On Markov selection for the compressible Navier–Stokes system

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Field equations

Phase variables

mass density .... $\varrho = \varrho(t,x)$

bulk velocity .... $\mathbf{u} = \mathbf{u}(t,x) \in R^N$

Mass conservation

$$\mathrm{d}\varrho + \mathrm{div}_x(\varrho\mathbf{u})\mathrm{d}t = 0$$

Balance of momentum

$$\mathrm{d}(\varrho\mathbf{u}) + \mathrm{div}_x(\varrho\mathbf{u}\otimes\mathbf{u})\mathrm{d}t + \nabla_x p(\varrho)\mathrm{d}t = \mathrm{div}_x\mathbb{S}(\nabla_x\mathbf{u})\mathrm{d}t + \boxed{\varrho\mathbf{g}(\varrho,\mathbf{u})\mathrm{d}W}$$

$$\mathbb{S}(\nabla_x\mathbf{u}) = \mu\left(\nabla_x\mathbf{u} + \nabla_x^t\mathbf{u} - \frac{2}{N}\mathrm{div}_x\mathbf{u}\mathbb{I}\right) + \lambda\mathrm{div}_x\mathbf{u}\mathbb{I}$$

Boundary and initial conditions

spatially periodic boundary conditions .... $Q = \left([-1,1]|_{\{-1,1\}}\right)^N$

$$\varrho(0,\cdot) = \varrho_0,\ \ \varrho\mathbf{u}(0,\cdot) = (\varrho\mathbf{u})_0$$

Stochastic forcing

Stochastic basis

$$\left\{ \Omega,\ \mathfrak{F},\ \{\mathfrak{F}_t\}_{t\geq 0},\ \mathcal{P} \right\}$$

Random driving force in the momentum equation

$$\mathbf{g}(\varrho, \mathbf{u})\mathrm{d}W \equiv \sum_{k=1}^{\infty} \mathbf{g}_k(\varrho, \mathbf{u})\mathrm{d}W_k$$

$$\mathbf{g}_k \in W^{1,\infty}(R^{N+1}, R^N),\ \|\mathbf{g}_k\|_{W^{1,\infty}} \leq \alpha_k \text{ where } \sum_{k=1}^{\infty} \alpha_k^2 < \infty$$

Deterministic initial conditions

$$\varrho(0,\cdot) = \varrho_0,\ (\varrho\mathbf{u})(0,\cdot) = (\varrho\mathbf{u})_0\ \mathcal{P} - \text{a.s.}$$

Weak martingale solution

Probability basis, forcing term

$$\left\{\Omega,\ \mathfrak{F},\ \{\mathfrak{F}_t\}_{t\geq 0},\ \mathcal{P}\right\},\ \{W_k\}_{k=1}^{\infty}$$

Field equations

$$\left[\int_Q \varrho\varphi\ \mathrm{d}x\right]_{t=0}^{t=\tau} = \int_0^\tau \int_Q \varrho\mathbf{u}\cdot\nabla_x\varphi\ \mathrm{d}x\mathrm{d}t,\ \int_Q \varrho(0,\cdot)\varphi\ \mathrm{d}x = \int_Q \varrho_0\varphi\ \mathrm{d}x$$

$$\left[\int_Q \varrho\mathbf{u}\cdot\boldsymbol{\varphi}\ \mathrm{d}x\right]_{t=0}^{t=\tau} = \int_0^\tau \int_Q \left[\varrho\mathbf{u}\otimes\mathbf{u} : \nabla_x\boldsymbol{\varphi} + p(\varrho)\mathrm{div}_x\boldsymbol{\varphi}\right]\ \mathrm{d}x\mathrm{d}t$$

$$-\int_0^\tau \int_Q \mathbb{S}(\nabla_x\mathbf{u}) : \nabla_x\boldsymbol{\varphi}\ \mathrm{d}x\mathrm{d}t + \sum_{k=1}^{\infty}\int_0^\tau \left(\int_Q \varrho\mathbf{g}_k(\varrho,\mathbf{u})\cdot\boldsymbol{\varphi}\ \mathrm{d}x\right)\mathrm{d}W_k$$

$$\int_Q \varrho\mathbf{u}(0,\cdot)\cdot\boldsymbol{\varphi}\ \mathrm{d}x = \int_Q (\varrho\mathbf{u})_0\cdot\boldsymbol{\varphi}\ \mathrm{d}x$$

