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# Artificial boundary conditions for linearized stationary incompressible viscous flow around rotating and translating body

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**Abstract**

We consider the linearized and nonlinear stationary incompressible flow around rotating and translating body in the exterior domain $\mathbb{R}^3 \setminus \overline{\mathcal{D}}$, where $\mathcal{D} \subset \mathbb{R}^3$ is open and bounded, with Lipschitz boundary. We derive the pointwise estimates for the pressure in both cases. Moreover, we consider the linearized problem in a truncation domain $\mathcal{D}_R := B_R \backslash \overline{\mathcal{D}}$ of the exterior domain $\mathbb{R}^3 \setminus \overline{\mathcal{D}}$ under certain artificial boundary conditions on the truncating boundary $\partial B_R$, and then compare this solution with the solution in the exterior domain $\mathbb{R}^3 \setminus \overline{\mathcal{D}}$ to get the truncation error estimate.

## 1 Introduction

We consider the systems of equations

$$\begin{aligned} -\Delta u(z) + (\tau e_1 - \varrho e_1 \times z) \cdot \nabla u(z) + \varrho e_1 \times u(z) \\ +\tau(u(z) \cdot \nabla)u(z) + \nabla \pi(z) = F(z) \\ \operatorname{div} u(z) = 0 \text{ for } z \in \mathbb{R}^3 \setminus \overline{\mathcal{D}} \end{aligned} \tag{1.1}$$

$$\begin{aligned} -\Delta u(z) + (\tau e_1 - \varrho e_1 \times z) \cdot \nabla u(z) + \varrho e_1 \times u(z) + \nabla \pi(z) = F(z) \\ \operatorname{div} u(z) = 0 \text{ for } z \in \mathbb{R}^3 \setminus \overline{\mathcal{D}} \end{aligned} \tag{1.2}$$

where $\mathcal{D} \subset \mathbb{R}^3$ is open and bounded, with Lipschitz boundary. Problems (1.1) and (1.2) together with some boundary conditions on $\partial\mathcal{D}$ constitute mathematical models (linear and non-linear, respectively) describing stationary flow of a viscous incompressible fluid around a rigid body which moves at a constant velocity and rotates at a constant angular velocity, where we consider that the rotation is parallel to the velocity at the infinity. For details concerning of deriving the model, see [11, 15]. The description and the analysis in the case when the rotation is not parallel to the velocity at infinity can be find in the following works, see [13, 17].

The aim of this paper is two folds:

First, we would like to derive the pointwise estimates for the pressure in the linear and also in the non-linear cases in order to complete the pointwise estimates for the velocity and its gradient from [7, 8] by the pointwise estimates of the pressure in order to get complete decay information of all parts $u$, $\pi$ of solutions to systems (1.1), (1.2). Let us mention that the decay of pressure was also investigated in the work of Galdi, Kyed [16] and in case of pure rotation see [12].

Second, to solve the linear system (1.2) in a truncation $\mathcal{D}_R := B_R \backslash \overline{\mathcal{D}}$ of the exterior domain $\mathbb{R}^3 \setminus \overline{\mathcal{D}}$ under certain artificial boundary conditions on the truncating boundary $\partial B_R$, and then compare this solution with the solution of (1.2) in the exterior domain, i.e. to get some sort of error estimates of the method of an artificial boundary condition. For this aim we use pointwise estimates of the velocity and of the pressure.

Mathematical analysis of the problem of the Navier-Stokes equations with artificial boundary condition was performed by many authors but without considering the rotation of body, see e.g. [1, 9, 3, 4]. The article can be seen as a first result in the case of motion of viscous fluids around rotating and translating body with artificial boundary condition.

The paper is organized as follows: In the rest of this section we introduce notation and give some auxiliary results. The next section 2 deals with pointwise estimates of the pressure of the linear system (1.2). In Section 3 we consider the linear system (1.2) with artificial boundary conditions. The error estimate of the velocity is derived comparing to the solution to the system given in the exterior domain. First let us introduce notation:

## Definitions and notation related to the rotational system

Define $s(y) := 1 + |y| - y_1$ for $y \in \mathbb{R}^3$,

$\mathcal{D}_R := B_R \setminus \overline{\mathcal{D}}$, where $B_R := \{x \in \mathbb{R}^3; |x| < R\}$ for $R > 0$.

Fix $\tau \in (0, \infty)$, $\rho \in \mathbb{R} \setminus \{0\}$, and put $e_1 := (1, 0, 0)$, $\Omega := \rho \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$,

so that $\Omega \times z = \rho e_1 \times z$ for $z \in \mathbb{R}^3$.

For $U \subset \mathbb{R}^3$ open, $u \in W^{2,1}_{\text{loc}}(U)^3$, $z \in U$, put

$$\begin{aligned}(Lu)(z) &:= -\Delta u(z) + \tau\partial_1 u(z) - (\rho e_1 \times z)\cdot\nabla u(z) + \rho e_1 \times u(z),\\ (L^* u)(z) &:= -\Delta u(z) - \tau\partial_1 u(z) + (\rho e_1 \times z)\cdot\nabla u(z) - \rho e_1 \times u(z).\end{aligned}$$

Put

$$\begin{aligned}
&K(z,t) := (4\pi t)^{-3/2} e^{-|z|^2/(4t)} \quad (z \in \mathbb{R}^3, t \in (0,\infty)),\\
&\Lambda(z,t) := \left(K(z,t)\delta_{jk} + \partial z_j \partial z_k \left(\int_{\mathbb{R}^3} (4\pi|z-y|)^{-1} K(y,t)\,dy\right)\right)_{1\le j,k\le 3}\\
&(z \in \mathbb{R}^3,\ t > 0),\\
&\Gamma(x,y,t) := \Lambda(x - \tau t e_1 - e^{-t\Omega} y, z)\cdot e^{-t\Omega},\\
&\widetilde{\Gamma}(x,y,t) := \Lambda(x + \tau t e_1 - e^{t\Omega} y, t)\cdot e^{t\Omega} \quad (x,y \in \mathbb{R}^3, t > 0),\\
&\mathcal{Z}(x,y) := \int_0^\infty \Gamma(x,y,t)\,dt,\ \widetilde{\mathcal{Z}}(x,y) := \int_0^\infty \widetilde{\Gamma}(x,y,t)\,dt,\\
&(x,y \in \mathbb{R}^3,\ x \neq y).
\end{aligned}$$

For $q \in (1,2)$, $f \in L^q(\mathbb{R}^3)^3$, put

$$\mathcal{R}(f)(x) := \int_{\mathbb{R}^3} \mathcal{Z}(x,y) f(y)\,dy \quad (x \in \mathbb{R}^3);$$

see [6, Lemma 3.1]. We will use the space

$D^{1,2}_0(\overline{\mathcal{D}}^c)^3 := \{v \in L^6(\overline{\mathcal{D}}^c)^3 \cap H^1_{loc}(\overline{\mathcal{D}}^c)^3 \ :\ \nabla v \in L^2(\overline{\mathcal{D}}^c)^9,\ v|\partial\mathcal{D} = 0\}$ equipped with the norm $\|\nabla u\|_2$.

For $p \in (1,\infty)$, define $M_p$ as the space of all pairs of functions $(u,\pi)$ such that $u \in W^{2,p}_{loc}(\overline{\mathcal{D}}^c)^3$, $\pi \in W^{1,p}_{loc}(\overline{\mathcal{D}}^c)$,

$$\begin{gathered} u|\mathcal{D}_T \in W^{1,p}(\mathcal{D}_T)^3, \quad \pi|\mathcal{D}_T \in L^p(\mathcal{D}_T), \quad u|\partial\mathcal{D} \in W^{2-1/p,p}(\partial\mathcal{D})^3,\\ \operatorname{div} u|\mathcal{D}_T \in W^{1,p}(\mathcal{D}_T), \quad L(u) + \nabla\pi|\mathcal{D}_T \in L^p(\mathcal{D}_T)^3\end{gathered}$$

for some $T \in (0,\infty)$ with $\overline{\mathcal{D}} \subset B_T$.

We write $C$ for generic constants. It should be clear from context which are the parameters these constants depend on. In order to lift possible ambiguities, we sometimes use the notation $C(\gamma_1, ..., \gamma_n)$ in order to indicate that the constant in question depends in particular on $\gamma_1, ..., \gamma_n \in (0,\infty)$, for some $n \in \mathbb{N}$. But the relevant constant may depend on other parameters as well.

## Auxiliary results to asymptotic behavior of the pressure

**Lemma 1.1** (Weyl's lemma). *Let $u \in \mathbb{N}$, $U \subset \mathbb{R}^n$ open, $u \in L^1_{\text{loc}}(U)$ with $\int_U u\cdot\Delta l\,dx = 0$ for $l \in C^\infty_0(U)$. Then $u \in C^\infty(U)$ and $\Delta u = 0$.*

*Proof:* An elementary proof is given in [19, Appendix] $\square$

For $q \in (1, 3/2)$, $h \in L^q(\mathbb{R}^3)$, put

$$\mathcal{N}(h)(x) := \int_{\mathbb{R}^3} -(4\pi|x-y|)^{-1} h(y)dy \quad (x \in \mathbb{R}^3).$$

For $q \in (1, 3)$, $h \in L^q(\mathbb{R}^3)$, put

$$\mathcal{S}(h)(x) := \left( \int_{\mathbb{R}^3} (4\pi|x-y|^3)^{-1}(x-y)_j \cdot h(y)dy \right)_{1 \le j \le 3} \quad (x \in \mathbb{R}^3).$$

For $q \in (1, 3)$, $h \in L^q(\mathbb{R}^3)^3$, put

$$\mathcal{P}(h)(x) := \int_{\mathbb{R}^3} (4\pi|x-y|^3)^{-1}((x-y) \cdot h(y))dy \quad (x \in \mathbb{R}^3).$$

Note that $S(h)$ is a vector-valued function with $h$ being scalar, whereas $P(h)$ is a scalar function with $h$ being vector-valued.

