Energy dissipative characteristic schemes for the diffusive Oldroyd-B viscoelastic fluid

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What is viscoelastic fluid?

viscoelastic = viscous + elastic

High Weissenberg Effect

Outline

- Part I: Introduction
- Part II: Energy stability
- Part III: Numerical methods

Navier-Stokes

$$\begin{cases} \rho(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u}) = -\nabla p + \nabla \cdot \boldsymbol{\tau} \\ \nabla \cdot \mathbf{u} = 0 \end{cases}$$

ρ: density
$\mathbf{u}$: velocity
p: pressure
$\boldsymbol{\tau}$: shear stress
Newtonian fluids:

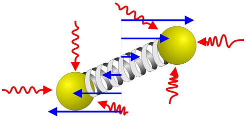
$$\boldsymbol{\tau} = 2\mu_0 \mathbf{D}(\mathbf{u}), \mathbf{D}(\mathbf{u}) = \frac{\nabla \mathbf{u} + (\nabla \mathbf{u})^T}{2}.$$

viscoelastic fluid is non-Newtonian:

$$\boldsymbol{\tau} = f(\mathbf{D}) \text{ is nonlinear}$$

Dilute theory

- Spring force: $\mathbf{F}(\mathbf{R}) = \mathbf{R}$ (Hooke law)
- Friction force from solvent: $\mathbf{f} = \zeta(\dot{\mathbf{r}} - \mathbf{v}(\mathbf{r}, t))$
- Stochastic force due to Brownian motion [1]: $\mathbf{B}_i = \sqrt{2kT\zeta} d\mathbf{W}_i/dt$



k Boltzmann constant, T absolute temperature, $\zeta = 6\pi\mu_s a$ friction coefficient, μ_s solvent viscosity, a radius of bead.

Fokker-Planck equation

$$\frac{\partial\psi}{\partial t} + \mathbf{u}\cdot\nabla\psi = \nabla_{\mathbf{R}}\cdot((-\nabla\mathbf{u}\cdot\mathbf{R} + \frac{1}{2We}\mathbf{F}(\mathbf{R}))\psi) + \frac{1}{2We}\Delta_{\mathbf{R}}\psi$$

Fokker-Planck equation

$$\frac{\partial\psi}{\partial t} + \mathbf{u}\cdot\nabla\psi = \nabla_{\mathbf{R}}\cdot((-\nabla\mathbf{u}\cdot\mathbf{R} + \frac{1}{2We}\mathbf{F}(\mathbf{R}))\psi) + \frac{1}{2We}\Delta_{\mathbf{R}}\psi + \eta\Delta_x\psi$$

Take the momentum of $\mathbf{R} \otimes \mathbf{R}$ and let $\boldsymbol{\sigma} = \int \psi \mathbf{R} \otimes \mathbf{R} d\mathbf{R}$

$$\frac{\delta \boldsymbol{\sigma}}{\delta t} = \frac{1}{We}(\mathbf{I} - \boldsymbol{\sigma}) + \eta \Delta \boldsymbol{\sigma}, \quad \frac{\delta \boldsymbol{\sigma}}{\delta t} := \frac{\partial \boldsymbol{\sigma}}{\partial t} + (\mathbf{u} \cdot \nabla)\boldsymbol{\sigma} - \nabla \mathbf{u} \boldsymbol{\sigma} - \boldsymbol{\sigma}(\nabla \mathbf{u})^T$$

Oldroyd-B model

$$(A) \begin{cases} Re(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u}) = -\nabla p + \nabla \cdot \boldsymbol{\tau} \\ \nabla \cdot \mathbf{u} = 0 \\ \frac{\partial \boldsymbol{\sigma}}{\partial t} + (\mathbf{u} \cdot \nabla)\boldsymbol{\sigma} - \nabla \mathbf{u} \boldsymbol{\sigma} - \boldsymbol{\sigma}(\nabla \mathbf{u})^T = \frac{1}{We}(\mathbf{I} - \boldsymbol{\sigma}) + \eta \Delta \boldsymbol{\sigma} \end{cases}$$

$$\boldsymbol{\tau} = \underbrace{2\alpha \mathbf{D}}_{viscous} + \underbrace{\frac{\beta}{We}\boldsymbol{\sigma}}_{elastic}$$

Re: Reynolds number $\quad$ We: Weissenberg number
$\boldsymbol{\sigma}$: conformation tensor, spd
$\alpha \in (0,1)$: ratio of Newtonian viscocity in total viscocity ($\beta = 1 - \alpha$)
$\eta \geq 0$: diffusive parameter

Non-diffusive case: global in time existence is open

Renardy, 1990.
  well-posedness for Dirichlet IBV

Guillope and Saut, 1990

Fernandez-Cara, Guillen, and Ortega, 2002.
  local existence and global existence for small initial data

Lin, Liu and Zhang, 2005.
  local existence and global existence for small solutions

Arada and Sequeira, 2003.
  strong steady solutions in bounded domain

Diffusive case:

Constantin and Kliegl, 2012
  regularity in 2D

Barrett and Boyaval, 2014
  global existence of weak solutions in 2D

Early studies

- FE: Crochet and Keunings (1982), Keunings (1986), Keunings and Shipman (1986), Marchal and Crochet (1987)
- FV: Xue, Phan-Thien and Tanner (1998)
- FD-FE: Crochet, Davies and Walters (1984)
- FE-FV: Wapperom and Webster (1998,1999), Aboubacar, Matallah, and Webster (2002)

High Weissenberg Number Problem!

