SECOND ORDER BOUNDARY VALUE PROBLEMS WITH SIGN-CHANGING NONLINEARITIES AND NONHOMOGENEOUS BOUNDARY CONDITIONS

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Abstract. The authors consider the boundary value problem with a two-parameter nonhomogeneous multi-point boundary condition

$$u'' + g(t)f(t,u) = 0, \quad t \in (0,1),$$
$$u(0) = \alpha u(\xi) + \lambda, \quad u(1) = \beta u(\eta) + \mu.$$

Criteria for the existence of nontrivial solutions of the problem are established. The nonlinear term $f(t,x)$ may take negative values and may be unbounded from below. Conditions are determined by the relationship between the behavior of $f(t,x)/x$ for x near 0 and $\pm\infty$, and the smallest positive characteristic value of an associated linear integral operator. The analysis mainly relies on topological degree theory. This work complements some recent results in the literature. The results are illustrated with examples.

Keywords: nontrivial solutions, nonhomogeneous boundary conditions, cone, Krein-Rutman theorem, Leray-Schauder degree

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1. Introduction

In this paper, we study the existence of nontrivial solutions of the boundary value problem (BVP) consisting of the equation

$$u'' + g(t)f(t,u) = 0, \quad t \in (0,1), \tag{1.1}$$

and the nonhomogeneous multi-point boundary condition (BC)

$$u(0) = \alpha u(\xi) + \lambda, \quad u(1) = \beta u(\eta) + \mu, \tag{1.2}$$

where $f\colon [0,1]\times\mathbb{R}\to\mathbb{R}$ and $g\colon [0,1]\to\mathbb{R}_+ := [0,\infty)$ are continuous with $g(t)\not\equiv 0$ on $[0,1]$, ξ, $\eta\in[0,1]$, and α, β, λ, $\mu\in\mathbb{R}_+$. Throughout this paper, we assume the following condition holds without further mention:

(H) $\alpha(1-\xi)<1,\ \beta\eta<1,\ \text{ and } \varrho := (1-\alpha)(1-\beta\eta)+(1-\beta)\alpha\xi>0.$

When f is positone, (i.e., $f\geqslant 0$), existence of solutions of BVP (1.1), (1.2), or some of its variations, has been extensively investigated. For example, papers [6], [13], [14] studied BVPs with one-parameter BCs and [8], [9], [10] studied BVPs with two-parameter BCs. For one-parameter problems, Ma [13] studied the BVP consisting of Eq. (1.1) and the BC

$$u(0)=0,\ u(1)=\beta u(\eta)+\mu. \tag{1.3}$$

Under certain assumptions, he showed that there exists $\mu^*>0$ such that BVP (1.1), (1.3) has at least one positive solution for $0<\mu<\mu^*$ and has no positive solution for $\mu>\mu^*$; later, Guo et al. [6] and Sun et al. [14] obtained similar results for the BVPs consisting of Eq. (1.1) and the BCs

$$u(0)=0,\ u(1)=\sum_{i=1}^{m}\beta_i u(\eta_i)+\mu \quad\text{and}\quad u'(0)=0,\ u(1)=\sum_{i=1}^{m}\beta_i u(\eta_i)+\mu,$$

respectively. As for the two-parameter problems, Kong and Kong [8], [9] studied a more general form of BVP (1.1), (1.2) with $\lambda,\mu\in\mathbb{R}$, and under certain assumptions, they proved that there exists a continuous decreasing curve Γ separating the (λ,μ)-plane into two disjoint connected regions Λ^E and Λ^N, with $\Gamma\subseteq\Lambda^E$, such that BVP (1.1), (1.2) has at least two solutions for $(\lambda,\mu)\in\Lambda^E\setminus\Gamma$, has at least one solution for $(\lambda,\mu)\in\Gamma$, and has no solutions for $(\lambda,\mu)\in\Lambda^N$.

However, very little has been done in the literature on BVPs with nonhomogeneous BCs when the nonlinearities are sign-changing functions. Here we will apply topological degree theory to derive several new criteria for the existence of nontrivial solutions of BVP (1.1), (1.2) when the nonlinear term f is a sign-changing function and not necessarily bounded from below. To the best of our knowledge, this is the first work to study BVPs with sign-changing nonlinearities and nonhomogeneous BCs. Some of our existence conditions are determined by the relationship between the behavior of the quotient $f(t,x)/x$ for x near 0 and $\pm\infty$ and the smallest positive characteristic value (given by (3.5) below) of a related linear operator M defined in (2.7) in Section 2. Our results complement some recent works on BVPs with nonhomogeneous BCs, especially those in papers [8], [9], [10] for BVP (1.1), (1.2).

The next section contains some preliminary lemmas, Section 3 contains our main results and several examples, and the proofs are presented in Section 4.

2. Preliminary results

We let the bold $\mathbf{0}$ stand for the zero element in any given Banach space.

Lemma 2.1 ([5, Lemma 2.5.1]). *Let Ω be a bounded open set in a real Banach space X with $\mathbf{0} \in \Omega$ and let $T\colon \overline{\Omega} \to X$ be a compact operator. If $Tu \neq \tau u$ for all $u \in \partial\Omega$ and $\tau \geqslant 1$, then the Leray-Schauder degree is $\deg(I - T, \Omega, \mathbf{0}) = 1$.*

Let $(X, \|\cdot\|)$ be a real Banach space and $L\colon X \to X$ a linear operator. We recall that λ is an eigenvalue of L with a corresponding eigenvector φ if φ is nontrivial and $L\varphi = \lambda\varphi$. The reciprocals of eigenvalues are called the characteristic values of L. Recall also that a cone P in X is called a total cone if $X = \overline{P - P}$. The following lemma is known as the Krein-Rutman theorem.

Lemma 2.2 ([1, Theorem 19.2]). *Assume that P is a total cone in a real Banach space X. Let $L\colon X \to X$ be a compact linear operator such that $L(P) \subseteq P$ and the spectral radius, r_L, of L satisfies $r_L > 0$. Then r_L is an eigenvalue of L with an eigenvector in P.*

Let X^* be the dual space of X, P a total cone in X, and P^* the dual cone of P, i.e., $P^* = \{l \in X^*\colon l(u) \geqslant 0 \text{ for all } u \in P\}$. Let L, $M\colon X \to X$ be two linear compact operators such that $L(P) \subseteq P$ and $M(P) \subseteq P$. If their spectral radii r_L and r_M are positive, then by Lemma 2.2 there exist φ_L, $\varphi_M \in P \setminus \{\mathbf{0}\}$ such that

$$L\varphi_L = r_L\varphi_L \quad \text{and} \quad M\varphi_M = r_M\varphi_M. \tag{2.1}$$

Assume there exists $h \in P^* \setminus \{\mathbf{0}\}$ such that

$$L^*h = r_M h, \tag{2.2}$$

where L^* is the dual operator of L. Choose $\delta > 0$ and define

$$P(h, \delta) = \{u \in P\colon h(u) \geqslant \delta\|u\|\}. \tag{2.3}$$

Then $P(h, \delta)$ is a cone in X.

