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Abstract The paper establishes necessary and sufficient conditions for the Coleman-Noll inequality for isotropic materials.

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Statement and proof

Towards the end of the formatting period of the nonlinear elasticity, in 1956, in his paper *Das ungelöste Hauptproblem der endlichen Elastizitätstheorie,* [11], Truesdell points out that the nonlinear elasticity misses conditions on the strain-energy that would guarantee reasonable behavior (existence of solutions, stability, reality of wave speeds, uniqueness etc.). These restrictions should be nonlinear counterparts of the well-known inequalities on Lamé constants in linear elasticity. After a debate that

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lasted more than two decades, the final solution of the *Hauptproblem* came in 1977 with J. M. Ball [1], who showed that Morrey's condition of quasiconvexity is exactly what is missing in the analysis of the nonlinear response in elasticity. The virtues and consequences of the quasiconvexity are well-known and will not be repeated here.

The early paper of Coleman & Noll [8] (1959) is, on the contrary, probably the first response to Truesdell's paper. The authors postulate an inequality that now bears their name. The subsequent discussion showed the untenability of the Coleman-Noll inequality. Nevertheless, in this note I discuss the consequences of the inequality for isotropic materials. This class of materials is treated in the original paper by Coleman & Noll [8; §12], where the authors show that the Coleman-Noll inequality *implies* that the free energy is a convex function of principal stretches, i.e., Condition (i) below. Here I complement that condition with Inequality (8) below to get conditions that are simultaneously necessary and sufficient.

We shall consider only an isothermal version, which deals with the material governed by the constitutive equations for the specific free energy and Piola-Kirchhoff stress of the form

$$\psi = \bar\psi(\boldsymbol{F}), \quad \boldsymbol{S} = \bar{\boldsymbol{S}}(\boldsymbol{F}),$$

where $\boldsymbol{F}$ is the deformation gradient. Throughout this note, it is assumed that the common domain of $\bar\psi$ and $\bar{\boldsymbol{S}}$ is the set Lin^+ of all second-order tensors with positive determinant, and that $\bar\psi$ is twice continuously differentiable and $\bar{\boldsymbol{S}}$ continuously differentiable.

The isothermal version of the Coleman-Noll inequality reads

$$\bar\psi(\boldsymbol{F}^*) \ge \bar\psi(\boldsymbol{F}) + \bar{\boldsymbol{S}}(\boldsymbol{F}) \cdot (\boldsymbol{F}^* - \boldsymbol{F}) \tag{1}$$

for all $\boldsymbol{F}^* \in \mathrm{Lin}^+$ such that

$$\boldsymbol{F}^* = \boldsymbol{G}\boldsymbol{F}$$

where $\boldsymbol{G}$ is a symmetric positive definite tensor. (We have also replaced the strict inequality by a non-strict one.)

If $\boldsymbol{H}$ is any symmetric second-order tensor, then $\boldsymbol{G}(t) := \mathbf{1} + t\boldsymbol{H}$ is positive definite tensor for all $t \in \mathbb{R}$ sufficiently close to 0. Hence if (1) holds for the given $\boldsymbol{F}$, the excess function

$$\phi(t) = \bar\psi\big(\boldsymbol{F}^*(t)\big) - \bar\psi(\boldsymbol{F}) - \bar{\boldsymbol{S}}(\boldsymbol{F}) \cdot \big(\boldsymbol{F}^*(t) - \boldsymbol{F}\big),$$

where $\boldsymbol{F}^*(t) = \boldsymbol{G}(t)\boldsymbol{F}$, has a local minimum at $t = 0$. The conditions $\dot\phi(0) = 0$ gives the stress relation

$$\bar{\boldsymbol{S}}(\boldsymbol{F}) = \partial_{\boldsymbol{F}}\bar\psi(\boldsymbol{F}) \tag{2}$$

while $\ddot\phi(0) \ge 0$ reads

$$\mathrm{D}^2\bar\psi(\boldsymbol{F})[\boldsymbol{HF}, \boldsymbol{HF}] \ge 0 \tag{3}$$

for every $\boldsymbol{F} \in \mathrm{Lin}^+$ and every $\boldsymbol{H} \in \mathrm{Sym}$. Here $\mathrm{D}^2\bar\psi(\boldsymbol{F})[\cdot,\cdot]$ is the second derivative of $\bar\psi$ interpreted as a quadratic form. Conversely, assume that (2) and (3) hold for all $\boldsymbol{F} \in \mathrm{Lin}^+$. If $\boldsymbol{G}$ is positive definite symmetric, then the tensor $\boldsymbol{F}(t) = \boldsymbol{F} + t\boldsymbol{H}$, where $\boldsymbol{H} = \boldsymbol{G} - \mathbf{1}$, belongs to Lin^+ for all $t \in [0, 1]$ and hence the replacement of $\boldsymbol{F}$ by $\boldsymbol{F}(t)$ in (3) gives $\mathrm{D}^2\bar\psi(\boldsymbol{F}(t))[\boldsymbol{HF}(t), \boldsymbol{HF}(t)] \ge 0$. Upon the integration with respect to t and upon the use of (2) this gives (1). Thus, *when postulated for all* $\boldsymbol{F} \in \mathrm{Lin}^+$, *Inequality* (1) *is equivalent to the joint satisfaction of* (2) *and* (3) *for all* $\boldsymbol{F} \in \mathrm{Lin}^+$.