Logarithm transformation: Fattal and Kupferman (FD, 2004, 2005)

$$\psi = \ln \boldsymbol{\sigma}$$

FE: Turek's group (2010, 2010); Pan's group (2007, 2007)
FV: Afonso, Oliveira, Pinho and Alves (2009)

Square-root: Balci, Thomases, Renardy, Doering (2011)

$$\boldsymbol{\psi} = \boldsymbol{\sigma}^{1/2}$$

Kernel: Afonso, Pinho and Alves (2012)

$$\boldsymbol{\psi} = \boldsymbol{\sigma}^k, \boldsymbol{\psi} = \log_a \boldsymbol{\sigma}.$$

Euler-Lagrangian method:
FD: Trebotich, Colella, and Miller (2005)
FE: Lee and Xu (2006), Lee, Xu, and Zhang(2011,2011)

$$\begin{aligned} \frac{\delta \boldsymbol{\sigma}}{\delta t} &= \frac{\partial \boldsymbol{\sigma}}{\partial t} + (\mathbf{u} \cdot \nabla)\boldsymbol{\sigma} - \nabla \mathbf{u} \cdot \boldsymbol{\sigma} - \boldsymbol{\sigma} \cdot (\nabla \mathbf{u})^T \\ &\approx \frac{\boldsymbol{\sigma}^{n+1} - \mathbf{F}(\boldsymbol{\sigma}^n \circ X)\mathbf{F}^T}{\Delta t}, \text{ where } \frac{d\mathbf{F}}{dt} = \nabla \mathbf{u} \mathbf{F}. \end{aligned}$$

For any matrix $\nabla \mathbf{u}$ and symmetric positive definite matrix $\boldsymbol{\sigma}$:

$$\nabla \mathbf{u} = \mathbf{B} + \boldsymbol{\Omega} + \mathbf{N}\boldsymbol{\sigma}^{-1}.$$

$\mathbf{N}, \boldsymbol{\Omega}$ are anti-symmetric, $\mathbf{B}$ is symmetric and commutes with $\boldsymbol{\sigma}$.

$$\frac{\partial \boldsymbol{\sigma}}{\partial t} + \mathbf{u} \cdot \nabla \boldsymbol{\sigma} - \nabla \mathbf{u} \boldsymbol{\sigma} - \boldsymbol{\sigma} (\nabla \mathbf{u})^T = \frac{1}{We}(\mathbf{I} - \boldsymbol{\sigma})$$

can be written as

$$\frac{\partial \boldsymbol{\sigma}}{\partial t} + (\mathbf{u} \cdot \nabla)\boldsymbol{\sigma} = \underbrace{\Omega\boldsymbol{\sigma} - \Omega\boldsymbol{\sigma}}_{rotation} + \underbrace{2\mathbf{B}\boldsymbol{\sigma}}_{stretch} + \frac{1}{We}(\mathbf{I} - \boldsymbol{\sigma}).$$

log-transform $\psi = \ln \boldsymbol{\sigma}$

$$\frac{\partial \psi}{\partial t} + \mathbf{u} \cdot \nabla \psi = \underbrace{\boldsymbol{\Omega}\psi - \psi\boldsymbol{\Omega}}_{rotation} + \underbrace{2\mathbf{B}}_{stretch} + \frac{1}{We}(e^{-\psi} - \mathbf{I}).$$

Stretch: exponential to polynomial.

Is "High Weissenberg Number Problem" solved?

Motivation

- the log-transform + the diffusive model
- construct numerical scheme such that
  - stable for high We
  - convergent

Diffusive Oldroyd-B model with LCR:

$$(B)\begin{cases} \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} = -\nabla p + \alpha\Delta\mathbf{u} + \frac{\beta}{We}\nabla\cdot(e^{\psi} - \mathbf{I}) \\ \nabla\cdot\mathbf{u} = 0 \\ \frac{\partial \psi}{\partial t} + \mathbf{u}\cdot\nabla\psi = \mathbf{\Omega}\psi - \psi\mathbf{\Omega} + 2\mathbf{B} + \frac{1}{We}(e^{-\psi} - \mathbf{I}) + \varepsilon\Delta\psi \end{cases}$$

Energy stability

Free-energy (Hu and Lelièvre, 2006)

$$F(\mathbf{u}, \boldsymbol{\sigma}) = Re\ Ke + \frac{\beta}{2We}\ En. \tag{1}$$

$$Ke = \frac{1}{2}\int_{\mathcal{T}} \mathbf{u}^2 \geq 0,$$

$$En = \int_{\mathcal{T}} \mathrm{tr}(\boldsymbol{\sigma} - \ln \boldsymbol{\sigma} - \mathbf{I}) \geq 0.$$

Theorem 1

(Energy estimate for the diffusive logarithmic Oldroyd-B model) Let $(\mathbf{u}, p, \psi)$ be a smooth solution to system (B), supplied with the homogeneous Dirichlet boundary condition for velocity, and with the zero Neumann boundary condition for ψ. Further, we assume that initially e^{ψ} is a symmetric positive-definite tensor. The free energy satisfies:

$$\frac{d}{dt}F(\mathbf{u}, e^{\psi}) + \alpha \int_{\mathcal{T}} |\nabla \mathbf{u}|^2 + \frac{\beta}{2We} \int_{\mathcal{T}} tr(e^{\psi} + e^{-\psi} - 2\mathbf{I}) \leq 0. \qquad (2)$$

From this estimate, we obtain that $F(\mathbf{u}, e^{\psi})$ decreases in time exponentially fast to zero.

