Mathematics of fluids in motion

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Some naive questions concerning models

Well posedness

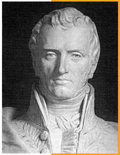
- Does the problem admit a solution for given data?
- In which sense the solution is understood?
- Is solution determined uniquely by the data?
- Are solutions stable under data perturbation?

Typical answers

We don't know yet. But in certain cases we do know...

Navier-Stokes system - Millenium Problem

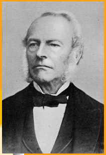
- $\mathbf{u} = \mathbf{u}(t,x)$ .... fluid velocity
- $\Pi = \Pi(t,x)$ .... pressure



Claude Louis Marie Henri Navier [1785-1836]

"Incompressibility"

$$\mathrm{div}_x \mathbf{u} = 0$$

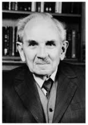


George Gabriel Stokes [1819-1903]

Balance of momentum

$$\partial_t \mathbf{u} + \mathrm{div}_x(\mathbf{u} \otimes \mathbf{u}) + \nabla_x \Pi = \Delta_x \mathbf{u}$$

State of the art for viscous fuids



Jean Leray - Royal society (1991)

Jean Leray [1906-1998]

Global existence of the so-called **weak** solutions for the incompressible Navier-Stokes system (3D)

Olga Aleksandrovna Ladyzhenskaya [1922-2004]

Global existence of classical solutions for the incompressible 2D Navier-Stokes system

Pierre-Louis Lions[*1956]

Global existence of weak solutions for the compressible barotropic Navier-Stokes system (2,3D)

and many, many others...

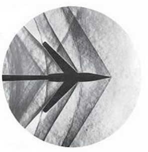
Things may go wrong...

Blow-up singularities - concentrations

Solutions become large (infinite) in a finite time.
There is too much energy pumped in the system

Shock waves - oscillations

Shocks are singularities in "derivatives".
Originally smooth solutions become discontinuous in a finite time



"Bad" nonlinearities

$$\partial_t U = \boxed{U^2}, \ \partial_t U + \boxed{U \partial_x U} = 0$$

Do we need mathematics?

However beautiful the strategy, you should occasionally look at the results...
Sir Winston Churchill
[1874-1965]

Euler system (compressible inviscid)

- $\mathbf{u} = \mathbf{u}(t, x)$ .... fluid velocity
- $\varrho = \varrho(t, x)$ .... density

Leonhard Paul Euler
[1707-1783]

Mass conservation

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

Balance of momentum

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = 0$$

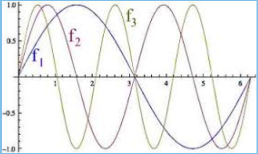
Oscillations vs. nonlinearity

Oscillatory solutions - velocity

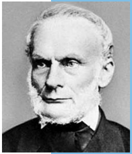
$$U(x) \approx \sin(nx),\ U \to 0 \text{ in the sense of avarages (weakly)}$$

Oscillatory solutions - kinetic energy

$$\frac{1}{2}|U|^2(x) \approx \frac{1}{2}\sin^2(nx) \to \frac{1}{4} \neq \frac{1}{2}0^2 \text{ in the sense of avarages (weakly)}$$



Do some solutions lose/produce energy?



Rudolph Clausius, [1822–1888]

First and Second law of thermodynamics

Die Energie der Welt ist constant; Die Entropie der Welt strebt einem Maximum zu

Mechanical energy balance for compressible fluid

$$\textbf{classical:}\ \frac{\mathrm{d}}{\mathrm{d}t}\int \frac{1}{2}\varrho|\mathbf{u}|^2 + P(\varrho)\ \mathrm{d}x = 0,\ \ P(\varrho) = \varrho\int_1^{\varrho}\frac{p(z)}{z^2}\ \mathrm{d}z$$

$$\textbf{weak:}\ \frac{\mathrm{d}}{\mathrm{d}t}\int \frac{1}{2}\varrho|\mathbf{u}|^2 + P(\varrho)\ \mathrm{d}x \boxed{\leq} 0$$

Compressible Euler system - summary

Existence

Global-in-time solutions (in general) do not exist. Weak solutions may exist but may not be uniquely determined by the initial data.

Mechanical energy

$$E = \frac{1}{2}\varrho|\mathbf{u}|^2 + P(\varrho)$$

Admissibility criteria - mechanical energy dissipation

$$\partial_t E + \mathrm{div}_x(E\mathbf{u} + p(\varrho)\mathbf{u}) \boxed{\leq} 0$$

Bad or good news for compressible Euler?

Camillo DeLellis [*1976]

Existence

Good news: There exists a global-in-time weak solution of compressible Euler system for "any" initial data

Bad news: There are infinitely many...

Admissible solutions?

Good news: Most of these "wild" solutions produce energy.

Bad news: There is a vast class of data for which there exist infinitely many admissible solutions

László Székelyhidi [*1977]

Known facts about compressible Euler...

Many admissible solutions [E.Chiodaroli, EF]

For any (or a large class) of initial densities, there exists an initial bounded velocity field such that the compressible Euler system admits infinitely many *physically admissible* solutions

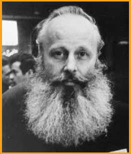
Many admissible solutions [E.Chiodaroli, C.DeLellis, O.Kreml]

There exists Lipschitz (regular) initial data for which the compressible Euler system admits infinitely many admissible solutions

Many admissible solutions [E.Chiodaroli, O.Kreml]

There exist initial data for the Riemann problem (in 2D) and a weak solution of the compressible Euler system that dissipates “globally” more energy than the Riemann solution

Do we compute the right object?



Young measures

$$U(t,x) \approx \nu_{t,x}[U]$$

$\nu(B), B \subset R^3$ probability that **U** belongs to the set B

Laurence Chisholm Young [1905-2000]

Siddhartha Mishra

Numerical results

Certain numerical solutions of "inviscid" problems exhibit scheme independent oscillatory behavior

The way out?

Possible solutions...

- Forget Euler and similar (inviscid) systems
- Take into account only viscosity solutions to Euler but what kind?
- Formulate a local condition of maximality of energy dissipation
- Add more equations...