**Lemma 1.2** *Let $q \in (1, 3/2)$, $h \in L^q(\mathbb{R}^3)$. Then $\mathcal{N}(h) \in W^{2,q}_{\text{loc}}(\mathbb{R}^3) \cap L^{(1/q-2/3)^{-1}}(\mathbb{R}^3)$, $\Delta\mathcal{N}(h) = h$. If $h \in W^{1,q}(\mathbb{R}^3)$, then $\partial_l \mathcal{N}(h) = \mathcal{N}(\partial_l h)$ $(1 \le l \le 3)$.*

*Let $q \in (1, 3)$, $h \in L^q(\mathbb{R}^3)$. Then $\mathcal{S}(h) \in W^{1,q}_{\text{loc}}(\mathbb{R}^3)^3$, $\operatorname{div} \mathcal{S}(h) = h$. If $q \in (1, 3/2)$, then $\nabla\mathcal{N}(h) = \mathcal{S}(h)$. If $h \in W^{1,q}(\mathbb{R}^3)$, then $\mathcal{S}(h) \in W^{2,q}_{\text{loc}}(\mathbb{R}^3)^3$.*

*Let $q \in (1, 3)$, $h \in L^q(\mathbb{R}^3)^3$. Then*

$$\mathcal{P}(h) \in W^{1,q}_{\text{loc}}(\mathbb{R}^3) \cap L^{(1/q-1/3)^{-1}}(\mathbb{R}^3),$$
$$\left( \int_{\mathbb{R}^3} \left( \int_{\mathbb{R}^3} |x-y|^{-2}|h(y)|dy \right)^{(1/q-1/3)^{-1}} dx \right)^{1/q-1/3} \le C\|h\|_q.$$

*Proof:* Well known (Hardy-Littlewood-Sobolev inequality, Calderon-Zygmund inequality, density arguments). $\square$

**Lemma 1.3** *[18, Lemma 2.2] Let $B \in \mathbb{R}$, $S \in (0, \infty)$. Then*

$$\int_{\partial B_R} s(x)^{-B} \, do_x \le C(S, B) \cdot R^{2-min\{1,B\}} \cdot \sigma(R) \tag{1.3}$$

*for $R \in [S, \infty)$, with $\sigma(R) := 1$ if $B \neq 1$, and $\sigma(R) = \ln(1 + \mathrm{R})$ if $B = 1$.*

# 2 Decay estimates

In first part of this section we recall some known results from [6] and [8] about the decay of the velocity part of the solution of the system (1.2), and in order to get the full decay characterization of the solution we derive the decay of the pressure part of solution of (1.2). In the second part of this section we extend the result for the pressure to the non-linear case of (1.1).

## Decay estimates in the linear case

Our starting point is a decay result from [8] for the velocity part $u$ of a solution to (1.2).

**Theorem 2.1** ([8, Theorem 3.12]) *Suppose that $\mathcal{D}$ is $C^2$-bounded. Let $p \in (1,\infty)$, $(u,\pi) \in M_p$. Put $F = L(u) + \nabla\pi$. Suppose there are numbers $S_1, S, \gamma \in (0,\infty)$, $A \in [2,\infty)$, $B \in \mathbb{R}$ such that $S_1 < S$,*

$$\overline{\mathcal{D}} \cup \operatorname{supp}(\operatorname{div} u) \subset B_{S_1}, \quad u|_{B_S^c} \in L^6(B_S^c)^3, \quad \nabla u|_{B_S^c} \in L^2(B_S^c)^9,$$

$$A + \min\{1,B\} \geq 3, \ |F(z)| \leq \gamma |z|^{-A} s(z)^{-B} \ \textit{for } z \in B_{S_1}^c.$$

*Then*

$$|u(y)| \leq C\,(|y|s(y))^{-1}\, l_{A,B}(y), \tag{2.1}$$

$$|\nabla u(y)| \leq C\,(|y|s(y))^{-3/2}\, s(y)^{\max(0,7/2-A-B)} \tag{2.2}$$

*for $y \in B_S^c$, where function $l_{A,B}$ is given by*

$$\begin{cases} 1 & \textit{if} \quad A + \min\{1,B\} > 3 \\ \max(1, \mathit{ln}(y)) & \textit{if} \quad A + \min\{1,B\} = 3. \end{cases}$$

The requirements $u|_{B_S^c} \in L^6(B_S^c)^3$, $\nabla u|_{B_S^c} \in L^2(B_S^c)^9$ should be interpreted as decay conditions on $u$.

It may be deduced from Theorem 2.1 that inequalities (2.1) and (2.2) hold under assumptions weaker than those stated in that theorem. We specify this more general situation in the ensuing corollary, which in addition indicates some properties of $F$ that will be useful in the following.

**Corollary 2.2** *Let $p \in (1,\infty)$, $\gamma$, $S_1$, $S \in (0,\infty)$ with $\overline{\mathcal{D}} \subset B_{S_1}$, $S_1 < S$, $A \in [2,\infty)$, $B \in \mathbb{R}$ with $A + \min\{1,B\} \geq 3$. Let $F : \overline{\mathcal{D}}^c \mapsto \mathbb{R}^3$ be measurable with $F|\mathcal{D}_{S_1} \in L^p(\mathcal{D}_{S_1})^3$ and $|F(z)| \leq \gamma |z|^{-A} s(z)^{-B}$ for $z \in B_{S_1}^c$.*

*Let $u \in W^{1,p}_{loc}(\overline{\mathcal{D}}^c)^3$ with $u|_{B_S^c} \in L^6(B_S^c)^3$, $\nabla u|_{B_S^c} \in L^2(B_S^c)^9$, $\operatorname{supp}(\operatorname{div} u) \subset B_{S_1}$,*

$$\int_{\overline{\mathcal{D}}^c} \big[\, \nabla u \cdot \nabla \varphi + \big(\, \tau\, \partial_1 u - (\varrho\, e_1 \times z) \cdot \nabla u + (\varrho\, e_1 \times u) - F \,\big) \cdot \varphi \,\big]\, dz \tag{2.3}$$
$$= 0 \quad \textit{for } \ \varphi \in C_0^\infty(\overline{\mathcal{D}}^c)^3 \ \ \textit{with } \ \operatorname{div} \varphi = 0.$$

*Then inequalities (2.1) and (2.2) hold for $y \in B_S^c$.*

*Moreover $F \in L^q(\overline{\mathcal{D}}^c)^3$ for $q \in (1,p]$. If $p \geq 6/5$, the function $F$ may be considered as a bounded linear functional on $\mathcal{D}_0^{1,2}(\overline{\mathcal{D}}^c)^3$, in the usual sense.*

*Proof:* For $z \in B^c_{S_1}$, we have

$$|F(z)| \leq \gamma\, C(S_1, A)\, |z|^{-2}\, s(z)^{-A+2-B} \leq \gamma\, C(S_1, A)\, |z|^{-2}\, s(z)^{-A+2-\min\{1,B\}}$$
$$\leq \gamma\, C(S_1, A)\, |z|^{-2}\, s(z)^{-1}.$$

Thus for $q \in (1, \infty)$, with Lemma 1.3,

$$\int_{B^c_{S_1}} |F(z)|^q\, dz \leq C \int_{S_1}^{\infty} r^{-2q} \int_{\partial B_r} s(z)^{-q}\, do_z\, dr \leq C \int_{S_1}^{\infty} r^{-2q+1}\, dr < \infty.$$

It follows that $F \in L^q(\overline{\mathcal{D}}^c)^3$ for $q \in (1, p]$. According to [14, Theorem II.6.1], the inequality $\|v\|_6 \leq C\, \|\nabla v\|_2$ holds for $v \in \mathcal{D}^{1,2}_0(\overline{\mathcal{D}}^c)^3$. Thus, if $p \geq 6/5$, hence $F \in L^{6/5}(\overline{\mathcal{D}}^c)^3$, this function $F$ may be considered as a linear bounded functional on $\mathcal{D}^{1,2}_0(\overline{\mathcal{D}}^c)^3$. The $L^p$-integrability of $F$ and the assumptions on $u$ imply that the function

$$G(z) := F(z) - \big(\tau\, e_1 - (\varrho\, e_1 \times z)\big) \cdot \nabla u(z) - \big(\varrho\, e_1 \times u(z)\big), \quad z \in \overline{\mathcal{D}}^c, \tag{2.4}$$

belongs to $L^p_{loc}(\overline{\mathcal{D}}^c)^3$. Fix some number $S_0 \in (0, S_1)$ with $\overline{\mathcal{D}} \cup \mathrm{supp}(\mathrm{div}\, u) \subset B_{S_0}$. This means in particular that $\mathrm{div}\,(u|\overline{B_{S_0}}^c) = 0$. This equation, (2.3), the relation $G \in L^p_{loc}(\overline{\mathcal{D}}^c)^3$ and interior regularity of solutions to the Stokes system (see [14, Theorem IV.4.1] for example) imply that $u|\overline{B_{S_0}}^c \in W^{2,p}_{loc}(\overline{B_{S_0}}^c)^3$ and there is $\pi \in W^{1,p}_{loc}(\overline{B_{S_0}}^c)$ with $L(u|\overline{B_{S_0}}^c) + \nabla\pi = F|\overline{B_{S_0}}^c$. Put $S_0' := (S_0 + S_1)/2,\ A_{S_0',R} := B_R \backslash B_{S_0'}$ for $R \in (S_0', \infty)$. Then $u|A_{S_0',R} \in W^{2,p}(A_{S_0',R})^3$ and $\pi|A_{S_0',R} \in W^{1,p}(A_{S_0',R})^3$ for $R \in (S_0', \infty)$, so $(u|A_{S_0',R},\, \pi|A_{S_0',R}) \in M_p$, with $B_{S_0'}$ in the role of $\mathcal{D}$. Note that $S_0 < S_o' < S_1 < S$. Thus the assumptions of Theorem 2.1 are satisfied with $\mathcal{D}$ replaced by $B_{S_0'}$. As a consequence inequalities (2.1) and (2.2) hold. □

