PROPERTY (A) OF n-TH ORDER ODE'S

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Abstract. The aim of this paper is to deduce oscillatory and asymptotic behavior of the solutions of the ordinary differential equation

$$L_n u(t) + p(t)u(t) = 0.$$

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Consider the n-th order $(n \geqslant 2)$ differential equation

$$L_n u(t) + p(t)u(t) = 0, \tag{1}$$

where

$$L_n u(t) = \left(\frac{1}{r_{n-1}(t)}\left(\frac{1}{r_{n-2}(t)}\dots\left(\frac{1}{r_1(t)}u'(t)\right)'\dots\right)'\right)',$$

p and $r_i : (t_0, \infty) \to \mathbb{R}^+ = (0, \infty)$ are continuous, $1 \leqslant i \leqslant n-1$. In the sequel we will suppose that $\displaystyle\int_{t_0}^{\infty} r_i(s)\,\mathrm{d}s = \infty$ for $1 \leqslant i \leqslant n-1$. It is usual to denote

$$\begin{aligned} D_0 u(t) &= u(t),\\ D_i u(t) &= \frac{1}{r_i(t)}\frac{\mathrm{d}}{\mathrm{d}t}D_{i-1}u(t), \quad 1 \leqslant i \leqslant n-1,\\ D_n u(t) &= \frac{\mathrm{d}}{\mathrm{d}t}D_{n-1}u(t). \end{aligned} \tag{2}$$

By a solution of Eq. (1) we mean a function $u : (T_u, \infty) \to \mathbb{R}$ such that

(i) $D_i u(t)$, $0 \leqslant i \leqslant n$ exist and are continuous on $[T_u, \infty)$;

(ii) $u(t)$ satisfies (1);

(iii) $\sup\{|u(s)| : t \leqslant s < \infty\} > 0$ for any $t \geqslant T_u$.

Such a solution of (1) is called oscillatory if it has arbitrarily large zeros; otherwise it is called nonoscillatory. An equation is said to be oscillatory if all its solutions are oscillatory.

It is well known (see e.g. [2] or [3]) that the set $\mathcal{N}$ of all nonoscillatory solutions of (1) can be divided into the following classes:

$$\begin{aligned} \mathcal{N} &= \mathcal{N}_0 \cup \mathcal{N}_2 \cup \ldots \cup \mathcal{N}_{n-1} \quad \text{for } n \text{ odd},\\ \mathcal{N} &= \mathcal{N}_1 \cup \mathcal{N}_3 \cup \ldots \cup \mathcal{N}_{n-1} \quad \text{for } n \text{ even}, \end{aligned}$$

where $u(t) \in \mathcal{N}_\ell$ if and only if

$$\begin{aligned} u(t)D_i u(t) &> 0, \quad 0 \leqslant i \leqslant \ell,\\ (-1)^{i-\ell} u(t) D_i u(t) &> 0, \quad \ell \leqslant i \leqslant n \end{aligned} \tag{3}$$

for all large t. Following Foster and Grimmer [3] we say that $u(t)$ is a function of degree ℓ if $u(t)$ satisfies (3).

For the class $\mathcal{N}_0$ of (1), it is shown in [4] that $\mathcal{N}_0 \neq \emptyset$ if n is odd. Therefore, we are interested in the following particular situation:

Definition 1. Equation (1) is said to have property (A) if for n even $\mathcal{N} = \emptyset$ (i.e. (1) is oscillatory) and for n odd $\mathcal{N} = \mathcal{N}_0$.

This definition can be found in [6]. There is much literature regarding property (A) of (1) (see enclosed references). Integral conditions have been given under which (1) enjoys property (A). The following result is due to Trench [18].

Define for $1 \leqslant k \leqslant n-1$ and $t, s \in [t_0, \infty)$

$$\begin{aligned} I_0 &= 1,\\ I_k(t, s; r_1, \ldots, r_k) &= \int_s^t r_1(x) I_{k-1}(x, s; r_2, \ldots, r_k)\,\mathrm{d}x,\\ J_k(t) &= I_k(t_0, t; r_1, \ldots, r_k),\\ N_k(t) &= I_k(t_0, t; r_{n-1}, \ldots, r_{n-k}). \end{aligned}$$

Theorem A. *Let n be even. Assume that for all $i \in \{1, 3, \ldots, n-1\}$*

$$\int^\infty J_{i-1}(t) N_{n-i-1}(t) p(t)\,\mathrm{d}t = \infty. \tag{4}$$

Then (1) has property (A).

