + C_{\mathrm{F}}^2 \gamma\right)^{\frac{1}{2}} \left(\|\mathbf{q} - \nabla u_h\|^2 + \frac{1}{\gamma} \|f + \operatorname{div} \mathbf{q}\|^2 \right)^{\frac{1}{2}} \quad \forall \mathbf{q} \in \mathbf{H}(\operatorname{div}, \Omega)$$

Proof. $v \in V$

$$\begin{aligned} (\nabla u - \nabla u_h, \nabla v) &= (f, v) - (\nabla u_h, \nabla v) + (\mathbf{q}, \nabla v) + (\operatorname{div} \mathbf{q}, v) \\ &= (\mathbf{q} - \nabla u_h, \nabla v) + (f + \operatorname{div} \mathbf{q}, v) \\ &\le \|\mathbf{q} - \nabla u_h\| \|\nabla v\| + \qquad \|f + \operatorname{div} \mathbf{q}\| \qquad \|v\| \end{aligned}$$

Guaranteed error bound

Theorem.

Let $u_h \in V$, $\gamma > 0$, and C_F be the Friedrichs constant. Then

$$\|\nabla u - \nabla u_h\| \le \left(1 + C_\mathrm{F}^2\gamma\right)^{\frac{1}{2}} \left(\|\mathbf{q} - \nabla u_h\|^2 + \frac{1}{\gamma}\|f + \operatorname{div}\mathbf{q}\|^2\right)^{\frac{1}{2}} \quad \forall \mathbf{q} \in \mathbf{H}(\operatorname{div}, \Omega)$$

Proof. $v \in V$

$$\begin{aligned}(\nabla u - \nabla u_h, \nabla v) &= (f, v) - (\nabla u_h, \nabla v) + (\mathbf{q}, \nabla v) + (\operatorname{div}\mathbf{q}, v)\\ &= (\mathbf{q} - \nabla u_h, \nabla v) + (f + \operatorname{div}\mathbf{q}, v)\\ &\le \|\mathbf{q} - \nabla u_h\|\|\nabla v\| + \frac{1}{\gamma^{1/2}}\|f + \operatorname{div}\mathbf{q}\|\gamma^{1/2}\|v\|\end{aligned}$$

Guaranteed error bound

Theorem.

Let $u_h \in V$, $\gamma > 0$, and C_{F} be the Friedrichs constant. Then

$$\|\nabla u - \nabla u_h\| \leq \left(1 + C_{\mathrm{F}}^2 \gamma\right)^{\frac{1}{2}} \left(\|\mathbf{q} - \nabla u_h\|^2 + \frac{1}{\gamma} \|f + \operatorname{div} \mathbf{q}\|^2 \right)^{\frac{1}{2}} \quad \forall \mathbf{q} \in \mathbf{H}(\operatorname{div}, \Omega)$$

Proof. $v \in V$

$$\begin{aligned}
(\nabla u - \nabla u_h, \nabla v) &= (f, v) - (\nabla u_h, \nabla v) + (\mathbf{q}, \nabla v) + (\operatorname{div} \mathbf{q}, v) \\
&= (\mathbf{q} - \nabla u_h, \nabla v) + (f + \operatorname{div} \mathbf{q}, v) \\
&\leq \|\mathbf{q} - \nabla u_h\| \|\nabla v\| + \frac{1}{\gamma^{1/2}} \|f + \operatorname{div} \mathbf{q}\| \gamma^{1/2} \|v\| \\
&\leq \left(\|\mathbf{q} - \nabla u_h\|^2 + \frac{1}{\gamma} \|f + \operatorname{div} \mathbf{q}\|^2 \right)^{\frac{1}{2}} \Big(\|\nabla v\|^2 + \gamma \underbrace{\|v\|^2}_{\leq C_{\mathrm{F}}^2 \|\nabla v\|^2} \Big)^{\frac{1}{2}}
\end{aligned}$$

Guaranteed error bound

Theorem.