**Remark 2.3** *Solutions as considered in Corollary 2.2 exist if, for example, Dirichlet boundary conditions are prescribed on $\partial\mathcal{D}$. In fact, as stated in [14, Theorem VIII.1.2], if $F$ is a bounded linear functional on the space $D^{1,2}_0(\overline{\mathcal{D}}^c)^3$, and if $b \in H^{1/2}(\partial\overline{\mathcal{D}}^c)^3$, then there is a function $u \in L^6(\overline{\mathcal{D}}^c)^3 \cap W^{1,1}_{loc}(\overline{\mathcal{D}}^c)^3$ such that $\nabla u \in L^2(\overline{\mathcal{D}}^c)^9$ and $u$ satisfies the equations (2.3) and $\mathrm{div}\, u = 0$ (weak form of (1.2)), as well as the boundary conditions $u|\partial\mathcal{D} = b$.*

The main result of this section, dealing with the asymptotics of the pressure, is stated in

**Theorem 2.4** *Let $p$, $\gamma$, $S_1$, $S$, $A$, $B$, $F$, $u$ be given as in Corollary 2.2, but with the stronger assumptions $A = 5/2,\ B \in (1/2, \infty)$ on $A$ and $B$. Let $\pi \in L^p_{loc}(\overline{\mathcal{D}}^c)$ with*

$$\int_{\overline{\mathcal{D}}^c} \big[\nabla u \cdot \nabla\varphi + \big(\tau\, \partial_1 u - (\varrho\, e_1 \times z) \cdot \nabla u + (\varrho\, e_1 \times u) - F\big) \cdot \varphi \tag{2.5}$$
$$-\pi\, \mathrm{div}\, \varphi\big]\, dz = 0 \quad \textit{for}\ \ \varphi \in C^\infty_0(\overline{\mathcal{D}}^c)^3.$$

*Then there is $c_0 \in \mathbb{R}$ such that*

$$|\pi(x) + c_0| \leq C\,|x|^{-2} \quad \textit{for } x \in B_S^c. \tag{2.6}$$

*Proof:* By Corollary 2.2 we have $F \in L^q(\overline{\mathcal{D}}^c)^3$ for $q \in (1,p]$. As in the proof of that corollary, we note that the function $G$ introduced there (see (2.4)) belongs to $L^p_{loc}(\overline{\mathcal{D}}^c)^3$. Also as in that proof, we fix some number $S_0 \in (0, S_1)$ with $\overline{\mathcal{D}} \cup \operatorname{supp}(\operatorname{div} u) \subset B_{S_0}$, and note that $\operatorname{div}(u|\overline{B_{S_0}}^c) = 0$. Thus, in view of (2.5) and because $G \in L^p_{loc}(\overline{\mathcal{D}}^c)^3$, interior regularity of solutions of the Stokes system ([14, Theorem IV.4.1]) yields that $u|\overline{B_{S_0}}^c \in W^{2,p}_{loc}(\overline{B_{S_0}}^c)^3$, $\pi|\overline{B_{S_0}}^c \in W^{1,p}_{loc}(\overline{B_{S_0}}^c)$, and $L(u|\overline{B_{S_0}}^c) + \nabla(\pi|\overline{B_{S_0}}^c) = F|\overline{B_{S_0}}^c$. Note that $S_0 < S_1 < S$. Take $\phi \in C^\infty(\mathbb{R}^3)$ with

$$\phi|B_{S_1+\frac{1}{4}(S-S_1)} = 0, \;\; \phi|B^c_{S_1+\frac{3}{4}(S-S_1)} = 1,$$

and put $\widetilde{u} := \phi \cdot u$, $\widetilde{\pi} := \phi \cdot \pi$, with $\widetilde{u}, \widetilde{\pi}$ to be considered as functions in $\mathbb{R}^3$. By the choice of $\phi$ and the properties of $u$ and $\pi$, we get $\widetilde{u} \in W^{2,q}_{\text{loc}}(\mathbb{R}^3)^3$, $\widetilde{\pi} \in W^{1,q}_{\text{loc}}(\mathbb{R}^3)$ for $q \in [1,p]$, $\widetilde{u}|B_S^c = u|B_S^c \in L^6(\mathbb{R}^3)^3$, $\nabla\widetilde{u}|B_S^c = \nabla u|B_S^c \in L^2(\mathbb{R}^3)^9$. Put

$$\begin{aligned} g_l(z) := &-\sum_{k=1}^{3}\big[\,\partial_k\phi(z)\partial_k u_l(z) + \Delta\phi(z)u_l(z)\,\big] + \tau\partial_1\phi(z)u_l(z) \\ &-\sum_{k=1}^{3}(\tau e_1 \times z)_k \cdot \partial_k\phi(z) \cdot u_l(z) + \partial_l\phi(z)\pi(z) \end{aligned}$$

for $z \in \mathbb{R}^3$, $1 \leq l \leq 3$, and set $\gamma := \operatorname{div}\widetilde{u}$. Then

$$\begin{aligned} &\operatorname{supp}(g) \subset \overline{B_{S_1+3(S-S_1)/4}} \setminus B_{S_1+(S-S_1)/4}, \;\; g \in L^q(\mathbb{R}^3)^3 \text{ for } q \in [1,p], \\ &L\widetilde{u} + \nabla\widetilde{\pi} = g + \phi \cdot F, \;\; \gamma = \nabla\phi \cdot u, \end{aligned} \tag{2.7}$$

in particular $\operatorname{supp}(\gamma) \subset \overline{B_{S_1+3(S-S_1)/4}} \setminus B_{S_1+(S-S_1)/4}$, $\gamma \in W^{2,q}(\mathbb{R}^3)$ for $q \in [1,p]$, $g+\phi\cdot F \in L^q(\mathbb{R}^3)^3$ for $q \in (1,p]$. Let $x \in \mathbb{R}^3$, $\varepsilon > 0$ with $\overline{B_\varepsilon(x)} \subset B_{S_1}$. Since $\widetilde{u}|B_{S_1} = 0$, $\widetilde{\pi}|B_{S_1} = 0$, it follows from [8, Theorem 3.11] with $\mathcal{D}$ replaced by $B_\varepsilon(x)$ that

$$\widetilde{u}(y) = \mathcal{R}(g + \phi F)(y) + \mathcal{S}(\gamma)(y) \text{ for } y \in \overline{B_\varepsilon(x)}^c.$$

Since this is true for any $x \in \mathbb{R}^3$, $\varepsilon > 0$ with $\overline{B_\varepsilon(x)} \subset B_{S_1}$, it follows that

$$\widetilde{u} = \mathcal{R}(g + \phi F) + \mathcal{S}(\gamma) \text{ in } \mathbb{R}^3. \tag{2.8}$$

But $\mathcal{S}(\gamma) \in W^{2,q}_{\text{loc}}(\mathbb{R}^3)^3$ for $q \in [1, \min\{3,p\})$ by Lemma 1.2, so from (2.8)

$$\mathcal{R}(g + \phi F) \in W^{2,q}_{\text{loc}}(\mathbb{R}^3)^3 \text{ for } q \in [1, \min\{3,p\}).$$

This relation and [6, (3.11) and the inequalities following (3.15)] imply

$$\sup_{z\in\mathbb{R}^3}\int_{\mathbb{R}^3}|\mathcal{Z}(z,y)\cdot(g+\phi F)(y)|dy<\infty. \tag{2.9}$$

Let $\psi\in C_0^\infty(\mathbb{R}^3)^3$. Due to (2.9), we may apply Fubini's theorem, to obtain

$$\begin{aligned} A &:= \int_{\mathbb{R}^3}\psi(x)(L\mathcal{R}(g+\phi F))(x)dx = \int_{\mathbb{R}^3}(L^*\psi)(x)\mathcal{R}(g+\phi F)(x)dx \\ &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}[(L^*\psi)(x)]^T\cdot\mathcal{Z}(x,y)\cdot(g+\phi F)(y)dydx \\ &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}[(L^*\psi)(z)]^T\cdot\mathcal{Z}(x,y)\cdot(g+\phi F)(y)dxdy. \end{aligned} \tag{2.10}$$

But for $a,b,x,y\in\mathbb{R}^3$ with $x\neq y$,

$$a^T\cdot\mathcal{Z}(x,y)\cdot b=\int_0^\infty a^T\Gamma(x,y,t)bdt,$$

hence with [6, Lemma 2.10],

$$\begin{aligned} a^T\cdot\mathcal{Z}(x,y)\cdot b &= \int_0^\infty a^T\cdot e^{-t\Omega}\Lambda(e^{t\Omega}x-\tau te_1-y,t)\cdot bdt \\ &= \int_0^\infty b^T[e^{-t\Omega}\Lambda(e^{t\Omega}x-\tau te_1-y,t)]^Tadt \\ &= \int_0^\infty b^T\cdot\Lambda(e^{t\Omega}x-\tau te_1-y,t)e^{t\Omega}\cdot adt \\ &= \int_0^\infty b^T\Lambda(y+\tau te_1-e^{t\Omega}x,t)e^{t\Omega}adt \\ &= \int_0^\infty b^T\widetilde{\Gamma}(y,x,t)\cdot adt = b^T\cdot\widetilde{Z}(y,x)\cdot a. \end{aligned}$$

