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Universal Proof Theory: Semi-analytic Rules and Craig Interpolation

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Abstract

In [6], Iemhoff introduced the notion of a centered axiom and a centered rule as the building blocks for a certain form of sequent calculus which she calls a centered proof system. She then showed how the existence of a terminating centered system implies the uniform interpolation property for the logic that the calculus captures. In this paper we first generalize her centered rules to semi-analytic rules, a dramatically powerful generalization, and then we will show how the semi-analytic calculi consisting of these rules together with our generalization of her centered axioms, lead to the feasible Craig interpolation property. Using this relationship, we first present a uniform method to prove interpolation for different logics from sub-structural logics \mathbf{FL}_e , \mathbf{FL}_{ec} , \mathbf{FL}_{ew} and \mathbf{IPC} to their appropriate classical and modal extensions, including the intuitionistic and classical linear logics. Then we will use our theorem negatively, first to show that so many sub-structural logics including L_n , G_n , BL , R and RM^e and almost all super-intuitionistic logics (except at most seven of them) do not have a semi-analytic calculus. To investigate the case that the logic actually has the Craig interpolation property, we will first define a certain specific type of semi-analytic calculus which we call PPF systems and we will then present a sound and complete PPF calculus for classical logic. However, we will show that all such PPF calculi are exponentially slower than the classical Hilbert-style proof system (or

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equivalently $\mathbf{LK} + \mathbf{Cut}$). We will then present a similar exponential lower bound for a certain form of complete PPF calculi, this time for any super-intuitionistic logic.

1 Introduction

Proof systems are the main tool in any proof theoretic investigation, from Gentzen's consistency proof of arithmetic and Kreisel's proof mining program to the characterization of all the admissible rules of propositional intuitionistic logic. These applications are based on an extensive investigation of the behavior of a certain appropriate type of proof systems, tailored for a specific purpose including proving decidability, consistency searches or program extractions. However, we believe that while investigating specific proof systems is a very important task in proof theory, proof systems deserve their own general theory aimed to study the mathematical properties of them in a universal generic manner, exactly in the same way that universal algebra and model theory studies the generic properties of algebras and first order structures, respectively.

Let's imagine such a generic theory and let's follow the terminology of universal algebra to call this theory, the *universal proof theory*.¹ Whatever this theory turns out to be, its agenda definitely includes the following fundamental problems:

- (i) The *existence problem* to investigate the existence of some sort of interesting proof systems such as terminating ones, normalizable ones, etc.
- (ii) The *equivalence problem* to investigate the natural notions of equivalence of proof systems. This can be seen as an approach to address the so-called Hilbert's twenty fourth problem of studying the equivalence of different mathematical proofs, rigorously.
- (iii) And finally, the *characterization problem* to investigate the possible characterizations of proof systems via a given equivalence relation.

In this paper we will address the first problem for sequent style proof systems for propositional and modal logics. We will study the calculi consisting of a certain form of rules called semi-analytic rules and a certain form of

¹We are grateful to Masoud Memarzadeh for this elegant terminological suggestion.

axioms called centered axioms that are significant generalizations of the centered axioms and centered rules introduced in [6]. Our approach is based on connecting the existence of this kind of proof systems to a strong version of Craig interpolation property called the feasible interpolation. In fact, we show that the existence of such a calculus implies the feasible interpolation and we will use this implication to develop a uniform method to prove Craig interpolation for so many logics from substructural logics \mathbf{FL}_e , \mathbf{FL}_{ec} , \mathbf{FL}_{ew} and \mathbf{IPC} to their appropriate classical or modal extensions via the usual K , 4 , D and $S4$ type of rules. These modal extensions expand the method also to cover the intuitionistic and classical linear logics in which the exponentials play the role of the modality.

Despite its possible widespread use, it is fair to say that developing a uniform method to prove Craig interpolation may seem less useful than what it appears to be. The reason is the common wisdom that the Craig interpolation property is a rare property for a logic to have. To justify this feeling, note that in the sub-structural setting, we know that there are a lot of relevant and semilinear logics ([9], [7]) that lack this property and in the super-intuitionistic case, there is a well-known result by Maksimova [8] stating that among super-intuitionistic logics, there are only seven specific logics that have Craig interpolation.

Using this insight, we will turn the relation between interpolation and the existence of proof systems to its negative side to propose the main contribution of this paper. We will prove that logics without Craig interpolation property do not have a calculus consisting only of semi-analytic rules and centered axioms. Together with the generality of these rules and axioms, the result excludes almost all logics to have a reasonable interesting proof system. To name a few concrete applications, we will then apply our result to some specific logics of the previous paragraph including \mathbf{L}_n , \mathbf{G}_n , \mathbf{BL} , \mathbf{R} and \mathbf{RM}^e in the substructural case and all super-intuitionistic logics except \mathbf{IPC} , \mathbf{LC} , \mathbf{KC} , \mathbf{Bd}_2 , \mathbf{Sm} , \mathbf{GSc} and \mathbf{CPC} in the super-intuitionistic case.

Now the natural question to ask is about the case in which the logic actually does have the Craig interpolation property. Since the mentioned relationship is one way, it seems that there is no way to investigate the existence of that sort of calculus for these logics. For this case, we move from the existence of the calculus to its effectiveness. More precisely, we will show that for the classical logic, the existence of a PPF calculus, a calculus consisting only of the PPF rules (a limited version of semi-analytic rules) and centered axioms, leads to a certain sort of monotone feasible interpolation. Then

using a lower bound in monotone circuit complexity, we will show that for any sound and complete PPF calculus for the classical logic, there are some **CPC**-valid sequents with short tree-like Hilbert-style proofs (or equivalently **LK** + **Cut**-proofs) whose proofs in any PPF calculus are exponentially long. Therefore, the result shows that first, the PPF calculi are either incomplete or feasibly incomplete for **CPC** and second, any PPF calculus for **CPC** is exponentially slower than the Hilbert-style proof system (or equivalently **LK** + **Cut**). It is possible to lower down the base theory even to have some **IPC**-provable sequents, but the cost is weakening the centered axioms to have a more constructively acceptable form. We will see the details later.

2 Preliminaries

In this section we will cover some of the preliminaries needed for the following sections. The definitions are similar to the same concepts in [6] and [7], but they have been changed whenever it is needed.

First, note that all of the finite objects that we will use here can be represented by a fixed reasonable binary string code. Therefore, by the length of any object O including formulas, proofs, etc. we mean the length of this string code and we will denote it by $|O|$.

Definition 2.1. Let \mathcal{L} and \mathcal{L}' be two languages. By a *translation* $t : \mathcal{L} \rightarrow \mathcal{L}'$, we mean an assignment which assigns a formula $\phi_C(\bar{p}) \in \mathcal{L}'$ to any logical connective $C(\bar{p}) \in \mathcal{L}$ such that any p_i has at most one occurrence in $\phi_C(\bar{p})$. It is possible to extend a translation from the basic connectives of the language to all of its formulas in an obvious compositional way. We will denote the translation of a formula ϕ by ϕ^t and the translation of a multiset Γ , by $\Gamma^t = \{\phi^t \mid \phi \in \Gamma\}$.

Note that for any translation t we have $|\psi^t| \leq O(1)|\psi|$ which shows that all translations are polynomially bounded.

In this paper, we will work with a fixed but arbitrary language \mathcal{L} that is augmented by a translation $t : \{\wedge, \vee, \rightarrow, *, 0, 1\} \cup \mathcal{L} \rightarrow \mathcal{L}$ that fixes all logical connectives in \mathcal{L} . For this reason and w.l.o.g, we will assume that the language already includes the connectives $\{\wedge, \vee, \rightarrow, *, 0, 1\}$. In addition, whenever we investigate the multi-conclusion systems we always assume that the translation expands to include $+$.

Example 2.2. The usual language of classical propositional logic is a valid language in our setting. In this case, there is a *canonical translation* that

sends fusion, addition, 1 and 0 to conjunction, disjunction, \top and \perp , respectively. In this paper, whenever we pick this language, we assume that we are working with this canonical translation.

2.1 Sequents

By a sequent, we mean an expression of the form $\Gamma \Rightarrow \Delta$, where Γ and Δ are multisets of formulas in the language, and it is interpreted as $*\Gamma \rightarrow +\Delta$. By a single-conclusion sequent $\Gamma \Rightarrow \Delta$ we mean a sequent that $|\Delta| \leq 1$, and we call it multi-conclusion otherwise. We denote multisets by capital Greek letters such as Σ , Γ , Π , Δ and Λ . However, sometimes we use the bar notation for multisets to make everything simpler. For instance, by $\bar{\phi}$, we mean a multiset consisting of formulas ϕ_i .

Meta-language is the language with which we define the sequent calculi. It extends our given language with the formula symbols (variables) such as ϕ and ψ . A meta-formula is defined as the following: Atomic formulas and formula symbols are meta-formulas and if $\bar{\phi}$ is a set of meta-formulas, then $C(\bar{\phi})$ is also a meta-formula, where $C \in \mathcal{L}$ is a logical connective of the language. Moreover, we have infinitely many variables for meta-multisets and we use capital Greek letters again for them, whenever it is clear from the context whether it is a multiset or a meta-multiset variable. A meta-multiset is a multiset of meta-formulas and meta-multiset variables. By a meta-sequent we mean a sequent where the antecedent and the succedent are both meta-multisets. We use meta-multiset variable and context, interchangeably.

For a meta-formula ϕ , by $V(\phi)$ we mean the meta-formula variables and atomic constants in ϕ . A meta-formula ϕ is called p -free, for an atomic formula or meta-formula variable p , when $p \notin V(\phi)$.

Let us recall some of the notions related to sequent calculi and some of the important systems that we will use throughout the paper.

For a sequent $S = (\Gamma \Rightarrow \Delta)$, by S^a we mean the antecedent of the sequent, which is Γ , and by S^s we mean the succedent of the sequent, which is Δ . And, the multiplication of two sequents S and T is defined as $S \cdot T = (S^a \cup T^a \Rightarrow T^s \cup S^s)$.

By a rule we mean an expression of the form

$$\frac{S_1, \dots, S_n}{S_0}$$

where S_i 's are meta-sequents. By an instance of a rule, we mean substituting multisets of formulas for its contexts and substituting formulas for its meta-formula variables. A rule is backward applicable to a sequent S , when the conclusion of the rule is S .

By a sequent calculus G , we mean a set of rules. A sequent S is derivable in G , denoted by $G \vdash S$, if there exists a tree with sequents as labels of the nodes such that the label of the root is S and in each node the set of the labels of the children of the node together with the label of the node itself, constitute an instance of a rule in the system. This tree is called the proof of S in G which is sometimes called a tree-like proof to emphasize its tree-like form.

Now consider the following set of rules:

Identity:

$$\overline{\phi \Rightarrow \phi}$$

Context-free Axioms:

$$\overline{\Rightarrow 1} \quad \overline{0 \Rightarrow}$$

Rules for 0 and 1:

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, 1 \Rightarrow \Delta} L1 \quad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow 0, \Delta} R0$$

Conjunction Rules:

$$\frac{\Gamma, \phi \Rightarrow \Delta}{\Gamma, \phi \wedge \psi \Rightarrow \Delta} L\wedge \quad \frac{\Gamma \Rightarrow \phi, \Delta \quad \Gamma \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \phi \wedge \psi, \Delta} R\wedge$$

Disjunction Rules:

$$\frac{\Gamma, \phi \Rightarrow \Delta \quad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \phi \vee \psi \Rightarrow \Delta} L\vee \quad \frac{\Gamma \Rightarrow \phi, \Delta}{\Gamma \Rightarrow \phi \vee \psi, \Delta} R\vee \quad \frac{\Gamma \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \phi \vee \psi, \Delta} R\vee$$

Fusion Rules:

$$\frac{\Gamma, \phi, \psi \Rightarrow \Delta}{\Gamma, \phi * \psi \Rightarrow \Delta} L* \quad \frac{\Gamma \Rightarrow \phi, \Delta \quad \Sigma \Rightarrow \psi, \Lambda}{\Gamma, \Sigma \Rightarrow \phi * \psi, \Delta, \Lambda} R*$$

Implication Rules:

$$\frac{\Gamma \Rightarrow \phi, \Delta \quad \Sigma, \psi \Rightarrow \Lambda}{\Gamma, \Sigma, \phi \rightarrow \psi \Rightarrow \Delta, \Lambda} L \rightarrow \quad \frac{\Gamma, \phi \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \phi \rightarrow \psi, \Delta} R \rightarrow$$

The system consisting of the single-conclusion version of all of these rules is \mathbf{FL}_e^- . If we also add the single-conclusion version of the following axioms, we will have the system \mathbf{FL}_e .

Contextual Axioms:

$$\overline{\Gamma \Rightarrow \top, \Delta} \quad \overline{\Gamma, \perp \Rightarrow \Delta}$$

In the multi-conclusion case define \mathbf{CFL}_e^- and \mathbf{CFL}_e with the same rules as \mathbf{FL}_e^- and \mathbf{FL}_e , this time in their full multi-conclusion version and add $+$ to the language and the following rules to the systems:

Rules for $+$:

$$\frac{\Gamma, \phi \Rightarrow \Delta \quad \Sigma, \psi \Rightarrow \Lambda}{\Gamma, \Sigma, \phi + \psi \Rightarrow \Delta, \Lambda} L+ \quad \frac{\Gamma \Rightarrow \phi, \psi, \Delta}{\Gamma \Rightarrow \phi + \psi, \Delta} R+$$

The system \mathbf{MALL} is defined as \mathbf{CFL}_e minus the implication rules. Moreover if we consider the following rules:

$$\frac{! \Gamma \Rightarrow \phi}{! \Gamma \Rightarrow ! \phi} \dagger \quad \frac{\Gamma, \phi \Rightarrow \Delta}{\Gamma, ! \phi \Rightarrow \Delta}$$

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, ! \phi \Rightarrow \Delta} \quad \frac{\Gamma, ! \phi, ! \phi \Rightarrow \Delta}{\Gamma, ! \phi \Rightarrow \Delta}$$

we can define \mathbf{ILL} as \mathbf{FL}_e plus the single-conclusion version of the above rules and \mathbf{CLL} as \mathbf{CFL}_e plus the above rules, themselves. Note that in both cases the \dagger rule is single-conclusion.

Moreover, we have the following additional rules that we will use later:

Weakening rules:

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, \phi \Rightarrow \Delta} Lw \quad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \phi, \Delta} Rw$$

Note that in the single-conclusion cases, in the rule (Rw) , Δ is empty.

Contraction rules:

$$\frac{\Gamma, \phi, \phi \Rightarrow \Delta}{\Gamma, \phi \Rightarrow \Delta} Lc \quad \frac{\Gamma \Rightarrow \Delta, \phi, \phi}{\Gamma \Rightarrow \phi, \Delta} Rc$$

The rule (*Rc*) is only allowed in multi-conclusion systems.

If we consider the logic \mathbf{FL}_e and add the weakening rules (contraction rules), the resulted system is called \mathbf{FL}_{ew} (\mathbf{FL}_{ec}). The same also goes for \mathbf{CFL}_{ew} and \mathbf{CFL}_{ec} .