Inner product of the Navier-Stokes equation with the velocity:

$$\frac{Re}{2}\frac{d}{dt}\int_{\mathcal{T}}\mathbf{u}^2+\alpha\int_{\mathcal{T}}|\nabla\mathbf{u}|^2+\frac{\beta}{We}\int_{\mathcal{T}}\nabla\mathbf{u}:e^{\psi}=0 \tag{3}$$

$(B)_3 : e^{\psi}$

$$\frac{d}{dt}\int_{\mathcal{T}}\mathrm{tr}(e^{\psi})=2\int_{\mathcal{T}}\nabla\mathbf{u}:e^{\psi}+\frac{1}{We}\mathrm{tr}(\mathbf{I}-e^{\psi})+\varepsilon\int_{\mathcal{T}}\Delta\psi:e^{\psi}. \tag{4}$$

$$\frac{d}{dt}\psi:e^{\psi}=\frac{d}{dt}\mathrm{tr}(e^{\psi}).$$

$$(\boldsymbol{\Omega}\psi-\psi\boldsymbol{\Omega}):e^{\psi}=\mathrm{tr}(\boldsymbol{\Omega}\psi e^{\psi})-\mathrm{tr}(\psi\boldsymbol{\Omega}e^{\psi})=0.$$

$$\nabla\mathbf{u}:e^{\psi}=\boldsymbol{\Omega}:e^{\psi}+\mathbf{B}:e^{\psi}+\mathbf{N}e^{-\psi}:e^{\psi}=\mathbf{B}:e^{\psi}.$$

(3) + $\frac{\beta}{2We}\times$ (4) $\Rightarrow$

$$\frac{d}{dt}\int_{\mathcal{T}}\left(\frac{Re}{2}\mathbf{u}^2+\frac{\beta}{2We}\mathrm{tr}(e^{\psi}-\mathbf{I})\right)+\int_{\mathcal{T}}\left(\alpha|\nabla\mathbf{u}|^2+\frac{\beta}{2We^2}\mathrm{tr}(e^{\psi}-\mathbf{I})\right)$$
$$=\frac{\varepsilon\beta}{2We}\int_{\mathcal{T}}\Delta\psi:e^{\psi}=-\frac{\varepsilon\beta}{2We}\int_{\mathcal{T}}\nabla\psi:\nabla e^{\psi}\leq 0. \tag{5}$$

$$\frac{\partial e^{\psi}}{\partial x}\neq e^{\psi}\frac{\partial\psi}{\partial x}.$$
$$\psi=Rdiag(\lambda_i)R^T,\boldsymbol{\sigma}=Rdiag(e^{\lambda_i})R^T.$$

We cannot show

$$e^{\psi}-\mathbf{I}\geq 0$$

but

$$e^{\psi}-\psi-\mathbf{I}\geq 0.$$

Taking the trace of $(B)_3$, $$\frac{d}{dt}\int_{\mathcal{T}} \mathrm{tr}(\boldsymbol{\psi}) = \frac{1}{We}\mathrm{tr}(e^{-\boldsymbol{\psi}} - \mathbf{I}). \quad (6)$$

$$\frac{d}{dt}\int_{\mathcal{T}}\left(\frac{Re}{2}\mathbf{u}^2 + \frac{\beta}{2We}\mathrm{tr}(e^{\boldsymbol{\psi}} - \boldsymbol{\psi} - \mathbf{I})\right) + \int_{\mathcal{T}}\left(\alpha|\nabla\mathbf{u}|^2 + \frac{\beta}{2We^2}\mathrm{tr}(e^{\boldsymbol{\psi}} + e^{-\boldsymbol{\psi}} - 2\mathbf{I})\right) \leq 0.$$

Poincaré inequality: $\int_{\mathcal{T}} \mathbf{u}^2 \leq C_p \int_{\mathcal{T}} |\nabla\mathbf{u}|^2$.

$$0 \leq \mathrm{tr}(e^{\boldsymbol{\psi}} - \boldsymbol{\psi} - \mathbf{I}) \leq \mathrm{tr}(e^{\boldsymbol{\psi}} - \boldsymbol{\psi} - \mathbf{I}) + \mathrm{tr}(e^{-\boldsymbol{\psi}} + \boldsymbol{\psi} - \mathbf{I}) = \mathrm{tr}(e^{\boldsymbol{\psi}} + e^{-\boldsymbol{\psi}} - 2\mathbf{I}).$$

$$\frac{d}{dt}F(\mathbf{u},\boldsymbol{\sigma}) \leq -cF(\mathbf{u},\boldsymbol{\sigma}) \leq 0 \quad (c = min(\frac{2\alpha}{Re\ C_p}, \frac{1}{We}) > 0).$$