In the following, Lemma 2.3 is a generalization of [7, Theorem 2.1]. It is proved in [12, Lemma 2.5] for the case when L and M are two specific linear operators, but the proof there also works for any general linear operators L and M satisfying (2.1) and (2.2). Lemma 2.4 generalizes [3, Lemma 3.5] and is proved in [4, Lemma 2.5]. From here on, for any $R > 0$, let $B(\mathbf{0}, R) = \{u \in X\colon \|u\| < R\}$ be the open ball of X centered at $\mathbf{0}$ with radius R.

Lemma 2.3. *Assume that the following conditions hold:*

(A1) *there exist* $\varphi_L, \varphi_M \in P \setminus \{\mathbf{0}\}$ *and* $h \in P^* \setminus \{\mathbf{0}\}$ *such that* (2.1) *and* (2.2) *hold and* $L(P) \subseteq P(h, \delta)$*;*

(A2) $H\colon X \to P$ *is a continuous operator satisfying* $\lim\limits_{\|u\| \to \infty} \|Hu\|/\|u\| = 0$*;*

(A3) $F\colon X \to X$ *is a bounded continuous operator and there exists* $u_0 \in X$ *such that* $Fu + Hu + u_0 \in P$ *for all* $u \in X$*;*

(A4) *there exist* $v_0 \in X$ *and* $\varepsilon > 0$ *such that* $LFu \geqslant r_M^{-1}(1 + \varepsilon)Lu - LHu - v_0$ *for all* $u \in X$*.*

Let $T = LF$*. Then there exists* $R > 0$ *such that the Leray-Schauder degree satisfies* $\deg(I - T, B(\mathbf{0}, R), \mathbf{0}) = 0$.

Remark 2.1. Let $K_1 = \delta^{-1} r_M (1 + \varepsilon^{-1})\|h\| + \|L\|$, $K_2 = \|Lu_0\| + \delta^{-1}(r_M h(u_0) + \varepsilon^{-1} h(v_0))$, and $\varsigma \in (0, 1/K_1)$. By carefully examining the proof of [12, Lemma 2.5], we see that, in the conclusion of Lemma 2.3, we can choose any R satisfying $R > K_2/(1 - \varsigma K_1)$.

Lemma 2.4. *Assume that* (A1) *and the following conditions hold:*

(A2)* $H\colon X \to P$ *is a continuous operator satisfying* $\lim\limits_{\|u\| \to 0} \|Hu\|/\|u\| = 0$*;*

(A3)* $F\colon X \to X$ *is a bounded continuous operator and there exists* $r_1 > 0$ *such that* $Fu + Hu \in P$ *for all* $u \in X$ *with* $\|u\| < r_1$*;*

(A4)* *there exist* $\varepsilon > 0$ *and* $r_2 > 0$ *such that* $LFu \geqslant r_M^{-1}(1 + \varepsilon)Lu$ *for all* $u \in X$ *with* $\|u\| < r_2$*.*

Let $T = LF$*. Then there exists* $0 < R < \min\{r_1, r_2\}$ *such that the Leray-Schauder degree satisfies* $\deg(I - T, B(\mathbf{0}, R), \mathbf{0}) = 0$.

The following lemma is a special case of [2, Lemma 2.2].

Lemma 2.5. *Let* $y \in C[0, 1]$*. Then a function* $u(t)$ *is a solution of the BVP consisting of the equation*

$$u'' + y(t) = 0, \quad t \in (0, 1),$$

and BC (1.2) *with* $\lambda = \mu = 0$ *if and only if*

$$u(t) = \int_0^1 K(t, s) y(s) \, \mathrm{d}s,$$

where

$$(2.4)\quad K(t,s)=\frac{1}{\varrho}\big[\alpha\big((\beta-1)t+(1-\beta\eta)\big)G(\xi,s)+\beta\big((1-\alpha)t+\alpha\xi\big)G(\eta,s)\big]+G(t,s),$$

with ϱ being defined in (H), *and*

$$(2.5)\qquad G(t,s)=\begin{cases} t(1-s), & 0\leqslant t\leqslant s\leqslant 1,\\ s(1-t), & 0\leqslant s\leqslant t\leqslant 1.\end{cases}$$

In the remainder of the paper, let the Banach space $X:=C[0,1]$ be equipped with the norm $\|u\|=\max\limits_{t\in[0,1]}|u(t)|$, and define a cone P in X by

$$(2.6)\qquad P=\{u\in X\colon\ u(t)\geqslant 0 \text{ on } \mathbb{R}\}.$$

Let linear operators L and $M\colon\ X\to X$ be defined by

$$(2.7)\qquad Lu(t)=\int_0^1 K(t,s)g(s)u(s)\,\mathrm{d}s \quad\text{and}\quad Mu(t)=\int_0^1 K(s,t)g(s)u(s)\,\mathrm{d}s.$$

The next lemma provides some information about the operators L and M.

Lemma 2.6. *The operators L and M map P into P and are compact. In addition:*

(a) *The spectral radius, r_L, of L satisfies $r_L>0$, and r_L is an eigenvalue of L with an eigenvector $\varphi_L\in P$.*

(b) *The spectral radius, r_M, of M satisfies $r_M>0$, and r_M is an eigenvalue of M with an eigenvector $\varphi_M\in P$.*

Proof. The proof that these operators are compact is standard and will be omitted. We will only prove (a) since the proof of (b) is similar. From (2.4), it is clear that there exist $t_1,t_2\in(0,1)$ such that $K(t,s)>0$ for $t,s\in[t_1,t_2]$. Choose $u\in C[0,1]$ such that $u(t)\geqslant 0$ on $[0,1]$, $u(t^*)>0$ for some $t^*\in[t_1,t_2]$, and $u(t)=0$ for $t\in[0,1]\setminus[t_1,t_2]$. Then, for $t\in[t_1,t_2]$, we have

$$Lu(t)\geqslant\int_{t_1}^{t_2}K(t,s)g(s)u(s)\,\mathrm{d}s>0.$$

Thus, there exists $c>0$ such that $cLu(t)\geqslant u(t)$ for $t\in[0,1]$. Now, from [11, Chapter 5, Theorem 2.1], it follows that $r_L>0$. Finally, in view of $r_L>0$ and the fact that the cone P defined by (2.6) is a total cone, the remainder of part (a) readily follows from Lemma 2.2 and the first statement in this lemma. $\square$

3. Main results

For convenience, we introduce the following notation:

$$f_0 = \liminf_{x\to 0^+} \min_{t\in[0,1]} \frac{f(t,x)}{x}, \quad f_\infty = \liminf_{x\to\infty} \min_{t\in[0,1]} \frac{f(t,x)}{x},$$
$$F_0 = \limsup_{x\to 0} \max_{t\in[0,1]} \left|\frac{f(t,x)}{x}\right|, \quad F_\infty = \limsup_{|x|\to\infty} \max_{t\in[0,1]} \left|\frac{f(t,x)}{x}\right|,$$

$$C = \frac{1}{k_1\int_0^1 \delta(s)g(s)\,\mathrm{d}s}, \quad D = \frac{k_1}{k_2^2\int_\theta^{1-\theta} \delta(s)g(s)\,\mathrm{d}s}, \tag{3.1}$$

where

$$\delta(s) = (\beta-\alpha)s + s(1-s) + 1 - \beta\eta + \alpha\xi, \tag{3.2}$$
$$k_1 = \max\left\{\frac{1}{\varrho}\alpha\xi(1-\xi),\ \frac{1}{\varrho}\beta\eta(1-\eta),\ 1\right\}, \tag{3.3}$$

$\theta \in (0,1/2)$ is a fixed constant, and

$$k_2 = \begin{cases} \theta(1-\theta), & \text{if } \alpha\xi(1-\xi)+\beta\eta(1-\eta)=0,\\ \theta(1-\theta)\min\{\varrho^{-1}\alpha\xi(1-\xi), \varrho^{-1}\beta\eta(1-\eta), 1\}, & \text{otherwise.}\end{cases} \tag{3.4}$$

In the rest of this paper, we also let

$$\mu_M = 1/r_M, \tag{3.5}$$

where r_M is given in Lemma 2.6 (b). Clearly, μ_M is the smallest positive characteristic value of M satisfying $\varphi_M = \mu_M M\varphi_M$, and by Lemma 4.1 in Section 4 below, $C \leqslant \mu_M \leqslant D$. We need the following assumptions.

