REFLECTION AND A MIXED BOUNDARY VALUE PROBLEM CONCERNING ANALYTIC FUNCTIONS

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Dedicated to Professor Dr.-Ing. Wolfgang Wendland on the occasion of his sixtieth birthday

Abstract. A mixed boundary value problem on a doubly connected domain in the complex plane is investigated. The solution is given in an integral form using reflection mapping. The reflection mapping makes it possible to reduce the problem to an integral equation considered only on a part of the boundary of the domain.

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Let D be a bounded domain (= open and connected set) with the boundary C in the complex plane $\mathbb{C} \equiv \mathbb{R}^2$ and let $A \subset D$ be a continuum with the boundary B such that $D \setminus A$ is connected. The following mixed boundary value problem is motivated by some problems in hydrodynamics (compare [8]). Given real-valued functions f_C, f_B continuous on C and B respectively, find a holomorphic function $f\colon D \setminus A \to \mathbb{C}$ and a constant $k \in \mathbb{R}$ such that

$$\zeta \in C \Longrightarrow \lim_{\substack{z \to \zeta \\ z \in D\setminus A}} \operatorname{Re} f(z) = f_C(\zeta), \tag{1}$$

$$\zeta \in B \Longrightarrow \lim_{\substack{z \to \zeta \\ z \in D\setminus A}} \operatorname{Im} f(z) = f_B(\zeta) + k. \tag{2}$$

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If B and C are rectifiable Jordan curves satisfying some appropriate regularity conditions then the unknown holomorphic function f can be expressed in terms of integrals of the Cauchy type with continuous real-valued densities φ_B on B and φ_C on C (compare [16], [22]). If D is a special domain admitting the so-called reflection mapping mirroring D with respect to the boundary C (compare [18]), then the above problem of finding two unknown functions φ_B, φ_C can be reduced to a problem with only one unknown function φ_B by using a suitable integral representation of the function f. Applications of this technique to boundary value problems in the theory of harmonic functions have been studied in [19], [20], [21], [3]–[5]. In this remark we shall try to find "minimal" assumptions concerning regularity of the boundary B of the continuum A guaranteeing that the reflection method can be used to obtain suitable integral representation of the holomorphic function f in the mixed boundary value problem described above.

Notation. In the following, $D \subset \mathbb{C}$ will be a fixed bounded domain with the boundary $\partial D = C$. As usual, points $(a, b) \in \mathbb{R}^2$ will be identified with the corresponding complex numbers $a + \mathrm{i}b \in \mathbb{C}$ (i is the imaginary unit).

If $M \subset \mathbb{C} \equiv \mathbb{R}^2$ then ∂M, $\operatorname{cl} M$ and M° denote the boundary, closure and interior of M, respectively. If further $z \in \mathbb{C}$ then $\operatorname{dist}(z, M)$ stands for the distance of the point z from the set M. The symbol ∂_j will denote the partial derivative with respect to the j-th variable ($j = 1, 2$).

Definition 1. A reflection mapping corresponding to D is the mapping $g\colon U \to \mathbb{C}$ defined on some neighbourhood U of the boundary C satisfying the following conditions:

(i) The complex conjugate $\overline{g}$ of g is one-to-one and holomorphic on U.
(ii) $g(\zeta) = \zeta$ for any $\zeta \in C$.
(iii) $g(U \cap D) = U \setminus \operatorname{cl} D$, $g(U \setminus \operatorname{cl} D) = U \cap D$.
(iv) $g\bigl(g(z)\bigr) = z$ for any $z \in U$.

Example. If

$$D = \{\, z \in \mathbb{C} \mid |z| < r \,\}$$

is the disc with radius $r > 0$ then the mapping

$$g(z) = \frac{r^2}{\overline{z}}$$

($\overline{z}$ denotes the conjugate of $z \in \mathbb{C}$) defined on $U = \mathbb{C} \setminus \{0\}$ is the reflection mapping corresponding to D. Reflection mappings corresponding to more general Jordan domains have been investigated in [18]. Let us start with some elementary observations.

Observation 1. If D has the reflection mapping g then each point $z_0 = x_0 + \mathrm{i}y_0 \in C$ has a neighbourhood in C which is a non-parametric simple arc given by equation $y = \psi_1(x)$ (where ψ_1 is an infinitely differentiable function on an open interval $I_1 \subset \mathbb{R}$ with $x_0 \in I_1$) or by equation $x = \psi_2(y)$ (where ψ_2 is an infinitely differentiable function on an open interval $I_2 \subset \mathbb{R}$ with $y_0 \in I_2$). The boundary $C = \partial D$ thus consists of finitely many infinitely smooth Jordan curves. (We will always suppose in the following that C is oriented positively with respect to D.)