Apply the Gronwall inequality,

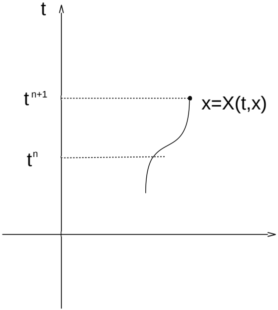
$$F(\mathbf{u},\boldsymbol{\sigma}) \leq F(\mathbf{u}(t=0), \boldsymbol{\sigma}(t=0))e^{-ct}.$$

Numerical methods

- Schemes
- Energy stability of the schemes

Characteristic method

$$\begin{cases} \frac{d}{dt}X = \mathbf{u}(X,t), \ \forall t \in [t^n, t^{n+1}], \\ X(t;x) = x. \end{cases}$$



Material derivative:

$$\frac{D\phi}{Dt} = \frac{\partial \phi}{\partial t} + \mathbf{u}\cdot\nabla\phi$$
$$\approx \frac{\phi(X(t^{n+1}), t^{n+1}) - \phi(X(t^n), t^n)}{t^{n+1} - t^n}$$

Nonlinear convection is avoided, symmetric coefficient matrix.
No CFL condition.

$$\frac{Re}{\Delta t}(\mathbf{u}_h^{n+1} - \mathbf{u}_h^n \circ X, \mathbf{v}_h) + 2\alpha(D(\mathbf{u}_h^{n+1}), D(\mathbf{v}_h))$$

$$-(p_h^{n+1}, \nabla \cdot \mathbf{v}_h) - (\nabla \cdot \mathbf{u}_h^{n+1}, q_h) + s_h(p_h^{n+1}, q_h) = -\frac{\beta}{We}(e^{\psi_h^{n+1}}, \nabla \mathbf{v}_h)$$

Stabilization $s_h(p,q) = -\delta \sum_K h_K^2 (\nabla p, \nabla q)_K, \quad \delta > 0.$

$$\frac{1}{\Delta t}(\psi_h^{n+1} - \psi_h^n \circ X^n, \phi) + \varepsilon(\nabla \psi_h^{n+1}, \nabla \phi_h)$$

$$= (\mathbf{\Omega}_h^{n+1}\psi_h^{n+1} - \psi_h^{n+1}\mathbf{\Omega}_h^{n+1} + 2B_h^{n+1}, \phi_h) + \frac{1}{We}(e^{-\psi_h^{n+1}} - \mathbf{I}, \phi_h)$$

$$X_h \equiv \{\mathbf{v}_h \in C^0(\bar{\mathcal{T}}_h)^d; \mathbf{v}_h|_K \in \mathcal{P}^1(K)^d, \forall K \in \mathcal{T}_h\},\ V_h \equiv X_h \cap H_0^1(\mathcal{T})^d,$$

$$M_h \equiv \{q_h \in C^0(\bar{\mathcal{T}}_h); q_h|_K \in \mathcal{P}^1(K), \forall K \in \mathcal{T}_h\},\ Q_h \equiv M_h \cap L_0^2(\mathcal{T}),$$

$$\Sigma_h \equiv \{\phi_h \in C^0(\bar{\mathcal{T}}_h)^{d\times d}; \phi_h|_K \in \mathcal{P}^1(K)^{d\times d}, \forall K \in \mathcal{T}_h\},\ W_h \equiv \Sigma_h \cap H^1(\mathcal{T})^{d\times d}.$$

Algorithm

Step-1 *Given* $\mathbf{u}_h^n, p_h^n, \psi_h^n$, *set* $\mathbf{u}_h^{n,0} = \mathbf{u}_h^n, \psi_h^{n,0} = \psi_h^n, p_h^{n,0} = p_h^n$.

Step-2 *FOR* $\ell = 0, 1, \cdots,$

solve the equations with explicit RHS:

$$\frac{Re}{\Delta t}(\mathbf{u}_h^{n,\ell+1}, \mathbf{v}_h) + 2\alpha(D(\mathbf{u}_h^{n,\ell+1}), D(\mathbf{v}_h)) - (p_h^{n,\ell+1}, \nabla \cdot \mathbf{v}_h) - (\nabla \cdot \mathbf{u}_h^{n,\ell+1}, q_h)$$
$$+ s_h(p_h^{n,\ell+1}, q_h) = \frac{Re}{\Delta t}(\mathbf{u}_h^n \circ X, \mathbf{v}_h) - \frac{\beta}{We}(e^{\psi_h^{n,\ell}}, \nabla \mathbf{v}_h)$$

$$\frac{1}{\Delta t}(\psi_h^{n,\ell+1}, \phi_h) + \varepsilon(\nabla \psi_h^{n,\ell+1}, \nabla \phi_h) = (\boldsymbol{\Omega}_h^{n,\ell}\psi_h^{n,\ell} - \psi_h^{n,\ell}\boldsymbol{\Omega}_h^{n,\ell} + 2B_h^{n,\ell}, \phi_h)$$
$$+ (\frac{1}{\Delta t}\psi_h^n \circ X_1(\mathbf{u}_h^{n,\ell}, \Delta t), \phi_h) + \frac{1}{We}(e^{-\psi_h^{n,\ell}} - \mathbf{I}, \phi_h)$$