We also have the following rule:

Context-sharing left implication:

$$\frac{\Gamma \Rightarrow \phi \quad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \phi \rightarrow \psi \Rightarrow \Delta}$$

Finally, note that Γ and Δ are multisets everywhere, therefore the exchange rule is built in and hence admissible in our system. Moreover, note that the calculi defined in this section are written in the given language which can be any extension of the language of the system itself. For instance, \mathbf{FL}_e is the calculus with the mentioned rules on our fixed language that can have more connectives than $\{\wedge, \vee, *, \rightarrow, \top, \perp, 1, 0\}$.

Definition 2.3. Let L and L' be two logics such that $\mathcal{L}_L \subseteq \mathcal{L}_{L'}$. We say L' is an *extension* of L if $L \vdash A$ implies $L' \vdash A$.

Definition 2.4. Let G and H be two sequent calculi such that $\mathcal{L}_G \subseteq \mathcal{L}_H$. We say H is an *extension* of G if $G \vdash \Gamma \Rightarrow \Delta$ implies $H \vdash \Gamma \Rightarrow \Delta$. It is called an *axiomatic extension*, if the provable sequents in G are considered as axioms of H , to which H adds some rules.

Definition 2.5. Let G be a sequent calculus and L be a logic with the same language as G 's. We say G is a *sequent calculus for the logic L* when:

$$G \vdash \Gamma \Rightarrow \Delta \quad \text{if and only if} \quad L \vdash (*\Gamma \rightarrow +\Delta).$$

Note that if the calculus is single-conclusion, by $+\Delta$, we mean Δ if Δ is a singleton, and 0 if Δ is empty. Therefore, in this case we do not need the $+$ operator.

Theorem 2.6. Let L be a logic and G a single-conclusion (multi-conclusion) sequent calculus for L . If L extends \mathbf{FL}_e (\mathbf{CFL}_e), then cut is admissible in G .

Proof. Assume that $G \vdash \Gamma \Rightarrow A, \Delta$ and $G \vdash \Gamma', A \Rightarrow \Delta'$. Hence $L \vdash * \Gamma \rightarrow A + (\dashv \Delta)$ and $L \vdash (* \Gamma') * A \rightarrow (\dashv \Delta')$. Since L extends \mathbf{FL}_e (\mathbf{CFL}_e) and in this theory the formula

$$[* \Gamma \rightarrow A + (\dashv \Delta)] * [(* \Gamma') * A \rightarrow (\dashv \Delta')]$$

implies the formula

$$[(* \Gamma) * (* \Gamma') \rightarrow (\dashv \Delta) + (\dashv \Delta')]$$

the last formula is provable in L which implies $G \vdash \Gamma, \Gamma' \Rightarrow \Delta, \Delta'$. \square

2.2 Logics

In this section we will recall the Craig interpolation property and also some useful substructural logics that we will need in the rest of the paper.

Definition 2.7. We say a logic L has Craig interpolation property if for any formulas ϕ and ψ if $L \vdash \phi \rightarrow \psi$, then there exists formula θ such that $L \vdash \phi \rightarrow \theta$ and $L \vdash \theta \rightarrow \psi$ and $V(\theta) \subseteq V(\phi) \cap V(\psi)$.

To recall some of the well known substructural logics and following [7], we have to introduce the semantical framework, first.

By a *pointed commutative residuated lattice* we mean an algebraic structure $\mathbf{A} = \langle A, \wedge, \vee, *, \rightarrow, 0, 1 \rangle$ with binary operations $\wedge, \vee, *, \rightarrow$, and constants $0, 1$ such that $\langle A, \wedge, \vee \rangle$ is a lattice with order \leq , $\langle A, *, 1 \rangle$ is a commutative monoid, and $x * y \leq z$ if and only if $x \leq y \rightarrow z$ for all $x, y, z \in A$.

For a single pointed commutative residuated lattice \mathbf{A} and a class of pointed commutative residuated lattices \mathbf{K} , denote $\mathcal{V}(\mathbf{A})$ and $\mathcal{V}(\mathbf{K})$ as the varieties generated by \mathbf{A} and \mathbf{K} , respectively.

In the following we will borrow the definition of some logics and also two tables directly from [7]. Table (1) gives a list of some important equational conditions for pointed commutative residuated lattices. And table (2) defines some of the logics that we are interested in. Since, in all the cases, both of the axioms (*prl*) and (*dis*) are present, we just mention the other axioms in the table.

For $n > 1$ define

$$L_n = \left\{ 0, \frac{1}{n-1}, \dots, \frac{n-2}{n-1}, 1 \right\}, \quad L_\infty = [0, 1]$$

Table 1: Equational conditions for pointed commutative residuated lattices.

Label	Name	Condition
(prl)	prelinearity	$1 \leq (x \rightarrow y) \vee (y \rightarrow x)$
(dis)	distributivity	$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$
(inv)	involutivity	$\neg\neg x = x$
(int)	integrality	$x \leq 1$
(bd)	boundedness	$0 \leq x$
(id)	idempotence	$x = x * x$
(fp)	fixed point negation	$0 = 1$
(div)	divisibility	$x * (x \rightarrow y) = y * (y \rightarrow x)$
(can)	cancellation	$x \rightarrow (x * y) = y$
(rcan)	restricted cancellation	$1 = \neg x \vee ((x \rightarrow (x * y)) \rightarrow y)$
(nc)	non-contradiction	$x \wedge \neg x \leq 0$

and the pointed commutative residuated lattices (again for $n > 1$)

$$\mathbf{L}_n = \langle L_n, \min, \max, *_L, \rightarrow_L, 1, 0 \rangle$$

and

$$\mathbf{G}_n = \langle L_n, \min, \max, *_G, \rightarrow_G, 1, 0 \rangle$$

where $x *_L y = \max(0, x + y - 1)$, $x \rightarrow_L y = \min(1, 1 - x + y)$, and $x \rightarrow_G y$ is y if $x > y$, otherwise 1. Then, for $n > 1$, \mathbf{L}_n and \mathbf{G}_n are the logics with equivalent algebraic semantics $\mathcal{V}(\mathbf{L}_n)$ and $\mathcal{V}(\mathbf{G}_n)$, respectively.

Define

$$\mathbf{P} = \langle [0, 1], \min, \max, *_P, \rightarrow_P, 1, 0 \rangle$$

$$\mathbf{CHL} = \langle (0, 1], \min, \max, *_P, \rightarrow_P, 1, 1 \rangle$$

where $*_P$ is the ordinary multiplication and $x \rightarrow_P y = y/x$ if $x > y$, otherwise it is equal to 1. Define P and CHL as the logics of $\mathcal{V}(\mathbf{P})$ and $\mathcal{V}(\mathbf{CHL})$, respectively.

R is the logic of a variety consisting of all distributive pointed commutative residuated lattices with the condition that $x * x \leq x$ for all x .

Now consider the following binary functions on the set of integers \mathbb{Z} , where \wedge and \vee are min and max, respectively, and $|x|$ is the absolute value

Table 2: Some semilinear logics and their equivalent algebraic semantics.

Label	Logic	Conditions
UL^-	unbounded uninorm logic	
IUL^-	unbounded involutive uninorm logic	(inv)
MTL	monoidal t-norm logic	$(int), (bd)$
$SMTL$	strict monoidal t-norm logic	$(int), (bd), (nc)$
$IMTL$	involutive monoidal t-norm logic	$(int), (bd), (inv)$
BL	basic fuzzy logic	$(int), (bd), (div)$
G	Gödel logic	$(int), (bd), (id)$
\mathbb{L}	Lukasiewicz logic	$(int), (bd), (div), (inv)$
P	product logic	$(int), (bd), (div), (rcan)$
CHL	cancellative hoop logic	$(int), (fp), (div), (can)$
UML^-	unbounded uninorm mingle logic	(id)
RM^e	R -mingle with unit	$(id), (inv)$
$IUML^-$	unbounded involutive uninorm mingle logic	$(id), (inv), (fp)$
A	abelian logic	$(inv), (fp), (can)$

of x :

$$x * y = \begin{cases} x \wedge y & \text{if } |x| = |y| \\ y & \text{if } |x| < |y| \\ x & \text{if } |y| < |x| \end{cases} \quad x \rightarrow y = \begin{cases} -(x) \vee y & \text{if } x \leq y \\ -(x) \wedge y & \text{otherwise} \end{cases}$$

And finally define the following algebras:

$$\mathbf{S}_{2m} = \langle \{-m, -m+1, \dots, -1, 1, \dots, m-1, m\}, \wedge, \vee, *, \rightarrow, 1, -1 \rangle \quad (m \geq 1)$$

$$\mathbf{S}_{2m+1} = \langle \{-m, -m+1, \dots, -1, 0, 1, \dots, m-1, m\}, \wedge, \vee, *, \rightarrow, 0, 0 \rangle \quad (m \geq 0)$$

and define RM_n^e as the logic of $\mathcal{V}(\mathbf{S}_n)$.

3 Semi-analytic Rules

In this section we will introduce a class of rules which we will investigate in the rest of this paper. First let us begin with the single-conclusion case in which all sequents have at most one succedent.

Definition 3.1. A rule is called a *left semi-analytic rule* if it is of the form

$$\frac{\langle \langle \Pi_j, \bar{\psi}_{js} \Rightarrow \bar{\theta}_{js} \rangle_s \rangle_j \quad \langle \langle \Gamma_i, \bar{\phi}_{ir} \Rightarrow \Delta_i \rangle_r \rangle_i}{\Pi_1, \dots, \Pi_m, \Gamma_1, \dots, \Gamma_n, \phi \Rightarrow \Delta_1, \dots, \Delta_n}$$

where Π_j , Γ_i and Δ_i 's are meta-multiset variables and

$$\bigcup_{i,r} V(\bar{\phi}_{ir}) \cup \bigcup_{j,s} V(\bar{\psi}_{js}) \cup \bigcup_{j,s} V(\bar{\theta}_{js}) \subseteq V(\phi)$$

and it is called a *right semi-analytic rule* if it is of the form

$$\frac{\langle\langle\Gamma_i, \bar{\phi}_{ir} \Rightarrow \bar{\psi}_{ir}\rangle_r\rangle_i}{\Gamma_1, \dots, \Gamma_n \Rightarrow \phi}$$

where Γ_i 's are meta-multiset variables and

$$\bigcup_{i,r} V(\bar{\phi}_{ir}) \cup \bigcup_{i,r} V(\bar{\psi}_{ir}) \subseteq V(\phi)$$

Moreover, a rule is called a *context-sharing semi-analytic rule* if it is of the form

$$\frac{\langle\langle\Gamma_i, \bar{\psi}_{is} \Rightarrow \bar{\theta}_{is}\rangle_s\rangle_i \quad \langle\langle\Gamma_i, \bar{\phi}_{ir} \Rightarrow \Delta_i\rangle_r\rangle_i}{\Gamma_1, \dots, \Gamma_n, \phi \Rightarrow \Delta_1, \dots, \Delta_n}$$

where Γ_i and Δ_i 's are meta-multiset variables and

$$\bigcup_{i,r} V(\bar{\phi}_{ir}) \cup \bigcup_{i,s} V(\bar{\psi}_{is}) \cup \bigcup_{i,s} V(\bar{\theta}_{is}) \subseteq V(\phi)$$

We will call the conditions for the variables in all the semi-analytic rules, the *occurrence preserving* conditions.

Note that in the left rule, for each i we have $|\Delta_i| \leq 1$, and since the size of the succedent of the conclusion of the rule must be at most 1, it means that at most one of Δ_i 's can be non-empty.

For the multi-conclusion case, we define a rule to be *left multi-conclusion semi-analytic* if it is of the form

$$\frac{\langle\langle\Gamma_i, \bar{\phi}_{ir} \Rightarrow \bar{\psi}_{ir}, \Delta_i\rangle_r\rangle_i}{\Gamma_1, \dots, \Gamma_n, \phi \Rightarrow \Delta_1, \dots, \Delta_n}$$

with the same occurrence preserving condition as above and the same condition that all Γ_i 's and Δ_i 's are meta-multiset variables. A rule is defined to be a *right multi-conclusion semi-analytic rule* if it is of the form

$$\frac{\langle\langle\Gamma_i, \bar{\phi}_{ir} \Rightarrow \bar{\psi}_{ir}, \Delta_i\rangle_r\rangle_i}{\Gamma_1, \dots, \Gamma_n \Rightarrow \phi, \Delta_1, \dots, \Delta_n}$$

again with the similar occurrence preserving condition and the same condition that all Γ_i 's and Δ_i 's are meta-multiset variables. Whenever it is clear from the context, we will omit the phrase “multi-conclusion”.

A rule is called *modal semi-analytic* if it has one of the following forms:

$$\frac{\Gamma \Rightarrow \phi}{\Box\Gamma \Rightarrow \Box\phi} K \quad \frac{\Gamma \Rightarrow \phi}{\Box\Gamma \Rightarrow \phi} D \quad \frac{\Box\Gamma \Rightarrow \phi}{\Box\Gamma \Rightarrow \Box\phi} RS4$$

with the conditions that first, Γ is a meta-multiset variable, secondly whenever the rule (D) is present, the rule (K) must be present, and thirdly, whenever the rule ($RS4$) is present in a system, the following rule, ($LS4$), must be present, as well:

$$\frac{\Gamma, \phi \Rightarrow \Delta}{\Gamma, \Box\phi \Rightarrow \Delta} LS4$$

where Γ and Δ are both meta-multiset variables. In the case of the modal rules, we use the convention that $\Box\emptyset = \emptyset$.

Moreover, consider the following modal rules that we do *not* consider as semi-analytic but we will address in our investigations. Like the previous case, we assume that whenever the rule ($4D$) is present in a system the modal rule (4) must be present, as well:

$$\frac{\Box\Gamma, \Gamma \Rightarrow \phi}{\Box\Gamma \Rightarrow \Box\phi} 4 \quad \frac{\Box\Gamma, \Gamma \Rightarrow \phi}{\Box\Gamma \Rightarrow \phi} 4D$$

where Γ is a meta-multiset variable. By the notation $\langle\langle\cdot\rangle_r\rangle_i$ we mean first considering the sequents ranging over r and then ranging over i . For instance, $\langle\langle\Gamma_i, \bar{\phi}_{ir} \Rightarrow \bar{\psi}_{ir}\rangle_r\rangle_i$ is short for the following set of sequents where $1 \leq r \leq m_i$ and $1 \leq i \leq n$:

$$\begin{aligned} \Gamma_1, \bar{\phi}_{11} \Rightarrow \bar{\psi}_{11}, \dots, \Gamma_1, \bar{\phi}_{1m_1} \Rightarrow \bar{\psi}_{1m_1}, \\ \Gamma_2, \bar{\phi}_{21} \Rightarrow \bar{\psi}_{21}, \dots, \Gamma_2, \bar{\phi}_{2m_2} \Rightarrow \bar{\psi}_{2m_2}, \\ \vdots \\ \Gamma_n, \bar{\phi}_{n1} \Rightarrow \bar{\psi}_{n1}, \dots, \Gamma_n, \bar{\phi}_{nm_n} \Rightarrow \bar{\psi}_{nm_n}. \end{aligned}$$

$\langle\langle\Gamma_i, \bar{\phi}_{ir} \Rightarrow \Delta_i\rangle_r\rangle_i$ and $\langle\langle\Pi_j, \bar{\psi}_{js} \Rightarrow \bar{\theta}_{js}\rangle_s\rangle_j$ are defined similarly.