(B1) There exist three nonnegative functions $a, b \in C[0,1]$ and $c \in C(\mathbb{R})$ such that $c(x)$ is even and nondecreasing on $\mathbb{R}_+$,

$$f(t,x) \geqslant -a(t) - b(t)c(x) \quad \text{for all } (t,x) \in [0,1]\times\mathbb{R}, \tag{3.6}$$

and

$$\lim_{x\to\infty} c(x)/x = 0. \tag{3.7}$$

(B2) There exist a constant $0 < r < 1$ and two nonnegative functions $d \in C[0,1]$ and $e \in C(\mathbb{R})$ such that e is even and nondecreasing on $\mathbb{R}^+$,

$$f(t,x) \geqslant -d(t)e(x) \quad \text{for all} \quad (t,x) \in [0,1] \times [-r,0], \tag{3.8}$$

and

$$\lim_{x\to 0} e(x)/x = 0. \tag{3.9}$$

Remark 3.1. Here we want to emphasize that in (B1) we assume that $f(t,x)$ is bounded from below by $-a(t) - b(t)c(x)$ for all $(t,x) \in [0,1] \times \mathbb{R}$; however, in (B2) we only require that $f(t,x)$ is bounded from below by $-d(t)e(x)$ for $t \in [0,1]$ and x in a small left-neighborhood of 0.

We now state our existence results. The first four results give conditions to guarantee that BVP (1.1), (1.2) has a nontrivial solution for $(\lambda,\mu) \in \mathbb{R}^2_+$ with $\lambda + \mu$ small.

Theorem 3.1. *Assume that* (B1) *holds and* $F_0 < \mu_M < f_\infty$. *Then, for each* $(\lambda,\mu) \in \mathbb{R}^2_+$ *with* $\lambda+\mu$ *sufficiently small, BVP* (1.1), (1.2) *has at least one nontrivial solution.*

Theorem 3.2. *Assume that* (B2) *holds and* $F_\infty < \mu_M < f_0$. *Then, for each* $(\lambda,\mu) \in \mathbb{R}^2_+$ *with* $\lambda+\mu$ *sufficiently small, BVP* (1.1), (1.2) *has at least one nontrivial solution.*

Corollary 3.1. *Assume that* (B1) *holds and* $F_0/C < 1 < f_\infty/D$. *Then the conclusion of Theorem* 3.1 *holds.*

Corollary 3.2. *Assume that* (B2) *holds and* $F_\infty/C < 1 < f_0/D$. *Then the conclusion of Theorem* 3.2 *holds.*

Theorem 3.3 below provides conditions for the existence of nontrivial solutions of BVP (1.1), (1.2) for all $(\lambda,\mu) \in \mathbb{R}^2$.

Theorem 3.3. *Assume* $F_\infty < C$. *Then, for each* $(\lambda,\mu) \in \mathbb{R}^2 \setminus \{0,0\}$, *BVP* (1.1), (1.2) *has at least one nontrivial solution. Moreover, for the case where* $(\lambda,\mu) = (0,0)$, *if* $f(t,0) \not\equiv 0$ *on* $[0,1]$, *then BVP* (1.1), (1.2) *has at least one nontrivial solution.*

R e m a r k 3.2. From the proof of Theorem 3.2 it can be seen that in Theorem 3.2 and Corollary 3.2, a set of values for (λ,μ) guaranteeing the existence of nontrivial solutions of BVP (1.1), (1.2) is given by $\{(\lambda,\mu)\in\mathbb{R}^2_+\colon\ \lambda\geqslant 0,\ \mu\geqslant 0,\ \lambda\|\varphi\|+\mu\|\psi\| \leqslant \zeta_1\}$, where φ and ψ are defined in (4.1) below and $0<\zeta_1<1$ is such that (4.16) holds.

In view of Remark 2.1 and Lemma 4.1, and from the proof of Theorem 3.1, we also can obtain explicit ranges of (λ,μ) in Theorem 3.1 and Corollary 3.1. Since these ranges involve relatively more equations and inequalities, for brevity we will not state them here.

R e m a r k 3.3. Under appropriate assumptions, results similar to Theorems 3.1–3.3 and Corollaries 3.1 and 3.2 can be obtained for the BVP consisting of Eq. (1.1) and the more general nonhomogeneous multi-point BC

$$u(0)=\sum_{i=1}^{m}\alpha_i u(\xi_i)+\lambda,\ u(1)=\sum_{i=1}^{m}\beta_i u(\eta_i)+\mu,$$

where $m\geqslant 1$ is an integer, ξ_i, $\eta_i\in[0,1]$, and α_i, β_i, λ, $\mu\in\mathbb{R}_+$ for $i=1,\dots,m$. We omit the discussions here.

R e m a r k 3.4. If the nonlinear term $f(t,x)$ is separable, say $f(t,x)=f_1(t)f_2(x)$, then conditions such as $\mu_M<f_\infty$ and $\mu_M<f_0$ imply that $f_1(t)>0$ on $[0,1]$. However, the function $g(t)$ in Eq. (1.1) may have zeros on $[0,1]$.

We conclude this section with several examples.

E x a m p l e 3.1. In (1.1) and (1.2), let

$$f(t,x)=\begin{cases}\sum\limits_{i=1}^{n}a_i(t)x^i, & x\in[-1,\infty),\\ \sum\limits_{i=1}^{n}(-1)^i a_i(t)-\tilde b(t)|x|^\kappa+\tilde b(t), & x\in(-\infty,-1),\end{cases}\tag{3.10}$$

$g(t)\equiv 1$ on $[0,1]$, $\alpha=\xi=\eta=1/2$, and $\beta=1$, where $n\geqslant 1$ is an integer, a_i, $\tilde b\in C[0,1]$ with $0\leqslant\|a_1\|<6/7$ and $a_n(t)>0$ on $[0,1]$, and $0\leqslant\kappa<1$. Then, for each $(\lambda,\mu)\in\mathbb{R}^2_+$ with $\lambda+\mu$ sufficiently small, BVP (1.1), (1.2) has at least one nontrivial solution.