IF $\|\mathbf{w}_h^{n,\ell+1} - \mathbf{w}_h^{n,\ell}\| \leq \xi \|\mathbf{w}_h^{n,\ell}\|$ *where* $\mathbf{w}_h \in \{\mathbf{u}_h, p_h, \boldsymbol{\sigma}_h\}, \xi$ *is small*

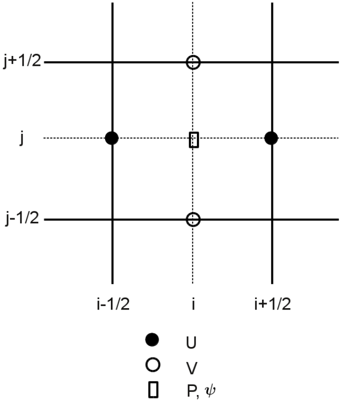
break

ENDIF

ENDFOR

Step-3 *Update solution:* $\mathbf{u}_h^{n+1} = \mathbf{u}_h^{n,\ell+1}, p_h^{n+1} = p_h^{n,\ell+1}, \psi_h^{n+1} = \psi_h^{n,\ell+1}$.

Staggered mesh



$$\frac{(\psi^{n+1} - \psi^n \circ \mathbf{X}^n)_{i,j}}{\Delta t} = (\boldsymbol{\Omega}^{n+1}\psi^{n+1} - \psi^{n+1}\boldsymbol{\Omega}^{n+1} + 2\mathbf{B}^{n+1})_{i,j} + \frac{1}{We}(e^{-\psi_{i,j}^{n+1}} - \mathbf{I}) + \varepsilon \Delta_h \psi_{i,j}^{n+1}. \tag{7a}$$

$$\begin{aligned} Re&\frac{(U^{n+1} - U^n \circ \mathbf{X}^n)_{i+1/2,j}}{\Delta t} \\ &= -Re\delta_x (U^{n+1})^2_{i+1/2,j} - Re\delta_y (U^{n+1}V^{n+1})_{i+1/2,j} - (\delta_x p^{n+1})_{i+1/2,j} \\ &+ \alpha \Delta_h U_{i+1/2,j}^{n+1} + (\delta_x \sigma_{11}^{n+1})_{i+1/2,j} + (\delta_y \sigma_{12}^{n+1})_{i+1/2,j}, \end{aligned} \tag{7b}$$

$$\begin{aligned} Re&\frac{(V^{n+1} - V^n \circ \mathbf{X}^n)_{i,j+1/2}}{\Delta t} \\ &= -Re\delta_x (U^{n+1}V^{n+1})_{i+1/2,j} - Re\delta_y (V^{n+1})^2_{i+1/2,j} - (\delta_y p^{n+1})_{i,j+1/2} \\ &+ \alpha \Delta_h V_{i,j+1/2}^{n+1} + (\delta_x \sigma_{21}^{n+1})_{i,j+1/2} + (\delta_y \sigma_{22}^{n+1})_{i,j+1/2}, \end{aligned} \tag{7c}$$

$$\nabla_h \cdot \mathbf{u}_{i,j}^{n+1} := \delta_x U_{i,j}^{n+1} + \delta_y V_{i,j}^{n+1} = 0. \tag{7d}$$

Chorin projection

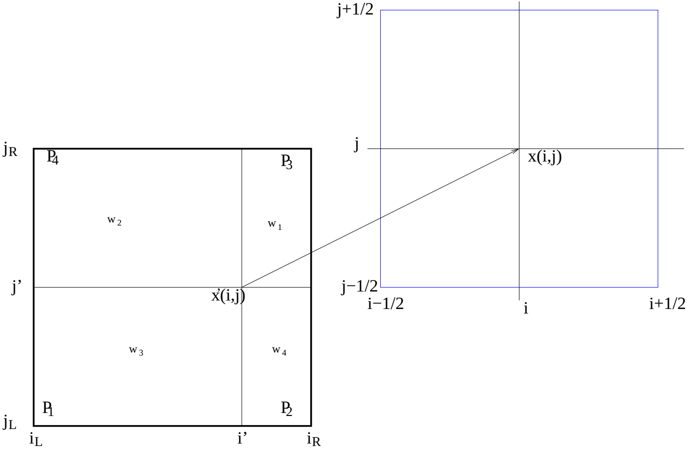
$$\mathbf{u}_t - \frac{\alpha}{Re}\Delta\mathbf{u} = -\mathbf{u}\cdot\nabla\mathbf{u} + \frac{1-\alpha}{Re}\frac{1}{We}\nabla\cdot\boldsymbol{\sigma}$$

$$\mathbf{u}_t = -\frac{1}{Re}\nabla p$$

$$\frac{1}{\Delta t}(\mathbf{u}^{n+1} - \mathbf{u}^*) = -\nabla p^{n+1}$$

a). $F = \nabla\cdot\mathbf{u}^*$
b). $-\Delta p^{n+1} = -\frac{1}{\Delta t}F$
c). $\mathbf{G} = \nabla p^{n+1}$
d). $\mathbf{u}^{n+1} = \mathbf{u}^* - \Delta t\mathbf{G}$.