Both in the single-conclusion and multi-conclusion case, a rule is called *semi-analytic*, if it is either a left semi-analytic rule, a right semi-analytic rule

or it is of the form of a semi-analytic modal rule. In all the semi-analytic rules, the meta-variables and atomic constants occurring in the meta-formulas of the premises of the rule, should also occur in the meta-formulas in the consequence. Because of this condition, we call these rules semi-analytic. This occurrence preserving condition is a weaker version of the analyticity property in the analytic rules, which demands the formulas in the premises to be sub-formulas of the formulas in the consequence.

Example 3.2. A generic example of a left semi-analytic rule is the following:

$$\frac{\Gamma, \phi_1, \phi_2 \Rightarrow \psi \quad \Gamma, \theta \Rightarrow \eta \quad \Pi, \mu_1, \mu_2, \mu_3 \Rightarrow \Delta}{\Gamma, \Pi, \alpha \Rightarrow \Delta}$$

where

$$V(\phi_1, \phi_2, \psi, \theta, \eta, \mu_1, \mu_2, \mu_3) \subseteq V(\alpha)$$

and a generic example of a context-sharing left semi-analytic rule is:

$$\frac{\Gamma, \theta \Rightarrow \eta \quad \Gamma, \mu_1, \mu_2, \mu_3 \Rightarrow \Delta}{\Gamma, \alpha \Rightarrow \Delta}$$

where

$$V(\theta, \eta, \mu_1, \mu_2, \mu_3) \subseteq V(\alpha)$$

Moreover, for a generic example of a right semi-analytic rule we can have

$$\frac{\Gamma, \phi \Rightarrow \psi \quad \Gamma, \theta_1, \theta_2 \Rightarrow \eta \quad \Pi, \mu_1, \mu_2, \Rightarrow \nu}{\Gamma, \Pi \Rightarrow \alpha}$$

where

$$V(\phi, \psi, \theta_1, \theta_2, \eta, \mu_1, \mu_2, \nu) \subseteq V(\alpha)$$

Here are some remarks. First note that in any left semi-analytic rule there are two types of premises; the type whose right hand-side includes meta-multi variables and the type whose right hand-side includes meta-formulas. This is a crucial point to consider. Any left semi-analytic rule allows any kind of combination of sharing/combining contexts in any type. However, between two types, we can only combine the contexts. The case in which we can share the contexts of the two types is called context-sharing semi-analytic rule. This should explain why our second example is called context-sharing left semi-analytic while the first one is not. The reason is the fact that the two types share the same context in the second rule while in the first one this situation happens in just one type. The second point is the presence of contexts. This is very crucial for almost all the arguments in this paper, that any sequent present in a semi-analytic rule should have meta-multiset variables as left contexts and in the case of left rules, at least one meta-multiset variable for the right hand-side must be present.

Example 3.3. Now for more concrete examples, note that all the usual conjunction, disjunction and implication rules for **IPC** are semi-analytic. The same also holds for all the rules in sub-structural logic **FL_e**, the weakening and the contraction rules and some of the well-known restricted versions of them including the following rules for exponentials in linear logic:

$$\frac{\Gamma, !\phi, !\phi \Rightarrow \Delta}{\Gamma, !\phi \Rightarrow \Delta} \quad \frac{\Gamma \Rightarrow \Delta}{\Gamma, !\phi \Rightarrow \Delta}$$

For a context-sharing semi-analytic rule, consider the following rule in the Dyckhoff calculus for **IPC** (see [3]):

$$\frac{\Gamma, \psi \rightarrow \gamma \Rightarrow \phi \rightarrow \psi \quad \Gamma, \gamma \Rightarrow \Delta}{\Gamma, (\phi \rightarrow \psi) \rightarrow \gamma \Rightarrow \Delta}$$

Example 3.4. For a concrete non-example consider the cut rule; it is not semi-analytic because it does not preserve the variable occurrence condition. Moreover, the following rule in the calculus of **KC**:

$$\frac{\Gamma, \phi \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \phi \rightarrow \psi, \Delta}$$

in which Δ should consist of negation formulas is not a multi-conclusion semi-analytic rule, simply because the context is not free for all possible substitutions. The rule of thumb is that any rule in which we have *side conditions on the contexts* is not semi-analytic.

Definition 3.5. A sequent is called a *centered axiom* if it has the following form:

- (1) Identity axiom: $(\phi \Rightarrow \phi)$
- (2) Context-free right axiom: $(\Rightarrow \bar{\alpha})$
- (3) Context-free left axiom: $(\bar{\beta} \Rightarrow)$
- (4) Contextual left axiom: $(\Gamma, \bar{\phi} \Rightarrow \Delta)$
- (5) Contextual right axiom: $(\Gamma \Rightarrow \bar{\phi}, \Delta)$

where Γ and Δ are meta-multiset variables and in (2) the variables in any pair of elements in $\bar{\alpha}$ are equal. The same condition also holds for any pair of elements in $\bar{\beta}$ in (3) or in $\bar{\phi}$ in (4) and (5). A sequent is called context-free centered axiom if it has the form (1), (2) or (3).

Example 3.6. It is easy to see that the axioms given in the preliminaries are examples of centered axioms. Here are some more examples:

$$\begin{aligned}
& \neg 1 \Rightarrow \quad , \quad \Rightarrow \neg 0 \\
& \phi, \neg \phi \Rightarrow \quad , \quad \Rightarrow \phi, \neg \phi \\
& \Gamma, \neg \top \Rightarrow \Delta \quad , \quad \Gamma \Rightarrow \Delta, \neg \perp
\end{aligned}$$

where the first four are context-free while the last two are contextual.

4 Interpolation

In this section we will investigate the relationship between the semi-analytic rules and the Craig interpolation property. Apart from its clear use in proving interpolation for different logics, it has a very interesting application to show that some of the natural sub-structural and super-intuitionistic logics can not have a calculus consisting only of semi-analytic rules and the centered axioms.

First, let us define the interpolation property for a sequent calculus.

Definition 4.1. (*essentially Maehara*) Let G and H be sequent calculi. G has H -interpolation if for any sequent $S = (\Sigma, \Lambda \Rightarrow \Delta)$ if S is provable in G by a tree-like proof π , then there exists a formula C such that $(\Sigma \Rightarrow C)$ and $(\Lambda, C \Rightarrow \Delta)$ are provable in H and $V(C) \subseteq V(\Sigma) \cap V(\Lambda \cup \Delta)$, where $V(A)$ is the set of the atoms of A . We say G has H -feasible interpolation if we also have the bound $|C| \leq |\pi|^{O(1)}$.

Moreover, we say G has *strong H -interpolation* if for any sequent $S = (\Sigma, \Lambda \Rightarrow \Theta, \Delta)$ if S is provable in G by a tree-like proof π , then there exists a formula C such that $(\Sigma \Rightarrow C, \Theta)$ and $(\Lambda, C \Rightarrow \Delta)$ are provable in H and $V(C) \subseteq V(\Sigma \cup \Theta) \cap V(\Lambda \cup \Delta)$. We say G has *strong H -feasible interpolation* if we also have the bound $|C| \leq |\pi|^{O(1)}$.

The following theorem shows that the interpolation property of a sequent calculus leads to the Craig interpolation of its logic.

Theorem 4.2. *If a logic L has a complete sequent calculus G with the G -interpolation property, then L has Craig interpolation.*

Proof. Let $L \vdash \phi \rightarrow \psi$. Since G is complete for L , we have $G \vdash \phi \Rightarrow \psi$. Since G has the interpolation property, there exists θ such that $G \vdash \phi \Rightarrow \theta$, $G \vdash \theta \Rightarrow \psi$ and $V(\theta) \subseteq V(\phi) \cap V(\psi)$. Again from the completeness of G , $L \vdash \phi \rightarrow \theta$ and $L \vdash \theta \rightarrow \psi$ which completes the proof. \square

The following theorem ensures that any set of centered axioms of a sequent calculus H , has H -interpolation property. It can also serve as an example to show how this notion of relative interpolation works.

Theorem 4.3. *Let G and H be two sequent calculi such that every provable sequent in G is also provable in H , and let G consist of only centered (context-free centered) axioms. Then:*

- (i) *If both G and H are single-conclusion and H extends \mathbf{FL}_e (\mathbf{FL}_e^-), G has H -feasible interpolation.*
- (ii) *If both of G and H are multi-conclusion and H extends \mathbf{CFL}_e (\mathbf{CFL}_e^-), G has strong H -feasible interpolation.*

Proof. To prove (i), note that a sequent S is provable in G if it is one of the centered axioms. We will check each case separately:

- (1) In this case the sequent S is of the form $(\phi \Rightarrow \phi)$. For any partition Σ and Λ that we have $(\Sigma, \Lambda \Rightarrow \phi)$ in G , we have to find a formula C such that $(\Sigma \Rightarrow C)$ and $(\Lambda, C \Rightarrow \phi)$ are provable in H . There are two cases to consider. First, if $\Sigma = \{\phi\}$ and $\Lambda = \emptyset$. For this case define C to be ϕ . Obviously both conditions hold since we have $(\phi \Rightarrow \phi)$ as an axiom. Second, if $\Sigma = \emptyset$ and $\Lambda = \{\phi\}$ define C as 1. We must have $(\Rightarrow 1)$ and $(1, \phi \Rightarrow \phi)$ in H . The first one is an axiom of G and hence provable in H , and the second is the consequence of an instance of the rule (L1) and the fact that $(\phi \Rightarrow \phi)$ is provable in H .
- (2) For the case $(\Rightarrow \bar{\alpha})$, consider C to be 1. Then since both Σ and Λ are empty sequents, we must have $(\Rightarrow 1)$ and $(1 \Rightarrow \bar{\alpha})$ in H . The first one is an axiom of G and hence provable in H , and the second is the consequence of an instance of the rule (L1) and the fact that $(\Rightarrow \bar{\alpha})$ is provable in H .
- (3) For the axiom $(\bar{\beta} \Rightarrow)$, there are three cases to consider:
 - (i) If $\bar{\beta} \subseteq \Lambda$. Then define $C = 1$. It is clear that $\Sigma = \emptyset$ and hence $\Sigma \Rightarrow 1$. Moreover, since we have $\Lambda = \bar{\beta}$, by the axiom and the rule (L1) we will have $\Lambda, 1 \Rightarrow$.
 - (ii) If $\bar{\beta} \subseteq \Sigma$, define $C = 0$. The reasoning is dual of the argument in (i).
 - (iii) If non of the above happens, there are at least one element in $\bar{\beta} \cap \Sigma$ and $\bar{\beta} \cap \Lambda$. Define $C = * \Sigma$. Then $\Sigma \Rightarrow C$ by (R*) and $\Lambda, C \Rightarrow$ holds by the axiom itself and (L*). For the variables, note

that if $p \in V(C)$, then p is clearly occurring in Σ . Moreover, we know that p is in one of the members in $\bar{\beta}$. Since there is at least one of $\bar{\beta}$'s in Λ and each pair of the elements of $\bar{\beta}$ have the same variables, $p \in V(\Lambda)$ which completes the proof.

(4) If S is of the form $\Gamma, \bar{\phi} \Rightarrow \Delta$, there are three cases to consider:

- (i) If $\bar{\phi} \subseteq \Lambda$. Then define $C = \top$. It is clear that $\Sigma \Rightarrow \top$. Moreover, if we substitute $\{\top\} \cup \Lambda - \bar{\phi}$ for the left context in the original axiom, we have $\top, \Lambda \Rightarrow \Delta$.
- (ii) If $\bar{\phi} \subseteq \Sigma$, define $C = \perp$. The reasoning is similar to (i).
- (iii) If non of the above happens, there are at least one element in $\bar{\phi} \cap \Sigma$ and $\bar{\phi} \cap \Lambda$. Define $C = *(\Sigma \cap \bar{\phi}) * \top^n$ where n is the cardinal of $\Sigma - \Sigma \cap \bar{\phi}$. First we have $\Sigma \Rightarrow C$, simply because for any $\phi_i \in \Sigma \cap \bar{\phi}$, $\phi_i \Rightarrow \phi_i$ and for any $\psi \in \Sigma - \Sigma \cap \bar{\phi}$ we have $\psi \Rightarrow \top$, and at the end we use the rule (R^*) . Secondly, $\Lambda, C \Rightarrow \Delta$. The reason is that the part of $\bar{\phi}$ which is occurred in Σ (and now in C) together with the part of $\bar{\phi}$ in Λ completes $\bar{\phi}$. Therefore, the left hand-side of $\Lambda, C \Rightarrow \Delta$ contains $\bar{\phi}$ and hence, the sequent is an instance of the axiom and it is valid. Finally, for the variables, note that if $p \in V(C)$ then p is clearly occurring in Σ . Moreover, p is in one of the members in $\bar{\phi}$. Since there is at least one of $\bar{\phi}$'s in Λ and each pair of the elements of $\bar{\phi}$ have the same variables, $p \in V(\Lambda)$ which completes the proof.

(5) If S is of the form $(\Gamma \Rightarrow \bar{\phi}, \Delta)$ define $C = \top$. Note that $\Sigma \Rightarrow \top$ is valid on the one hand and $C, \Lambda \Rightarrow \bar{\phi}, \Delta$ on the other. The latter is an instance of the axiom itself and hence valid.

It is easy to check that in each case the length of C is bounded by the length of the sequent itself. For instance in case (4)(iii), the length of $*(\Sigma \cap \bar{\phi}) * \top^n$ is bounded by the length of Σ which is bounded by the length of $(\Gamma, \bar{\phi})$ in the sequent $\Gamma, \bar{\phi} \Rightarrow \Delta$. Hence C is polynomially bounded. Proving (ii) is similar. \square

Now we are ready to prove that semi-analytic rules respect the interpolation property. More precisely:

Theorem 4.4. *Let G and H be two sequent calculi such that H extends \mathbf{FL}_e .*

- (i) *Suppose H is an axiomatic extension of G with semi-analytic rules, and in the case that we add also the modal rule 4 or 4D in H , the left weakening rule for boxed formulas is admissible in H . Then if G has H -interpolation (H -feasible interpolation), so does H .*

(ii) Suppose H is an axiomatic extension of G with semi-analytic rules and context-sharing semi-analytic rules and moreover, the rules left weakening and right weakening and left context-sharing implication are admissible in H . Then if G has H -interpolation (H -feasible interpolation), so does H .

Proof. First we prove the interpolation property and then we will investigate the feasibility case. The proof uses induction on the H -length of π (note that by the H -length we mean counting just the new rules that H adds to the provable sequents in G that H considers as axioms). For the zero H -length, the proof is in G and the existence of the interpolation is proved by the assumption. For the rest, we will consider the last rule used in the proof and there are several cases to investigate.

First we will prove (i).