To see this, we first note that $f\in C([0,1]\times\mathbb{R})$ and assumption (H) is satisfied. Let $a(t)=\sum\limits_{i=1}^{n}|a_i(t)|+|\tilde b(t)|$, $b(t)=|\tilde b(t)|$, and $c(x)=|x|^\kappa$. Then it is easy to see that (B1) holds. From (3.1) with $\theta=1/4$ and by a simple calculation, we have

$C = 6/7$ and $D = 32768/177$. Moreover, (3.10) implies that

$$F_0 = \limsup_{x\to 0} \max_{t\in[0,1]} \left|\frac{f(t,x)}{x}\right| = \|a_1\| < \frac{6}{7} \quad \text{and} \quad f_\infty = \liminf_{x\to\infty} \min_{t\in[0,1]} \frac{f(t,x)}{x} = \infty.$$

Hence, $F_0/C < 1 < f_\infty/D$. The conclusion then follows from Corollary 3.1.

Example 3.2. In BC (1.2), choose α, β, ξ, $\eta \in \mathbb{R}_+$ such that assumption (H) holds. Let μ_M be defined by (3.5). In Eq. (1.1), let

$$f(t,x) = \begin{cases} \mu_M\left((t^2+1)x^{1/3} + 2t^2 + 3\right), & x \in (-\infty, -1), \\ \mu_M(t^2+2)x^{2/3}, & x \in [-1,1], \\ \mu_M(t^2 - x^{1/2} + 3), & x \in (1,\infty), \end{cases} \tag{3.11}$$

and $g(t) = 1 - \sin(2\pi t)$ on $[0,1]$. Then, for each $(\lambda,\mu) \in \Lambda$, BVP (1.1), (1.2) has at least one nontrivial solution, where

$$\Lambda = \left\{(\lambda,\mu) \in \mathbb{R}_+^2 \colon\ \lambda \geqslant 0,\ \mu \geqslant 0,\ \lambda\|\varphi\| + \mu\|\psi\| \leqslant 1/2\right\}$$

with φ and ψ being defined in (4.1) below.

To see this, we first note that $f \in C([0,1]\times\mathbb{R})$, $g \in C[0,1]$, $g(t) > 0$ a.e. on $[0,1]$, and assumption (H) is satisfied. Let $d(t) = t^2 + 1$ and $e(x) = x^{2/3}$. Then, from (3.11), it is easy to see that (B2) is satisfied for any $0 < r < 1$, and

$$F_\infty = \limsup_{|x|\to\infty} \max_{t\in[0,1]} \left|\frac{f(t,x)}{x}\right| = 0 \quad \text{and} \quad f_0 = \liminf_{x\to 0^+} \min_{t\in[0,1]} \frac{f(t,x)}{x} = \infty.$$

Then, for C and D defined in (3.1), we have $F_\infty/C < 1 < f_0/D$. Moreover, from (3.11) we see that we can choose $\zeta_1 = 1/2$ in (4.16). The conclusion then follows from Corollary 3.2 and Remark 3.2.

Example 3.3. In Eq. (1.1), let

$$f(t,x) = -(t+1)|x|^{1/2} + 3 \quad \text{and} \quad g(t) = (t-1/2)^2,$$

and in BC (1.2), choose α, β, ξ, $\eta \in \mathbb{R}_+$ such that assumption (H) holds. Then, for each $(\lambda,\mu) \in \mathbb{R}^2$, BVP (1.1), (1.2) has at least one nontrivial solution.

To see this, we first note that $f \in C([0,1]\times\mathbb{R})$, $g \in C[0,1]$, $g(t) > 0$ a.e. on $[0,1]$, and assumption (H) is satisfied. Moreover, for C defined in (3.1) we have

$$F_\infty = \limsup_{|x|\to\infty} \max_{t\in[0,1]} \left|\frac{f(t,x)}{x}\right| = 0 < C.$$

Note that $f(t,0) = 3$ on $[0,1]$. The conclusion then follows from Theorem 3.3.

R e m a r k 3.5. In the above examples, the nonlinearity $f(t,x)$ may take negative values and is unbounded from below. To the best of our knowledge, no known criteria can be applied to these examples.

4. Proofs of the main results

Let

$$\varphi(t)=\frac{1}{\varrho}\big[(\beta-1)t+(1-\beta\eta)\big] \quad \text{and} \quad \psi(t)=\frac{1}{\varrho}\big[(1-\alpha)t+\alpha\xi\big]. \tag{4.1}$$

Clearly, $\varphi(t)\geqslant 0$ and $\psi(t)\geqslant 0$ on $[0,1]$, and

$$\begin{aligned}&\varphi''=0, \quad t\in(0,1),\\ &\varphi(0)=\alpha\varphi(\xi)+1,\ \varphi(1)=\beta\varphi(\eta),\end{aligned}$$

and

$$\begin{aligned}&\psi''=0, \quad t\in(0,1),\\ &\psi(0)=\alpha\psi(\xi),\ \psi(1)=\beta\psi(\eta)+1.\end{aligned}$$

For any fixed $(\lambda,\mu)\in\mathbb{R}^2$, let $v:=u-\lambda\varphi-\mu\psi$. Then BVP (1.1), (1.2) becomes the BVP consisting of the equation

$$v''+g(t)f(t,v+\lambda\varphi+\mu\psi)=0,\ t\in(0,1), \tag{4.2}$$

and the homogeneous BC

$$v(0)=\alpha v(\xi), v(1)=\beta v(\eta). \tag{4.3}$$

Moreover, if $v(t)$ is a solution of BVP (4.2), (4.3), then $u(t)=v(t)+\lambda\varphi(t)+\mu\psi(t)$ is a solution of BVP (1.1), (1.2).

Let X,P,L,M be defined by (2.6) and (2.7). By Lemma 2.6, L and M map P into P and are compact. Define operators $F_{\lambda,\mu}$, $T\colon X\to X$ by

$$F_{\lambda,\mu}v(t)=f(t,v+\lambda\varphi+\mu\psi) \tag{4.4}$$

and

$$Tv(t)=LF_{\lambda,\mu}v(t)=\int_0^1 K(t,s)g(s)F_{\lambda,\mu}v(s)\,\mathrm{d}s, \tag{4.5}$$

where K is defined by (2.4). Then $F_{\lambda,\mu}\colon X\to X$ is bounded and $T\colon X\to X$ is compact. Moreover, by Lemma 2.5, a solution of BVP (4.2), (4.3) is equivalent to a fixed point of T in X.

P r o o f of Theorem 3.1. We first verify that conditions (A1)–(A4) of Lemma 2.3 are satisfied. By Lemma 2.6, there exist $\varphi_L, \varphi_M \in P \setminus \{\mathbf{0}\}$ such that (2.1) holds. To show (2.2), we let

$$h(v) = \int_0^1 \varphi_M(t) g(t) v(t)\,\mathrm{d}t, \ v \in X. \tag{4.6}$$

Then $h \in P^* \setminus \{\mathbf{0}\}$, and from (2.1), (2.7), and (4.6),

$$\begin{aligned}(L^*h)(v) = h(Lv) &= \int_0^1 \varphi_M(t) g(t) \left(\int_0^1 K(t,s) g(s) v(s)\,\mathrm{d}s\right)\mathrm{d}t \\ &= \int_0^1 g(s) v(s) \left(\int_0^1 K(t,s) g(t) \varphi_M(t)\,\mathrm{d}t\right)\mathrm{d}s = \int_0^1 g(s) v(s) M\varphi_M(s)\,\mathrm{d}s = r_M h(v),\end{aligned}$$

i.e., h satisfies (2.2). Note from (2.4) and (2.5) that $K(t,0) = K(t,1) = 0$ for $t \in [0,1]$. Then, from $\varphi_M = \mu_M M \varphi_M$ and (2.7), we see that $\varphi_M(0) = \varphi_M(1) = 0$ and $\varphi_M(t) > 0$ on $(0,1)$, which in turn implies that $\varphi'_M(0) > 0$ and $\varphi'_M(1) < 0$. Thus,

$$\lim_{s\to 0^+} \frac{\varphi_M(s)}{s(1-s)} = \varphi'_M(0) > 0 \quad \text{and} \quad \lim_{s\to 1^-} \frac{\varphi_M(s)}{s(1-s)} = -\varphi'_M(1) > 0.$$