A question naturally arises what will happen when conditions (4) are violated. In fact, Theorem A cannot cover an important class of Euler's equation of the form

$$\frac{\mathrm{d}^m}{\mathrm{d}t^m} t^{\alpha+m} \frac{\mathrm{d}^m x}{\mathrm{d}t^m} + c t^{\alpha-m} x = 0, \quad t \geqslant 1, \tag{5}$$

where α and $c > 0$ are constants with $\alpha + m \leqslant 1$, since in this case the integrals in (4) converge.

Trench's result has been later improved by Kusano, Naito and Tanaka in [6] and [7], where (1) is compared with a set of second order differential equations and property (A) of (1) is reduced to the oscillation of a set of second order differential equations. On the other hand, Chanturia and Kiguradze [1] have improved (4) for the particular case of (1), namely for the differential equation

$$y^{(n)}(t) + p(t)y(t) = 0. \tag{6}$$

They have compared (1) with Euler's equation $t^n y^{(n)} + cy = 0$ to obtain the integral criterion

$$\liminf_{t\to\infty} t^{n-1} \int_t^{\infty} p(s)\,\mathrm{d}s = \frac{M^*}{n-1}$$

for property (A) of (6).

Our concern in this paper is to replace condition (4) by a similar one that is applicable also to (5). Our results complement and extend the above-mentioned results and also some other ones given in [16], [14], [10] and [8].

We consider a set of ℓ-th order ($n-1 \geqslant \ell \geqslant 1$) differential inequalities

$$\bigl\{M_{\ell+1}u(t) + q_{\ell+1}(t)u(t)\bigr\} \operatorname{sgn} u(t) \leqslant 0, \tag{$E_{\ell+1}$}$$

where $q_{\ell+1}$ is positive and continuous and

$$M_{\ell+1}u(t) = \left(\frac{1}{r_\ell(t)}\left(\frac{1}{r_{\ell-1}(t)} \cdots \left(\frac{1}{r_1(t)}u'(t)\right)' \cdots\right)'\right)',$$

that is $M_{\ell+1}u(t) = r_{\ell+1}(t)D_{\ell+1}u(t)$ for $\ell < n$, and $M_n u(t) = D_n u(t)$.

Let us put

$$J_{1,\ell}(t) = J_\ell(t) \quad \text{and} \quad J_{2,\ell}(t) = I_{\ell-1}(t, t_0; r_2, \ldots, r_\ell).$$

Our main results are based on the following theorem:

Theorem 1. *Let $1 \leqslant \ell \leqslant n-1$. Assume that*

$$(7_\ell) \qquad \int^{\infty} \left(J_{1,\ell}(t) q_{\ell+1}(t) - \frac{r_1(t) J_{2,\ell}(t)}{4 J_{1,\ell}(t)} \right) \mathrm{d}t = \infty.$$

Then $(E_{\ell+1})$ has no solutions of degree ℓ.

P r o o f. Assume that $(E_{\ell+1})$ possesses a positive nonoscillatory solution $u(t)$ such that $u(t)$ is of degree ℓ, that is

$$D_0 u(t) > 0,\ D_1 u(t) > 0, \ldots, D_\ell u(t) > 0,\ \bigl(D_\ell u(t)\bigr)' < 0, \quad t \geqslant t_0.$$

Let

$$z(t) = \frac{J_{1,\ell}(t) D_\ell u(t)}{u(t)}, \qquad t \geqslant t_0.$$

Then $z(t) > 0$ and

$$(8) \qquad z'(t) = \frac{r_1(t) J_{2,\ell}(t)}{J_{1,\ell}(t)} z(t) + \frac{J_{1,\ell}(t)(D_\ell u(t))'}{u(t)} - z(t) \frac{r_1(t) D_1 u(t)}{u(t)}.$$

