Institute of Mathematics
Czech Academy of Sciences

Combinatorial homotopy theory for operads

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Operads, algebras, resolutions & polytopes

Operads	P
Algebras	$Alg_{\mathcal{E}}(P)$

Operads, algebras, resolutions & polytopes

Operads	P	*As*
Algebras	$Alg_{\mathcal{E}}(P)$	associative algebras

$$(ab)c = a(bc)$$

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$$\alpha_{a,b,c} : (ab)c \to a(bc)$$

Operads, algebras, resolutions & polytopes

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Algebras	$Alg_{\mathcal{E}}(P)$	associative algebras	A_∞-algebras	operads

$$(a \circ_i b) \circ_{j+i-1} c = a \circ_i (b \circ_j c) \qquad (a \circ_i b) \circ_{j+k-1} c = (a \circ_j c) \circ_i b$$

Operads, algebras, resolutions & polytopes

Operads	P	*As*	A_∞-operad	$\mathcal{O}$	$\mathcal{O}_\infty$-operad
Algebras	$Alg_{\mathcal{E}}(P)$	associative algebras	A_∞-algebras	operads	strongly homotopy operads

$\beta^{i,j}_{a,b,c} : (a \circ_i b) \circ_{j+i-1} c \to a \circ_i (b \circ_j c)$ $\qquad$ $\theta^{i,j}_{a,b,c} : (a \circ_i b) \circ_{j+k-1} c \to (a \circ_j c) \circ_i b$

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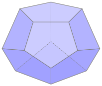
J. D. Stasheff
Homotopy associativity of H-spaces, I, II
Trans Amer Math Soc Vol. 108, No. 2, 275-312, 1963



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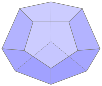

Goal: Provide a combinatorial description of the $\mathcal{O}_\infty$-operad.

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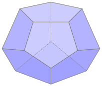

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P. Van der Laan
Coloured Koszul duality and strongly homotopy operads
PhD Thesis, arXiv:math/0312147v2, 2003

Hypergraph polytopes ⊇ Operadic polytopes

K. Došen, Z. Petrić
Hypergraph polytopes
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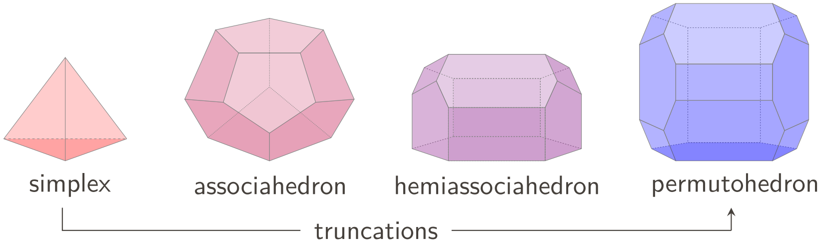

Hypergraph terminology

$\mathbf{H} = (H, \mathbf{H})$ H - vertices $\mathbf{H} \subseteq \mathcal{P}(H) \setminus \emptyset$ - hyperedges $\bigcup \mathbf{H} = H$

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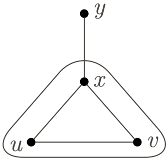

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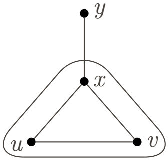

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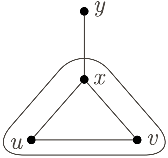

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$$Sat(\mathbf{H}) = \mathbf{H} \cup \{\{x, y, u\}, \{x, y, v\}, \{x, y, u, v\}\}$$

A polytope from a hypergraph

Hemiassociahedron

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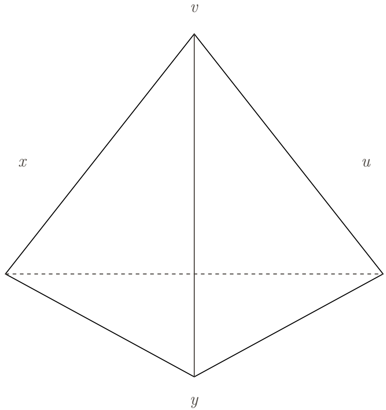