- Consider the case where the last rule used in the proof is a left semi-analytic rule and the main formula, ϕ , is in Λ in the Definition 4.1 (or informally, ϕ appears in the same sequent as Δ appears). Hence, the sequent S is of the form $(\Gamma', \Gamma'', \Pi', \Pi'', \phi \Rightarrow \Delta)$ and we have to find a formula C that satisfies $(\Gamma', \Pi' \Rightarrow C)$ and $(\Gamma'', \Pi'', \phi, C \Rightarrow \Delta)$, where $\Sigma = \{\Gamma', \Pi'\}$ and $\Lambda = \{\Gamma'', \Pi'', \phi\}$. Therefore, we must have had the following instance of the rule

$$\frac{\langle\langle\Pi'_j, \Pi''_j, \bar{\psi}_{js} \Rightarrow \bar{\theta}_{js}\rangle_s\rangle_j \quad \langle\langle\Gamma'_i, \Gamma''_i, \bar{\phi}_{ir} \Rightarrow \Delta_i\rangle_r\rangle_i}{\Pi', \Pi'', \Gamma', \Gamma'', \phi \Rightarrow \Delta}$$

Using the induction hypothesis for the premises we have

$$\Pi'_j \Rightarrow C_{js} \quad , \quad \Pi''_j, \bar{\psi}_{js}, C_{js} \Rightarrow \bar{\theta}_{js}$$

$$\Gamma'_i \Rightarrow D_{ir} \quad , \quad \Gamma''_i, \bar{\phi}_{ir}, D_{ir} \Rightarrow \Delta_i$$

Using the rules $(R\wedge)$ and $(L\wedge)$ we have

$$\Pi'_j \Rightarrow \bigwedge_s C_{js} \quad , \quad \Pi''_j, \bar{\psi}_{js}, \bigwedge_s C_{js} \Rightarrow \bar{\theta}_{js}$$

$$\Gamma'_i \Rightarrow \bigwedge_r D_{ir} \quad , \quad \Gamma''_i, \bar{\phi}_{ir}, \bigwedge_r D_{ir} \Rightarrow \Delta_i$$

For the left sequents, using the rule (R^*) we have

$$\Pi', \Gamma' \Rightarrow \left(\bigwedge_j^* C_{js} \right) * \left(\bigwedge_r^* D_{ir} \right)$$

And if we substitute the left sequents in the original rule and using the rule (L^*), we conclude

$$\Pi'', \Gamma'', \left(\bigwedge_j^* C_{js} \right) * \left(\bigwedge_r^* D_{ir} \right), \phi \Rightarrow \Delta$$

Therefore, we let C be $\left(\bigwedge_j^* C_{js} \right) * \left(\bigwedge_r^* D_{ir} \right)$ and we have proved $(\Gamma', \Pi' \Rightarrow C)$ and $(\Gamma'', \Pi'', \phi, C \Rightarrow \Delta)$.

To check $V(C) \subseteq V(\Sigma) \cap V(\Lambda \cup \Delta)$, note that an atom is in C if and only if it is in one of C_{js} or D_{ir} . If it is in C_{js} , by induction hypothesis, it is either in Π'_j (which means it is in Σ), or it is in $\{\Pi''_j, \bar{\psi}_{js}, \bar{\theta}_{js}\}$. If it is in Π''_j , then it is in Λ and if it is in either $\bar{\psi}_{js}$ or $\bar{\theta}_{js}$, since the rule is occurrence preserving, it also appears in ϕ which means it appears in Λ .

If the atom is in D_{ir} , we reason in the similar way, and it either appears in Γ'_i (and hence in Σ) or it appears in $\{\Gamma''_i, \bar{\phi}_{ir}, \Delta_i\}$ and hence in $\Lambda \cup \Delta$.

- Consider the case where the last rule used in the proof is a left semi-analytic rule and the main formula, ϕ , is this time in Σ in the Definition 4.1. Hence, the sequent S is again of the form $(\Gamma', \Gamma'', \Pi', \Pi'', \phi \Rightarrow \Delta)$ and we have to find a formula C that satisfies $(\Gamma', \Pi', \phi \Rightarrow C)$ and $(\Gamma'', \Pi'', C \Rightarrow \Delta)$, where $\Sigma = \{\Gamma', \Pi', \phi\}$ and $\Lambda = \{\Gamma'', \Pi''\}$. W.l.o.g. suppose that for $i \neq 1$ we have $\Delta_i = \emptyset$ and $\Delta_1 = \Delta$. Therefore, we must have had the following instance of the rule

$$\frac{\langle \langle \Pi'_j, \Pi''_j, \bar{\psi}_{js} \Rightarrow \bar{\theta}_{js} \rangle_s \rangle_j \quad \langle \langle \Gamma'_i, \Gamma''_i, \bar{\phi}_{ir} \Rightarrow \rangle_r \rangle_{i \neq 1} \quad \langle \Gamma'_1, \Gamma''_1, \bar{\phi}_{1r} \Rightarrow \Delta \rangle_r}{\Pi', \Pi'', \Gamma', \Gamma'', \phi \Rightarrow \Delta}$$

Using the induction hypothesis for the premises we have (for $i \neq 1$)

$$\Pi'_j, \bar{\psi}_{js}, C_{js} \Rightarrow \bar{\theta}_{js} \quad , \quad \Pi''_j \Rightarrow C_{js}$$

$$\Gamma'_i, \bar{\phi}_{ir}, D_{ir} \Rightarrow \quad , \quad \Gamma''_i \Rightarrow D_{ir}$$

$$\Gamma'_1, \bar{\phi}_{1r} \Rightarrow D_{1r} \quad , \quad \Gamma''_1, D_{1r} \Rightarrow \Delta$$

Using the rules $(L\wedge)$, $(R\wedge)$, $(R\vee)$ and $(L\vee)$, we have (for $i \neq 1$)

$$\Pi'_j, \bar{\psi}_{js}, \bigwedge_s C_{js} \Rightarrow \bar{\theta}_{js} \quad , \quad \Pi''_j \Rightarrow \bigwedge_s C_{js}$$

$$\Gamma'_i, \bar{\phi}_{ir}, \bigwedge_r D_{ir} \Rightarrow \quad , \quad \Gamma''_i \Rightarrow \bigwedge_r D_{ir}$$

$$\Gamma'_1, \bar{\phi}_{1r} \Rightarrow \bigvee_r D_{1r} \quad , \quad \Gamma''_1, \bigvee_r D_{1r} \Rightarrow \Delta$$

If we substitute the left sequents in the original rule, we get (for $i \neq 1$)

$$\Pi', \Gamma', \bigwedge_s C_{js}, \bigwedge_r D_{ir}, \phi \Rightarrow \bigvee_r D_{1r}$$

First, using the rule $(L*)$ and then $(R \rightarrow)$ we get

$$\Pi', \Gamma', \phi \Rightarrow \left(\bigstar_{i \neq 1} \bigwedge_r D_{ir} \right) * \left(\bigstar_j \bigwedge_s C_{js} \right) \rightarrow \bigvee_r D_{1r}$$

On the other hand, using the rules $(R*)$ and $(L \rightarrow)$ for the right sequents we have

$$\Pi'', \Gamma'', \left(\bigstar_{i \neq 1} \bigwedge_r D_{ir} \right) * \left(\bigstar_j \bigwedge_s C_{js} \right) \rightarrow \bigvee_r D_{1r} \Rightarrow \Delta$$

It is enough to take C as $\left(\bigstar_{i \neq 1} \bigwedge_r D_{ir} \right) * \left(\bigstar_j \bigwedge_s C_{js} \right) \rightarrow \bigvee_r D_{1r}$ to finish the proof of this case.

To check $V(C) \subseteq V(\Sigma) \cap V(\Lambda \cup \Delta)$, note that an atom is in C if and only if it is either in one of C_{js} or D_{ir} for $(i \neq 1)$ or in D_{1r} . By induction hypothesis if it is in C_{js} , it is both in $\{\Pi'_j, \bar{\psi}_{js}, \bar{\theta}_{js}\}$ and in Π''_j . If it is in D_{ir} for $(i \neq 1)$, then it is both in $\{\Gamma'_i, \bar{\phi}_{ir}\}$ and in Γ''_i . And if it is in D_{1r} , then it is both in $\{\Gamma'_1, \bar{\phi}_{1r}\}$ and in $\{\Gamma''_1, \Delta\}$. One can easily check that therefore, the atom will be both in $\Sigma = \{\Gamma', \Pi', \phi\}$ and in $\Lambda \cup \Delta = \{\Gamma'', \Pi'', \Delta\}$. Note that in the reasoning we will need the occurrence preserving property, as well.

- Consider the case where the last rule used in the proof is a right semi-analytic rule. Hence, the sequent S is of the form $(\Gamma', \Gamma'' \Rightarrow \phi)$ and we have to find a formula C that satisfies $(\Gamma'' \Rightarrow C)$ and $(\Gamma', C \Rightarrow \phi)$, where $\Sigma = \Gamma''$, $\Lambda = \Gamma'$ and $\Delta = \phi$. Therefore, we must have had the following instance of the rule

$$\frac{\langle\langle\Gamma'_i, \Gamma''_i, \bar{\phi}_{ir} \Rightarrow \bar{\psi}_{ir}\rangle_r\rangle_i}{\Gamma', \Gamma'' \Rightarrow \phi}$$

Using the induction hypothesis we get

$$\Gamma'_i, C_{ir}, \bar{\phi}_{ir} \Rightarrow \bar{\psi}_{ir} \quad , \quad \Gamma''_i \Rightarrow C_{ir}$$

Using the rules $(L\wedge)$ and $(R\wedge)$ we have

$$\Gamma'_i, \bigwedge_r C_{ir}, \bar{\phi}_{ir} \Rightarrow \bar{\psi}_{ir} \quad , \quad \Gamma''_i \Rightarrow \bigwedge_r C_{ir}$$

Substituting the left sequent in the original rule and then using the rule $(L*)$, we conclude

$$\Gamma', *_{i} \left(\bigwedge_r C_{ir} \right) \Rightarrow \phi.$$

On the other hand, using the rule $(R*)$ for the sequents $\Gamma''_i \Rightarrow \bigwedge_r C_{ir}$, we get $\Gamma'' \Rightarrow *_{i} \left(\bigwedge_r C_{ir} \right)$ which means that the sequent $*_{i} \left(\bigwedge_r C_{ir} \right)$ serves as the formula C .

To check $V(C) \subseteq V(\Sigma) \cap V(\Lambda \cup \Delta)$, note that an atom is in C if and only if it is either in one of C_{ir} . Then by induction hypothesis it is both in $\{\Gamma'_i, \bar{\phi}_{ir}, \bar{\psi}_{ir}\}$ and in Γ''_i . It is easy to check that it meets the conditions needed.

- And finally, consider the case where the last rule used in the proof is a modal rule. We will investigate K and D together first, and second 4 and $4D$ together, and at last, we will investigate the rule $RS4$.

Consider the case where the last rule used in the proof is either K or D . Then, the sequent S is of the form $\Box\Gamma', \Box\Gamma'' \Rightarrow \Box\Delta$, where $|\Delta| \leq 1$ and we have to find a formula C that satisfies $\Box\Gamma' \Rightarrow C$ and $C, \Box\Gamma'' \Rightarrow \Box\Delta$. Therefore, we must have had the following instance of the rule

$$\frac{\Gamma', \Gamma'' \Rightarrow \Delta}{\Box\Gamma', \Box\Gamma'' \Rightarrow \Box\Delta}$$

Using the induction hypothesis there exists D such that

$$\Gamma' \Rightarrow D \quad , \quad \Gamma'', D \Rightarrow \Delta$$

Then, using the rule K for both of them (or if $\Delta = \emptyset$, use the rule D for $(\Gamma'', D \Rightarrow)$), we get

$$\Box\Gamma' \Rightarrow \Box D \quad , \quad \Box\Gamma'', \Box D \Rightarrow \Box\Delta$$

Let $\Box D$ be the formula C and we are done. And since $V(D) \subseteq V(\Gamma') \cap V(\Gamma'' \cup \Delta)$ we have $V(C) \subseteq V(\Box\Gamma') \cap V(\Box\Gamma'' \cup \Box\Delta)$, because the set of atoms of $\Box\Pi$ for a multiset Π is the same as atoms in Π .

Now, consider the case that the last rule used in the proof is 4. Then, the sequent S is of the form $\Box\Gamma', \Box\Gamma'' \Rightarrow \Box\phi$, and we have to find a formula C that satisfies $\Box\Gamma' \Rightarrow C$ and $C, \Box\Gamma'' \Rightarrow \Box\phi$. Therefore, we must have had the following instance of the rule

$$\frac{\Gamma', \Gamma'', \Box\Gamma', \Box\Gamma'' \Rightarrow \phi}{\Box\Gamma', \Box\Gamma'' \Rightarrow \Box\phi}$$

Using the induction hypothesis there exists D such that

$$\Gamma', \Box\Gamma' \Rightarrow D \quad , \quad \Gamma'', \Box\Gamma'', D \Rightarrow \phi$$

If we use the rule 4 on the left sequent and using the left weakening rule on the right sequent (adding $\Box D$ to the left hand side of the sequent) and then using the rule 4, we get

$$\Box\Gamma' \Rightarrow \Box D \quad , \quad \Box\Gamma'', \Box D \Rightarrow \Box\phi$$

If we take $C = \Box D$, then the claim follows. Checking the atoms is similar as before.

For the proof of the case $4D$ is identical to the proof of the rule 4, if we ignore ϕ and $\Box\phi$ everywhere.

If the last rule used in the proof is the rule $RS4$, then the sequent S is of the form $\Box\Gamma', \Box\Gamma'' \Rightarrow \Box\phi$, and we have to find a formula C that satisfies $\Box\Gamma' \Rightarrow C$ and $C, \Box\Gamma'' \Rightarrow \Box\phi$. Therefore, we must have had the following instance of the rule

$$\frac{\Box\Gamma', \Box\Gamma'' \Rightarrow \phi}{\Box\Gamma', \Box\Gamma'' \Rightarrow \Box\phi}$$

Using the induction hypothesis there exists D such that

$$\Box\Gamma' \Rightarrow D \quad , \quad \Box\Gamma'', D \Rightarrow \phi$$

On the left sequent, use the rule *RS4*. On the right sequent, use the rule *LS4* (since the rule *LS4* is present in the system, whenever we have *RS4*) and then use the rule *RS4*. We get

$$\Box\Gamma' \Rightarrow \Box D \quad , \quad \Box\Gamma'', \Box D \Rightarrow \Box\phi$$

It is easy to see that $C = \Box D$ works in this case.

Now, we will prove part (ii). We have discussed the cases of left and right semi-analytic and modal rules in the previous part. It only remains to investigate the case of context-sharing semi-analytic rules.