Assume that $\ell > 1$. The identity $D_\ell u(t) = \dfrac{1}{r_\ell(t)} \bigl(D_{\ell-1} u(t)\bigr)'$ implies that

$$\begin{aligned} D_{\ell-1} u(t) &= D_{\ell-1} u(t_0) + \int_{t_0}^{t} r_\ell(s) D_\ell u(s) \,\mathrm{d}s \\ &\geqslant D_\ell u(t) \int_{t_0}^{t} r_\ell(s) \,\mathrm{d}s. \end{aligned}$$

Hence, after $(\ell-3)$-fold integration, we arrive at

$$D_1 u(t) \geqslant J_{2,\ell}(t) D_\ell u(t), \qquad t \geqslant t_0.$$

Therefore, combining (8) with the last inequality, one gets

$$(9) \qquad \frac{J_{1,\ell}(t)(D_\ell u(t))'}{u(t)} \geqslant z'(t) + \frac{r_1(t) J_{2,\ell}(t)}{J_{1,\ell}(t)} \bigl(z^2(t) - z(t)\bigr).$$

Note that $z^2(t) - z(t) \geqslant -\frac{1}{4}$. Multiplying $(E_{\ell+1})$ by $J_{1,\ell}(t)$ and dividing the resulting equality by $u(t)$, we see in view of (9) that $z(t)$ is a positive solution of the differential inequality

$$(10) \qquad z'(t) - \frac{r_1(t) J_{2,\ell}(t)}{4 J_{1,\ell}(t)} + J_{1,\ell}(t) q_{\ell+1}(t) \leqslant 0.$$

That (10) also holds for $\ell = 1$ follows from (8) and (M_2) (note that $J_{2,1}(t) \equiv 1$). An integration of (10) yields

$$z(t) + \int_{t_0}^{t} \left(J_{1,\ell}(s) q_{\ell+1}(s) - \frac{r_1(s) J_{2,\ell}(s)}{4 J_{1,\ell}(s)} \right) \mathrm{d}s \leqslant z(t_0).$$

Letting $t \to \infty$, we get a contradiction with (7_ℓ). The proof is complete. $\square$

The following result can be found in [5, Corollary 1].

Theorem B. *The equation* (1) *has a solution of degree* $n-1$ *if and only if the inequality* (E_n) *has a solution of degree* $n-1$.

For the particular case of (1) with $n=2$ and $n=3$ we have the following corollaries.

Corollary 1. *Denote* $R(t)=\int_{t_0}^{t} r(s)\,\mathrm{d}s$. *Assume that*

$$\int^{\infty}\left(R(s)p(s)-\frac{r(s)}{4R(s)}\right)\mathrm{d}s=\infty. \tag{11}$$

Then the second order differential equation

$$\left(\frac{1}{r(t)}u'\right)'+p(t)u=0 \tag{12}$$

is oscillatory.

P r o o f. By Theorem B, Eq. (12) is oscillatory if and only if (E_2) with $q_2=p$ and $r_1=r$ has no solution of degree 1. Since (7_1) reduces to (11), the assertion of this corollary follows from Theorem 1. $\square$

Corollary 2. *Assume that*

$$\int^{\infty}\left(J_{1,2}(s)p(s)-\frac{r_1(s)J_{2,2}(s)}{4J_{1,2}(s)}\right)\mathrm{d}s=\infty. \tag{13}$$

Then the third order differential equation

$$\left(\frac{1}{r_2(t)}\left(\frac{1}{r_1(t)}u'\right)'\right)'+p(t)u=0$$

has property (A).

P r o o f. The proof of this corollary is analogous to that of Corollary 1 (noting that (7_2) reduces to (13)) and can be omitted. $\square$

E x a m p l e 1. Consider the equation

$$\left(\frac{1}{t}u''\right)'+\frac{a}{t^4}u=0,\qquad a>0,\quad t\geqslant 1.$$

By Corollary 2, this equation has property (A) provided $a>4.5$.