Hence, there exists $\delta_1 > 0$ such that

$$\varphi_M(s) \geqslant \delta_1 s(1-s) \quad \text{for } s \in [0,1]. \tag{4.7}$$

From (2.5) we have $G(t,s) \leqslant s(1-s)$ for $t, s \in [0,1]$. Then (2.4) implies that

$$K(t,s) \leqslant \frac{1}{\varrho}\big[\alpha\big((\beta-1)t + (1-\beta\eta)\big) + \beta\big((1-\alpha)t + \alpha\xi\big) + 1\big] s(1-s) \leqslant ks(1-s),$$

where $k = \max\limits_{t\in[0,1]} \big[\alpha\big((\beta-1)t + (1-\beta\eta)\big) + \beta\big((1-\alpha)t + \alpha\xi\big) + 1\big]/\varrho > 0$. Combining the above inequality with (4.7) yields

$$\varphi_M(s) \geqslant \delta_1 k^{-1} K(t,s) \quad \text{for } t, s \in [0,1]. \tag{4.8}$$

Let $\delta = r_M \delta_1 k^{-1}$. For any $v \in P$ and $t \in [0,1]$, (2.7), (4.6), and (4.8) imply

$$\begin{aligned}h(Lv) = r_M h(v) &= r_M \int_0^1 \varphi_M(s) g(s) v(s)\,\mathrm{d}s \\ &\geqslant \frac{r_M \delta_1}{k} \int_0^1 K(t,s) g(s) v(s)\,\mathrm{d}s = \delta Lv(t).\end{aligned}$$

Hence, $h(Lv) \geqslant \delta\|Lv\|$, i.e., $L(P) \subseteq P(h,\delta)$. Therefore, (A1) of Lemma 2.3 holds.

Since c is nondecreasing on $\mathbb{R}^+$, we have $c(v(t)) \leqslant c(\|v\|)$ for all $v \in P$ and $t \in [0,1]$. Then, from the fact that c is even, it follows that $c(v(t)) \leqslant c(\|v\|)$ for all $v \in X$ and $t \in [0,1]$. Thus, $\|c(v)\| \leqslant c(\|v\|)$ for all $v \in X$. From (3.7) we see that $\lim\limits_{\|v\|\to\infty} \|c(v)\|/\|v\| = 0$ for any $v \in X$. Let $Hv(t) = \bar{b}c(v(t))$ for $v \in X$, where $\bar{b} = \max\limits_{t\in[0,1]} b(t)$. Then (A2) of Lemma 2.3 holds.

Let $(\lambda,\mu) \in \mathbb{R}^2_+$, $F_{\lambda,\mu}$ be defined by (4.4), and $u_0(t) = a(t)$. Then, from (3.6), we have $F_{\lambda,\mu}v + Hv + u_0 \in P$ for all $v \in X$. Hence, (A3) of Lemma 2.3 with $F = F_{\lambda,\mu}$ holds.

Since $f_\infty > \mu_M$, there exist $\varepsilon > 0$ and $N > 0$ such that

$$f(t,x) \geqslant \mu_M(1+\varepsilon)x \quad \text{for } (t,x) \in [0,1] \times [N,\infty).$$

Then, in view of (3.6), there exists $\zeta > 0$ such that

$$f(t,x) \geqslant \mu_M(1+\varepsilon)x - \bar{b}c(x) - \zeta \quad \text{for } (t,x) \in [0,1] \times [N,\infty).$$

From (3.5) and (4.4) we have

$$\begin{aligned} F_{\lambda,\mu}v(t) &\geqslant \mu_M(1+\varepsilon)(v(t) + \lambda\varphi(t) + \mu\psi(t)) - \bar{b}c(v(t)) - \zeta \\ &\geqslant r_M^{-1}(1+\varepsilon)v(t) - Hv(t) - \zeta \quad \text{for all } v \in X. \end{aligned}$$

Thus,

$$LF_{\lambda,\mu}v(t) \geqslant r_M^{-1}(1+\varepsilon)Lv(t) - LHv(t) - L\zeta \quad \text{for all } v \in X.$$

Then (A4) of Lemma 2.3 holds with $F = F_{\lambda,\mu}$ and $v_0 = L\zeta$.

All conditions of Lemma 2.3 hold, so there exists $R_1 > 0$ such that

$$\deg(I - T, B(\mathbf{0}, R_1), \mathbf{0}) = 0. \tag{4.9}$$

Next, since $F_0 < \mu_M$, there exist $0 < \nu < 1$ and $0 < R_2 < R_1$ such that

$$|f(t,x)| \leqslant \mu_M(1-\nu)|x| \quad \text{for } (t,x) \in [0,1] \times [-2R_2, 2R_2]. \tag{4.10}$$

In what follows, let $(\lambda,\mu) \in \mathbb{R}^2_+$ satisfy

$$\lambda\|\varphi\| + \mu\|\psi\| < R_2 \tag{4.11}$$

and

$$C_1 := \mu_M(1-\nu)\left(\lambda\|\varphi\| + \mu\|\psi\|\right) \max_{t\in[0,1]} \int_0^1 K(t,s)g(s)\,\mathrm{d}s < \nu R_2. \tag{4.12}$$

We claim that

$$Tv \neq \tau v \quad \text{for all } v \in \partial B(\mathbf{0}, R_2) \text{ and } \tau \geqslant 1. \tag{4.13}$$

If this is not the case, then there exist $\overline{v} \in \partial B(\mathbf{0}, R_2)$ and $\overline{\tau} \geqslant 1$ such that $T\overline{v} = \overline{\tau}\overline{v}$. It follows that $\overline{v} = \bar{s}T\overline{v}$, where $\bar{s} = 1/\overline{\tau}$. Clearly, $\bar{s} \in (0, 1]$. From (4.4), (4.10), and (4.11), we have

$$|F_{\lambda,\mu}\overline{v}(t)| \leqslant \mu_M(1-\nu)|\overline{v}(t)+\lambda\varphi(t)+\mu\psi(t)| \leqslant \mu_M(1-\nu)(|\overline{v}(t)|+\lambda\|\varphi\|+\mu\|\psi\|). \tag{4.14}$$