Proof. Let us suppose that g is the reflection mapping corresponding to D and defined on U. It is seen from conditions (ii), (iii) that C is the set of all points $z \in U$ with $g(z) = z$. If $g = g_1 + \mathrm{i}g_2$ (i.e. g_1, g_2 are the real and the imaginary part of g) then the condition $g(z) = z$ can be rewritten in the form

$$F_1(x, y) = 0, \qquad F_2(x, y) = 0,$$

where

$$F_1(x, y) = g_1(x, y) - x, \qquad F_2(x, y) = g_2(x, y) - y.$$

The functions F_1, F_2 are infinitely differentiable on U. Since $\overline{g}$ is holomorphic,

$$\partial_1 g_1 = -\partial_2 g_2$$

and we see that at no point of U it may occur that

$$\partial_1 F_1 = 0 \quad \text{and at the same time} \quad \partial_2 F_2 = 0.$$

Given $z_0 = x_0 + \mathrm{i}y_0 \in C$ suppose, for example, that

$$\partial_2 F_2(x_0, y_0) \neq 0.$$

By the implicit function theorem there are $\delta_1, \delta_2 > 0$ and a function

$$\psi_1 \colon]x_0 - \delta_1, x_0 + \delta_1[\, \to \,]y_0 - \delta_2, y_0 + \delta_2[$$

such that

$$F_2\big(x, \psi_1(x)\big) = 0$$

for each $x \in \,]x_0 - \delta_1, x_0 + \delta_1[$; points of the form $(x, \psi_1(x))$ are the only solutions of the equation $F_2(x, y) = 0$ belonging to $Q = \,]x_0 - \delta_1, x_0 + \delta_1[\, \times \,]y_0 - \delta_2, y_0 + \delta_2[$ (we may suppose that δ_1, δ_2 are such that $Q \subset U$) and ψ_1 is infinitely differentiable.

Denote by

$$K := \big\{\, (x, y) \;\big|\; x \in \,]x_0 - \delta_1, x_0 + \delta_1[, y = \psi_1(x) \,\big\}$$

the graph of ψ_1. Clearly $C \cap Q \subset K$. Now it suffices to show that $C \cap Q = K$. Let us suppose that there is a point $(x, y) \in K$ such that $(x, y) \notin C$. Then any two points from $Q \setminus C$ can be joined in Q by an arc not meeting C. Since $C \cap Q$ contains (x_0, y_0) we have $C \cap Q \neq \emptyset$ and

$$D \cap Q = Q \setminus C.$$

But this is not possible since any neighbourhood of $z_0 = (x_0, y_0)$ (as well as a neighbourhood of any point from C in general) contains points from $U \setminus \operatorname{cl} D$ [if $z_n \in D$, $z_n \to z_0$ then $g(z_n) \to g(z_0) = z_0$ and $g(z_n) \in U \setminus \operatorname{cl} D$ due to (iii)]. The case $\partial_1 F_1(x_0, y_0) \neq 0$ is similar. □

O b s e r v a t i o n 2. Let D be a bounded domain with reflection mapping g defined on an open neighbourhood U of the boundary $\partial D = C$. If C consists of more than one Jordan curve then U cannot be connected and each component of U contains at most one Jordan curve composing C.

P r o o f. For simplicity let us suppose that $C = C_1 \cup C_2$, where C_1, C_2 are Jordan curves and that, for example, $D = \operatorname{Int} C_1 \cap \operatorname{Ext} C_2$, where $\operatorname{Int} C_1$ denotes the bounded complementary domain of C_1 and $\operatorname{Ext} C_2$ is the unbounded complementary domain of C_2. Assume that there is a component U_1 of U such that $C \subset U_1$. First observe that $U_1 \cap \operatorname{cl} D$ is connected. Indeed, if $U_1 \cap \operatorname{cl} D = M_1 \cup M_2$ where M_1, M_2 are separated, then each of the curves C_1, C_2 is contained in M_1 or in M_2; suppose, for example, that $C_1 \subset M_1$, $C_2 \subset M_2$. Then

$$U_1 = \big[M_1 \cup (U_1 \cap \operatorname{Ext} C_1)\big] \cup \big[M_2 \cup (U_1 \cap \operatorname{Int} C_2)\big],$$

where the two sets on the right-hand side are evidently separated—a contradiction; similar reasoning applies in the other cases. In view of smoothness of C it follows that $U_1 \cap D$ is connected, too. Let $z_1 \in C_1$, $z_2 \in C_2$. Then there is a simple arc $L \subset U_1 \cap \operatorname{cl} D$ with end-points z_1, z_2 such that

$$L \setminus \{z_1, z_2\} \subset D \cap U_1.$$

Then $g(L)$ is also a simple arc with the same end-points and condition (iii) implies

$$g(L) \setminus \{z_1, z_2\} \subset \mathbb{C} \setminus \operatorname{cl} D,$$

which is impossible. □

O b s e r v a t i o n 3. If there is a compact $A \subset D$ such that $D \setminus A$ is connected and $D \setminus A \subset U$ then C consists of only one Jordan curve (i.e. D is a Jordan domain).