Therefore from (2.10)

$$A=\int_{\mathbb{R}^3}(g+\phi F)(y)^T\cdot\int_{\mathbb{R}^3}\widetilde{\mathcal{Z}}(y,x)\cdot(L^*\psi)(x)dxdy. \tag{2.11}$$

Since $\psi\in C_0^\infty(\mathbb{R}^3)^3$, we may choose $x_0\in\mathbb{R}^3$, $\varepsilon>0$ such that

$$\overline{B_\varepsilon(x_0)}\subset\mathbb{R}^3\setminus\operatorname{supp}(\psi).$$

Thus we get from [5, Theorem 4.3] with $\mathcal{D},U,\omega$ replaced by $B_\varepsilon(x_0)$, $\tau e_1$, $-\rho e_1$, respectively, and with $\pi=0$, that

$$\int_{\mathbb{R}^3}\widetilde{\mathcal{Z}}(y,x)\cdot(L^*\psi)(x)dx=\psi(y)-\mathcal{S}(\operatorname{div}\psi)(y)$$

for $y \in \mathbb{R}^3 \setminus \overline{B_\varepsilon(x_0)}$. Since this is true for any $x_0 \in \mathbb{R}^3$, $\varepsilon > 0$ with $\overline{B_\varepsilon(x_0)} \subset \mathbb{R}^3 \setminus \mathrm{supp}(\psi)$, the preceding equation holds for any $y \in \mathbb{R}^3$. It follows from (2.11)

$$\int_{\mathbb{R}^3} \psi(x)(L\mathcal{R}(g+\phi F))(x)dx = \int_{\mathbb{R}^3} (g+\phi F)(y) \cdot (\psi(y) - \mathcal{S}(\operatorname{div}\psi)(y))dy \tag{2.12}$$

Again recalling that $g + \phi F \in L^q(\mathbb{R}^3)^3$ for $q \in (1,p]$, we get with Lemma 1.2 that

$$\int_{\mathbb{R}^3} \psi(x)\nabla\mathcal{P}(g+\phi F)(x)dx = \int_{\mathbb{R}^3} -\operatorname{div}\psi(x) \cdot \mathcal{P}(g+\phi F)(x)dx. \tag{2.13}$$

Put $q_0 := \min\{6/5, p\}$, and note that $q_0 \in (1, 3/2)$, $q_0 \le p$.

Thus, by Hölder's inequality and Lemma 1.2,

$$\begin{aligned}
&\int_{\mathbb{R}^3} |\operatorname{div}\psi(x)\,\mathcal{P}(g+\varphi\,F)(x)|\,dx \\
&\le \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |\operatorname{div}\psi(x)(4\pi|x-y|^3)^{-1}(x-y)\cdot(g+\phi F)(y)|dydx \\
&\le \|\operatorname{div}\psi\|_{(4/3-1/q_0)^{-1}} \\
&\quad \left(\int_{\mathbb{R}^3}\left(\int_{\mathbb{R}^3}(4\pi|x-y|^2)^{-1}|(g+\phi F)(y)|dy\right)^{(1/q_0-1/3)^{-1}} dx\right)^{1/q_0-1/3} \\
&\le C\cdot\|\operatorname{div}\psi\|_{(4/3-1/q_0)^{-1}}\cdot\|g+\phi F\|_{q_0} < \infty.
\end{aligned}$$

As a consequence, we may apply Fubini's theorem to deduce from (2.13) that

$$\begin{aligned}
&\int_{\mathbb{R}^3} \psi(x)\nabla\mathcal{P}(g+\phi F)(x)dx \\
&= -\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} (\operatorname{div}\psi)(x)(4\pi|x-y|^3)^{-1}(x-y)\cdot(g+\phi F)(y)dxdy \\
&= \int_{\mathbb{R}^3} (g+\phi F)(y)\cdot\mathcal{S}(\operatorname{div}\psi)(y)dy.
\end{aligned} \tag{2.14}$$

From (2.12) and (2.14),

$$\begin{aligned}
&\int_{\mathbb{R}^3} \psi(x)((L\mathcal{R}(g+\phi F))(x) + \nabla\mathcal{P}(g+\phi F)(x))dx \\
&= \int_{\mathbb{R}^3} \psi(x)(g+\phi F)(x)dx.
\end{aligned}$$

Since this is true for any $\psi \in C_0^\infty(\mathbb{R}^3)^3$, we have found that

$$L\mathcal{R}(g+\phi F) + \nabla\mathcal{P}(g+\phi F) = g + \phi F. \tag{2.15}$$

On the other hand, by (2.8) and (2.7)

$$L\mathcal{R}(g+\phi F) + L\mathcal{S}(\gamma) + \nabla\widetilde{\pi} = L\widetilde{u} + \nabla\widetilde{\pi} = g + \phi F.$$

By subtracting this equation from (2.15), we get

$$\nabla\mathcal{P}(g+\phi F) - L\mathcal{S}(\gamma) - \nabla\widetilde{\pi} = 0. \tag{2.16}$$

Next we consider the term $\operatorname{div}\big(L\mathcal{S}(\gamma)\big)$. Recall that $q_0 < 3/2$ and $\gamma \in W^{2,q}(\mathbb{R}^3)$ for $q \in [1,p]$, so by Lemma 1.2

$$\begin{cases} \mathcal{S}(\gamma) \in W^{2,q_0}_{\mathrm{loc}}(\mathbb{R}^3)^3,\ \operatorname{div}\mathcal{S}(\gamma) = \gamma,\ \mathcal{N}(\gamma) \in W^{2,q_0}_{\mathrm{loc}}(\mathbb{R}^3), \\ \nabla\mathcal{N}(\gamma) = \mathcal{S}(\gamma). \end{cases} \tag{2.17}$$

Since $e_1 \times \mathcal{S}(\gamma) = (0, -\mathcal{S}_3(\gamma), \mathcal{S}_2(\gamma))$, we may conclude

$$\operatorname{div}(e_1 \times \mathcal{S}(\gamma)) = -\partial_2\mathcal{S}_3(\gamma) + \partial_3\mathcal{S}_2(\gamma) = -\partial_2\partial_3\mathcal{N}(\gamma) + \partial_3\partial_2\mathcal{N}(\gamma) = 0.$$

Moreover, for $z \in \mathbb{R}^3$, $1 \le j \le 3$,

$$(e_1 \times z)\cdot\nabla\mathcal{S}_j(\gamma)(z) = -z_3\partial_2\mathcal{S}_j(\gamma)(z) + z_2\partial_3\mathcal{S}_j(\gamma)(z),$$

hence with (2.17),

$$\begin{aligned} &\operatorname{div}_z((e_1 \times z)\cdot\nabla\mathcal{S}(\gamma)(z)) \\ &= -z_3\partial_2\gamma(z) + z_2\partial_3\gamma(z) - \partial_2\mathcal{S}_3(\gamma)(z) + \partial_3\mathcal{S}_2(\gamma)(z) \\ &= -z_3\partial_2\gamma(z) + z_2\partial_3\gamma(z) - \partial_2\partial_3\mathcal{N}(\gamma)(z) + \partial_3\partial_2\mathcal{N}(\gamma)(z) = \varphi(z), \end{aligned}$$

where $\varphi(z) := -z_3\partial_2\gamma(z) + z_2\partial_3\gamma(z)$. Let $\psi \in C_0^\infty(\mathbb{R}^3)$. Then it follows that

$$\begin{aligned} &\int_{\mathbb{R}^3} \nabla\psi\cdot(\rho e_1 \times \mathcal{S}(\gamma))dx = 0, \\ &\int_{\mathbb{R}^3} \nabla\psi(z)[(\rho e_1 \times z)\cdot\nabla\mathcal{S}(\gamma)(z)]dz = \int_{\mathbb{R}^3} \psi(z)(-\varphi(z))dz. \end{aligned}$$

Obviously, again with (2.17),

$$\int_{\mathbb{R}^3} \nabla\psi\cdot\Delta\mathcal{S}(\gamma)dx = \int_{\mathbb{R}^3} \nabla\Delta\psi\cdot\mathcal{S}(\gamma)dx = -\int_{\mathbb{R}^3} \Delta\psi\cdot\gamma dx = \int_{\mathbb{R}^3} \psi\cdot(-\Delta\gamma)dx,$$

and similarly,

$$\int_{\mathbb{R}^3} \nabla\psi(\tau\partial_1\mathcal{S}(\gamma)) = \int_{\mathbb{R}^3} \psi(-\tau\partial_1\gamma)dx.$$

Combining these equations, we get

$$\int_{\mathbb{R}^3} \nabla\psi \cdot L\mathcal{S}(\gamma)dx = \int_{\mathbb{R}^3} \psi(\varphi + \Delta\gamma - \tau\partial_1\gamma)dx.$$

Now from (2.16)

$$\int_{\mathbb{R}^3} \nabla\psi[\nabla\mathcal{P}(g+\phi F) - \nabla(\phi\pi)]dx = \int_{\mathbb{R}^3} \psi(\varphi + \Delta\gamma - \tau\partial_1\gamma)dx. \tag{2.18}$$