- Consider the case where the last rule used in the proof is a context-sharing semi-analytic rule and the main formula, ϕ , is in Λ in the Definition 4.1 (or informally, ϕ appears in the same sequent as Δ appears). Hence, the sequent S is of the form $(\Gamma', \Gamma'', \phi \Rightarrow \Delta)$ and we have to find a formula C that satisfies $(\Gamma' \Rightarrow C)$ and $(\Gamma'', \phi, C \Rightarrow \Delta)$, where $\Sigma = \{\Gamma'\}$ and $\Lambda = \{\Gamma'', \phi\}$. Therefore, we must have had the following instance of the rule

$$\frac{\langle\langle\Gamma'_i, \Gamma''_i, \bar{\psi}_{is} \Rightarrow \bar{\theta}_{is}\rangle_s\rangle_i \quad \langle\langle\Gamma'_i, \Gamma''_i, \bar{\phi}_{ir} \Rightarrow \Delta_i\rangle_r\rangle_i}{\Gamma', \Gamma'', \phi \Rightarrow \Delta}$$

Using the induction hypothesis for the premises we have

$$\Gamma'_i \Rightarrow C_{is} \quad , \quad \Gamma''_i, \bar{\psi}_{is}, C_{is} \Rightarrow \bar{\theta}_{is}$$

$$\Gamma'_i \Rightarrow D_{ir} \quad , \quad \Gamma''_i, \bar{\phi}_{ir}, D_{ir} \Rightarrow \Delta_i$$

Using the rules (*R* \wedge) and (*L* \wedge) we have

$$\Gamma'_i \Rightarrow \bigwedge_s C_{is} \quad , \quad \Gamma''_i, \bar{\psi}_{is}, \bigwedge_s C_{is} \Rightarrow \bar{\theta}_{is}$$

$$\Gamma'_i \Rightarrow \bigwedge_r D_{ir} \quad , \quad \Gamma''_i, \bar{\phi}_{ir}, \bigwedge_r D_{ir} \Rightarrow \Delta_i$$

We want to make the contexts of the above sequents in the right the same, so that we can use them in the original rule. Therefore, using the rule (*L* \wedge) we have

$$\Gamma''_i, \bar{\psi}_{is}, (\bigwedge_s C_{is}) \wedge (\bigwedge_r D_{ir}) \Rightarrow \bar{\theta}_{is} \quad , \quad \Gamma''_i, \bar{\phi}_{ir}, (\bigwedge_r D_{ir}) \wedge (\bigwedge_s C_{is}) \Rightarrow \Delta_i$$

Now, we can substitute them in the original rule and conclude

$$\Gamma'', \langle (\bigwedge_r D_{ir}) \wedge (\bigwedge_s C_{is}) \rangle_i, \phi \Rightarrow \Delta$$

And using the rule (L^*) we get

$$\Gamma'', *[(\bigwedge_r D_{ir}) \wedge (\bigwedge_s C_{is})], \phi \Rightarrow \Delta$$

On the other hand, considering the sequents $(\Gamma'_i \Rightarrow \bigwedge_s C_{is})$ and $(\Gamma'_i \Rightarrow \bigwedge_r D_{ir})$ and using the rule ($R\wedge$) for every i , we get

$$\Gamma'_i \Rightarrow (\bigwedge_r D_{ir}) \wedge (\bigwedge_s C_{is})$$

and then using the rule (R^*) we have

$$\Gamma' \Rightarrow *[(\bigwedge_r D_{ir}) \wedge (\bigwedge_s C_{is})]$$

and we can see that $*[(\bigwedge_r D_{ir}) \wedge (\bigwedge_s C_{is})]$ serves as C .

To check $V(C) \subseteq V(\Sigma) \cap V(\Lambda \cup \Delta)$, note that an atom is in C if and only if it is either in one of C_{is} or D_{ir} . By induction hypothesis, if it is in C_{is} , then it is both in Γ'_i and in $\{\Gamma''_i, \bar{\psi}_{is}, \bar{\theta}_{is}\}$ and if it is in D_{ir} , then it is both in Γ'_i and in $\{\Gamma''_i, \bar{\phi}_{ir}, \Delta_i\}$. It is easy to check that it meets the conditions.

- Consider the case where the last rule used in the proof is a context-sharing semi-analytic rule and the main formula, ϕ , is this time in Σ in the Definition 4.1. Hence, the sequent S is of the form $(\Gamma', \Gamma'', \phi \Rightarrow \Delta)$ and we have to find a formula C that satisfies $(\Gamma', \phi \Rightarrow C)$ and $(\Gamma'', C \Rightarrow \Delta)$, where $\Sigma = \{\Gamma', \phi\}$ and $\Lambda = \{\Gamma''\}$. W.l.o.g. suppose that for $i \neq 1$ we have $\Delta_i = \emptyset$ and $\Delta_1 = \Delta$. Therefore, we must have had the following instance of the rule

$$\frac{\langle \langle \Gamma'_i, \Gamma''_i, \bar{\psi}_{is} \Rightarrow \bar{\theta}_{is} \rangle_s \rangle_i \quad \langle \langle \Gamma'_i, \Gamma''_i, \bar{\phi}_{ir} \Rightarrow \rangle_r \rangle_{i \neq 1} \quad \langle \Gamma'_1, \Gamma''_1, \bar{\phi}_{1r} \Rightarrow \Delta \rangle_r}{\Gamma', \Gamma'', \phi \Rightarrow \Delta}$$

Using the induction hypothesis for the premises we have (for $i \neq 1$)

$$\Gamma'_i, \bar{\psi}_{is}, C_{is} \Rightarrow \bar{\theta}_{is} \quad , \quad \Gamma''_i \Rightarrow C_{is}$$

$$\Gamma'_i, \bar{\phi}_{ir}, D_{ir} \Rightarrow \quad , \quad \Gamma''_i \Rightarrow D_{ir}$$

$$\Gamma'_1, \bar{\phi}_{1r} \Rightarrow D_{1r} \quad , \quad \Gamma''_1, D_{1r} \Rightarrow \Delta$$

Using the rules $(L\wedge)$, $(R\wedge)$, $(R\vee)$ and $(L\vee)$, we have (for $i \neq 1$)

$$\Gamma'_i, \bar{\psi}_{is}, \bigwedge_s C_{is} \Rightarrow \bar{\theta}_{is} \quad , \quad \Gamma''_i \Rightarrow \bigwedge_s C_{is}$$

$$\Gamma'_i, \bar{\phi}_{ir}, \bigwedge_r D_{ir} \Rightarrow \quad , \quad \Gamma''_i \Rightarrow \bigwedge_r D_{ir}$$

$$\Gamma'_1, \bar{\phi}_{1r} \Rightarrow \bigvee_r D_{1r} \quad , \quad \Gamma''_1, \bigvee_r D_{1r} \Rightarrow \Delta$$

$$\Gamma'_1, \bar{\psi}_{1s}, \bigwedge_s C_{1s} \Rightarrow \bar{\theta}_{1s} \quad , \quad \Gamma''_1 \Rightarrow \bigwedge_s C_{1s}$$

Now, we want to make the contexts of the sequents in the left the same, so that we can use them in the original rule. For ($i \neq 1$) use the rule $(L\wedge)$ to make the context $\{\Gamma'_i, (\bigwedge_s C_{is}) \wedge (\bigwedge_r D_{ir})\}$ and for ($i = 1$) use the left weakening rule to make the context $\{\Gamma'_1, \bigwedge_s C_{1s}\}$. If we substitute the updated left sequents in the original rule, we get (for $i \neq 1$)

$$\Gamma', \langle (\bigwedge_s C_{is}) \wedge (\bigwedge_r D_{ir}) \rangle_{i \neq 1}, \bigwedge_s C_{1s}, \phi \Rightarrow \bigvee_r D_{1r}$$

First, using the rule $(L*)$ and then $(R \rightarrow)$ we get

$$\Gamma', \phi \Rightarrow (*_{i \neq 1} [(\bigwedge_s C_{is}) \wedge (\bigwedge_r D_{ir})] * \bigwedge_s C_{1s}) \rightarrow \bigvee_r D_{1r}.$$

On the other hand, using the rule $(R\wedge)$ for every ($i \neq 1$) we have $\Gamma''_i \Rightarrow (\bigwedge_s C_{is}) \wedge (\bigwedge_r D_{ir})$. Together with the sequent $\Gamma''_1 \Rightarrow \bigwedge_s C_{1s}$, and using the rule $(R*)$ we get

$$\Gamma'' \Rightarrow (*_{i \neq 1} [(\bigwedge_s C_{is}) \wedge (\bigwedge_r D_{ir})] * \bigwedge_s C_{1s}).$$

We have $\Gamma''_1, \bigvee_r D_{1r} \Rightarrow \Delta$. Use the left weakening rule to get $\Gamma'', \bigvee_r D_{1r} \Rightarrow \Delta$. Now, we can use the rule left sharing implication to get

$$\Gamma'', (\bigstar_{i \neq 1} [(\bigwedge_s C_{is}) \wedge (\bigwedge_r D_{ir})] * \bigwedge_s C_{1s}) \rightarrow \bigvee_r D_{1r} \Rightarrow \Delta.$$

We can see that $(\bigstar_{i \neq 1} [(\bigwedge_s C_{is}) \wedge (\bigwedge_r D_{ir})] * \bigwedge_s C_{1s}) \rightarrow \bigvee_r D_{1r}$ serves as C and we are done.

To check $V(C) \subseteq V(\Sigma) \cap V(\Lambda \cup \Delta)$, note that an atom is in C if and only if it is either in one of C_{is} or D_{ir} . By induction hypothesis, if it is in C_{is} , then it is both in $\{\Gamma'_i, \bar{\psi}_{is}, \bar{\theta}_{is}\}$ and in Γ''_i and if it is in D_{ir} for ($i \neq 1$), then it is both in $\Gamma'_i, \bar{\phi}_{ir}$, and in $\{\Gamma''_i\}$. If it is in D_{1r} , then it is both in $\Gamma'_1, \bar{\phi}_{1r}$, and in $\{\Gamma''_1, \Delta\}$. It is easy to check that it meets the conditions.

It is easy to check that in both cases of (i) and (ii), if G has H -feasible interpolation, then so does H . By the assumption, we know that there exists a number m (which only depends on the proof system G) such that $|C| \leq |\pi|^m$. Now for the proofs in H we will claim that our previously constructed interpolant C has the property $|C| \leq |\pi|^M$ where $M = \max\{m, 2\}$ and we will prove it by induction on the H -length of π .

If the H -length of the proof is 0, then there is no new rule of H in the proof π , and since G has H -feasible interpolation, by Definition $|C| \leq |\pi|^m$ and hence $|C| \leq |\pi|^M$. For the rest, note that in each of the above cases, the number of the formulas which appear in C (we have shown them by C_{js} and D_{ir}) is equal to the number of premises of the last rule used in the proof. The rest of the symbols appeared in C are connectives, and the number of them is less than or equal to $N_{\mathcal{R}}$, where $N_{\mathcal{R}}$ is the number of the premises of the rule \mathcal{R} , which is the last rule used in the proof. Since the sequent S is the conclusion of a rule in H , the H -lengths of the proofs of its premises are less than the H -length of π and we can use the induction hypothesis for them. Then $|C| \leq \sum_{j,s} |C_{js}| + \sum_{i,r} |D_{ir}| + N_{\mathcal{R}}$. By induction hypothesis we have $|C_{js}| \leq |\pi_{j,s}|^M$ and $|C_{ir}| \leq |\pi_{i,r}|^M$, where $\pi_{j,s}$ (or $\pi_{i,r}$) is the proof of the sequent whose interpolant is C_{js} (or C_{ir}). But since the proof is tree-like, we have $\sum_{j,s} |\pi_{j,s}| + \sum_{i,r} |\pi_{i,r}| + 1 \leq |\pi|$. It is easy to see that $|C| \leq \sum_{j,s} |\pi_{j,s}|^M + \sum_{i,r} |\pi_{i,r}|^M + N_{\mathcal{R}} \leq (\sum_{j,s} |\pi_{j,s}| + \sum_{i,r} |\pi_{i,r}| + 1)^M \leq |\pi|^M$, and the claim follows. The last inequality uses the fact $M \geq 2$ and

$$N_{\mathcal{R}} \leq \sum_{j,s} |\pi_{j,s}| + \sum_{i,r} |\pi_{i,r}|$$

The latter is an easy consequence of the fact that the number of $\pi_{j,s}$ and $\pi_{i,r}$ in total is $N_{\mathcal{R}}$. \square

In the following we will generalize the Theorem 4.4 to the multi-conclusion case.

Theorem 4.5. *Let G and H be two multi-conclusion sequent calculi such that H extends \mathbf{CFL}_e . Suppose H is an axiomatic extension of G with multi-conclusion semi-analytic rules and if we also add the modal rule 4 or 4D in H , then the left weakening rule for boxed formulas is admissible in H . Then if G has strong H -interpolation (strong H -feasible interpolation), so does H .*

Proof. The proof is similar to the proof of Theorem 4.4 and again it uses induction on the H -length of π . For the zero H -length, the proof is in G and the existence of the interpolation is proved by the assumption. For the rest, we will consider the last rule used in the proof and there are several cases to investigate. Throughout the proof we use the convention $A = A_1, \dots, A_k$ for different sequents A and different numbers k , for simplicity.

- Consider the case where the last rule used in the proof is a left multi-conclusion semi-analytic rule and the main formula, ϕ , is in Λ in the Definition 4.1. Hence, the sequent S is of the form $(\Gamma', \Gamma'', \phi \Rightarrow \Delta', \Delta'')$ and we have to find a formula C that satisfies $(\Gamma' \Rightarrow C, \Delta')$ and $(\Gamma'', \phi, C \Rightarrow \Delta'')$. Therefore, we must have had the following instance of the rule

$$\frac{\langle \langle \Gamma'_i, \Gamma''_i, \bar{\phi}_{ir} \Rightarrow \bar{\psi}_{ir}, \Delta'_i, \Delta''_i \rangle_r \rangle_i}{\Gamma', \Gamma'', \phi \Rightarrow \Delta', \Delta''}$$

Using the induction hypothesis for the premises we have for every i and r

$$\Gamma'_i \Rightarrow C_{ir}, \Delta'_i \quad , \quad \Gamma''_i, \bar{\phi}_{ir}, C_{ir} \Rightarrow \bar{\psi}_{ir}, \Delta''_i$$

Using the rule $(R\wedge)$ and $(L\wedge)$ we have for every i

$$\Gamma'_i \Rightarrow \bigwedge_r C_{ir}, \Delta'_i \quad , \quad \Gamma''_i, \bar{\phi}_{ir}, \bigwedge_r C_{ir} \Rightarrow \bar{\psi}_{ir}, \Delta''_i$$

Using the rule $(R*)$ for the left sequents we get

$$\Gamma' \Rightarrow * \bigwedge_i \bigwedge_r C_{ir}, \Delta'$$

and, if we substitute the right sequents in the original rule, and then using the rule (L^*) , we get

$$\Gamma'', \phi, * \bigwedge_i \bigwedge_r C_{ir} \Rightarrow \Delta''$$

Hence, we take C as $* \bigwedge_i \bigwedge_r C_{ir}$ and we are done.

To check $V(C) \subseteq V(\Gamma' \cup \Delta') \cap V(\{\Gamma'' \cup \{\phi\}\} \cup \Delta'')$, note that an atom is in C if and only if it is in one of C_{ir} 's. Then, by induction hypothesis, it is in $(\Gamma'_i \cup \Delta'_i)$ and in $\{\Gamma''_i, \bar{\phi}_{ir}, \bar{\psi}_{ir}, \Delta''_i\}$. It can be easily seen that the claim holds; the only thing to remember is that if the atom is in either $\bar{\phi}_{ir}$ or in $\bar{\psi}_{ir}$, since the rule is occurrence preserving, it also appears in ϕ .