Let $u_h \in V$, $\gamma > 0$, and C_{F} be the Friedrichs constant. Then

$$\|\nabla u - \nabla u_h\| \le \left(1 + C_{\mathrm{F}}^2 \gamma\right)^{\frac{1}{2}} \left(\|\mathbf{q} - \nabla u_h\|^2 + \frac{1}{\gamma} \|f + \operatorname{div} \mathbf{q}\|^2 \right)^{\frac{1}{2}} \quad \forall \mathbf{q} \in \mathbf{H}(\operatorname{div}, \Omega)$$

Proof. $v \in V, \quad v = u - u_h$

$$\begin{aligned}
(\nabla u - \nabla u_h, \nabla v) &= (f, v) - (\nabla u_h, \nabla v) + (\mathbf{q}, \nabla v) + (\operatorname{div} \mathbf{q}, v) \\
&= (\mathbf{q} - \nabla u_h, \nabla v) + (f + \operatorname{div} \mathbf{q}, v) \\
&\le \|\mathbf{q} - \nabla u_h\| \|\nabla v\| + \frac{1}{\gamma^{1/2}} \|f + \operatorname{div} \mathbf{q}\| \gamma^{1/2} \|v\| \\
&\le \left(\|\mathbf{q} - \nabla u_h\|^2 + \frac{1}{\gamma} \|f + \operatorname{div} \mathbf{q}\|^2 \right)^{\frac{1}{2}} \underbrace{\left(\|\nabla v\|^2 + \gamma \|v\|^2 \right)^{\frac{1}{2}}}_{\left(1 + C_{\mathrm{F}}^2 \gamma\right)^{\frac{1}{2}} \|\nabla v\|}
\end{aligned}$$

Guaranteed error bound

Theorem.

Let $u_h \in V$, $\gamma > 0$, and C_{F} be the Friedrichs constant. Then

$$\|\nabla u - \nabla u_h\| \leq \left(1 + C_{\mathrm{F}}^2 \gamma\right)^{\frac{1}{2}} \left(\|\mathbf{q} - \nabla u_h\|^2 + \frac{1}{\gamma}\|f + \operatorname{div}\mathbf{q}\|^2\right)^{\frac{1}{2}}$$

$$\forall \mathbf{q} \in \mathbf{H}(\operatorname{div}, \Omega)$$

Find $\mathbf{q}_h \in \mathbf{W}_h \subset \mathbf{H}(\operatorname{div}, \Omega)$ minimizing

$$\|\mathbf{q}_h - \nabla u_h\|^2 + \frac{1}{\gamma}\|f + \operatorname{div}\mathbf{q}_h\|^2$$

Guaranteed error bound

Theorem.

Let $u_h \in V$, $\gamma > 0$, and C_{F} be the Friedrichs constant. Then

$$\|\nabla u - \nabla u_h\| \leq \left(1 + C_{\mathrm{F}}^2 \gamma\right)^{\frac{1}{2}} \left(\|\mathbf{q} - \nabla u_h\|^2 + \frac{1}{\gamma} \|f + \operatorname{div} \mathbf{q}\|^2 \right)^{\frac{1}{2}} \quad \forall \mathbf{q} \in \mathbf{H}(\operatorname{div}, \Omega)$$

Find $\mathbf{q}_h \in \mathbf{W}_h \subset \mathbf{H}(\operatorname{div}, \Omega)$ minimizing

$$\|\mathbf{q}_h - \nabla u_h\|^2 + \frac{1}{\gamma} \|f + \operatorname{div} \mathbf{q}_h\|^2$$

Equivalent to: Find $\mathbf{q}_h \in \mathbf{W}_h$ such that

$$\frac{1}{\gamma}(\operatorname{div} \mathbf{q}_h, \operatorname{div} \mathbf{w}_h) + (\mathbf{q}_h, \mathbf{w}_h) = (\nabla u_h, \mathbf{w}_h) - \frac{1}{\gamma}(f, \operatorname{div} \mathbf{w}_h) \quad \forall \mathbf{w}_h \in \mathbf{W}_h$$