$$\nabla\cdot\mathbf{u}^{n+1} = \nabla\cdot(\mathbf{u}^* - \Delta t\mathbf{G}) = F - \Delta t\nabla\cdot(\nabla p^{n+1}) = 0.$$



$$\psi^n \circ X^n = \psi^n(\mathbf{x}') = \sum w_k \psi(\mathbf{x}(P_k)), \quad \mathbf{x}' = \mathbf{x} - \mathbf{u}\Delta t.$$

Lemma 2

Let $(\mathbf{u}_h^n, p_h^n, \psi_h^n)_{0\leq n\leq N_T}$ be a solution to the characteristic FEM scheme, supplied with homogeneous Dirichlet boundary condition for velocity, and with the zero Neumann boundary condition for ψ_h. Further, we assume that initially $e^{\psi_\mathbf{h}}$ is symmetric positive-definite. Then the free energy

$$F_h^n = F(\mathbf{u}_h^n, e^{\psi_h^n}) = \frac{Re}{2}\int_\Omega |\mathbf{u}_h^n|^2 + \frac{\beta}{2We}\int_\Omega tr(e^{\psi_h^n} - \psi_h^n - \mathbf{I})$$

satisfies

$$F_h^{n+1} + \Delta t \int_\Omega 2\alpha C_k |\nabla \mathbf{u}_h^{n+1}|^2 + \frac{\beta}{2We^2} tr(e^{\psi_h^n} + e^{-\psi_h^n} - 2\mathbf{I}) \leq F_h^n. \tag{8}$$

In particular, the sequence $(F_h^n)_{0\leq n\leq N_T}$ is non-increasing.

Lemma 3

Let $(U^n_{i+1/2,j}, V^n_{i,j+1/2}, p^n_{i,j}, \psi^n_{i,j})_{0\le n\le N_T}$ be a solution of the discrete characteristic FD scheme, supplied with homogeneous Dirichlet boundary condition for velocity, and with the zero Neumann boundary condition for ψ. Further, we assume that initially e^{ψ} is symmetric positive-definite. Then the free energy

$$F^n_h = \frac{Re}{2}\left(\sum_{i=1}^{M-1}\sum_{j=1}^{N}(U^n_{i+1/2,j})^2 + \sum_{i=1}^{M}\sum_{j=1}^{N-1}(V^n_{i,j+1/2})^2\right) + \frac{\beta}{2We}\sum_{i=1}^{M}\sum_{j=1}^{N} tr\left(e^{\psi^n} - \psi^n - \mathbf{I}\right)_{i,j}$$

satisfies

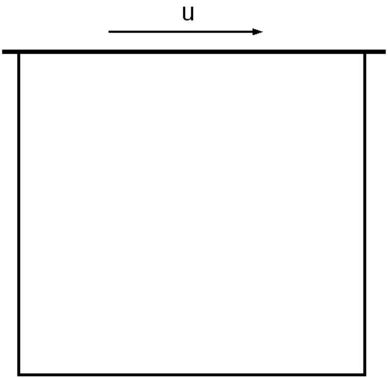
$$F^{n+1}_h + \alpha\Delta t\left(\sum_{i=1}^{M-1}\sum_{j=1}^{N}|\nabla_h U^{n+1}_{i+1/2,j}|^2 + \sum_{i=1}^{M}\sum_{j=1}^{N-1}|\nabla_h V^{n+1}_{i,j+1/2}|^2\right) + \frac{\Delta t\beta}{2We^2}\sum_{i=1}^{M}\sum_{j=1}^{N} tr(e^{\psi^{n+1}} + e^{-\psi^{n+1}} - 2\mathbf{I})_{i,j} \le F^n_h. \quad (9)$$

In particular, the sequence $(F^n_h)_{0\le n\le N_T}$ is non-increasing.

Initial values $\mathbf{u} = \mathbf{0}, \boldsymbol{\sigma} = \mathbf{I}$. Boundary conditions

$$\mathbf{u} = \begin{cases} (16x^2(1-x)^2, 0)^T, & \text{if } \; y = 1, x \in (0,1), \\ \mathbf{0}, & \text{else.} \end{cases}$$