Dissipative martingale solutions

Energy inequality

$$\left[\psi \int_Q \left[\frac{1}{2}\varrho|\mathbf{u}|^2 + P(\varrho)\right] \, \mathrm{d}x\right]_{t=0}^{t=\tau} + \int_0^\tau \psi \int_Q \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} \, \mathrm{d}x\mathrm{d}t$$

$$\leq \int_0^\tau \partial_t \psi \int_Q \left[\frac{1}{2}\varrho|\mathbf{u}|^2 + P(\varrho)\right] \, \mathrm{d}x\mathrm{d}t$$

$$+ \int_0^\tau \psi \int_Q \frac{1}{2} \sum_{k=1}^\infty \varrho|\mathbf{g}_k(\varrho,\mathbf{u})|^2 \, \mathrm{d}x\mathrm{d}t \int_0^\tau \psi \sum_{k=1}^\infty \left(\int_Q \varrho\mathbf{u} \cdot \mathbf{g}_k(\varrho,\mathbf{u}) \, \mathrm{d}x\right) \mathrm{d}W_k$$

Pressure potential

$$P(\varrho) = \varrho \int_1^\varrho \frac{p(z)}{z^2} \, \mathrm{d}z$$

Available results

Existence for finite energy data

- Global existence for any finite energy data, $p(\varrho) \approx \varrho^{\gamma}$, $\gamma > \frac{N}{2}$ **[Breit, Hofmanová 2015]**
- Local (up to a stopping time) existence of unique strong solutions **[Breit, EF, Hofmanová 2017], [Kim 2011]**

Weak–strong uniqueness

- Pathwise weak–strong uniqueness
- Weak–strong uniqueness in law **[Breit, EF, Hofmanová 2016]**

Augmented system

Mass conservation

$$\mathrm{d}\varrho + \mathrm{div}_x(\varrho \mathbf{u})\mathrm{d}t = 0$$

Balance of momentum

$$\mathrm{d}(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u})\mathrm{d}t + \nabla_x p(\varrho)\mathrm{d}t = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u})\mathrm{d}t + \boxed{\varrho \mathbf{g}(\varrho, \mathbf{u})\mathrm{d}W}$$

$$\mathbb{S}(\nabla_x \mathbf{u}) = \mu \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{N} \mathrm{div}_x \mathbf{u} \mathbb{I} \right) + \lambda \mathrm{div}_x \mathbf{u} \mathbb{I}$$

Velocity "potential"

$$\mathrm{d}\mathbf{v} = \mathbf{u} \ \mathrm{d}t, \ \mathbf{v}(0) = \mathbf{v}_0$$

Canonical (trajectory) space

Label space

$$X = \cup_{M>0} X_M$$

$$X_M \equiv \left\{ [r, \mathbf{m}, \mathbf{w}] \ \middle| \ r \geq 0, \ \int_Q \frac{1}{2} \frac{|\mathbf{m}|^2}{r} + P(r) \, \mathrm{d}x < M, \ \mathbf{w} \in W^{1,2}(Q; R^N) \right\}$$

Trajectory space

$$\Omega^{[0,\infty)} = \left\{ \boldsymbol{\xi} = [\xi_1, \boldsymbol{\xi}_2, \boldsymbol{\xi}_3] \ \middle| \ \boldsymbol{\xi} \in C_{\rm loc}([0,\infty); W^{-k,2}(Q; R^{2N+1})) \right\}$$