P r o o f. If C contains two Jordan curves C_1, C_2 then denote by U_1 the component of U containing C_1 and put $U_2 = U \setminus U_1$ $(\supset C_2)$. Then

$$(D \setminus A) \cap U_1, \qquad (D \setminus A) \cap U_2$$

are separated and clearly non-empty. $\square$

Now let us suppose that $D \subset \mathbb{C}$ is a bounded domain and $A \neq \emptyset$ is a continuum contained in D. We will need the following known assertion.

Proposition 1. *If $D \setminus A$ is a domain and $f\colon D \setminus A \to \mathbb{C}$ is a holomorphic function such that for some $a, b \in \mathbb{R}$*

$$\zeta \in C \ (= \partial D) \Longrightarrow \lim_{\substack{z \to \zeta \\ z \in D \setminus A}} \operatorname{Re} f(z) = a, \tag{3}$$

$$\zeta \in B \ (= \partial A) \Longrightarrow \lim_{\substack{z \to \zeta \\ z \in D \setminus A}} \operatorname{Im} f(z) = b, \tag{4}$$

then f is constant on $D \setminus A$ (and $f \equiv a + \mathrm{i}b$ there).

P r o o f. For convenience of the reader we include the following argument which is merely a simplification of a more general consideration from [22]. Suppose that f is not constant and put

$$R(a) := \{\, \zeta \in \mathbb{C} \mid \operatorname{Re} \zeta = a \,\}, \qquad I(b) := \{\, \zeta \in \mathbb{C} \mid \operatorname{Im} \zeta = b \,\}.$$

It follows from (3), (4) that

$$\partial f(D \setminus A) \subset R(a) \cup I(b). \tag{5}$$

Indeed, given $\zeta \in \partial f(D \setminus A)$ there is a sequence of points $z_n \in D \setminus A$ [which can be supposed to be convergent, $z_n \to z_0 \in \operatorname{cl}(D \setminus A)$] such that $\lim_{n\to\infty} f(z_n) = \zeta$. In the case $z_0 \in D \setminus A$ the point $f(z_0) = \zeta$ would be an interior point of $f(D \setminus A)$, which is impossible. Hence either $z_0 \in C = \partial D$ [in which case $\lim_{n\to\infty} \operatorname{Re} f(z_n) = a = \operatorname{Re} \zeta$] or $z_0 \in B = \partial A$ [and then $\lim_{n\to\infty} \operatorname{Im} f(z_n) = b = \operatorname{Im} \zeta$] and $\zeta \in R(a) \cup I(b)$ is verified.

Under our assumptions $f(D \setminus A)$ is a domain. We can thus fix $\tilde{z} \in D \setminus A$ such that $f(\tilde{z}) \notin R(a) \cup I(b)$. Choose $\varepsilon > 0$,

$$\varepsilon < \operatorname{dist}\bigl(f(\tilde{z}), R(a) \cup I(b)\bigr).$$

Then there is a $\delta > 0$ such that

$$\Bigl(z \in D \setminus A, \operatorname{dist}\bigl(z, \partial(D \setminus A)\bigr) < \delta\Bigr) \Longrightarrow \operatorname{dist}\bigl(f(z), R(a) \cup I(b)\bigr) < \varepsilon.$$

This means that the compact $Q := \{z \in D \setminus A \mid \operatorname{dist}(z, \partial(D \setminus A)) \geqslant \delta\}$ contains $\widetilde{z}$ and if $R > 0$ is sufficiently large then $f(Q)$ is contained in

$$B_R(a + \mathrm{i}b) := \{\, z \in \mathbb{C} \bigm| |z - (a + \mathrm{i}b)| < R \,\}.$$

Thus

$$f(D \setminus A) \subset B_R(a + \mathrm{i}b) \cup \{\, w \in \mathbb{C} \bigm| \operatorname{dist}\bigl(w, R(a) \cup I(b)\bigr) < \varepsilon \,\}.$$

Consider the half-line

$$P = \{\, a + \mathrm{i}b + t[f(\widetilde{z}) - (a + \mathrm{i}b)] \bigm| t \geqslant 1 \,\}.$$