Since $\gamma \in W^{2,q}(\mathbb{R}^3)$ for $q \in [1,p]$ and $\mathrm{supp}(\gamma) \subset B_S \backslash B_{S_1}$, we have $\varphi + \Delta\gamma - \tau\partial_1\gamma \in L^q(\mathbb{R}^3)$ for $q \in [1,p]$, so we may consider $\mathcal{N}(\varphi + \Delta\gamma - \tau\partial_1\gamma)$. Lemma 1.2 yields

$$\begin{aligned} &\mathcal{N}(\varphi + \Delta\gamma - \tau\partial_1\gamma) \in W^{2,q_0}_{\mathrm{loc}}(\mathbb{R}^3), \\ &\Delta\mathcal{N}(\varphi + \Delta\gamma - \tau\partial_1\gamma) = \varphi + \Delta\gamma - \tau\partial_1\gamma. \end{aligned}$$

Therefore from (2.18)

$$\int_{\mathbb{R}^3} \nabla\psi[\nabla\mathcal{P}(g+\phi F) - \nabla\mathcal{N}(\varphi + \Delta\phi - \tau\partial_1\gamma) - \nabla(\phi\pi)]dx = 0$$

Lemma 1.1 now yields

$$Q := \mathcal{P}(g+\phi F) - \mathcal{N}(\varphi + \Delta\gamma - \tau\partial_1\gamma) - \phi\pi \in C^\infty(\mathbb{R}^3), \quad \Delta Q = 0. \tag{2.19}$$

Now we again apply Lemma 1.2. Since $g + \phi \cdot F \in L^{q_0}(\mathbb{R}^3)^3$, we have

$$\mathcal{P}(g+\phi F) \in L^{(1/q_0 - 1/3)^{-1}}(\mathbb{R}^3).$$

Moreover $\varphi + \Delta\gamma - \tau\partial_1\gamma \in L^{q_0}(\mathbb{R}^3)$, so

$$\mathcal{N}(\varphi + \Delta\gamma - \tau\partial_1\gamma) \in L^{(1/q_0 - 2/3)^{-1}}(\mathbb{R}^3).$$

Since $q_0 \le p$, and in view of our remarks at the beginning of this proof we know that $u|\overline{B_{S_0}}^c \in W^{2,q_0}_{loc}(\overline{B_{S_0}}^c)$, $\pi|\overline{B_{S_0}}^c \in W^{1,q_0}_{loc}(\overline{B_{S_0}}^c)$, $L(u|\overline{B_{S_0}}^c) + \nabla(\pi|\overline{B_{S_0}}^c) = F|\overline{B_{S_0}}^c$, $\mathrm{div}(u|\overline{B_{S_0}}^c) = 0$ and $F \in L^{q_0}(\mathbb{R}^3)^3$. By the choice of $u$ in Corollary 2.2, we have $u|B_S^c \in L^6(B_S^c)^3$. Now [8, Theorem 2.1] yields there is $c_0 \in \mathbb{R}$ such that

$$\pi + c_0|B_{2S}^c \in L^{3q_0/(3-q_0)}(B_{2\,S}^c) + L^3(B_{2\,S}^c).$$

But by (2.19),

$$Q - c_0 = \mathcal{P}(g+\phi F) - \mathcal{N}(\varphi + \Delta\gamma - \tau\partial_1\gamma) - \phi\,(\pi + c_0) + (\phi - 1)\,c_0,$$

where $\text{supp}(\phi - 1) \subset B_S$ and $\text{supp}(\phi) \subset B^c_{S_1}$. Therefore we may conclude that

$$Q - c_0 \in L^{(1/q_0-1/3)^{-1}}(\mathbb{R}^3) + L^{(1/q_0-2/3)^{-1}}(\mathbb{R}^3) + L^{q_0}(\mathbb{R}^3) \qquad (2.20)$$
$$+ L^{3q_0/(3-q_0)}(\mathbb{R}^3) + L^3(\mathbb{R}^3).$$

Let $\varepsilon \in (0, \infty)$, and let $(Q - c_0)_\varepsilon$ be the usual Friedrich's mollifier of $Q - c_0$ associated with $\varepsilon$.

Due to (2.19), (2.20) and by standard properties Friedrich's mollifier, the function $(Q - c_0)_\varepsilon$ is bounded and $\Delta(Q - c_0)_\varepsilon = 0$. Now Liouville's theorem yields $(Q - c_0)_\varepsilon = 0$. Since this is true for any $\varepsilon > 0$, we may conclude that $Q - c_0 = 0$, that is,

$$\phi\,(\pi + c_0) = \mathcal{P}(g + \phi F) - \mathcal{N}(\varphi + \Delta\gamma - \tau\partial_1\gamma) + (\phi - 1)\,c_0,$$

hence

$$\pi + c_0|B^c_S = \mathcal{P}(g + \phi F) - \mathcal{N}(\varphi + \Delta\gamma - \tau\partial_1\gamma)|B^c_S, \qquad (2.21)$$

where we used that $\text{supp}(\phi - 1) \subset B_S$ and $\phi|B^c_S = 1$. Since $\text{supp}(g) \subset \overline{B_{S_1+3(S-S_1)/4}}$, we have

$$|\mathcal{P}(g)(x)| \le c \cdot |x|^{-2} \text{ for } x \in B^c_S. \qquad (2.22)$$

Due to the assumptions $A = 5/2,\ B \in (1/2, \infty)$ and because $\phi F|B_{S_1+(S-S_1)/4} = 0$ and $S_1 < S_1 + (S - S_1)/4 < S$, we get by [10, Theorem 3.2] or [18, Theorem 3.4][1] that

$$|\mathcal{P}(\phi\, F)(x)| \le c\,|x|^{-2} \text{ for } x \in B^c_S. \qquad (2.23)$$

Define $\zeta(x) = -x_3\gamma(x)$, $\tilde{\zeta}(x) := x_2\gamma(x)$ for $x \in \mathbb{R}^3$. Then $\text{supp}(\zeta) \cup \text{supp}(\tilde{\zeta}) \subset B_S \setminus \overline{B_{S_1}}$,

$$\zeta, \tilde{\zeta} \in W^{2,q}(\mathbb{R}^3) \text{ for } q \in [1, p],$$
$$\varphi = \partial_2\zeta + \partial_3\tilde{\zeta}.$$

It follows with Lemma 1.2 that

$$\mathcal{N}(\varphi) = \partial_2\mathcal{N}(\zeta) + \partial_3\mathcal{N}(\tilde{\zeta}) = \mathcal{S}_2(\zeta) + \mathcal{S}_3(\tilde{\zeta}).$$

Similarly, since $\text{supp}(\gamma) \subset B_S \setminus \overline{B_{S_1}}$, $\gamma \in W^{2,q}(\mathbb{R}^3)$ for $q \in [1, p]$,

$$\mathcal{N}(\Delta\gamma - \tau\partial_1\gamma) = \sum_{k=1}^{3} \mathcal{S}_k(\partial_k\gamma) - \tau\mathcal{S}_1(\gamma).$$

Together

$$\mathcal{N}(\varphi + \Delta\gamma - \tau\partial_1\gamma) = \mathcal{S}_2(\zeta) + \mathcal{S}_3(\tilde{\zeta}) + \sum_{k=1}^{3} \mathcal{S}_k(\partial_k\gamma) - \tau\mathcal{S}_1(\gamma).$$

[1] The result stated in both these theorems is not optimal, with respect of logarithmic terms: No logarithmic term appears e.g. in the case $A = 5/2, B = 1$. Nevertheless we need not this case.

Since $\mathrm{supp}(\zeta) \cup \mathrm{supp}(\tilde{\zeta}) \cup \mathrm{supp}(\gamma) \subset B_{S_1+3(S-S_1)/4}$, we may conclude that

$$|\mathcal{N}(\varphi + \Delta\gamma - \tau\partial_1\gamma)(x)| \le C\,|x|^{-2} \text{ for } |\xi| \in B_S^c. \tag{2.24}$$

Inequality (2.6) follows from (2.21)–(2.24). □

We remark that Theorem 2.4 remains valid if the assumptions on $A$ and $B$ are replaced by the conditions $A \ge 5/2,\ A + \min\{1, B\} > 1$, which are weaker than those in Corollary 2.2. This observation is made precise by the ensuing corollary. Its proof is obvious, but this modified version of Theorem 2.4 still is interesting because its requirements on $A$ and $B$ are closer to the ones in Corollary 2.2 than those stated in Theorem 2.4.