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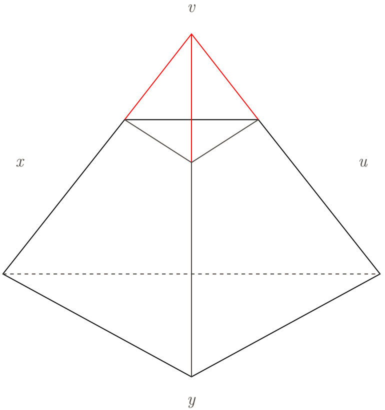

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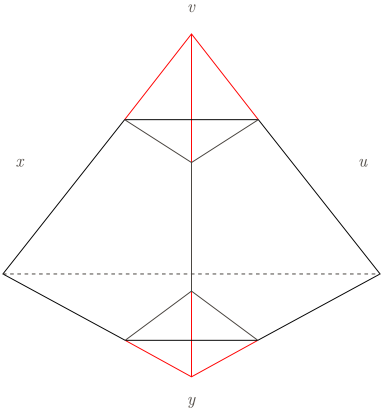

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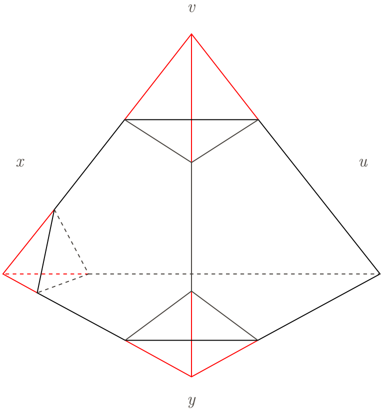

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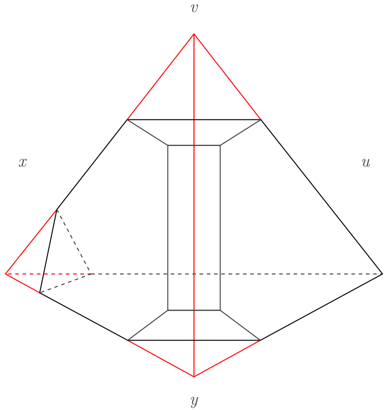

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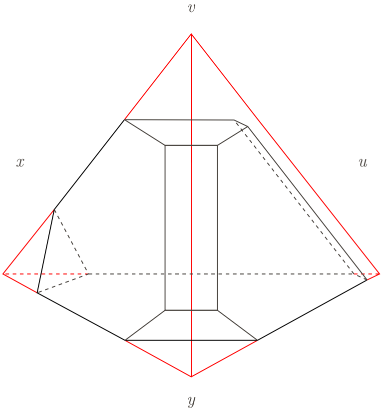

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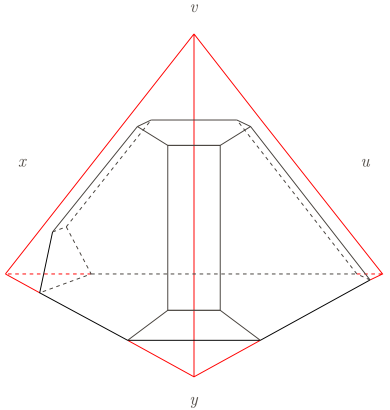

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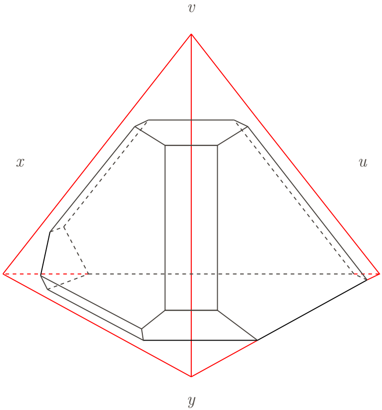

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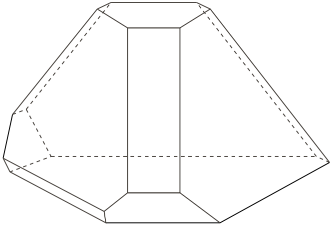

Combinatorial description of hypergraph polytopes

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- Elements: constructs $C : \mathbf{H}$. Pick $\emptyset \neq Y \subseteq H$.
  - If $Y = H$, then $H : \mathbf{H}$;
  - If $Y \subsetneq H$, $\mathbf{H} \backslash Y \rightsquigarrow H_1, \ldots, H_n$ and $T_1 : \mathbf{H}_1, \ldots, T_n : \mathbf{H}_n$, then $Y\{T_1, \ldots, T_n\} : \mathbf{H}$.