2. Main course.

Lehmann–Goerisch lower bound on eigenvalues

Laplace eigenvalue problem

$$-\Delta u_n = \lambda_n u_n \quad \text{in } \Omega \subset \mathbb{R}^d$$
$$u_n = 0 \quad \text{on } \partial\Omega$$

Weak formulation: $\lambda_n \in \mathbb{R},\ u_n \in V \setminus \{0\}$:

$$(\nabla u_n, \nabla v) = \lambda_n (u_n, v) \quad \forall v \in V$$

Finite element method: $\lambda_{h,n} \in \mathbb{R},\ u_{h,n} \in V_h \setminus \{0\}$:

$$(\nabla u_{h,n}, \nabla v_h) = \lambda_{h,n} (u_{h,n}, v_h) \quad \forall v_h \in V_h$$

Guaranteed upper bound

$$\lambda_n \leq \lambda_{h,n}$$

Laplace eigenvalue problem

$$-\Delta u_n = \lambda_n u_n \quad \text{in } \Omega \subset \mathbb{R}^d$$
$$u_n = 0 \quad \text{on } \partial\Omega$$

Weak formulation: $\lambda_n \in \mathbb{R},\ u_n \in V \setminus \{0\}$:

$$(\nabla u_n, \nabla v) = \lambda_n (u_n, v) \quad \forall v \in V$$

Finite element method: $\lambda_{h,n} \in \mathbb{R},\ u_{h,n} \in V_h \setminus \{0\}$:

$$(\nabla u_{h,n}, \nabla v_h) = \lambda_{h,n} (u_{h,n}, v_h) \quad \forall v_h \in V_h$$

Can we have a lower bound?

$$? \leq \lambda_n \leq \lambda_{h,n}$$

Laplace eigenvalue problem

$$-\Delta u_n + \gamma u_n = (\lambda_n + \gamma) u_n \quad \text{in } \Omega \subset \mathbb{R}^d,\ \gamma > 0$$
$$u_n = 0 \quad \text{on } \partial\Omega$$

Weak formulation: $\lambda_n \in \mathbb{R},\ u_n \in V \setminus \{0\}$:

$$(\nabla u_n, \nabla v) + \gamma(u_n, v) = (\lambda_n + \gamma)(u_n, v) \quad \forall v \in V$$

Finite element method: $\lambda_{h,n} \in \mathbb{R},\ u_{h,n} \in V_h \setminus \{0\}$:

$$(\nabla u_{h,n}, \nabla v_h) + \gamma(u_{h,n}, v_h) = (\lambda_{h,n} + \gamma)(u_{h,n}, v_h) \quad \forall v_h \in V_h$$

Can we have a lower bound?

$$? \le \lambda_n \le \lambda_{h,n}$$

Lehmann–Goerisch method

Theorem (Lehmann–Goerisch)

Let $\rho \leq \lambda_{N+1} + \gamma$, $\gamma > 0$

- $u_{h,1}, u_{h,2}, \dots, u_{h,N} \in V$ be linearly independent
- $A_{0,ij} = (\nabla u_{h,i}, \nabla u_{h,j}) + \gamma(u_{h,i}, u_{h,j})$
- $A_{1,ij} = (u_{h,i}, u_{h,j})$
- $\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \dots, \boldsymbol{\sigma}_N \in \mathbf{H}(\operatorname{div}, \Omega)$ be arbitrary
  $\hat{A}_{2,ij} = (\boldsymbol{\sigma}_i, \boldsymbol{\sigma}_j) + \frac{1}{\gamma}(u_{h,i} + \operatorname{div} \boldsymbol{\sigma}_i, u_{h,j} + \operatorname{div} \boldsymbol{\sigma}_j)$
- $\mu_1 \leq \mu_2 \leq \dots \leq \mu_N : \quad (\rho A_1 - A_0)\mathbf{x} = \mu(A_0 - 2\rho A_1 + \rho^2 \hat{A}_2)\mathbf{x}$