Assume $R_2 = \|\overline{v}\| = |\overline{v}(\bar{t})|$ for some $\bar{t} \in [0, 1]$. Then, from (2.7), (3.5), (4.5), (4.12), and (4.14), we obtain that

$$\begin{aligned}
R_2 = |\overline{v}(\bar{t})| = \bar{s}|T\overline{v}(\bar{t})| &\leqslant \int_0^1 K(\bar{t}, s))g(s)|F_{\lambda,\mu}\overline{v}(s)| \,\mathrm{d}s \\
&\leqslant \mu_M(1-\nu)\int_0^1 K(\bar{t}, s)g(s)|\overline{v}(s)| \,\mathrm{d}s \\
&\quad + \mu_M(1-\nu)\left(\lambda\|\varphi\| + \mu\|\psi\|\right)\int_0^1 K(\bar{t}, s)g(s)\,\mathrm{d}s \\
&= \mu_M(1-\nu)L|\overline{v}(\bar{t})| + \mu_M(1-\nu)\left(\lambda\|\varphi\| + \mu\|\psi\|\right)\int_0^1 K(\bar{t}, s)g(s)\,\mathrm{d}s \\
&\leqslant \mu_M(1-\nu)LR_2 + C_1 = r_M^{-1}(1-\nu)LR_2 + C_1.
\end{aligned}$$

Consequently,

$$\begin{aligned}
h(R_2) &\leqslant r_M^{-1}(1-\nu)h(LR_2) + h(C_1) = r_M^{-1}(1-\nu)(L^*h)(R_2) + h(C_1) \\
&= r_M^{-1}(1-\nu)r_M h(R_2) + h(C_1) = (1-\nu)h(R_2) + h(C_1).
\end{aligned}$$

Thus,

$$(C_1 - \nu R_2)h(1) \geqslant 0.$$

Since $h(1) > 0$, we have $C_1 \geqslant \nu R_2$. But this contradicts (4.12). Thus, (4.13) holds. Now, Lemma 2.1 implies

$$\deg(I - T, B(\mathbf{0}, R_2), \mathbf{0}) = 1. \tag{4.15}$$

By the additivity property of the Leray-Schauder degree, (4.9), and (4.15), we have

$$\deg(I - T, B(\mathbf{0}, R_1) \setminus \overline{B(\mathbf{0}, R_2)}) = -1.$$

Then, from the solution property of the Leray-Schauder degree, T has at least one fixed point v in $B(\mathbf{0}, R_1)\setminus\overline{B(\mathbf{0}, R_2)}$, which is a solution of BVP (4.2), (4.3). Therefore, we have shown that, for $(\lambda,\mu)\in\mathbb{R}^2_+$ satisfying (4.11) and (4.12), BVP (4.2), (4.3) has at least one solution $v(t)$ satisfying $\|v\|\geqslant R_2$. Thus, for each $(\lambda,\mu)\in\mathbb{R}^2_+$ with $\lambda+\mu$ sufficiently small, BVP (1.1), (1.2) has at least one solution $u(t)=v(t)+\lambda\varphi(t)+\mu\psi(t)$ satisfying

$$\|u\|\geqslant\|v\|-\|\lambda\varphi+\mu\psi\|\geqslant R_2-\|\lambda\varphi+\mu\psi\|>0.$$

This completes the proof of the theorem. $\square$

P r o o f of Theorem 3.2. We first verify that conditions (A1) and (A2)*–(A4)* of Lemma 2.4 are satisfied. As in the proof of Theorem 3.1, there exist $\varphi_L,\varphi_M\in P\setminus\{\mathbf{0}\}$ and $h\in P^*\setminus\{\mathbf{0}\}$ defined by (4.6) such that (A1) holds.

From the fact that e is even and nondecreasing on $\mathbb{R}^+$, it is easy to see that $e(v(t))\leqslant e(\|v\|)$ for all $v\in X$ and $t\in[0,1]$. Thus, $\|e(v)\|\leqslant e(\|v\|)$ for all $v\in X$. This, together with (3.9), implies that $\lim\limits_{\|v\|\to0}\|e(v)\|/\|v\|=0$ for any $v\in X$. Let $Hv(t)=\bar d c(v(t))$ for $v\in X$, where $\bar d=\max\limits_{t\in[0,1]}d(t)$. Then (A2)* of Lemma 2.4 holds.

Since $f_0>\mu_M$, there exist $\varepsilon>0$ and $0<\zeta_1<1$ such that

$$f(t,x)\geqslant\mu_M(1+\varepsilon)x=r_M^{-1}(1+\varepsilon)x\geqslant0\quad\text{for }(t,x)\in[0,1]\times[0,2\zeta_1]. \tag{4.16}$$

Let $(\lambda,\mu)\in\mathbb{R}^2_+$ satisfy

$$\lambda\|\varphi\|+\mu\|\psi\|\leqslant\zeta_1 \tag{4.17}$$

and let $F_{\lambda,\mu}$ be defined by (4.4). Then, from (4.16), we have

$$\begin{aligned}F_{\lambda,\mu}v(t)&\geqslant\mu_M(1+\varepsilon)(v(t)+\lambda\varphi(t)+\mu\psi(t))\\&\geqslant\mu_M(1+\varepsilon)v(t)=r_M^{-1}(1+\varepsilon)v(t)\quad\text{for all }v\in P\text{ with }\|v\|\leqslant\zeta_1.\end{aligned} \tag{4.18}$$

Let r be given in (B2). Now, in view of (3.8) and (4.18), we see that (A3)* of Lemma 2.4 holds with $F=F_{\lambda,\mu}$ and $r_1=\min\{r,\zeta_1\}$.

By (3.9) there exists $0<\zeta_2<\min\{r,\zeta_1\}$ such that $-e(x)\geqslant\bar d^{-1}r_M^{-1}(1+\varepsilon)x$ for $x\in[-\zeta_2,0]$. Then, from (3.8), we obtain

$$f(t,x)\geqslant d(t)\bar d^{-1}r_M^{-1}(1+\varepsilon)x\geqslant r_M^{-1}(1+\varepsilon)x\quad\text{for }(t,x)\in[0,1]\times[-\zeta_2,0]. \tag{4.19}$$

From (4.16) and (4.19) it is easy to see that $F_{\lambda,\mu}v(t)\geqslant\mu_M(1+\varepsilon)(v(t)+\lambda\varphi(t)+\mu\psi(t))\geqslant\mu_M(1+\varepsilon)v(t)=r_M^{-1}(1+\varepsilon)v(t)$ for all $v\in X$ with $\|v\|\leqslant\zeta_2$, which clearly implies that $LF_{\lambda,\mu}v(t)\geqslant r_M^{-1}(1+\varepsilon)Lv(t)$ for all $v\in X$ with $\|u\|<\zeta_2$. Hence, (A4)* of Lemma 2.4 holds with $F=F_{\lambda,\mu}$ and $r_2=\zeta_2$.