**Corollary 2.5** *Let $p$, $\gamma$, $S_1$, $S$, $A$, $B$, $F$, $u$ be given as in Corollary 2.2, but with the stronger assumptions $A \ge 5/2,\ A + \min\{1, B\} > 3$ on $A$ and $B$. Let $\pi \in L^p_{loc}(\overline{\mathcal{D}}^c)$ such that (2.5) holds. Then there is $c_0 \in \mathbb{R}$ such that inequality (2.6) is valid.*

*Proof:* Put $B' := A - 5/2 + \min\{1, B\}$. Since $A + \min\{1, B\} > 3$, we have $B' \in (1/2, \infty)$. Moreover, since $A \ge 5/2$, we find for $x \in B^c_{S_1}$ that

$$|F(z)| \le \gamma\, C(S_1, A)\, |z|^{-5/2}\, s(z)^{-A+5/2-B} \le \gamma\, C(S_1, A)\, |z|^{-5/2}\, s(z)^{-B'}.$$

Thus the assumptions of Theorem 2.4 are satisfied with $B$ replaced by $B'$ and with a modified parameter $\gamma$. This implies the conclusion of Theorem 2.4. □

## Decay estimates in the non-linear case

Let us assume now the non-linear case, i.e. the system (1.1). First, recall the result about the decay properties of the velocity in this non-linear case:

**Theorem 2.6** *[7, Theorem 1.1] Let $\gamma, S_1 \in (0, \infty)$, $p_0 \in (1, \infty)$, $A \in (2, \infty)$, $B \in [0, 3/2]$ with $\overline{\mathcal{D}} \subset B_{S_1}$, $A + \min\{B, 1\} > 3$, $A + B \ge 7/2$. Take $F : \mathbb{R}^3 \mapsto \mathbb{R}^3$ measurable with $F|B_{S_1} \in L^{p_0}(B_{S_1})^3$,*

$$|F(y)| \le \gamma \cdot |y|^{-A} \cdot s(y)^{-B} \ \textit{for}\ y \in B^c_{S_1}.$$

*Let $u \in L^6(\overline{\mathcal{D}}^c)^3 \cap W^{1,1}_{loc}(\overline{\mathcal{D}}^c)^3, \pi \in L^2_{loc}(\overline{\mathcal{D}}^c)$ with $\nabla u \in L^2(\overline{\mathcal{D}}^c)^9, \mathrm{div}\, u = 0$ and*

$$\int_{\overline{\mathcal{D}}^c} [\nabla u \cdot \nabla\varphi + ((\tau e_1 - \rho e_1 \times z) \cdot \nabla u + \rho e_1 \times u$$
$$+\tau(u \cdot \nabla)u - F) \cdot \varphi - \pi\, \mathrm{div}\, \varphi]\, \mathrm{d}x = 0$$

*for $\varphi \in C_0^\infty(\overline{\mathcal{D}}^c)^3$. Let $S \in (S_1, \infty)$. Then*

$$|\partial^\alpha u(x)| \le C\, (|x| s(x))^{-1-|\alpha|/2} \ \textit{for}\ x \in B^c_S,\ \alpha \in \mathbb{N}_0^3 \ \textit{with}\ |\alpha| \le 1. \tag{2.25}$$

Now, using Theorems 2.4 and 2.6, we are in the position to prove the result on the decay of the pressure in the non-linear case:

**Theorem 2.7** *Consider the situation in Theorem 2.6. Suppose in addition that $A \geq 5/2$. Then there is $c_0 \in \mathbb{R}$ such that inequality (2.6) holds.*

*Proof:* Observe that $(u \cdot \nabla)u \in L^{3/2}(\overline{\mathcal{D}}^c)^3$. Thus, putting $p := \min\{3/2,\, p_0\}$, $\widetilde{F} := F - \tau\,(u \cdot \nabla)u$, we get $\widetilde{F}|\mathcal{D}_{S_1} \in L^p(\mathcal{D}_{S_1})^3$. Put $B' := \min\{5/2,\, A + B - 5/2\}$. Since $A \geq 5/2$, we have

$$|F(z)| \leq \gamma\, C(S_1, A)\, |z|^{-5/2}\, s(z)^{-B'} \quad \text{for } z \in B^c_{S_1}.$$

On the other hand, by Theorem 2.6 with $(S_1 + S)/2$ in the place of $S$,

$$|\big(u(z) \cdot \nabla)u(z)\big)| \leq C\, |z|^{-5/2}\, s(z)^{-5/2} \leq C\, |z|^{-5/2}\, s(z)^{-B'}$$

for $z \in B^c_{(S_1+S)/2}$. In this way we get $|\widetilde{F}| \leq C\, |z|^{-5/2}\, s(z)^{-B'}$ for $z \in B^c_{(S_1+S)/2}$.

We further note that $B' \in (1/2, \infty)$. This is obvious in the case $B' = 5/2$. If $B' < 5/2$, we have $B' = A + B - 5/2$. Due to the assumption $A + \min\{1, B\} > 3$ in Theorem 2.6, we thus get $B' \in (1/2, \infty)$. (The requirement $A + B \geq 7/2$ in Theorem 2.6 even yields $B' \geq 1$, but if this requirement is weakened in a suitable way, pointwise decay of $u$ and $\nabla u$ could still be proved. However, this point is not elaborated in [7], and therefore is not reflected in Theorem 2.6. But we still take account of it here by avoiding to use the assumption $A + B \geq 7/2$.)

We further have $u \in W^{1,p}_{loc}(\overline{\mathcal{D}}^c)^3$, $\pi \in W^p_{loc}(\overline{\mathcal{D}}^c)$, and equation (2.5) holds with $F$ replaced by $\widetilde{F}$. Since in addition $u|B^c_S \in L^6(B^c_S)^3$, $\nabla u|B^c_S \in L^2(B^c_S)^9$ and $\operatorname{div} u = 0$, we see that the assumptions of Theorem 2.4 are satisfied with $p$ as defined above and with $(S_1 + S)/2$, $B'$, $\widetilde{F}$ in the role of $S_1$, $B$ and $F$, respectively. Thus Theorem 2.4 implies the conclusion of Theorem 2.7. $\square$

# 3 Formulation of the problem with artificial boundary conditions

Recall that we defined $\mathcal{D}_R = B_R \setminus \overline{\mathcal{D}}$. We introduce the subspace $W_R$ of $H^1(\mathcal{D}_R)$ denoting

$$W_R := \{v \in H^1(\mathcal{D}_R)^3 : v|_{\partial\mathcal{D}} = 0\}.$$

**Lemma 3.1** ([3, Lemma 4.1]) *The estimate*

$$\|u\|_2 \leq C\,(R\,\|\nabla u\|_2 + R^{1/2}\,\|u|_{\partial B_R}\|_2)$$

*holds for $R \in (0, \infty)$ with $\overline{\mathcal{D}} \subset B_R$ and for $u \in W_R$.*

We introduce an inner product $(\cdot,\,\cdot)^{(R)}$ in $W_R$ by defining

$$(v,w)^{(R)} = \ \int_{\mathcal{D}_R} \nabla v\cdot\nabla w\,dx + \int_{\partial B_R}(\tau/2)v\cdot w\,do_x \text{ for } \ v,w\in W_R.$$

The space $W_R$ equipped with this inner product is a Hilbert space. The norm generated by this scalar product $(\cdot,\cdot)^{(R)}$ is denoted by $|\cdot|^{(R)}$, that is

$$|v|^{(R)} := (\|\nabla v\|_2^2 + (\tau/2)\|v_{|\partial B_R}\|_2^2)^{1/2} \text{ for } v\in W_R.$$

We define the bilinear forms

$$\begin{aligned}
a_R &: H^1(\mathcal{D}_R)^3\times H^1(\mathcal{D}_R)^3 \to \mathbb{R},\\
\beta_R &: H^1(\mathcal{D}_R)^3\times L^2(\mathcal{D}_R) \to \mathbb{R},\\
\delta_R &: H^1(\mathcal{D}_R)^3\times H^1(\mathcal{D}_R)^3 \to \mathbb{R},
\end{aligned}$$

$$\begin{aligned}
a_R(u,w) &:= \int_{\mathcal{D}_R}[\nabla u\cdot\nabla w + \tau D_1 u\cdot w]dx\\
&\quad + \frac{\tau}{2}\int_{\partial B_R}(u(x)\cdot w(x))\left(1-\frac{x_1}{R}\right)do_x,\\
\beta_R(w,\sigma) &:= -\int_{\mathcal{D}_R}(\operatorname{div} w)\,\sigma dx,\\
\delta_R(u,w) &:= \int_{\mathcal{D}_R}[-\big(\,(\varrho e_1\times x)\cdot\nabla\,\big)u + (\varrho e_1\times u)]\cdot w\,dx
\end{aligned}$$

for $\ u,w\in H^1(\mathcal{D}_R)^3$, $\sigma\in L^2(\mathcal{D}_R)$, $\ R\in(0,\infty)$ with $\overline{\mathcal{D}}\subset B_R$.

**Lemma 3.2** *Let $R\in(0,\infty)$ with $\overline{\mathcal{D}}\subset B_R$. Then*

$$|a_R(u,w)+\delta_R(u,w)| \le C(R)\,|u|^{(R)}\,|w|^{(R)}$$

*for $u,w\in H^1(\mathcal{D}_R)^3$.*

*Proof:* Use of Lemma 3.1.

The key observation in this section is stated in the following lemma, which is the basis of the theory presented in this section.