If $A_0 - 2\rho A_1 + \rho^2 \hat{A}_2$ is positive definite
then for all $n = 1, 2, \dots, N$ such that $\mu_n > 0$ we have

$$\ell_n = \rho - \gamma - \frac{\rho}{1 + \mu_n} \leq \lambda_n.$$

[Behnke, Mertins, Plum, Wieners 2000]

How to choose $\boldsymbol{\sigma}_i$?

Goerisch matrix: $$\hat{A}_{2,ij} = (\boldsymbol{\sigma}_i, \boldsymbol{\sigma}_j) + \frac{1}{\gamma}(u_{h,i} + \operatorname{div} \boldsymbol{\sigma}_i, u_{h,j} + \operatorname{div} \boldsymbol{\sigma}_j)$$

How to choose $\boldsymbol{\sigma}_i$?

Goerisch matrix: $$\hat{A}_{2,ij} = (\boldsymbol{\sigma}_i, \boldsymbol{\sigma}_j) + \frac{1}{\gamma}(u_{h,i} + \operatorname{div} \boldsymbol{\sigma}_i, u_{h,j} + \operatorname{div} \boldsymbol{\sigma}_j)$$

Lehmann matrix: $$A_{2,ij} = (\nabla w_i, \nabla w_j) + \gamma(w_i, w_j)$$

where $w_i \in V$: $$(\nabla w_i, \nabla v) + \gamma(w_i, v) = (u_{h,i}, v) \quad \forall v \in V$$

How to choose $\boldsymbol{\sigma}_i$?

Goerisch matrix: $\hat{A}_{2,ij} = (\boldsymbol{\sigma}_i, \boldsymbol{\sigma}_j) + \frac{1}{\gamma}(u_{h,i} + \operatorname{div} \boldsymbol{\sigma}_i, u_{h,j} + \operatorname{div} \boldsymbol{\sigma}_j)$

Lehmann matrix: $A_{2,ij} = (\nabla w_i, \nabla w_j) + \gamma(w_i, w_j)$

where $w_i \in V$: $(\nabla w_i, \nabla v) + \gamma(w_i, v) = (u_{h,i}, v) \quad \forall v \in V$

Note: $(u_{h,i}, v) \approx \left(\frac{\nabla u_{h,i}}{\lambda_{h,i} + \gamma}, v\right) + \gamma \left(\frac{u_{h,i}}{\lambda_{h,i} + \gamma}, v\right)$

$$\Rightarrow w_i \approx \frac{u_{h,i}}{\lambda_{h,i} + \gamma}$$

How to choose $\boldsymbol{\sigma}_i$?

Goerisch matrix: $\hat{A}_{2,ij} = (\boldsymbol{\sigma}_i, \boldsymbol{\sigma}_j) + \frac{1}{\gamma}(u_{h,i} + \operatorname{div} \boldsymbol{\sigma}_i, u_{h,j} + \operatorname{div} \boldsymbol{\sigma}_j)$

Lehmann matrix: $A_{2,ij} = (\nabla w_i, \nabla w_j) + \gamma(w_i, w_j)$

where $w_i \in V$: $(\nabla w_i, \nabla v) + \gamma(w_i, v) = (u_{h,i}, v) \quad \forall v \in V$

Note: $(u_{h,i}, v) \approx \left(\frac{\nabla u_{h,i}}{\lambda_{h,i} + \gamma}, v \right) + \gamma \left(\frac{u_{h,i}}{\lambda_{h,i} + \gamma}, v \right)$

$$\Rightarrow w_i \approx \frac{u_{h,i}}{\lambda_{h,i} + \gamma}$$

Observe

$$\boldsymbol{\sigma}_i \approx \nabla w_i \approx \frac{\nabla u_{h,i}}{\lambda_{h,i} + \gamma} \quad \text{and} \quad \operatorname{div} \boldsymbol{\sigma}_i \approx \gamma w_i - u_{h,i} \approx \frac{-\lambda_{h,i} u_{h,i}}{\lambda_{h,i} + \gamma}$$

How to choose $\boldsymbol{\sigma}_i$?