  Constructions: constructs with singleton nodes.

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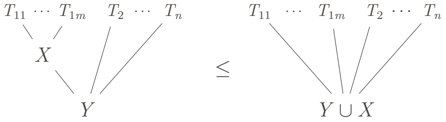
  Constructions: constructs with singleton nodes.
- Partial order: edge contraction

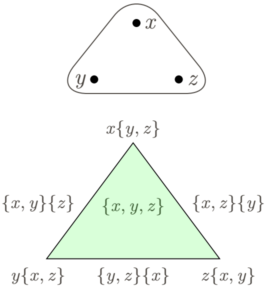


$$X\{Y\{T_{11}, \ldots, T_{1m}, T_2, \ldots, T_n\}\} \leq (X \cup Y)\{T_{11}, \ldots, T_{1m}, T_2, \ldots, T_n\}$$

Examples

2-dim simplex

$$\mathbf{H} = \{\{x\},\{y\},\{z\},\{x,y,z\}\}$$



truncating $x\{y,z\}$

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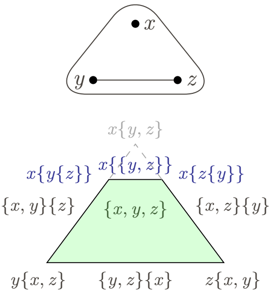
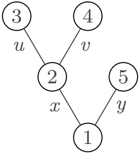


Operadic polytopes



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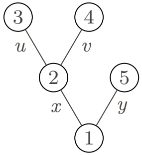

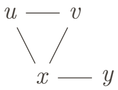

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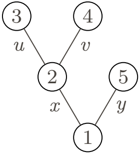

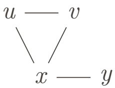

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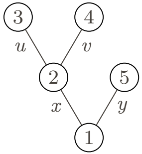

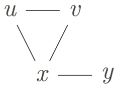

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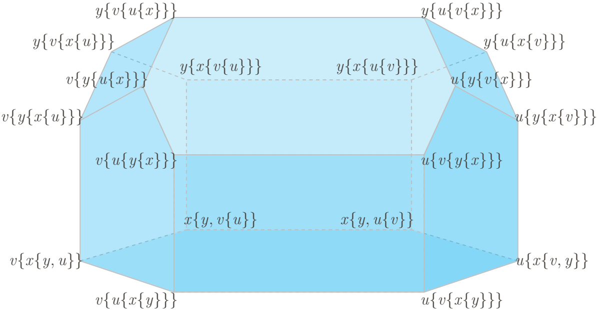

The $\mathcal{O}_\infty$-operad

$$\mathcal{O}_\infty = \mathbb{K}\left(\bigoplus_{\mathcal{T}} \mathcal{A}(\mathbf{H}_{\mathcal{T}})\right) \qquad (\mathcal{T}_1, C_1) \circ_i (\mathcal{T}_2, C_2) = (-1)^{\varepsilon}(\mathcal{T}_1 \bullet_i \mathcal{T}_2, C_1 \bullet_i C_2)$$

$$d(\mathcal{T}, C) = \sum_{\substack{V \in V(C) \\ |V| \geq 2}} \sum_{\substack{X \cup Y = V \\ X(Y):\mathbf{H}_{\mathcal{T}_V}}} (-1)^{\delta}(\mathcal{T}, C[X(Y)/V])$$