- Consider the case where the last rule used in the proof is a left multi-conclusion semi-analytic rule and the main formula, ϕ , is in Σ in the Definition 4.1. Hence, the sequent S is again of the form $(\Gamma', \Gamma'', \phi \Rightarrow \Delta', \Delta'')$ and we have to find a formula C that satisfies $(\Gamma', \phi \Rightarrow C, \Delta')$ and $(\Gamma'', C \Rightarrow \Delta'')$. Therefore, we must have had the following instance of the rule

$$\frac{\langle \langle \Gamma'_i, \Gamma''_i, \bar{\phi}_{ir} \Rightarrow \bar{\psi}_{ir}, \Delta'_i, \Delta''_i \rangle_r \rangle_i}{\Gamma', \Gamma'', \phi \Rightarrow \Delta', \Delta''}$$

Using the induction hypothesis for the premises we have for every i and r

$$\Gamma'_i, \bar{\phi}_{ir} \Rightarrow \bar{\psi}_{ir}, C_{ir}, \Delta'_i \quad , \quad \Gamma''_i, C_{ir} \Rightarrow \Delta''_i$$

Using the rules $(R\vee)$ and $(L\vee)$, we have for every i

$$\Gamma'_i, \bar{\phi}_{ir} \Rightarrow \bar{\psi}_{ir}, \bigvee_r C_{ir}, \Delta'_i \quad , \quad \Gamma''_i, \bigvee_r C_{ir} \Rightarrow \Delta''_i$$

If we substitute the left sequents in the original rule, we get

$$\Gamma', \phi \Rightarrow \bigvee_r C_{ir}, \Delta'$$

and, using the rule $(R+)$ we get

$$\Gamma', \phi \Rightarrow \bigoplus_i \bigvee_r C_{ir}, \Delta'$$

On the other hand, using the rule ($L+$) for the right sequents we have

$$\Gamma'', \bigoplus_i \bigvee_r C_{ir} \Rightarrow \Delta''$$

It is enough to take C as $\bigoplus_i \bigvee_r C_{ir}$ to finish the proof of this case.

To check $V(C) \subseteq V(\{\Gamma' \cup \{\phi\}\} \cup \Delta') \cap V(\Gamma'' \cup \Delta'')$, note that an atom is in C if and only if it is in one of C_{ir} 's. Then, by induction hypothesis, it is in $\{\Gamma'_i, \bar{\phi}_{ir}, \bar{\psi}_{ir}, \Delta'_i\}$ and in $(\Gamma''_i \cup \Delta''_i)$. It can be easily seen that the claim holds; the only thing to remember is that if the atom is in either $\bar{\phi}_{ir}$ or in $\bar{\psi}_{ir}$, since the rule is occurrence preserving, it also appears in ϕ .

- Consider the case where the last rule used in the proof is a modal multi-conclusion one. The case where it is the rule D or $4D$ is similar to the proof of the same cases in the Theorem 4.4. Let the last rule used in the proof be the rule K . Then, S is of the form $\Box\Gamma', \Box\Gamma'' \Rightarrow \Box\phi$. Therefore, there can be two cases based on the partition of the right side of the sequent. In the first one, we have to show that there exists C such that $\Box\Gamma' \Rightarrow C$ and $\Box\Gamma'', C \Rightarrow \Box\phi$ hold. In the second one, we have to show that there exists C such that $\Box\Gamma' \Rightarrow C, \Box\phi$ and $\Box\Gamma'', C \Rightarrow$ hold. Since the proof of the first case is similar to the proof in Theorem 4.4, we will investigate the second case. Hence, we must have had the following instance of the rule

$$\frac{\Gamma', \Gamma'' \Rightarrow \phi}{\Box\Gamma', \Box\Gamma'' \Rightarrow \Box\phi}$$

Using the induction hypothesis for the premise, there exists D such that we have

$$\Gamma' \Rightarrow D, \phi \quad , \quad D, \Gamma'' \Rightarrow$$

Using the rule ($L \rightarrow$) together with the axiom ($\Rightarrow 0$) on the one hand and on the other, using the rule ($R0$) and ($R \rightarrow$) we have

$$\Gamma', \neg D \Rightarrow, \phi \quad , \quad \Gamma'' \Rightarrow \neg D$$

Use the rule K to derive

$$\Box\Gamma', \Box\neg D \Rightarrow \Box\phi \quad , \quad \Box\Gamma'' \Rightarrow \Box\neg D$$

And we can derive

$$\Box\Gamma' \Rightarrow \neg\Box\neg D, \Box\phi \quad , \quad \neg\Box\neg D, \Box\Gamma'' \Rightarrow$$

which means we have to take $C = \neg\Box\neg D$. The atom check is easy.

Now, consider the case where the last rule used in the proof is the rule 4. Then S is of the form $\Box\Gamma', \Box\Gamma'' \Rightarrow \Box\phi$ and there are the exact two cases as above, in the case of the rule K , and again since the second case is new (the proof of the other one is similar to the proof in Theorem 4.4), we will investigate that one. Hence, we have to show that there exists C such that $\Box\Gamma' \Rightarrow C, \Box\phi$ and $\Box\Gamma'', C \Rightarrow$ hold. Therefore we must have had the following instance of the rule

$$\frac{\Gamma', \Gamma'', \Box\Gamma', \Box\Gamma'' \Rightarrow \phi}{\Box\Gamma', \Box\Gamma'' \Rightarrow \Box\phi}$$

Using the induction hypothesis for the premise, there exists D such that we have

$$\Gamma', \Box\Gamma' \Rightarrow D, \phi \quad , \quad D, \Gamma'', \Box\Gamma'' \Rightarrow$$

Using the rule $(L \rightarrow)$ together with the axiom $(\Rightarrow 0)$ on the one hand and on the other, using the rule $(R0)$ and $(R \rightarrow)$ we have

$$\Gamma', \Box\Gamma', \neg D \Rightarrow \phi \quad , \quad \Gamma'', \Box\Gamma'' \Rightarrow \neg D$$

Use the left weakening rule for the left sequent (to add $\Box\neg D$ to the left side of the sequent) and then apply the rule 4 to get

$$\Box\Gamma', \Box\neg D \Rightarrow \Box\phi \quad , \quad \Box\Gamma'' \Rightarrow \Box\neg D$$

And we can derive

$$\Box\Gamma' \Rightarrow \neg\Box\neg D, \Box\phi \quad , \quad \neg\Box\neg D, \Box\Gamma'' \Rightarrow$$

If we take $C = \neg\Box\neg D$, we are done. And it is easy to check the condition for atoms.

In the case of the rule $(RS4)$, we have exactly the same cases as in the rule K :

$$\Box\Gamma' \Rightarrow C \quad , \quad \Box\Gamma'', C \Rightarrow \Box\phi$$

and

$$\Box\Gamma' \Rightarrow C, \Box\phi \quad , \quad \Box\Gamma'', C \Rightarrow$$

Only the second case is new (the proof for the first one is the same as the proof of the same case in theorem 4.4). The proof of the second case is the same as the case for the rule K in the above, and $C = \neg\Box\neg D$ works here, as well.

The cases where the last rule in the proof is a right multi-conclusion semi-analytic one is similar and we do not investigate them here. The proof for the feasibility part is easy and similar to the proof in the Theorem 4.4. \square

Therefore combining the Theorems 4.3, 4.4 and 4.5 we will have:

Theorem 4.6. (i) *For any \mathbf{FL}_e -extension (\mathbf{FL}_e^- -extension) single-conclusion H consisting of semi-analytic rules and centered axioms (context-free centered axioms), H has H -feasible interpolation.*

(ii) *For any \mathbf{IPC} -extension single-conclusion H consisting of semi-analytic rules, context-sharing semi-analytic rules and centered axioms, H has H -feasible interpolation.*

(iii) *For any \mathbf{CFL}_e -extension (\mathbf{CFL}_e^- -extension) multi-conclusion H consisting of semi-analytic rules and centered axioms (context-free centered axioms), H has H -feasible interpolation.*

Combining Theorem 4.6 with Theorem 4.2, we have:

Corollary 4.7. (i) *If $\mathbf{FL}_e \subseteq L$, ($\mathbf{FL}_e^- \subseteq L$) and L has a single-conclusion sequent calculus consisting of semi-analytic rules and centered axioms (context-free centered axioms), then L has Craig interpolation.*

(ii) *If $\mathbf{IPC} \subseteq L$ and L has a single-conclusion sequent calculus consisting of semi-analytic rules, context-sharing semi-analytic rules and centered axioms, then L has Craig interpolation.*

(iii) *If $\mathbf{CFL}_e \subseteq L$, ($\mathbf{CFL}_e^- \subseteq L$) and L has a multi-conclusion sequent calculus consisting of semi-analytic rules and centered axioms (context-free centered axioms), then L has Craig interpolation.*

The following are the application of the main corollary of this section i.e., Corollary 4.7. To begin, let us consider the positive application:

Corollary 4.8. *The logics \mathbf{FL}_e , \mathbf{FL}_{ec} , \mathbf{FL}_{ew} , \mathbf{CFL}_e , \mathbf{CFL}_{ew} , \mathbf{CFL}_{ec} , \mathbf{ILL} , \mathbf{CLL} , \mathbf{IPC} , \mathbf{CPC} and their \mathbf{K} , \mathbf{KD} and $\mathbf{S4}$ versions have the Craig interpolation property. The same also goes for $\mathbf{K4}$ and $\mathbf{K4D}$ extensions of \mathbf{IPC} and \mathbf{CPC} .*

Proof. Note that the usual cut-free sequent calculus for all of these logics consists of semi-analytic rules and centered axioms. Therefore, by the Corollary 4.7 we can prove the Craig interpolation property for all of them. \square

For the negative applications, we use the results in [7] and [9] to ensure that the following logics do not have Craig interpolation. Then we will use the Corollary 4.7 to prove that these logics do not have a semi-analytic calculus consisting only of the centered axioms and semi-analytic rules.

Corollary 4.9. *Non of the logics UL^- , IUL^- , MTL , $SMTL$, $IMTL$, BL , L_∞ , L_n for $n \geq 3$, P , CHL and A have a single-conclusion sequent calculus consisting only of single-conclusion semi-analytic rules and context-free centered axioms.*

Corollary 4.10. *Non of the logics R , IUL^- , $IMTL$, BL , L_∞ , L_n for $n \geq 3$ and A have a single-conclusion (multi-conclusion) sequent calculus consisting only of single-conclusion (multi-conclusion) semi-analytic rules and context-free centered axioms.*

Corollary 4.11. *Except G , $G3$ and \mathbf{CPC} , non of the consistent BL -extensions have a single-conclusion sequent calculus consisting only of single-conclusion semi-analytic rules and context-free centered axioms.*

Corollary 4.12. *The only $IMTL$ -extension with a calculus consisting of single-conclusion (multi-conclusion) semi-analytic rules and context-free centered axioms, is \mathbf{CPC} .*

Corollary 4.13. *Except RM^e , $IUML^-$, \mathbf{CPC} , RM_3^e , RM_4^e , $\mathbf{CPC} \cap IUML^-$, $RM_4^e \cap IUML^-$, and $\mathbf{CPC} \cap RM_3^e$, non of the consistent extensions of RM^e have a single-conclusion (multi-conclusion) sequent calculus consisting only of single-conclusion (multi-conclusion) semi-analytic rules and context-free centered axioms. This category includes:*

- (i) RM_n^e for $n \geq 5$,
- (ii) $RM_{2m}^e \cap RM_{2n+1}^e$ for $n \geq m \geq 1$ with $n \geq 2$,
- (iii) $RM_{2m}^e \cap IUML^-$ for $m \geq 3$.

Corollary 4.14. *Except \mathbf{IPC} , \mathbf{LC} , \mathbf{KC} , \mathbf{Bd}_2 , \mathbf{Sm} , \mathbf{GSc} and \mathbf{CPC} , non of the consistent super-intuitionistic logics have a single-conclusion sequent calculus consisting only of single-conclusion semi-analytic rules, context-sharing semi-analytic rules and centered axioms.*

5 centered Calculi

In the previous section we have seen an interesting relationship between the semi-analytic rules and centered axioms on the one hand and the Craig interpolation property, on the other. In this section, we investigate a more specific form of semi-analytic rules called polarity preserving centered rules, PPF, for abbreviation. This time, the problem that we are interested in is not only the completeness of these rules for a given logic, but also the effectiveness of such a possible completeness. Our main result in this area is an exponential lower bound on the length of proofs in these PPF calculi for **CPC**-valid sequents (or in some cases even **IPC**-valid ones). However, the sequents that we will use have polynomially short tree-like proofs in the usual Hilbert-style proof system or equivalently **LK** + **Cut**. This slowness of PPF systems compared to the classical Hilbert-style system (or **LK** + **Cut**), highlights the weakness of semi-analytic rules again, this time in the complexity theoretic sense of the word.

Definition 5.1. A multi-conclusion semi-analytic rule is called *centered* if it has one of the following forms:

$$\frac{\langle\langle\Gamma_i, \bar{\phi}_{ir} \Rightarrow \Delta_i\rangle_r\rangle_i}{\Gamma_1, \dots, \Gamma_n, \phi \Rightarrow \Delta_1, \dots, \Delta_n} \quad \frac{\langle\langle\Gamma_i \Rightarrow \bar{\phi}_{ir}, \Delta_i\rangle_r\rangle_i}{\Gamma_1, \dots, \Gamma_n \Rightarrow \Delta_1, \dots, \Delta_n, \phi}$$

A sequent is called a *strongly centered* axiom if it has one of the following forms:

- (1) $\phi \Rightarrow \phi$
- (2) $\Rightarrow \bar{\alpha}$
- (3) $\bar{\beta} \Rightarrow$
- (4) $\Gamma, \bar{\phi} \Rightarrow \Delta$
- (5) $\Gamma \Rightarrow \bar{\phi}, \Delta$

where in (2) and (5), $\bar{\alpha}$ and $\bar{\phi}$ have no variable and Γ and Δ are meta-multiset variables.

Example 5.2. The conjunction, disjunction, fusion and addition rules in **CFL_e** are all centered. For the strongly centered axioms, note that all the axioms of **FL_e** are strongly centered. An example of a centered axiom which is not strongly centered is $(\Rightarrow \phi, \neg\phi)$. Since otherwise it would have been an instance of either 2 or 5, which is not possible. The reason is that ϕ can have a variable which must not appear in the right side of the sequent.

Definition 5.3. A logic L is called sub-classical if **CPC** extends L . In the same way, a calculus G is called sub-classical if **CPC** extends G .

First let us investigate the power of centered axioms. The natural question to ask is that whether it is possible to have a calculus consisting only of centered axioms and centered rules, complete for some given logic. For the multi-conclusion version of the original definition of centered axioms introduced in [6] as $(\Gamma, r \Rightarrow r)$, $(\Gamma, \perp \Rightarrow \phi)$ or $(\Gamma \Rightarrow \top)$, where r is a positive atom, the answer is negative. In the following corollary we will show how.