Goerisch matrix: $\hat{A}_{2,ij} = (\boldsymbol{\sigma}_i, \boldsymbol{\sigma}_j) + \frac{1}{\gamma}(u_{h,i} + \operatorname{div}\boldsymbol{\sigma}_i, u_{h,j} + \operatorname{div}\boldsymbol{\sigma}_j)$

Lehmann matrix: $A_{2,ij} = (\nabla w_i, \nabla w_j) + \gamma(w_i, w_j)$

where $w_i \in V$: $(\nabla w_i, \nabla v) + \gamma(w_i, v) = (u_{h,i}, v) \quad \forall v \in V$

Note: $(u_{h,i}, v) \approx \left(\frac{\nabla u_{h,i}}{\lambda_{h,i} + \gamma}, v\right) + \gamma\left(\frac{u_{h,i}}{\lambda_{h,i} + \gamma}, v\right)$

$$\Rightarrow w_i \approx \frac{u_{h,i}}{\lambda_{h,i} + \gamma}$$

Observe

$$\boldsymbol{\sigma}_i \approx \nabla w_i \approx \frac{\nabla u_{h,i}}{\lambda_{h,i} + \gamma} \quad \text{and} \quad \operatorname{div}\boldsymbol{\sigma}_i \approx \gamma w_i - u_{h,i} \approx \frac{-\lambda_{h,i} u_{h,i}}{\lambda_{h,i} + \gamma}$$

Thus, we look for $\boldsymbol{\sigma}_{h,i} \in \mathbf{H}(\operatorname{div}, \Omega)$ such that

$$\left\| \boldsymbol{\sigma}_{h,i} - \frac{\nabla u_{h,i}}{\lambda_{h,i} + \gamma} \right\|^2 + \frac{1}{\gamma} \left\| \frac{\lambda_{h,i} u_{h,i}}{\lambda_{h,i} + \gamma} + \operatorname{div}\boldsymbol{\sigma}_{h,i} \right\|^2 \quad \text{is small}$$

3. Dessert.

Surprise.

Comparison

Boundary value problem:

Find $\mathbf{q}_h \in \mathbf{W}_h \subset \mathbf{H}(\mathrm{div}, \Omega)$ minimizing

$$\|\mathbf{q}_h - \nabla u_h\|^2 + \frac{1}{\gamma}\|f + \mathrm{div}\,\mathbf{q}_h\|^2$$

Eigenvalue problem

Find $\boldsymbol{\sigma}_{h,i} \in \mathbf{W}_h \subset \mathbf{H}(\mathrm{div}, \Omega)$ minimizing

$$\left\|\boldsymbol{\sigma}_{h,i} - \frac{\nabla u_{h,i}}{\lambda_{h,i} + \gamma}\right\|^2 + \frac{1}{\gamma}\left\|\frac{\lambda_{h,i} u_{h,i}}{\lambda_{h,i} + \gamma} + \mathrm{div}\,\boldsymbol{\sigma}_{h,i}\right\|^2$$

Thus, if $f = \lambda_{h,i} u_{h,i}$ then

$$\boldsymbol{\sigma}_{h,i} = \frac{\mathbf{q}_h}{\lambda_{h,i} + \gamma}$$

Take home message

The same flux reconstruction can be used for both

1. guaranteed upper bounds on error for boundary value problems
2. lower bounds of eigenvalues for eigenvalue problems

Thank you for your attention

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