$$\frac{\partial \psi}{\partial \mathbf{n}} = 0.$$



Model (A): Standard-methods fail with $We > 1$

Model (B): Available for high We

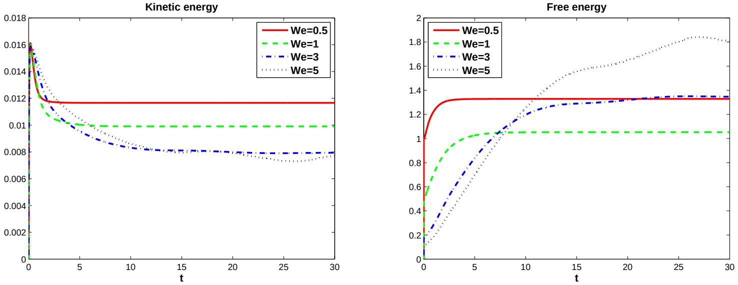
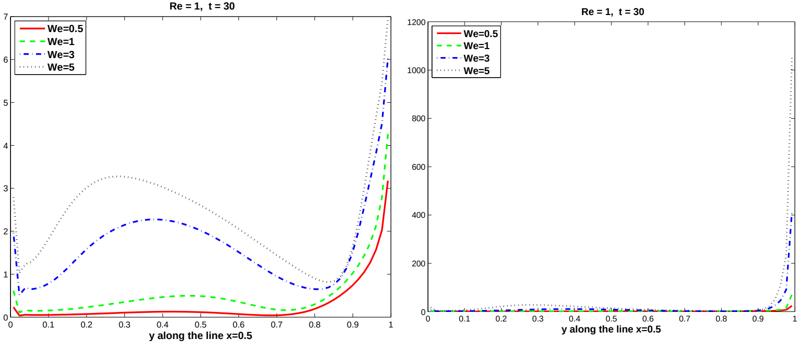


Figure : Kinetic and free energy.



(a) ψ_{11} **(b)** σ_{11}

Figure : Conformation tensor component along the mid-line $x = 0.5$ at $t = 30$, $Re = 1$.

Table : L^2-norm error with respect to mesh refinement of the non-diffusive Oldroyd-B model for σ_{11}: $\|\sigma_{11}(h) - \sigma_{11}(h/2)\|$.

mesh size h	$We = 0.5$	$We = 1$	$We = 3$
$1/32$	0.3502	1.5846	4.8967
$1/64$	0.5006	3.3141	10.8242
$1/128$	0.7181	5.3517	18.5389

Results do not converge !!

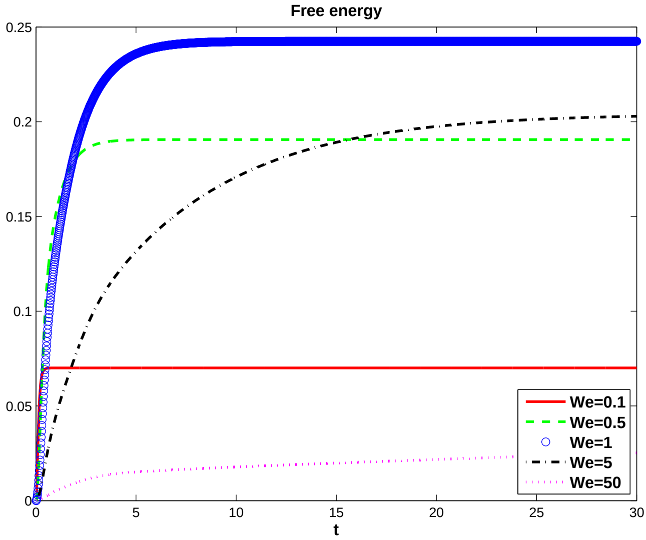


Figure : Free energy of the diffusive Oldroyd-B model (B), $\varepsilon = 0.01$.

Numerical methods - test

Diffusive model

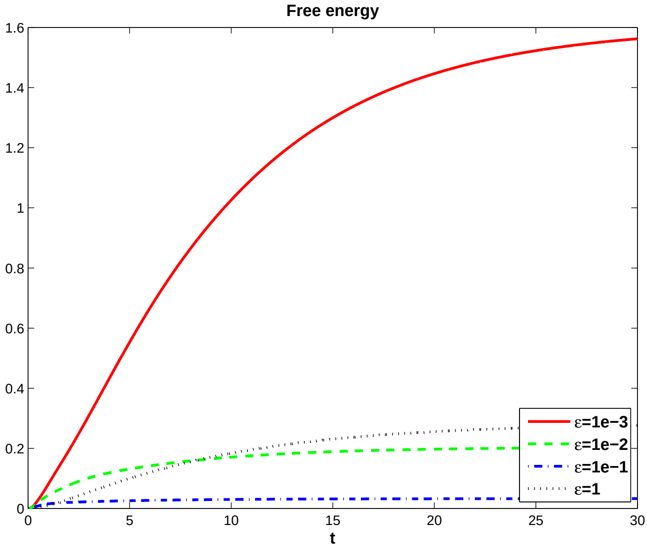


Figure : Free energy of the diffusive Oldroyd-B model, $We = 5$.

Error						
h	$e(\mathbf{u}_h)$	$e(p_h)$	$e(\boldsymbol{\sigma}_h)$	$e_1(u)$	$e_1(p_h)$	$e_1(\boldsymbol{\sigma}_h)$
1/8	1.56e-02	1.84e-01	2.12e+00	5.85e-01	5.55e-01	3.15e+01
1/16	4.69e-03	9.74e-02	1.06e+00	3.19e-01	2.19e-01	1.86e+01
1/32	1.08e-03	3.07e-02	3.27e-01	1.51e-01	8.18e-02	8.05e+00
1/64	3.16e-04	1.21e-02	1.31e-01	7.54e-02	3.68e-02	4.12e+00
1/128	9.00e-05	4.61e-03	4.32e-02	3.85e-02	1.72e-02	1.98e+00
EOC						
1/8	1.73	0.91	0.99	0.88	1.34	0.76
1/16	2.11	1.67	1.70	1.07	1.42	1.21
1/32	1.78	1.34	1.32	1.01	1.15	0.97
1/64	1.81	1.39	1.60	0.97	1.10	1.05

Table : Error norms and EOC for diffusive Oldroyd-B model (B), $\varepsilon = 0.01$, We=5, computed by characteristic FEM scheme.