We shall discuss Condition (3) for isotropic materials, i.e., for materials that satisfy

$$\bar{\psi}(\boldsymbol{QFR}) = \bar{\psi}(\boldsymbol{F}) \tag{4}$$

for every $\boldsymbol{F} \in \mathrm{Lin}^+$ and every proper orthogonal tensors $\boldsymbol{Q}$ and $\boldsymbol{R}$. Equation (4) is equivalent to the representation theorem

$$\bar{\psi}(\boldsymbol{F}) = f(v_1, v_2, v_3) \tag{5}$$

for every $\boldsymbol{F} \in \mathrm{Lin}^+$ where v_1, v_2, v_3 are the principal stretches of $\boldsymbol{F}$, i.e., the eigenvalues of $\sqrt{\boldsymbol{FF}^{\mathrm{T}}}$, and where $f : (0,\infty)^3 \to \mathbb{R}$ is a function given by

$$f(w_1, w_2, w_3) = \bar{\psi}(\mathrm{diag}(w_1, w_2, w_3))$$

for any $(w_1, w_2, w_3) \in (0,\infty)^3$. It follows that f is invariant under any permutation of (w_1, w_2, w_3). Our hypothesis implies that f is twice continuously differentiable. For isotropic materials it suffices to examine Condition (3) only for diagonal tensors $\boldsymbol{F}$; Equation (4) then implies that Condition (3) holds for all $\boldsymbol{F} \in \mathrm{Lin}^+$. We now apply the following assertion [3–6, 2, 9–10]: *if* $\boldsymbol{F} = \mathrm{diag}(v_1, v_2, v_3) \in \mathrm{Lin}$ *then*

$$\mathrm{D}\bar{\psi}(\boldsymbol{F}) = \mathrm{diag}(f_1, f_2, f_3),$$

where the subscripts attached to f indicate the partial derivatives of f evaluated at (v_1, v_2, v_3). Furthermore, if the components v_1, v_2, v_3 are distinct, then

$$\mathrm{D}^2\bar{\psi}(\boldsymbol{F})[\boldsymbol{B},\boldsymbol{B}] = \sum_{i,j=1}^{3} f_{ij}B_{ii}B_{jj} + \sum_{1\le i\ne j\le 3} \left(M_{ij}^{+}B_{ij}^2 + M_{ij}^{-}B_{ij}B_{ji}\right) \tag{6}$$

for any second-order tensor $\boldsymbol{B}$, where

$$M_{ij}^{+} = \frac{v_i f_i - v_j f_j}{v_i^2 - v_j^2}, \quad M_{ij}^{-} = \frac{v_j f_i - v_i f_j}{v_i^2 - v_j^2}.$$

Observing that $(\boldsymbol{HF})_{ij} = H_{ij}v_j$ we see from (6) that Condition (3) reads

$$\mathrm{D}^2\bar{\psi}(\boldsymbol{F})[\boldsymbol{HF},\boldsymbol{HF}] = \sum_{i,j=1}^{3} f_{ij}v_iv_jH_{ii}H_{jj} + \sum_{1\le i\ne j\le 3} C_{ij}H_{ij}^2 \tag{7}$$

where

$$C_{ij} = M_{ij}^{+}v_j^2 + M_{ij}^{-}v_iv_j = \frac{2v_j^2v_if_i - v_j^3f_j - v_i^2v_jf_j}{v_i^2 - v_j^2}.$$

For the given $\boldsymbol{F} = \mathrm{diag}(v_1, v_2, v_3)$ with distinct diagonal elements, the independence of the diagonal and off-diagonal elements of $\boldsymbol{H}$ and the symmetry of $\boldsymbol{H}$ show that Inequality (7) is is equivalent to the following two assertions:

(i) *the matrix $[f_{ij}]_{i,j=1}^3$ is positive-semidefinite at (v_1, v_2, v_3);*

(ii) *one has $C_{ij} + C_{ji} \ge 0$ whenever $1 \le i \ne j \le 3$, i.e.,*

$$\frac{(3v_j^2 + v_i^2)v_if_i - (3v_i^2 + v_j^2)v_jf_j}{v_i^2 - v_j^2} \ge 0 \tag{8}$$

where the partial derivatives are evaluated at (v_1, v_2, v_3).