Theorem 5.4. *Let L be a sub-classical logic that extends **IPC**. Then there is no calculus for L consisting only of centered rules and the axioms $(\Gamma, r \Rightarrow r, \Delta)$, $(\Gamma, \perp \Rightarrow \Delta)$ or $(\Gamma \Rightarrow \top, \Delta)$.*

Proof. Similar to the proof of the Theorem 4.4 we can use the induction on the length of the proof in the sequent calculus to assign a formula C to any provable sequent $\Gamma \Rightarrow \Delta$ such that $\Gamma \Rightarrow C$ and $C \Rightarrow \Delta$ and $V(C) \subseteq V(\Gamma) \cap V(\Delta)$. We can see that the interpolant for any derived sequent should be monotone (negation and implication-free) because the interpolant for the axioms are either \perp and \top or positive atoms and the rest were constructed via monotone operations. However, the interpolant for $\neg p \Rightarrow \neg p$ is $\neg p$ and this means that $\neg p$ should be L -equivalent to a monotone formula. Since L is sub-classical, $\neg p$ is **CPC**-equivalent to a monotone formula, which is not the case. Hence the claim follows. \square

Theorem 5.5. *There is no calculus for **IPC** consisting of only centered rules, the axioms $(\Gamma, \perp \Rightarrow \Delta)$ and $(\Gamma \Rightarrow \top, \Delta)$ and the atomic and negative atomic instances of the axiom $\Gamma, \phi \Rightarrow \phi, \Delta$.*

Proof. Similar to the proof of the Corollary 5.4, any interpolant is constructed as a repeated use of monotone operations on atomic and negated atomic formulas and \perp and \top . However, since the interpolant for the sequent $(p \rightarrow q) \Rightarrow (p \rightarrow q)$ is $p \rightarrow q$, and it is not **IPC**-equivalent to any formula of the mentioned form, the claim follows. \square

These simple observations show that the notion of a centered axiom introduced in [6] is not powerful enough in capturing logical systems. Expanding this notion to our definition of centered axioms in the preliminaries, we can make our proof systems powerful enough to capture at least some classical logics. These presentations can be considered as witnesses for the power and naturalness of centered axioms and rules.

Theorem 5.6. ***CFL_e**, **CFL_{ew}** and **CPC** have a sequent calculus consisting only of centered rules and centered axioms.*

Proof. Consider a sequent calculus with the following axioms:

Axioms:

$$\frac{}{\phi \Rightarrow \phi} \quad \frac{}{\phi, \neg\phi \Rightarrow} \quad \frac{}{\Rightarrow \phi, \neg\phi}$$

$$\frac{}{\neg 1 \Rightarrow} \quad \frac{}{\Rightarrow \neg 0}$$

$$\frac{}{\Gamma \Rightarrow \neg \perp, \Delta} \quad \frac{}{\Gamma, \neg \top \Rightarrow \Delta}$$

The usual left and right rules for disjunction and conjunction and the following rules for implication:

$$\frac{\Gamma \Rightarrow \neg\phi, \psi, \Delta}{\Gamma \Rightarrow \phi \rightarrow \psi, \Delta} \quad \frac{\Gamma_1, \neg\phi \Rightarrow \Delta_1 \quad \Gamma_2, \psi \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2, \phi \rightarrow \psi \Rightarrow \Delta_1, \Delta_2}$$

And finally, for any combination $\neg\vee$, $\neg\wedge$, $\neg*$, $\neg+$ and $\neg\neg$ we have the corresponding right and left rules, using De Morgan's laws. For instance, we have

$$\frac{\Gamma \Rightarrow \neg\phi, \Delta}{\Gamma \Rightarrow \neg(\phi \wedge \psi), \Delta} R^{\neg\wedge}$$

It is easy to check that all the rules of this sequent calculus is centered and the system is equivalent to the usual sequent calculus of \mathbf{CFL}_e . The proof of the second part is based on the observation that if $\Gamma, \Gamma' \Rightarrow \Delta, \Delta'$ is provable in the usual calculus for classical logic, then $\Gamma, \neg\Delta \Rightarrow \neg\Gamma', \Delta'$ is provable in the new calculus. The proof is an easy application of induction on the length of the usual proof of $\Gamma, \Gamma' \Rightarrow \Delta, \Delta'$. The idea for \mathbf{CFL}_{ew} and \mathbf{CPC} is similar. \square

So far, we have seen some calculi consisting only of centered axioms and rules. Now, it is time to examine that how effective such a characterization can be. For this purpose, from now on we will restrict our investigations to the usual language of \mathbf{CPC} and two natural sub-classes of centered rules as polarity preserving centered rules, PPF rules, and monotonicity preserving centered rules, MPF rules.

Definition 5.7. Let \mathcal{P} be a set of meta-formula variables or atomic constants. A meta-formula ψ is called \mathcal{P} -monotone if for any $\phi \in \mathcal{P}$, all occurrences of ϕ in ψ is positive, i.e., ϕ does not occur in the scope of negations or in the precedents of implications. A multiset Γ of meta-formulas is called \mathcal{P} -monotone if all of its elements are \mathcal{P} -monotone. It is called *monotone* if it is $V(\Gamma)$ -monotone.

Definition 5.8. A centered rule is called *polarity preserving*, PPF, if it preserves \mathcal{P} -monotonicity backwardly for any \mathcal{P} , i.e., if the antecedent of the consequence is \mathcal{P} -monotone, then the antecedents of all the premises are also \mathcal{P} -monotone. It is *monotonicity preserving*, MPF, if it is centered and preserves monotonicity backwardly, in the same way.

Example 5.9. All analytic centered rules in the language of **CPC**, the centered rules in which any formula in the premises is a subformula of a formula in the consequence, are both PPF and MPF.

The following theorem is our first example of the mentioned ineffectiveness of the combination of centered axioms and PPF rules. It shows that non of the combinations of centered axioms and PPF rules can simulate the cut rule in a feasible way.

Theorem 5.10. *There is no calculus G consisting of only centered axioms and PPF rules, sound and feasibly complete for **CPC**. More precisely, if G is a calculus for **CPC**, then there exists a sequence of **CPC**-valid sequents $\phi_n \Rightarrow \psi_n$, with polynomially short tree-like proofs in the Hilbert-style system or equivalently in **LK** + **Cut** such that $\|\phi_n \Rightarrow \psi_n\|_G$, the shortest tree-like G -proof of $\phi_n \Rightarrow \psi_n$, is exponential in n . Therefore, the PPF rules together with centered axioms are either incomplete or feasibly incomplete for **CPC**.*

Proof. Assume that G is a calculus for **CPC** consisting of PPF rules and centered axioms. Let $\text{Clique}_n^k(\bar{p}, \bar{r}_1)$ be the proposition asserting that \bar{r}_1 is a clique of size k on the graph represented by \bar{p} and, $\text{Color}_n^m(\bar{p}, \bar{r}_2)$ be the proposition asserting that \bar{r}_2 is an m -coloring of the same graph represented by \bar{p} , where $\bar{p} = p_1, \dots, p_n$. Note that by the formalization of the Clique formula, every occurrence of \bar{p} in $\text{Clique}_n^k(\bar{p}, \bar{r}_1)$ is positive (which means it is monotone in \bar{p}). We know that for $m < k$, the formula $\neg \text{Clique}_n^k(\bar{p}, \bar{r}_1) \vee \neg \text{Color}_n^m(\bar{p}, \bar{r}_2)$ is a tautology in classical logic which implies that

$$\text{Clique}_n^k(\bar{p}, \bar{r}_1) \Rightarrow \neg \text{Color}_n^m(\bar{p}, \bar{r}_2)$$

is **CPC**-valid.

First observe that by the Craig interpolation theorem for **CPC** and the fact that the precedent is monotone in \bar{p} , there exists a monotone interpolant $I(\bar{p})$ such that

$$\text{Clique}_n^k(\bar{p}, \bar{r}_1) \Rightarrow I(\bar{p}) \Rightarrow \neg \text{Color}_n^m(\bar{p}, \bar{r}_2)$$

which means that if the graph H represented by \bar{p} has a k -clique then $I(\bar{p}) = 1$ and if H has an m -coloring then $I(\bar{p}) = 0$. In other words, if $I(\bar{p}) \neq 0$ then

H does not have an m -coloring and if $I(\bar{p}) \neq 1$ then H does not have a k -clique. By the result in [1], every such monotone interpolant I must have exponential length in n for suitable polynomially bounded choices for k and m .

Secondly, define $\phi_n(\bar{p}, \bar{r}_1) = \text{Clique}_n^k(\bar{p}, \bar{r}_1)$ and $\psi_n(\bar{p}, \bar{r}_2) = \neg \text{Color}_n^m(\bar{p}, \bar{r}_2)$. We will show that this family of sequents, $\phi_n(\bar{p}, \bar{r}_1) \Rightarrow \psi_n(\bar{p}, \bar{r}_2)$, serve as the **CPC**-valid sequents mentioned in the theorem. The idea is simple. First note that the fact that the sequent

$$\text{Clique}_n^k(\bar{p}, \bar{r}_1) \Rightarrow \neg \text{Color}_n^m(\bar{p}, \bar{r}_2)$$

has a tree-like proof of the size $n^O(1)$ in the classical Hilbert-style proof system or equivalently **LK** + **Cut** is a folklore well-known fact in the proof complexity community. Now pick π_n as the shortest tree-like proof of the sequent in G . Note that the antecedent of our sequent, $\text{Clique}_n^k(\bar{p}, \bar{r}_1)$, is \bar{p} -monotone and since all the rules are PPF rules, the antecedent of all axioms in the proof π_n are also \bar{p} -monotone. Secondly, note that the interpolant of the axioms except the axiom $\gamma \Rightarrow \gamma$ are variable-free and hence monotone. For the identity axiom $\gamma \Rightarrow \gamma$ note that since $\bar{r}_1 \cap \bar{r}_2 = \emptyset$, the common variables of the two sides of the sequent are \bar{p} . Now since the rules are occurrence preserving, the use of the identity axioms are just on formulas consisting only of the variables in \bar{p} . Since the interpolant for the axiom $\gamma \Rightarrow \gamma$ is γ and the occurrence of \bar{p} is positive in γ , the interpolant for this axiom is also \bar{p} -monotone. Therefore, the interpolant for any axiom in π_n is monotone.

Note that in the case that all the semi-analytic rules are centered, it is possible to rewrite the proof of the Theorem 4.5 to assign a formula C to any provable sequent $\Gamma \Rightarrow \Delta$ such that $\Gamma \Rightarrow C$ and $C \Rightarrow \Delta$ and $V(C) \subseteq V(\Gamma) \cap V(\Delta)$. This process needs G to be an extension of **MALL** which is the case here because G is equivalent to **CPC** and **CPC** admits **MALL** via the canonical translation that sends fusion, addition, 1 and 0 to conjunction, disjunction, \top and \perp , respectively.

Now, we can see that again the interpolants are constructed by monotone operations. Since the interpolant for the axioms are monotone, the interpolant for the sequent

$$\phi_n(\bar{p}, \bar{r}_1) \Rightarrow \psi_n(\bar{p}, \bar{r}_2)$$

will be monotone. However, G captures **CPC**. Therefore, the whole process provides a classical monotone interpolant for the sequent

$$\text{Clique}_n^k(\bar{p}, \bar{r}_1) \Rightarrow \neg \text{Color}_n^m(\bar{p}, \bar{r}_2)$$

which we will call C_n . Note that $|C_n| \leq |\pi_n|^{O(1)}$ by the feasibility condition. However, any such C_n should be exponentially long in n as we explained before. Therefore, the shortest proof π_n for our sequent is exponentially long. \square

It is also possible to lower down the previous exponential lower bound to the level of the **IPC**-valid sequents. For that purpose we need a new form of interpolation and its preservation theorem.

Definition 5.11. Let G and H be two sequent calculi. G has *H-monotone feasible interpolation* if for any k and any sequent $S = (\Sigma \Rightarrow \Lambda_1, \dots, \Lambda_k)$ if S is provable in G by a tree-like proof π , then there exists formulas $|C_j| \leq |\pi|^{O(1)}$ for $1 \leq j \leq k$ such that $(\Sigma \Rightarrow C_1, \dots, C_k)$ and $(C_j \Rightarrow \Lambda_j)$ are provable in H and $V(C_j) \subseteq V(\Sigma) \cap V(\Lambda_j)$, where $V(A)$ is the set of the atoms of A . Moreover, if Σ is monotone, then C_j is also monotone for all $1 \leq j \leq k$. We call C_j 's, the interpolants of the partition $\Lambda_1, \dots, \Lambda_k$ of the sequent S .

Theorem 5.12. Let G and H be two sequent calculi such that G is a set of strongly centered axioms, H extends **MALL** and any sequent in G is provable in H . Then G has *H-monotone feasible interpolation*.

Proof. We will consider the strongly centered axioms one by one:

- (1) In this case the sequent S is of the form $(\phi \Rightarrow \phi)$. W.l.o.g assume $\Lambda_1 = \phi$ and $\Lambda_j = \emptyset$ for $j \neq 1$. Pick $C_1 = \phi$ and $C_j = 0$. Using the axiom 0 and the rule (R0), we can easily see that these C_j 's work. For monotonicity, note that since ϕ is monotone, C_1 is also monotone.
- (2) For the case $(\Rightarrow \bar{\alpha})$, consider C_j to be $\vdash \Lambda_j$. We can easily see that these C_j 's work, using the left and right rules for \vdash . For the variables, since $V(\bar{\alpha}) = \emptyset$, we have $V(C_j) \subseteq V(\emptyset) \cap V(\Lambda_j)$. And for the monotonicity, since $V(C_j) = \emptyset$, then C_j is monotone.
- (3) For the case $(\bar{\beta} \Rightarrow)$, pick $C_j = 0$. The reasoning is similar to (1). For the monotonicity, since $V(C_j) = \emptyset$, it is monotone.
- (4) If S is of the form $\Gamma, \bar{\phi} \Rightarrow \Delta$ define $C_j = \perp$. First note that we have $\Gamma, \bar{\phi} \Rightarrow \perp, \perp, \dots, \perp$ where in the right hand-side we have k many \perp 's. The reason is that this sequent is an instance of the axiom (4) itself. Moreover, for every j we have $\perp \Rightarrow \Lambda_j$ since it is an instance of the axiom \perp . And again $V(C_j) = \emptyset$.

- (5) If S is of the form $(\Gamma \Rightarrow \bar{\phi}, \Delta)$ define $C_j = \vdash(\Lambda_j \cap \bar{\phi}) + n_j \perp$ where $n_j = |\Lambda_j - (\Lambda_j \cap \bar{\phi})|$ and $n_j \perp$ means the addition of \perp to itself n_j times, and if $n_j = 0$, then $n_j \perp$ does not appear in the definition of C_j . It is easy to see that this C_j works. Because, if $n_j \neq 0$, then $C_j \Rightarrow \Lambda_j$ is an instance of the axiom \perp and if $n_j = 0$, then it means that $\Lambda_j = \Lambda_j \cap \bar{\phi}$ and using the rule $(R\vdash)$, we are done. We also have $\Gamma \Rightarrow C_1, \dots, C_k$, since in the right hand-side we will have the formula $\bar{\phi}$ (together with some other formulas which we will treat as the context) and it will become an instance of the same axiom. Note that since $V(\bar{\phi}) = \emptyset$, there is nothing to check for the variables. For the monotonicity, note that $V(C_j) = \emptyset$, therefore C_j is monotone.

□

The next theorem shows that MPF rules preserve the monotone feasible interpolation property. We will use this theorem later in the lower bound result that we have promised before.

Theorem 5.13. (*monotone feasible interpolation*) *Let G and H be two sequent calculi such that H extends **MALL** and axiomatically extends G by MPF rules. Then if G has H -monotone feasible interpolation property, so does H .*

Proof. First let us prove the interpolation property, then we will check the feasibility part. The proof uses induction on the H -length of π . If the H -length of π is zero, it means that the proof is in G . Hence the claim is clear by the assumption. There are three cases to consider based on the last rule of the proof.