Error						
h	$e(\mathbf{u}_h)$	$e(p_h)$	$e(\boldsymbol{\sigma}_h)$	$e_1(\mathbf{u}_h)$	$e_1(p_h)$	$e_1(\boldsymbol{\sigma}_h)$
1/8	2.08e-02	3.76e-01	3.32e+00	2.26e-01	3.08e+00	3.42e+01
1/16	6.99e-03	1.89e-01	2.01e+00	9.11e-02	1.70e+00	2.13e+01
1/32	2.84e-03	8.78e-02	1.03e+00	3.80e-02	8.29e-01	1.02e+01
1/64	1.10e-03	3.16e-02	3.90e-01	1.35e-02	2.78e-01	3.71e+00
EOC						
1/8	1.58	0.99	0.72	1.31	0.86	0.68
1/16	1.30	1.11	0.97	1.26	1.03	1.06
1/32	1.37	1.47	1.40	1.49	1.57	1.46

Table : Error norms and EOC for diffusive Oldroyd-B model (B), $\varepsilon = 0.01$, We=5, computed by characteristic FD scheme.

Conclusion

- Energy stability for the diffusive Oldroyd-B model with LCR
- Numerical methods
  - Scheme 1: Characteristic FEM
  - Scheme 2: Characteristic FD
- Energy stability for the characteristic schemes on the discrete level
- Observed mesh convergence.

Thank you for your attention!

Appendix

For any symmetric positive-definite matrix $\boldsymbol{\sigma}(t) \in (C^1([0, T]))^{\frac{d(d+1)}{2}}$, we have $\forall t \in [0, T)$:

$$(\frac{d}{dt}\boldsymbol{\sigma}) : \boldsymbol{\sigma}^{-1} = \mathrm{tr}(\boldsymbol{\sigma}^{-1}\frac{d}{dt}\boldsymbol{\sigma}) = \frac{d}{dt}\mathrm{tr}(\ln\boldsymbol{\sigma}),$$

$$(\frac{d}{dt}\ln\boldsymbol{\sigma}) : \boldsymbol{\sigma} = \mathrm{tr}(\boldsymbol{\sigma}\frac{d}{dt}\ln\boldsymbol{\sigma}) = \frac{d}{dt}\mathrm{tr}\boldsymbol{\sigma}.$$

Since $\boldsymbol{\sigma}(t) \in (C^1([0, T)))^{\frac{d(d+1)}{2}}$ is a symmetric positive-definite matrix, $\det\boldsymbol{\sigma}$ is positive and $C^1([0, T))$. Thus, we get the classical Jacobi formula

$$\frac{d}{dt}\mathrm{tr}(\ln\boldsymbol{\sigma}) = \frac{d}{dt}\ln(\det\boldsymbol{\sigma}) = (1/\det\boldsymbol{\sigma})\frac{d}{dt}\det\boldsymbol{\sigma} = \mathrm{tr}(\boldsymbol{\sigma}^{-1}\frac{d}{dt}\boldsymbol{\sigma}) = (\frac{d}{dt}\boldsymbol{\sigma}) : \boldsymbol{\sigma}^{-}$$

For the proof of the second equation, we set $\psi = \ln\boldsymbol{\sigma}$ and then we can show

$$(\frac{d}{dt}\ln\boldsymbol{\sigma}) : \boldsymbol{\sigma} = \mathrm{tr}(\boldsymbol{\sigma}\frac{d}{dt}\ln\boldsymbol{\sigma}) = \mathrm{tr}(e^{\psi}\frac{d}{dt}\psi) = \mathrm{tr}(\frac{d}{dt}e^{\psi}) = \frac{d}{dt}\mathrm{tr}\boldsymbol{\sigma}.$$

Appendix

Let $\boldsymbol{\sigma}, \boldsymbol{\tau} \in R^{d\times d}$ be symmetric positive-definite matrices, f_1 be an increasing function, and f_2 be a decreasing function, we have:

$$
\begin{aligned}
(\boldsymbol{\sigma} - \boldsymbol{\tau}) : (f_1(\boldsymbol{\sigma}) - f_1(\boldsymbol{\tau})) &\geq 0, \\
(\boldsymbol{\sigma} - \boldsymbol{\tau}) : (f_2(\boldsymbol{\sigma}) - f_2(\boldsymbol{\tau})) &\leq 0, \\
\nabla\boldsymbol{\sigma} : \nabla\boldsymbol{\sigma}^{-1} &\leq 0, \\
\nabla(\ln\boldsymbol{\sigma}) : \nabla\boldsymbol{\sigma} &\geq 0.
\end{aligned}
$$