**Lemma 3.3** *Let $R\in(0,\infty)$ with $\overline{\mathcal{D}}\subset B_R$, and let $w\in W_R$. Then the equation $(|w|^{(R)})^2 = a_R(w,w)+\delta_R(w,w)$ holds.*

*Proof:*

$$\begin{aligned}
&a_R(w,w)+\delta_R(w,w)\\
&=\int_{\mathcal{D}_R}\left[|\nabla w|^2+\tau\partial_1\left(\frac{|w|^2}{2}\right)-(\varrho e_1\times x)\cdot\nabla\left(\frac{|w|^2}{2}\right)\right]dx\\
&+\frac{\tau}{2}\int_{\partial B_R}|w(x)|^2\left(1-\frac{x_1}{R}\right)do_x\\
&=\int_{\mathcal{D}_R}|\nabla w|^2\,dx+\int_{\partial B_R}\left(\frac{\tau}{2}|w(x)|^2\frac{x_1}{R}-\frac{1}{2}(\varrho e_1\times x)\cdot\frac{x}{R}|w(x)|^2\right)do_x\\
&+\frac{\tau}{2}\int_{\partial B_R}|w(x)|^2\left(1-\frac{x_1}{R}\right)do_x\\
&=\int_{\mathcal{D}_R}|\nabla w|^2\,dx+\frac{\tau}{2}\int_{\partial B_R}|w(x)|^2=(|w|^{(R)})^2.
\end{aligned}$$

We applied that

$$\frac{1}{2}(\omega\times x)\cdot x=0\text{ for }\ x,\,\omega\in\mathbb{R}^3.$$

As in [3], we obtain that the bilinear form $\beta_R$ is stable:

**Theorem 3.4** ([3, Corollary 4.3]) *Let $R>0$ with $\overline{\mathcal{D}}\subset B_R$. Then*

$$\inf_{\rho\in L^2(\mathcal{D}_R),\rho\neq 0}\ \sup_{v\in W_R,v\neq 0}\frac{\beta_R(v,\rho)}{|v|^{(R)}\|\rho\|_2}\geq C(R).$$

We introduce an extension operator $\mathfrak{E}:H^{1/2}(\partial\mathcal{D})^3\mapsto H^1(\overline{\mathcal{D}}^c)^3$ such that $\operatorname{div}\mathfrak{E}(b)=0$:

**Theorem 3.5** *There is an operator $\mathfrak{E}:H^{1/2}(\partial\mathcal{D})^3\mapsto W^{1,1}_{loc}(\overline{\mathcal{D}}^c)^3$ such that $\nabla\mathfrak{E}(b)\in L^2(\overline{\mathcal{D}}^c)^9$, $\mathfrak{E}(b)|\partial\mathcal{D}=b$, $\operatorname{div}\mathfrak{E}(b)=0$ and $\mathfrak{E}(b)|\mathcal{D}_R\in H^1(\mathcal{D}_R)^3$ for $b\in H^{1/2}(\partial\mathcal{D})^3$, $R\in(0,\infty)$ with $\overline{\mathcal{D}}\subset B_R$.*

*Proof:* By [14, Exercise III.3.8], there is an operator $\mathfrak{E}:H^{1/2}(\partial\mathcal{D})^3\mapsto W^{1,1}_{loc}(\overline{\mathcal{D}}^c)^3$ such that $\nabla\mathfrak{E}(b)\in L^2(\overline{\mathcal{D}}^c)^9$, $\mathfrak{E}(b)|\partial\mathcal{D}=b$, $\operatorname{div}\mathfrak{E}(b)=0$ and $\|\nabla\mathfrak{E}(b)\|_2\leq C\,\|b\|_{1/2,2}$ for $b\in H^{1/2}(\partial\mathcal{D})^3$. (The latter inequality is not needed here.) Due to [14, Lemma II.6.1], we may conclude that $\mathfrak{E}(b)|\mathcal{D}_R\in H^1(\mathcal{D}_R)^3$ for $R\in(0,\infty)$ with $\overline{\mathcal{D}}\subset B_R$. $\Box$

In view of Lemma 3.2 and 3.3 and Theorem 3.5 and 3.4, the theory of mixed variational problems yields

**Theorem 3.6** *Let $S>0$ with $\overline{\mathcal{D}}\subset B_S$, $R\in[2S,\infty)$, $F\in L^{6/5}(\mathcal{D}_R)^3$, $b\in H^{1/2}(\partial\mathcal{D})^3$. Then there is a uniquely determined pair of functions $(\widetilde{V},P)=\big(\widetilde{V}(R,F,b),\,P(R,F,b)\big)\in$*

$W_R \times L^2(\mathcal{D}_R)$ *such that*

$$a_R(\widetilde{V}, g) + \delta_R(\widetilde{V}, g) + \beta_R(g, P) \tag{3.1}$$
$$= \int_{\mathcal{D}_R} F \cdot g \, dx - a_R\big(\,\mathfrak{E}(b)|\mathcal{D}_R,\, g\,\big) - \delta_R\big(\,\mathfrak{E}(b)|\mathcal{D}_R,\, g\,\big) \;\; \text{for} \;\; g \in W_R,$$

$$\beta_R(\widetilde{V}, \sigma) = 0 \;\; \text{for} \;\; \sigma \in L^2(\mathcal{D}_R), \tag{3.2}$$

*where the operator* $\mathfrak{E}$ *was introduced in Theorem 3.5.*

Let us interpret variational problem (3.1), (3.2) as a boundary value problem. Define the expression used in the boundary condition on the artificial boundary $\partial B_R$ :

$$\mathcal{L}_R(u, \pi)(x) := \left( \sum_{j=1}^{3} \partial_j u_k(x) \frac{x_j}{R} - \pi(x) \delta_{jk} \frac{x_j}{R} + \frac{\tau}{2} \left( 1 - \frac{x_1}{R} \right) u_k(x) \right)_{1 \le k \le 3}$$

for $x \in \partial B_R,\; R \in (0, \infty)$ with $\overline{\mathcal{D}} \subset B_R,\; u \in W^{6/5,\,2}(\mathcal{D}_R)^3,\; \pi \in W^{1,\,6/5}(\mathcal{D}_R)$.

**Lemma 3.7** *Assume that* $\mathcal{D}$ *is* $\mathcal{C}^2$*-bounded. Let* $R \in [2S, \infty)$ *with* $\overline{\mathcal{D}} \subset B_S$, $F \in L^{6/5}(\mathcal{D}_R)^3$ *and* $b \in W^{7/6,\,6/5}(\partial\mathcal{D})^3$*. Put* $V := \widetilde{V}(R, F, b) + \mathfrak{E}(b)|\mathcal{D}_R$*, with* $V(R, F, b)$ *from Theorem 3.6 and* $\mathfrak{E}(b)$ *from Theorem 3.5. Suppose that* $V \in W^{2,6/5}(\mathcal{D}_R)^3$ *and* $P = P(R, F, b) \in W^{1,\,6/5}(\mathcal{D}_R)$*, with* $P(R, F, b)$ *also introduced in Theorem 3.6. Then*

$$\begin{gathered} -\Delta V(z) + (\tau e_1 - \varrho e_1 \times z) \cdot \nabla V(z) + \varrho e_1 \times V(z) + \nabla P(z) = F(z), \\ \operatorname{div} V(z) = 0 \end{gathered} \tag{3.3}$$

*for* $z \in \mathcal{D}_R$*, and* $V|\partial\mathcal{D} = b, \quad \mathcal{L}_R(V, P) = 0.$

The proof of Lemma 3.7 is obvious. This lemma means that a solution of variational problem (3.1), (3.2) may be considered as a weak solution of the modified Oseen system with rotation in $\mathcal{D}_R$, under the Dirichlet boundary condition on $\partial D$ and under the artificial boundary condition $\mathcal{L}_R(V, P) = 0$ on $\partial B_R$. The solution of (3.1), (3.2) will be now compared to the exterior modified Oseen flow introduced in Corollary 2.2:

**Theorem 3.8** *Suppose that* $\mathcal{D}$ *is* $C^2$*-bounded. Let* $\gamma,\, S_1 \in (0, \infty)$ *with* $\overline{\mathcal{D}} \subset B_{S_1}$, $A \in [5/2,\, \infty)$, $B \in \mathbb{R}$ *with* $A + \min\{1, B\} > 3$*. Let* $F : \overline{\mathcal{D}}^c \mapsto \mathbb{R}^3$ *be measurable with* $F|\mathcal{D}_{S_1} \in L^{6/5}(\mathcal{D}_{S_1})^3$ *and* $|F(z)| \le \gamma\, |z|^{-A} s(z)^{-B}$ *for* $z \in B^c_{S_1}$*.*

*Let* $b \in W^{7/6,\,6/5}(\partial\mathcal{D})^3$, $u \in W^{1,1}_{loc}(\overline{\mathcal{D}}^c)^3 \cap L^6(\overline{\mathcal{D}}^c)^3$ *such that* $\nabla u \in L^2(\overline{\mathcal{D}}^c)^9$, $\operatorname{div} u = 0$, $u|\partial\mathcal{D} = b$ *and equation (2.3) is satisfied.*

*For* $R \in [2S_1,\, \infty)$*, put* $V_R := \widetilde{V}(R, F, b) + \mathfrak{E}(b)$*, with* $\mathfrak{E}(b)$ *from Theorem 3.5, and* $\widetilde{V}(R, F, b)$ *from Theorem 3.6. Then*

$$|u|_{\mathcal{D}_R} - V_R|^{(R)} \le C\, R^{-1} \quad \text{for} \;\; R \in [2S, \infty).$$

We note that since $W^{2,\,6/5}(\mathcal{D}) \subset H^1(\mathcal{D})$ by a Sobolev inequality, we have $W^{7/6,\,6/5}(\partial\mathcal{D}) \subset H^{1/2}(\partial\mathcal{D})$, as follows with the usual lifting property. As a consequence, $b \in H^{1/2}(\partial\mathcal{D})^3$, so the term $\mathfrak{E}(b)$ is well defined. We further remark that by Corollary 2.2 with $p = 6/5$, the function $F$ may be considered as a bounded linear functional on $\mathcal{D}_0^{1,2}(\overline{\mathcal{D}}^c)^3$. Therefore, as explained in Remark 2.3, a function $u$ with properties as stated in Theorem 3.8 does in fact exist.

*Proof of Theorem 3.8:* All conditions in Corollary 2.2 are verified if $\gamma$, $S_1$, $A$, $B$, $F$, $u$ are given as in Theorem 3.8, and if $p = 6/5$ and $S = 2\,S_1$. Note in this respect that the conditions on $u$ in Theorem 3.8 obviously imply $u \in W^{1,\,6/5}_{loc}(\overline{\mathcal{D}}^c)^3$. Corollary 2.2 now yields that $F \in L^{6/5}(\overline{\mathcal{D}}^c)^3$ and that the function $u$ satisfies inequalities (2.1) and (2.2) with $S = 2\,S_1$.