- Suppose that the last rule of the proof π is a left centered rule. Then it is of the following form:

$$\frac{\langle\langle\Gamma_i, \bar{\phi}_{ir} \Rightarrow \Delta_i\rangle_r\rangle_i}{\Gamma, \phi \Rightarrow \Delta}$$

where $\Gamma = \Gamma_1, \dots, \Gamma_n$ and $\Delta = \Delta_1, \dots, \Delta_n$. And, $\Lambda_1, \dots, \Lambda_k$ are given such that $\bigcup_{j=1}^k \Lambda_j = \Delta$. By induction hypothesis for the premises, there exist formulas D_{ir1}, \dots, D_{irk} such that for every i and r

$$\Gamma_i, \bar{\phi}_{ir} \Rightarrow D_{ir1}, \dots, D_{irk} \quad , \quad D_{irj} \Rightarrow \Lambda_{ij} \quad (\text{for } 1 \leq j \leq k)$$

where $\Lambda_{ij} = \Delta_i \cap \Lambda_j$. Using the rules $(R\vee)$ and $(L\vee)$, we get for every i

$$\Gamma_i, \bar{\phi}_{ir} \Rightarrow \bigvee_r D_{ir1}, \dots, \bigvee_r D_{irk} \quad , \quad \bigvee_r D_{irj} \Rightarrow \Lambda_{ij} \text{ (for } 1 \leq j \leq k) \quad (\dagger)$$

If we substitute the left sequents in the original centered rule, we get

$$\Gamma, \phi \Rightarrow \bigvee_r D_{ir1}, \dots, \bigvee_r D_{irk}$$

Using the rule $(R+)$, we have

$$\Gamma, \phi \Rightarrow \bigoplus_i \bigvee_r D_{ir1}, \dots, \bigoplus_i \bigvee_r D_{irk}$$

On the other hand, if we use the rule $(L+)$ on the right sequents of (\dagger) , we get for $1 \leq j \leq k$

$$\bigoplus_i \bigvee_r D_{irj} \Rightarrow \Lambda_j$$

If we take C_j to be $\bigoplus_i \bigvee_r D_{irj}$, we are done. It only remains to check the variables. If a variable is in C_j , then it is in one of D_{irj} 's. By induction hypothesis we have $V(D_{irj}) \subseteq V(\{\Gamma_i \cup \{\bar{\phi}_{ir}\}\}) \cap V(\Lambda_{ij}) \subseteq V(\{\Gamma \cup \{\phi\}\}) \cap V(\Lambda_j)$, since the rule is occurrence preserving, and this is what we wanted.

- If the last rule used in the proof is a right centered one, then it is of the following form:

$$\frac{\langle\langle \Gamma_i \Rightarrow \bar{\phi}_{ir}, \Delta_i \rangle_r \rangle_i}{\Gamma \Rightarrow \phi, \Delta}$$

where $\Gamma = \Gamma_1, \dots, \Gamma_n$ and $\Delta = \Delta_1, \dots, \Delta_n$. And, again $\Lambda_1, \dots, \Lambda_k$ are given such that $\bigcup_{j=1}^k \Lambda_j = \Delta \cup \{\phi\}$. W.l.o.g. suppose $\phi \in \Lambda_1$ and we denote $\Lambda_1 - \{\phi\}$ by Λ'_1 . By induction hypothesis for the premises, there exist formulas D_{ir1}, \dots, D_{irk} such that for every i, r and $j \neq 1$

$$D_{ir1} \Rightarrow \bar{\phi}_{ir}, \Lambda'_{i1} \quad , \quad D_{irj} \Rightarrow \Lambda_{ij} \quad , \quad \Gamma_i \Rightarrow D_{ir1}, \dots, D_{irk}$$

where $\Lambda_{ij} = \Delta_i \cap \Lambda_j$ and $\Lambda'_{i1} = \Delta_i \cap \Lambda'_1$. Using the rules $(R\vee)$, $(L\vee)$, $(R\wedge)$ and $(L\wedge)$, we get for every i and $j \neq 1$

$$\bigwedge_r D_{ir1} \Rightarrow \bar{\phi}_{ir}, \Lambda'_{i1} \quad , \quad \bigvee_r D_{irj} \Rightarrow \Lambda_{ij} \quad , \quad \Gamma_i \Rightarrow \bigwedge_r D_{ir1}, \bigvee_r D_{ir2}, \dots, \bigvee_r D_{irk}$$

Note that in the right sequent, we first use $(R\vee)$ to get $\Gamma_i \Rightarrow D_{ir1}, \bigvee_r D_{ir2}, \dots, \bigvee_r D_{irk}$, and then we can use the rule $(R\wedge)$. Now, we can substitute the left sequents in the original rule to get

$$\bigwedge_r D_{ir1} \Rightarrow \phi, \Lambda'_1$$

and using the rule (L^*) we have

$$*_i \bigwedge_r D_{ir1} \Rightarrow \phi, \Lambda'_1$$

We denote $*_i \bigwedge_r D_{ir1}$ by C_1 . Using the rule $(L+)$ for the sequents $\bigvee_r D_{irj} \Rightarrow \Lambda_{ij}$ we get

$$\dagger_i \bigvee_r D_{irj} \Rightarrow \Lambda_j$$

and we denote $\dagger_i \bigvee_r D_{irj}$ by C_j . We can see that first using the rule $(R+)$ and after that using the rule (R^*) we get

$$\Gamma \Rightarrow *_i \bigwedge_r D_{ir1}, \dagger_i \bigvee_r D_{ir2}, \dots, \dagger_i \bigvee_r D_{irk}$$

which is

$$\Gamma \Rightarrow C_1, \dots, C_k$$

It only remains to check the variables. If a variable is in C_j , then it is in one of D_{irj} 's. By induction hypothesis we have $V(D_{ir1}) \subseteq V(\Gamma_1) \cap V(\{\{\phi_{ir}\} \cup \Lambda'_{i1}\}) \subseteq V(\Gamma) \cap V(\{\{\phi\} \cup \Lambda'_1\})$ and $V(D_{irj}) \subseteq V(\{\Gamma_i\}) \cap V(\Lambda_{ij}) \subseteq V(\Gamma) \cap V(\Lambda_j)$, since the rule is occurrence preserving, and this is what we wanted.

For the monotonicity part, since the extending rules are MPF, it is easy to prove that if the antecedent of the consequence is monotone, then all the antecedents, everywhere in the proof up to the sequents in G , are also monotone. Since G has H -monotone feasible interpolation property, the interpolants in the base case are monotone. Finally, since the translation is monotone, conjunctions, disjunctions, fusions and additions do not change monotonicity. Hence, our constructed interpolants are also monotone.

The proof for the upper bound for the length of the interpolant is similar to the proof of the Theorem 4.4. \square

Lemma 5.14. [4] Let $A(\bar{p}, \bar{r}_1)$ and $B(\bar{q}, \bar{r}_2)$ be propositional formulas and \bar{p} , \bar{q} , \bar{r}_1 and \bar{r}_2 be mutually disjoint. Let $\bar{p} = p_1, \dots, p_n$ and $\bar{q} = q_1, \dots, q_n$. Assume that A is monotone in \bar{p} or B is monotone in \bar{q} and $A(\bar{p}, \bar{r}_1) \vee B(\neg\bar{p}, \bar{r}_2)$ is a classical tautology. Then

$$\bigwedge_{i=1}^n (p_i \vee q_i) \Rightarrow \neg\neg A(\bar{p}, \bar{r}_1), \neg\neg B(\bar{q}, \bar{r}_2)$$

is **IPC**-valid.

Proof. For the details the reader is referred to [4]. \square

Theorem 5.15. Let G and H be two sequent calculi such that H is sub-classical, H extends **MALL**, H axiomatically extends G by MPF rules and G has H -monotone feasible interpolation property. Then there exists a family of **IPC**-valid sequents $\phi_n \Rightarrow \psi_n$ with the length of $\phi_n \Rightarrow \psi_n$ bounded by a polynomial in n such that either there exists some n such that $H \not\vdash \phi_n \Rightarrow \psi_n$ or $\|\phi_n \Rightarrow \psi_n\|_H$, the shortest tree-like H -proof of $\phi_n \Rightarrow \psi_n$, is exponential in n . Therefore, the MPF rules together with strongly centered axioms are either incomplete or feasibly incomplete for **IPC**.

Proof. The proof is similar and also inspired by the lower bound proof given in [4]. Similar to the proof of Theorem 5.10, consider the **CPC**-valid sequent

$$\text{Clique}_n^k(\bar{p}, \bar{r}_2) \Rightarrow \neg \text{Color}_n^m(\bar{p}, \bar{r}_1)$$

which is equivalent to

$$\Rightarrow \neg \text{Clique}_n^k(\bar{p}, \bar{r}_2), \neg \text{Color}_n^m(\bar{p}, \bar{r}_1)$$

Then, using the Lemma 5.14, if we rewrite $\neg \text{Clique}_n^k(\bar{p}, \bar{r}_2)$ as $B(\neg\bar{p}, \bar{r}_2)$ and $\neg \text{Color}_n^m(\bar{p}, \bar{r}_1)$ as $A(\bar{p}, \bar{r}_1)$, we can easily see that A is monotone in \bar{p} and $A(\bar{p}, \bar{r}_1) \vee B(\neg\bar{p}, \bar{r}_2)$ is a classical tautology. Hence, we can transfer the **CPC**-valid sequent

$$\Rightarrow \neg \text{Clique}_n^k(\bar{p}, \bar{r}_2), \neg \text{Color}_n^m(\bar{p}, \bar{r}_1)$$

to a sequent of the form

$$\bigwedge_i (p_i \vee q_i) \Rightarrow \neg\neg A(\bar{p}, \bar{r}_1), \neg\neg B(\bar{q}, \bar{r}_2)$$

valid in **IPC**. Now, let

$$\phi_n(\bar{p}, \bar{q}) \Rightarrow \psi_n(\bar{p}, \bar{r}_1), \theta_n(\bar{q}, \bar{r}_2)$$

be this sequent. We will show that this family of sequents, $\phi_n(\bar{p}, \bar{q}) \Rightarrow \psi_n(\bar{p}, \bar{r}_1), \theta_n(\bar{q}, \bar{r}_2)$, serve as the **IPC**-valid sequents mentioned in the theorem.

If for some n we have $H \not\vdash \phi_n \Rightarrow \psi_n$, then the claim follows. Therefore, suppose that for every n we have $H \vdash \phi_n \Rightarrow \psi_n$. Let π_n be the shortest tree-like proof of the sequent $\phi_n \Rightarrow \psi_n$ in H . By Theorem 5.13, for every n , there exists monotone formulas $C_n(\bar{p}, \bar{q})$ and $D_n(\bar{p}, \bar{q})$ such that $|C_n| \leq |\pi_n|^{O(1)} = n^{O(1)}$ and $|D_n| \leq |\pi_n|^{O(1)} = n^{O(1)}$ and the followings are provable in H : $(\phi_n \Rightarrow C_n, D_n)$, $(C_n \Rightarrow \psi_n)$, $(D_n \Rightarrow \theta_n)$. Since H captures a sub-classical logic we have $(\phi_n \Rightarrow C_n, D_n)$, $(C_n \Rightarrow \psi_n)$, $(D_n \Rightarrow \theta_n)$ in **CPC**. Define $E_n(\bar{p}) = C_n(\bar{p}, 1, \dots, 1)$ and $F_n(\bar{q}) = D_n(1, \dots, 1, \bar{q})$. Since $(\phi_n \Rightarrow C_n, D_n)$ is valid in classical logic, we have $C_n(\bar{p}, \neg\bar{p}) \vee D_n(\bar{p}, \neg\bar{p}) = 1$. Since C_n and D_n are monotone, we can increase $\neg\bar{p}$ in $C_n(\bar{p}, \neg\bar{p})$ and \bar{p} in $D_n(\bar{p}, \neg\bar{p})$ without changing the valuation of the their disjunction. Hence, $E_n(\bar{p}) \vee F_n(\neg\bar{p}) = 1$. On the other hand, since A does not depend on q and C_n depends both on p and q , and $C_n \Rightarrow A_n$ classically, then we know that $E_n(\bar{p}) = 1$ implies $A(\bar{p}, \bar{r}_1) = 1$. Similarly, we have that $F_n(\bar{q}) = 1$ implies $B(\bar{q}, \bar{r}_2) = 1$. We Claim that $E_n(\bar{p})$ interpolates $\neg B(\neg\bar{p}, \bar{r}_2) \Rightarrow A(\bar{p}, \bar{r}_1)$. One direction is proved. For the other direction, note that if $B(\neg\bar{p}, \bar{r}_2) = 0$ then $F_n(\neg\bar{p}) = 0$ and since $E_n(\bar{p}) \vee F_n(\neg\bar{p}) = 1$ we have $E_n(\bar{p}) = 1$. Hence the monotone formula E_n interpolates $\neg B(\neg\bar{p}, \bar{r}_2) \Rightarrow A(\bar{p}, \bar{r}_1)$ or equivalently the sequent

$$\text{Clique}_n^k(\bar{p}, \bar{r}_2) \Rightarrow \neg \text{Color}_n^m(\bar{p}, \bar{r}_1)$$

However, in the proof of the Theorem 5.10, we mentioned that any such interpolant must have exponential length. Together with the fact that $|E_n(\bar{p})| \leq |\pi_n|^{O(1)}$, we can conclude that $\|\phi_n \Rightarrow \psi_n\|_H$ is exponential in n which implies the claim. \square

Corollary 5.16. *There is no calculus consisting only of strongly centered axioms and MPF rules, sound and feasibly complete for super-intuitionistic logics.*

Proof. This is an obvious consequence of the Theorem 5.15 and the Theorem 5.12. The only point that we have to explain is that if a calculus G consisting only of strongly centered axioms and MPF rules is sound and complete for a super-intuitionistic logic, then G extends **MALL** via the canonical translation. The reason is that G is complete for a super-intuitionistic logic and any calculus complete even for **IPC** extends **MALL** via the canonical translation in an obvious way. \square

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References

- [1] N. Alon, R. Boppana, The monotone circuit complexity of Boolean functions, *Combinatorica* 7 (1) (1987) 1 – 22.
- [2] Došen, K. Modal translations in substructural logics. *Journal of Philosophical Logic* (1992) 21: 283.
- [3] Dyckhoff, R. Contraction-Free Sequent Calculi for Intuitionistic Logic. *Journal of Symbolic Logic* 57 (3): 795807 (1992).
- [4] Hrubes, P. On lengths of proofs in non-classical logics. *Ann. Pure Appl. Logic* 157(2-3): 194-205 (2009).
- [5] Iemhoff, R. (2016) Uniform interpolation and sequent calculi in modal logic.
- [6] Iemhoff, R. (2017) Uniform interpolation and the existence of sequent calculi.
- [7] E. Marchioni, G. Metcalfe, Craig interpolation for semilinear substructural logics, *Mathematical Logic Quarterly*, Volume 58, Issue 6 November 2012 Pages 468-481.
- [8] Maxsimova, L.L. Craigs Theorem in superintuitionistic logics and amalgamated varieties of pseudo-boolean algebras. *Algebra Logika* 16 (6): 643681 (1977).
- [9] Urquhart, A. Failure of interpolation in relevant logics. *Journal of Philosophical Logic* 22, 449479 (1993).