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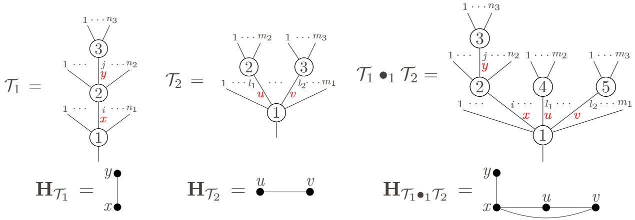

$$(\mathcal{T}_1, x\{y\}) \circ_1 (\mathcal{T}_2, v\{u\}) = (\mathcal{T}_1 \bullet_1 \mathcal{T}_2, x\{y, v\{u\}\})$$

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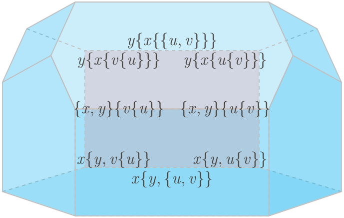
$$d(\{x, y\}(\{u, v\})) = x\{y, \{u, v\}\} + y\{x\{\{u, v\}\}\} - \{x, y\}\{v\{u\}\} - \{x, y\}\{u\{v\}\}$$



Minimal models for operads (Markl)

Theorem

The $\mathcal{O}_\infty$-operad is the minimal model of the $\mathbb{N}$-coloured operad $\mathcal{O}$, i.e.

- $\mathcal{O}_\infty \simeq \mathcal{T}_{\mathbb{N}}(E)$,
- there exists a quasi-isomorphism $\alpha_{\mathcal{O}} : \mathcal{O}_\infty \to \mathcal{O}$, and
- $d_{\mathcal{O}_\infty}(E) \subseteq \mathcal{T}_{\mathbb{N}}(E)^{(\geq 2)}$.

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Proof.

$$\mathcal{O}_\infty \simeq \mathcal{T}_{\mathbb{N}}\left(\bigoplus_{n_1,\ldots,n_k \geq 1} \bigoplus_{\mathcal{T} \in \mathcal{O}(n;n_1,\ldots,n_k)} \{E(\mathcal{T})\} \right)$$

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- $d_{\mathcal{O}_\infty}(E) \subseteq \mathcal{T}_{\mathbb{N}}(E)^{(\geq 2)}$.

Proof.

$$\mathcal{O}_\infty \simeq \mathcal{T}_{\mathbb{N}}\left(\bigoplus_{n_1,\dots,n_k \geq 1} \bigoplus_{\mathcal{T} \in \mathcal{O}(n;n_1,\dots,n_k)} \{E(\mathcal{T})\} \right)$$

$$\text{Ker } d_m \mathcal{O}_\infty(n; n_1, \dots, n_k)/\text{Im } d_{m+1}\mathcal{O}_\infty(n; n_1, \dots, n_k) = \begin{cases} \{0\}, & m \neq 0 \\ \mathcal{O}(n; n_1, \dots, n_k), & m = 0 \end{cases}$$

Minimal models for operads (Markl)

Theorem

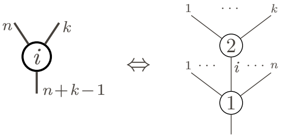
The $\mathcal{O}_\infty$-operad is the minimal model of the $\mathbb{N}$-coloured operad $\mathcal{O}$, i.e.

- $\mathcal{O}_\infty \simeq \mathcal{T}_{\mathbb{N}}(E)$,
- there exists a quasi-isomorphism $\alpha_{\mathcal{O}} : \mathcal{O}_\infty \to \mathcal{O}$, and
- $d_{\mathcal{O}_\infty}(E) \subseteq \mathcal{T}_{\mathbb{N}}(E)^{(\geq 2)}$.

Proof.

$$\mathcal{O}_\infty \simeq \mathcal{T}_{\mathbb{N}}\left(\bigoplus_{n_1,\ldots,n_k \geq 1} \ \bigoplus_{\mathcal{T} \in \mathcal{O}(n;n_1,\ldots,n_k)} \{E(\mathcal{T})\} \right)$$

$$\text{Ker } d_m \mathcal{O}_\infty(n; n_1, \ldots, n_k)/\text{Im } d_{m+1}\mathcal{O}_\infty(n; n_1, \ldots, n_k) = \begin{cases} \{0\}, & m \neq 0 \\ \mathcal{O}(n; n_1, \ldots, n_k), & m = 0 \end{cases}$$



Thank you!

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