Solution space

$U(x)$ − the law of a dissipative martingale solution $[\varrho, \mathbf{u}, \mathbf{v}]$ on $\Omega^{[0,\infty)}$

with the initial data $\delta_{[\xi_1, \boldsymbol{\xi}_2, \boldsymbol{\xi}_3]=x}$

$$x = [\varrho_0, (\varrho \mathbf{u})_0, \mathbf{v}_0] \in X$$

$U(x)$ − non-empty, convex, compact for all $x \in X$

Markovian family – conditional probability

Markovian family

$$x \in X \mapsto U(x) \in \mathrm{prob}\left[\Omega^{[0,\infty)}\right] \text{ Borel measurable}$$

$$U(x)\{\boldsymbol{\xi}(0) = x\} = 1$$

Conditional probability

$$U(x)\left\{\boldsymbol{\xi}|_{[0,T]} \in A \ \wedge \ \boldsymbol{\xi}|_{[T,\infty)} \in B\right\}$$
$$= \int_{\boldsymbol{\xi} \in A} U([\boldsymbol{\xi}(T)])\{S_T \circ B\} \, \mathrm{d}U(x)[\boldsymbol{\xi}]$$

A Borel subset of $\Omega^{[0,T]}$, B Borel subset of $\Omega^{[T,\infty)}$

for a.a. $T > 0$, where the exceptional set may depend on $U(x)$

$$S_T(\boldsymbol{\xi})(t) = \boldsymbol{\xi}(t + T),$$
$$\Omega^I = \left\{\boldsymbol{\xi} \ \middle| \ \boldsymbol{\xi} \in C_{\mathrm{loc}}(I; W^{-k,2}(Q; R^{2N+1}))\right\}$$

Pre-Markovian family - disintegration

Families of measures

$$\mathcal{U}(x) = \left\{ U(x) \;\middle|\; U(x) \text{ a law of a solution with the initial value } \delta_x \right\}$$

Disintagration

$$U(x)\left\{ \boldsymbol{\xi}|_{[0,T]} \in A \ \wedge \ \boldsymbol{\xi}|_{[T,\infty)} \in B \right\} = \int_{\boldsymbol{\xi} \in A} V_T[\boldsymbol{\xi}]\{B\} \, \mathrm{d}U(x)[\boldsymbol{\xi}]$$

$$\Rightarrow$$

$$\boxed{V_T[\boldsymbol{\xi}] \in \mathcal{U}(\boldsymbol{\xi}(T)) \circ S_T}, \ U(x) \text{ a.s.}$$

for a.a. $T > 0$, where the exceptional set may depend on $U(x)$

Pre-Markovian family – reconstruction

Conditional probability

$$V \in \mathcal{U}(x)$$

$$T > 0$$

$$y \in X \mapsto \widetilde{U}(y) \in \mathcal{U}(y)$$

$$\Rightarrow$$

$$\text{there exists } \boxed{U(x) \in \mathcal{U}(x)}, \; U|_{[0,T]} = V$$

$$U(x)\left\{\boldsymbol{\xi}|_{[0,T]} \in A \;\wedge\; \boldsymbol{\xi}|_{[T,\infty)} \in B\right\}$$
$$= \int_{\boldsymbol{\xi} \in A} \widetilde{U}(\boldsymbol{\xi}(T))\{S_T \circ B\} \, \mathrm{d}U(x)[\boldsymbol{\xi}]$$

Markov selection

Sufficient conditions for the existence of Markov selection [Krylov, Stroock–Varadhan, Flandoli–Romito]

- $Y \hookrightarrow X$ dense embedding, X and Y Polish spaces
- there exists a pre–Markovian family $\{\mathcal{U}(y)\}_{y \in Y}$
- for each $y \in Y$, the family of measures $\mathcal{U}(y)$ is non–void, convex, and compact

$$\Rightarrow$$

There exists a Markov selection

$$y \in Y \mapsto U(y) \in \mathrm{prob}\left[\Omega^{[0,\infty)}\right]$$