All conditions of Lemma 2.4 hold, so there exists $R_3 > 0$ such that

$$\deg(I - T, B(\mathbf{0}, R_3), \mathbf{0}) = 0. \tag{4.20}$$

Next, since $F_\infty < \mu_M$, there exist $0 < \tilde{\nu} < 1$ and $\tilde{R} > R_3$ such that

$$|f(t,x)| \leqslant \mu_M(1-\tilde{\nu})|x| = r_M^{-1}(1-\tilde{\nu})|x| \quad \text{for } (t,|x|) \in [0,1] \times (\tilde{R}, \infty). \tag{4.21}$$

Let

$$C_2 = r_M^{-1}(1-\tilde{\nu})(\lambda\|\varphi\| + \mu\|\psi\|) \max_{t\in[0,1]} \int_0^1 K(t,s)g(s)\,\mathrm{d}s \tag{4.22}$$

and

$$C_3 = \max_{t\in[0,1],|x|\leqslant \tilde{R}} |f(t,x)| \max_{t\in[0,1]} \int_0^1 K(t,s)g(s)\,\mathrm{d}s. \tag{4.23}$$

Then $0 < C_2,\ C_3 < \infty$. Choose R_4 large enough so that

$$R_4 > \max\{\overline{R},\ \tilde{\nu}^{-1}(C_2 + C_3)\}. \tag{4.24}$$

We claim that

$$Tv \neq \tau v \quad \text{for all } v \in \partial B(\mathbf{0}, R_4) \text{ and } \tau \geqslant 1. \tag{4.25}$$

If this is not the case, then there exist $\tilde{v} \in \partial B(\mathbf{0}, R_4)$ and $\tilde{\tau} \geqslant 1$ such that $T\tilde{v} = \tilde{\tau}\tilde{v}$. It follows that $\tilde{v} = \tilde{s}T\tilde{v}$, where $\tilde{s} = 1/\tilde{\tau}$. Clearly, $\tilde{s} \in (0,1]$. Assume $R_4 = \|\tilde{v}\| = |\tilde{v}(\tilde{t})|$ for some $\tilde{t} \in [0,1]$. Let $J_1(\tilde{v}) = \{t \in [0,1] \colon |\tilde{v}(t) + \lambda\varphi(t) + \mu\psi(t)| > \tilde{R}\}$, $J_2(\tilde{v}) = [0,1] \setminus J_1(\tilde{v})$, $p(\tilde{v}(t)) = \min\{|\tilde{v}(t) + \lambda\varphi(t) + \mu\psi(t)|, \tilde{R}\}$ for $t \in [0,1]$. Then, from (2.7), (4.4), (4.5), and (4.21)–(4.23), it follows that

$$\begin{aligned}
R_4 = |\tilde{v}(\tilde{t})| = \tilde{s}|T\tilde{v}(\tilde{t})| &\leqslant \int_0^1 K(\tilde{t},s)g(s)|F_{\lambda,\mu}\tilde{v}(s)|\,\mathrm{d}s \\
&= \int_{J_1(\tilde{v})} K(\tilde{t},s)g(s)|F_{\lambda,\mu}\tilde{v}(s)|\,\mathrm{d}s + \int_{J_2(\tilde{v})} K(\tilde{t},s)g(s)|F_{\lambda,\mu}\tilde{v}(s)|\,\mathrm{d}s \\
&\leqslant r_M^{-1}(1-\tilde{\nu}) \int_0^1 K(\tilde{t},s)g(s)|\tilde{v}(s)|\,\mathrm{d}s + \int_0^1 K(\tilde{t},s)g(s)|F_{\lambda,\mu}p(\tilde{v}(s))|\,\mathrm{d}s \\
&\quad + r_M^{-1}(1-\tilde{\nu})(\lambda\|\varphi\| + \mu\|\psi\|) \int_0^1 K(\tilde{t},s)g(s)\,\mathrm{d}s \\
&\leqslant r_M^{-1}(1-\tilde{\nu})L|v(\tilde{t})| + C_2 + C_3 = r_M^{-1}(1-\tilde{\nu})LR_4 + C_2 + C_3,
\end{aligned}$$

since $L|v(\tilde{t})| = \int_0^1 K(\tilde{t}, s)g(s)|v(s)|\,\mathrm{d}s$. Hence, for h defined by (4.6), we have

$$\begin{aligned} h(R_4) &\leqslant r_M^{-1}(1-\tilde{\nu})h(LR_4) + h(C_2+C_3) = r_M^{-1}(1-\tilde{\nu})(L^*h)(R_4) + h(C_2+C_3) \\ &= r_M^{-1}(1-\tilde{\nu})r_M h(R_4) + h(C_2+C_3) = (1-\tilde{\nu})h(R_4) + h(C_2+C_3), \end{aligned}$$

which implies $(\tilde{\nu}R_4 - C_2 - C_3)h(1) \leqslant 0$. In view of the fact that $h(1) > 0$, it follows that $R_4 \leqslant \tilde{\nu}^{-1}(C_2+C_3)$. This contradicts (4.24) and so (4.25) holds. By Lemma 2.1 we have

$$\deg(I-T, B(\mathbf{0}, R_4), \mathbf{0}) = 1. \tag{4.26}$$

By the additivity property of the Leray-Schauder degree, (4.20), and (4.26), we obtain

$$\deg(I-T, B(\mathbf{0}, R_4) \setminus \overline{B(\mathbf{0}, R_3)}) = 1.$$

Thus, from the solution property of the Leray-Schauder degree, T has at least one fixed point v in $B(\mathbf{0}, R_4)\setminus\overline{B(\mathbf{0}, R_3)}$, which is a solution of BVP (4.2), (4.3). Therefore, we have shown that, for $(\lambda, \mu) \in \mathbb{R}_+^2$ satisfying (4.17), BVP (4.2), (4.3) has at least one solution $v(t)$ satisfying $\|v\| \geqslant R_3$. Thus, for each $(\lambda, \mu) \in \mathbb{R}_+^2$ with $\lambda + \mu$ sufficiently small, BVP (1.1), (1.2) has at least one solution $u(t) = v(t) + \lambda\varphi(t) + \mu\psi(t)$ satisfying

$$\|u\| \geqslant \|v\| - \|\lambda\varphi + \mu\psi\| \geqslant R_3 - \|\lambda\varphi + \mu\psi\| > 0.$$

This completes the proof of the theorem. $\square$

Lemma 4.1. *Let μ_M be defined by* (3.5). *Then $C \leqslant \mu_M \leqslant D$, where C and D are given by* (3.1).

Proof. From (2.4) we have

$$\begin{aligned} K(s,t) = \frac{1}{\varrho}\big[\alpha\big((\beta-1)s + (1-\beta\eta)\big)G(\xi,t) \\ + \beta\big((1-\alpha)s + \alpha\xi\big)G(\eta,t)\big] + G(s,t). \end{aligned} \tag{4.27}$$

We first show that

$$K(s,t) \leqslant k_1\delta(s) \quad \text{for } t, s \in [0,1] \tag{4.28}$$

and

$$K(s,t) \geqslant k_2\delta(s) \quad \text{for } (t,s) \in [\theta, 1-\theta] \times [0,1], \tag{4.29}$$

where $\delta(s)$, k_1, and k_2 are defined by (3.2)–(3.4), respectively.