On the other hand, since $u \in W^{1,\,6/5}_{loc}(\overline{\mathcal{D}}^c)^3$, the function $G$ already considered in the proof of Corollary 2.2 (see (2.4)) belongs to $L^{6/5}_{loc}(\overline{\mathcal{D}}^c)^3$. Therefore, by interior regularity of solutions to the Stokes system (see [14, Theorem IV.4.1]), we may deduce from the equations (2.3) and $\operatorname{div} u = 0$ that $u \in W^{2,\,6/5}_{loc}(\overline{\mathcal{D}}^c)^3$ and that there is $\pi \in W^{1,\,6/5}_{loc}(\overline{\mathcal{D}}^c)^3$ with $L(u) + \nabla\pi = F$. In particular the pair $(u, \pi)$ verifies (2.5). In view of our assumptions on $A$ and $B$, we thus see that the requirements in Corollary 2.5 are fulfilled for $\gamma$, $S_1$, $A$, $B$, $F$, $u$ as in Theorem 3.8 and for $p = 6/5$ and $S = 2\,S_1$. As a consequence, Corollary 2.5 yields that there is $c_0 \in \mathbb{R}$ such that (2.6) holds with $S = 2\,S_1$.

Take $R \in [2\,S_1,\,\infty)$. Since $u \in W^{2,6/5}_{loc}(\overline{\mathcal{D}}^c)^3$, we have $u|\partial\mathcal{D} \in W^{7/5,\,6/5}(\partial\mathcal{D})^3$. Combining this relation with the assumption $b|\partial\mathcal{D} \in W^{7/5,\,6/5}(\partial\mathcal{D})^3$ and the boundary condition $u|\partial\mathcal{D} = b$, we get $u|\partial\mathcal{D}_R \in W^{7/5,\,6/5}(\partial\mathcal{D}_R)^3$. Moreover our requirements on $u$ yield that $u|\mathcal{D}_R \in W^{1,\,6/5}(\mathcal{D}_R)^3$. Since $F \in L^{6/5}(\overline{\mathcal{D}}^c)^3$, as already mentioned, we get $G|\mathcal{D}_R \in L^{6/5}(\mathcal{D}_R)^3$, with $G$ from (2.4). Recalling that $\mathcal{D}$ is supposed to be $C^2$-bounded, we may now apply the result in [14, Lemma IV.6.1] on boundary regularity of solutions to the Stokes system. This reference yields that $u|\mathcal{D}_R \in W^{2,\,6/5}(\mathcal{D}_R)^3$, $\pi|\mathcal{D}_R \in W^{1,\,6/5}(\mathcal{D}_R)$ and that the pair $(u, \pi)$ solves (1.2).

Let $P_R := P(R, F, b)$ be given as in Theorem 3.6, and put $w := u - V_R$, $\kappa := \pi - P_R$, and let $g \in W_R$. Note that by Theorem 3.6, we have $a_R(V_R, g) + \delta_R(V_R, g) + \beta_R(g, P_R) =$

$\int_{\mathcal{D}_R} F \cdot g\, dx$. Thus

$$\begin{aligned}
& a_R(w,g) + \delta_R(w,g) + \beta_R(g,\kappa) \\
& = a_R(u|_{\mathcal{D}_R}, g) + \delta_R(u|_{\mathcal{D}_R}, g) + \beta_R(g, \pi|_{\mathcal{D}_R}) - \big( \underbrace{a_R(V_R, g) + \delta_R(V_R, g) + \beta_R(g, P_R)}_{= \int_{\mathcal{D}_R} F \cdot g dx} \big) \\
& = \int_{\mathcal{D}_R} \big( \nabla u \cdot \nabla g + \tau \partial_1 u \cdot g - (\varrho e_1 \times x) \cdot \nabla u \cdot g + (\varrho e_1 \times u) \cdot g \\
& \qquad\qquad - \pi \operatorname{div} g - F \cdot g \big)\, dx \\
& \quad + \frac{\tau}{2} \int_{\partial B_R} u(x) \cdot g(x) \Big( 1 - \frac{x_1}{R} \Big)\, do_x \\
& = \int_{\mathcal{D}_R} [ -\Delta u \cdot g + \tau \partial_1 u \cdot g - (\varrho e_1 \times x) \cdot \nabla u \cdot g + (\varrho e_1 \times u) \cdot g \\
& \qquad\qquad + \nabla \pi \cdot g - F \cdot g ]\, dx \\
& \quad \underbrace{+ \int_{\partial B_R} \Big( \sum_{j,k=1}^{3} \big[ \partial_j u_k(x)\, g_k(x) \frac{x_j}{R} - \pi(x) \delta_{jk} g_k(x) \frac{x_j}{R} \big] + \frac{\tau}{2} u(x) \cdot g(x) (1 - \frac{x_1}{R}) \Big) do_x}_{= \int_{\partial B_R} \mathcal{L}_R(u,\pi) \cdot g \;\, do}
\end{aligned}$$

Since the pair $(u, \pi)$ solves (1.2), we now get

$$a_R(w,g) + \delta_R(w,g) + \beta_R(g,\kappa) = \int_{\partial B_R} \mathcal{L}_R(u,\pi)(x) \cdot g(x)\, do_x. \tag{3.4}$$

Let $c \in \mathbb{R}$ be an arbitrary constant. For $g := w$ we get with Lemma 3.3 that

$$\begin{aligned}
(|w|^{(R)})^2 &= a_R(w,w) + \delta_R(w,w) + \beta_R(w,\kappa) \\
&= \int_{\partial B_R} \mathcal{L}_R(u, \pi + c)(x) \cdot w(x)\, do_x\,,
\end{aligned} \tag{3.5}$$

$$\text{because: } \int_{\partial B_R} \Big[ \sum_{j,k=1}^{3} c \delta_{jk} w_k(x) \frac{x_j}{R} \Big] do_x = \int_{\partial \mathcal{D}_R} c\, w \cdot n\, do_x + \int_{\mathcal{D}_R} c \operatorname{div} w\, dx = 0.$$

Let $c_0$ be the constant introduced above as part of estimate (2.6). Because

$$\int_{\partial B_R} \mathcal{L}_R(u, \pi + c_0)(x) \cdot w(x)\, do_x \le \|\mathcal{L}_R(u, \pi + c_0)\|_2\, \|w|_{\partial \mathcal{D}}\|_2 \le C\, \|\mathcal{L}_R(u, \pi + c_0)\|_2 \cdot |w|^{(R)},$$

we get from (3.5)

$$|w|^{(R)} \le C \|\mathcal{L}_R(u, \pi + c_0)\|_2.$$

The last step is estimation: $\|\mathcal{L}_R(u, \pi + c_0)\|_2 \leq C \cdot R^{-1}$.

$$\|\mathcal{L}_R(u, \pi + c_0)\|_2 \\ \leq C\Big[\|\nabla u|_{\partial B_R}\|_2 + \|[\pi(x) + c_0]|_{\partial B_R}\|_2 + \Big(\int_{\partial B_R}\Big(1 - \frac{x_1}{R}\Big)^2 |u(x)|^2 do_x\Big)^{1/2}\Big].$$

As explained above, inequalities (2.1), (2.2) and (2.6) are valid with $S = 2\,S_1$. According to (2.1) and (2.6), we have $|u(x)| \leq C\,(|x|s(x))^{-1}$, and $|\pi(x) + c_0| \leq C\,|x|^{-2}$ for $x \in B^c_{2\,S_1}$. Inequality (2.2) yields $|\nabla u(x)| \leq C\,|x|^{-3}s(x)^{-B'}$ for $x$ as before, with $B' := 3/2 - \max\{0,\, 7/2 - A - B\}$. If $B \geq 1$, we recall that $A \geq 5/2$, getting $A + B \geq 7/2$, hence $B' = 3/2$. On the other hand, if $B < 1$, then $\max\{1, B\} = B$, so that the assumption $A + \max\{1, B\} > 3$ becomes $A + B > 3$, hence $B' > 1$. Thus we get in any case that $B' > 1 > 1/2$. In view of these observations, and with Lemma 1.3, we obtain

$$\begin{aligned}
\|\mathcal{L}_R(u, \pi + c_0)\|_2 \;&\leq\; C\Bigg[\Big(\int_{\partial B_R} |x|^{-3}s(x)^{-2\,B'}\,do_x\Big)^{1/2} + \Big(\int_{\partial B_R} |\pi(x) + c_0|^2 do_x\Big)^{1/2} \\
&\qquad + \Big(\frac{1}{R^2}\int_{\partial B_R} (|x| - x_1)^2 |u(x)|^2 do_x\Big)^{1/2}\Bigg] \\
&\leq\; C\Bigg[\Big(\frac{1}{R^3}\int_{\partial B_R} s(x)^{-2\,B'} do_x\Big)^{1/2} + \Big(\frac{1}{R^4}\int_{\partial B_R} 1\,do_x\Big)^{1/2} \\
&\qquad + \Big(\frac{1}{R^2}\int_{\partial B_R} s(x)^2(|x|s(x))^{-2} do_x\Big)^{1/2}\Bigg] \\
&\leq\; C\Bigg[\Big(\frac{1}{R^2}\Big)^{1/2} + \Big(\frac{1}{R^2}\Big)^{1/2} + \Big(\frac{1}{R^4}\int_{\partial B_R} 1\,do_x\Big)^{1/2}\Bigg] \leq CR^{-1}.
\end{aligned}$$

This completes the proof of Theorem 3.8. □

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