Conversely, if Conditions (i) and (ii) hold at all (v_1, v_2, v_3) with distinct elements then (3) holds for all $\boldsymbol{F} \in \mathrm{Lin}^+$ with distinct principal stretches. Hence for all $\boldsymbol{F} \in \mathrm{Lin}^+$ by continuity and density. Thus for an isotropic material, *Inequality* (1) *holds for all* $\boldsymbol{F} \in \mathrm{Lin}^+$ *if and only if*

$$\bar{\boldsymbol{S}}(\boldsymbol{F}) = \mathrm{diag}(f_1, f_2, f_3) \tag{9}$$

for ever $\boldsymbol{F} \in \mathrm{Lin}^+$ *and Conditions* (i) *and* (ii) *hold for all* (v_1, v_2, v_3) *with distinct elements.*

Recall that the partial derivatives f_i are called the principal forces, while the quantities $t_i = v_i f_i / v_1 v_2 v_3$, i.e., the eigenvalues of the Cauchy stress tensor, are called the principal stresses. Consequence (i), i.e., the convexity of f, was derived by Coleman & Noll, together with a well-known consequence for symmetric convex functions, the ordered-force inequality

$$(v_i - v_j)(f_i - f_j) \ge 0. \tag{10}$$

In terms of the principal stresses, Condition (ii) reads

$$\frac{(3v_j^2 + v_i^2)t_i - (3v_i^2 + v_j^2)t_j}{v_i^2 - v_j^2} \ge 0. \tag{11}$$

Let us now mention some consequences. Hence assume that (i) holds for all $(v_1, v_2, v_3) \in (0, \infty)^3$ and that (ii) holds for all $(v_1, v_2, v_3) \in (0, \infty)^3$ with distinct components. We have the following assertions.

Consequence A (Nonnegativity of pressure) If the stress reduces to the hydrostatic pressure, i.e., if $t_1 = t_2 = t_3 = -p$ then (11) reduces to $p \ge 0$.

Consequence B (Coleman-Noll inequality and Baker-Ericksen inequalities) Recall that the Baker-Ericksen inequalities read

$$(v_i - v_j)(t_i - t_j) \ge 0 \tag{12}$$

whenever $1 \le i \neq j \le 3$. To discuss the relationship to the Coleman-Noll inequality, assume that $v_i > v_j$ (so that $f_i \ge f_j$ by (10)) and (12) reduces to

$$t_i - t_j \ge 0. \tag{13}$$

Note first that

$$t_j \ge 0$$

then (13) holds. Second, note that if (13) is violated, i.e., if

$$t_i - t_j < 0, \tag{14}$$

then

$$t_i < t_j < 0. \tag{15}$$

Proof For $v_i > v_j$ Inequality (11) reduces to

$$(3v_j^2 + v_i^2)t_i - (3v_i^2 + v_j^2)t_j \ge 0. \tag{16}$$

Thus if $t_j \ge 0$ then $(3v_i^2 + v_j^2)t_j \ge (3v_j^2 + v_i^2)t_j$ and hence (16) gives (13). Similarly, if (14) holds then $(3v_j^2 + v_i^2)t_i < (3v_j^2 + v_i^2)t_j$ and one finds that (11) gives $(2v_j^2 - 2v_i^2)t_j > 0$, i.e., $t_j < 0$, and (15) follows by combination with (14). □

Consequence C (Coleman-Noll inequality and fluids) Fluids satisfy the requirement

$$\bar{\psi}(\boldsymbol{QFH}) = \bar{\psi}(\boldsymbol{F}) \tag{17}$$

for every $\boldsymbol{F} \in \mathrm{Lin}^+$, every proper orthogonal tensors $\boldsymbol{Q}$, and every second-order tensor $\boldsymbol{H}$ with $\det \boldsymbol{H} = 1$. Hence, in particular, (4) holds, i.e., fluids are isotropic materials. Equation (17) is satisfied if and only if

$$\bar{\psi}(\boldsymbol{F}) = \varphi(v)$$

for every $\boldsymbol{F} \in \mathrm{Lin}^+$ and some function $\varphi : (0, \infty) \to \mathbb{R}$, where

$$v = \det \boldsymbol{F} = v_1 v_2 v_3$$

is the specific volume. Thus we have (5) with $f(v_1, v_2, v_3) = \varphi(v_1 v_2 v_3)$. The Coleman-Noll inequality is equivalent to the stress relation (9) and Conditions (i) and (ii). The stress relation reduces to

$$t_1 = t_2 = t_3 = -p \quad \text{with} \quad p = -\varphi'.$$

One finds that $f_{ij} = \left((1 - \delta_{ij}) v\varphi' + v^2 \varphi''\right)/v_i v_j$ and hence the positive definite character of the matrix $[f_{ij}]_{i,j=1}^3$ is expressed by the inequality

$$(v\varphi' + v^2\varphi'')\Big(\sum_{i=1}^{3} x_i\Big)^2 - v\varphi'|x|^2 \geq 0 \tag{18}$$

for every $x \in \mathbb{R}^3$. The choice $x = (1,1,1)$ gives

$$2\varphi' + 3v\varphi'' \geq 0. \tag{19}$$

It is easily found that actually (19) is sufficient for (18) and these conditions are, in turn, equivalent to the assertion that $\varphi(v)$ is a convex function of $\sqrt[3]{v}$. If the pressure $p = -\varphi'$ is expressed in term of the density $\rho = 1/v$, i.e., $p = p(\rho)$, we have [7]

$$p(\rho) \geq 0, \qquad \frac{dp(\rho)}{d\rho} \geq \frac{2}{3} p(\rho).$$

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