In fact, from (2.5),

$$t(1-t)s(1-s) \leqslant G(s,t) \leqslant s(1-s) \quad \text{for } t, s \in [0,1].$$

This, together with (4.27), implies that

$$\begin{aligned} K(s,t) &\leqslant \frac{1}{\varrho}\alpha\xi(1-\xi)\big[(\beta-1)s+(1-\beta\eta)\big] + \frac{1}{\varrho}\beta\eta(1-\eta)\big[(1-\alpha)s+\alpha\xi\big] + s(1-s) \\ &\leqslant \max\left\{\frac{1}{\varrho}\alpha\xi(1-\xi), \frac{1}{\varrho}\beta\eta(1-\eta), 1\right\}\big[(\beta-\alpha)s+s(1-s)+1-\beta\eta+\alpha\xi\big] \\ &= k_1\delta(s) \end{aligned}$$

for $t, s \in [0,1]$, and

$$\begin{aligned} K(s,t) \geqslant{}& \frac{1}{\varrho}\alpha\xi(1-\xi)\theta(1-\theta)\big[(\beta-1)s+(1-\beta\eta)\big] \\ &+ \frac{1}{\varrho}\beta\eta(1-\eta)\theta(1-\theta)\big[(1-\alpha)s+\alpha\xi\big] + \theta(1-\theta)s(1-s) \\ \geqslant{}& \big[(\beta-\alpha)s+s(1-s)+1-\beta\eta+\alpha\xi\big] \\ &\times\begin{cases} \theta(1-\theta), & \text{if } \alpha\xi(1-\xi)+\beta\eta(1-\eta)=0, \\ \theta(1-\theta)\min\{\varrho^{-1}\alpha\xi(1-\xi), \varrho^{-1}\beta\eta(1-\eta), 1\}, & \text{otherwise,} \end{cases} \\ ={}& k_2\delta(s) \end{aligned}$$

for $(t,s) \in [\theta, 1-\theta] \times [0,1]$. Thus, (4.28) and (4.29) hold.

Let φ_M be given as in Lemma 2.6 (b). Then $\varphi_M(t) = \mu_M M\varphi_M(t)$, i.e.,

$$\varphi_M(t) = \mu_M \int_0^1 K(s,t)g(s)\varphi_M(s)\,\mathrm{d}s. \tag{4.30}$$

Thus, from (4.28) and (4.29),

$$\varphi_M(t) \leqslant \mu_M k_1 \int_0^1 \delta(s)g(s)\varphi_M(s)\,\mathrm{d}s \quad \text{for } t \in [0,1]$$

and

$$\varphi_M(t) \geqslant \mu_M k_2 \int_0^1 \delta(s)g(s)\varphi_M(s)\,\mathrm{d}s \quad \text{for } t \in [\theta, 1-\theta].$$

Hence,

$$\varphi_M(t) \geqslant \frac{k_2}{k_1}\|\varphi_M\| \quad \text{for } t \in [\theta, 1-\theta].$$

This, together with (4.29) and (4.30), implies that

$$\begin{aligned}\varphi_M(t) &\geqslant \mu_M \int_\theta^{1-\theta} K(s,t) g(s)\varphi_M(s)\,\mathrm{d}s \\ &\geqslant \mu_M \frac{k_2^2}{k_1}\|\varphi_M\| \int_\theta^{1-\theta} \delta(s) g(s)\,\mathrm{d}s \quad \text{for } t \in [\theta, 1-\theta].\end{aligned}$$

As a result,

$$\mu_M \leqslant \frac{k_1}{k_2^2 \int_\theta^{1-\theta} \delta(s) g(s)\,\mathrm{d}s} = D.$$

On the other hand, from (4.28) and (4.30) we have

$$\varphi_M(t) \leqslant \mu_M k_1 \|\varphi_M\| \int_0^1 \delta(s) g(s)\,\mathrm{d}s \quad \text{for } t \in [0,1].$$

Thus,

$$\mu_M \geqslant \frac{1}{k_1 \int_0^1 \delta(s) g(s)\,\mathrm{d}s} = C.$$

This completes the proof of the lemma. □

P r o o f of Corollary 3.1. The conclusion follows from Theorem 3.1 and Lemma 4.1. □

P r o o f of Corollary 3.2. The conclusion follows from Theorem 3.2 and Lemma 4.1. □

P r o o f of Theorem 3.3. Let $(\lambda, \mu) \in \mathbb{R}^2$ be fixed. Since $F_\infty < C$, there exist $0 < C_4 < C$ and $\tau_1 > 0$ such that

$$|f(t,x)| \leqslant C_4 |x| \quad \text{for } (t, |x|) \in [0,1] \times (\tau_1, \infty). \tag{4.31}$$

Let

$$N_1 = \max_{t\in[0,1],\, |x| \in [0,\tau_1]} |f(t,x)|.$$

Then

$$|f(t,x)| \leqslant N_1 \quad \text{for } (t,|x|) \in [0,1] \times [0,\tau_1]. \tag{4.32}$$

In view of the fact that $C_4 < C$ and the definition of C in (3.1), we have

$$k_1 C_4 \int_0^1 \delta(s) g(s)\,\mathrm{d}s < 1.$$

Thus, we can choose $\tau_2 > \tau_1$ large enough so that

$$k_1[N_1 + C_4(\tau_2 + \lambda\|\varphi\| + \mu\|\psi\|)] \int_0^1 \delta(s)g(s)\,\mathrm{d}s \leqslant \tau_2. \tag{4.33}$$

Let

$$S = \{v \in X\colon\ \|v\| \leqslant \tau_2\}.$$

For $v \in S$, define

$$I_1^v = \{t \in [0,1]\colon\ |v(t) + \lambda\varphi(t) + \mu\psi(t)| \leqslant \tau_1\}$$

and

$$I_2^v = \{t \in [0,1]\colon\ |v(t) + \lambda\varphi(t) + \mu\psi(t)| > \tau_1\}.$$

Clearly, $I_1^v \cup I_2^v = [0,1]$ and $I_1^v \cap I_2^v = \emptyset$. Now, (4.4) and (4.31) imply that

$$\begin{aligned} |F_{\lambda,\mu}v(t)| &\leqslant C_4|v(t) + \lambda\varphi(t) + \mu\psi(t)| \\ &\leqslant C_4(\tau_2 + \lambda\|\varphi\| + \mu\|\psi\|) \quad \text{for } t \in I_2^v. \end{aligned} \tag{4.34}$$

From (4.5), (4.28), (4.32)–(4.34) it follows that

$$\begin{aligned} |Tv(t)| &\leqslant k_1\biggl(\int_{I_1^v} \delta(s)g(s)|F_{\lambda,\mu}v(s)|\,\mathrm{d}s + \int_{I_2^v} \delta(s)g(s)|F_{\lambda,\mu}v(s)|\,\mathrm{d}s\biggr) \\ &\leqslant k_1\biggl(N_1\int_{I_1^v} \delta(s)g(s)\,\mathrm{d}s + C_4(\tau_2 + \lambda\|\varphi\| + \mu\|\psi\|)\int_{I_2^v} \delta(s)g(s)\,\mathrm{d}s\biggr) \\ &\leqslant k_1[N_1 + C_4(\tau_2 + \lambda\|\varphi\| + \mu\|\psi\|)] \int_0^1 \delta(s)g(s)\,\mathrm{d}s \leqslant \tau_2 \end{aligned}$$

for $t \in [0,1]$. Thus, $T(S) \subseteq S$. By the Schauder fixed point theorem, T has at least one fixed point v in S, which is a solution of BVP (4.2), (4.3). Therefore, we have shown that, for any fixed $(\lambda,\mu) \in \mathbb{R}^2$, BVP (4.2), (4.3) has at least one solution $v(t)$. Consequently, BVP (1.1), (1.2) has at least one solution $u(t) = v(t) + \lambda\varphi(t) + \mu\psi(t)$. Clearly, if $(\lambda,\mu) \neq (0,0)$, from (1.2) we see that $u(t)$ is nontrivial, and if $(\lambda,\mu) = (0,0)$, from (1.1) and the assumption that $f(t,0) \not\equiv 0$ on $[0,1]$, it also follows that $u(t)$ is nontrivial. This completes the proof of the theorem. $\square$

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