Now we extend our previous results to (1) with $n > 3$. For all large t and $i \in \{1, \dots, n-1\}$ define

$$\begin{aligned}
K_1(t;p) &= \int_t^\infty p(s)\,\mathrm{d}s,\\
K_2(t;r_{n-1},p) &= \int_t^\infty r_{n-1}(x)K_1(x;p)\,\mathrm{d}x,\\
K_i(t;r_{n-i+1},\dots,r_{n-1},p) &= \int_t^\infty r_{n-i+1}(x)K_{i-1}(x;r_{n-i+2},\dots,r_{n-1},p)\,\mathrm{d}x,\\
q_n(t) &= p(t),\\
q_i(t) &= r_i(t)K_{n-i}(t;r_{i+1},\dots,r_{n-1},p).
\end{aligned}$$

Theorem 2. *Assume that for all $\ell \in \{1, \dots, n-1\}$ with $n+\ell$ odd, conditions (7_ℓ) are satisfied. Then* (1) *has property* (A).

P r o o f. Since (7_1) with $n = 2$ reduces to (11) and (7_2) with $n = 3$ reduces to (13) the assertion of the theorem for $n = 2$ and $n = 3$ follows from Corollaries 1 and 2.

Now assume that $n > 3$. We want to show that $\mathcal{N}_\ell = \emptyset$ for all $\ell \in \{1, \dots, n-1\}$ with $n+\ell$ odd. Note that by Theorem 1, condition (7_{n-1}) implies that differential inequality (E_n) has no solution of degree $n-1$. By Theorem (B), Eq. (1) has no solution of degree $n-1$, either (i.e. $\mathcal{N}_{n-1} = \emptyset$).

Let $1 \leqslant \ell \leqslant n-2$. Assume that (1) has a positive nonoscillatory solution $u(t)$ and $u(t)$ is of degree ℓ. From (1) and $u'(t) > 0$ it follows that

$$D_{n-1}u(\infty) - D_{n-1}u(t) + \int_t^\infty p(s)u(s)\,\mathrm{d}s = 0, \quad t \geqslant t_0.$$

That is,

$$-D_{n-1}u(t) + u(t)\int_t^\infty p(s)\,\mathrm{d}s \leqslant 0.$$

Hence, after $(n-\ell-2)$-fold integration we arrive at

$$M_{\ell+1}u(t) + q_{\ell+1}(t)u(t) \leqslant 0.$$

That is, $u(t)$ is a solution of $(E_{\ell+1})$, but as $u(t)$ is of degree ℓ, it contradicts the assertions of Theorem 1. The proof is complete. □

Corollary 3. *Assume that*

$$\int^\infty \left((t-t_0)^{n-1}p(t) - \frac{(n-1)(n-1)!}{4(t-t_0)}\right)\mathrm{d}t = \infty. \tag{14}$$

Then the n-th order differential equation

$$u^{(n)} + p(t)u = 0 \tag{15}$$

has property (A).

Proof. To prove that (15) has property (A), it suffices (see Theorem 1.1 in [1]) to show that (15) has no solution of degree $n-1$. This fact follows from Theorems A and 1. $\square$

It is interesting to compare Corollary 3 with the following result which is due to Chanturia and Kiguradze [1].

Lemma A. *The condition*

$$\int^{\infty} t^{n-1} p(t)\,\mathrm{d}t = \infty \tag{16}$$

is necessary for (15) *to have property* (A).

Note that the stronger condition (13) guarantees property (A) of (14), while (16) is not enough.

Remark. Using suitable comparison theorems, our results can be extended to more general differential equations. In fact, it is known [5] that the delay differential equation

$$L_n u(t) + p(t)u(\tau(t)) = 0, \tag{17}$$

where L_n and p are the same as in (1) and τ satisfies

$$\tau \in C^1, \quad \tau(t) \leqslant t, \quad \tau(t) \to \infty \text{ as } t \to \infty, \tag{18}$$

has property (A) if so does the differential equation without delay

$$L_n u(t) + \frac{p(\tau^{-1}(t))}{\tau'(\tau^{-1}(t))} u(t) = 0, \tag{19}$$

where τ^{-1} is the inverse function to τ. Applying Theorem 2 to (19) we immediately have a sufficient condition for (17) to have property (A). We illustrate this by the following result.

Corollary 4. *Assume that* (18) *holds. Further assume that*

$$\int^{\infty} \left((\tau(t) - t_0)^{n-1} p(t) - \frac{(n-1)(n-1)!}{4(t-t_0)} \tau'(t) \right) \mathrm{d}t = \infty.$$

Then the delay differential equation

$$u^{(n)}(t) + p(t)u(\tau(t)) = 0$$

has property (A).

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