Clearly

$$P \cap \{\, w \in \mathbb{C} \bigm| \operatorname{dist}\bigl(w, R(a) \cup I(b)\bigr) < \varepsilon \,\} = \emptyset.$$

The origin of P (corresponding to $t = 1$) belongs to $f(D \setminus A)$, while the points of P corresponding to large $t > 1$ lie outside of $f(D \setminus A)$. Thus P has to meet $\partial f(D \setminus A)$ which contradicts (5); this contradiction proves that f is constant on $D \setminus A$. $\square$

Notation. Let $\mathscr{C}_0^{(1)}$ stand for the set of all real-valued continuously differentiable functions in $\mathbb{R}^2$ with compact support, and for a compact $Q \subset \mathbb{R}^2$ let

$$\mathscr{C}^{(1)}(Q) = \{\, \varphi|_Q \bigm| \varphi \in \mathscr{C}_0^{(1)} \,\}$$

be the linear space over $\mathbb{R}$ consisting of restrictions to Q of functions from $\mathscr{C}_0^{(1)}$.

Recall that ∂_j stands for the partial derivative with respect to the j-th variable ($j = 1, 2$); we put $\overline{\partial} = \frac{1}{2}(\partial_1 + \mathrm{i}\partial_2)$. If $A \subset \mathbb{C}$ is a Lebesgue measurable set with compact boundary $B = \partial A$ then to each $\varphi \in \mathscr{C}^{(1)}(B)$ we assign a function $\mathcal{K}^A\varphi(z)$ of the complex variable $z \in \mathbb{C} \setminus B$ in the following way:

Given $z \in \mathbb{C} \setminus B$ choose $\psi_\varphi \in \mathscr{C}_0^{(1)}$ such that $\psi_\varphi = \varphi$ on B and $z \notin \operatorname{spt} \psi_\varphi$ (thus ψ_φ vanishes on some neighbourhood of z) and put

$$\mathcal{K}^A\varphi(z) := \frac{2}{\pi \mathrm{i}} \int_{\mathbb{C} \setminus A} \frac{\overline{\partial}\psi_\varphi(\zeta)}{\zeta - z} \, \mathrm{d}\lambda_2(\zeta),$$

where λ_2 is the Lebesgue measure on $\mathbb{R}^2 \equiv \mathbb{C}$. (Note that in distinction to [13] the integral is multiplied by 2.) The value of $\mathcal{K}^A\varphi(z)$ is independent of the choice of ψ_φ satisfying the above conditions (compare Lemma 2.1 in [10]) and the function

$$z \mapsto \mathcal{K}^A\varphi(z)$$

is holomorphic on $\mathbb{C} \setminus \partial A$.

(In the case when A is a Jordan domain bounded by a rectifiable Jordan curve ∂A which is positively oriented with respect to A the Green formula yields

$$\mathcal{K}^A\varphi(z) = \frac{1}{\pi}\int_{\partial A}\frac{\varphi(\zeta)}{\zeta - z}\,\mathrm{d}\zeta,$$

which is the usual Cauchy type integral.)

If

$$h_z(\zeta) = \frac{1}{2\pi}\ln\frac{1}{|z-\zeta|}, \qquad \zeta \in \mathbb{C}\setminus\{z\},$$

is the fundamental harmonic function with the pole at z and if for $u, v \in \mathbb{C} \equiv \mathbb{R}^2$ we denote by $\langle u, v\rangle = \operatorname{Re} u\overline{v}$ the scalar product of the vectors u, v (by $\overline{v}$ we mean the complex conjugate to $v \in \mathbb{C}$) then for $z \in \mathbb{C}\setminus B$

$$\operatorname{Im}\mathcal{K}^A\varphi(z) \equiv W^A\varphi(z) := 2\int_{\mathbb{R}^2\setminus A}\bigl\langle \operatorname{grad}\psi_\varphi(\zeta), \operatorname{grad} h_z(\zeta)\bigr\rangle\,\mathrm{d}\lambda_2(\zeta)$$

(compare (16_{W}) in [13]); the function $z \mapsto W^A\varphi(z)$ is harmonic on $\mathbb{C}\setminus B$.

Further, for $r > 0$ and $z \in \mathbb{C}$ put

$$B_r(z) := \bigl\{\, \zeta \in \mathbb{C} \bigm| |\zeta - z| < r \,\bigr\}$$

and define the upper density

$$\overline{d}(A,z) = \limsup_{r\downarrow 0}\frac{\lambda_2[B_r(z)\cap A]}{\lambda_2[B_r(z)]},$$
$$\partial_{\mathrm{es}}A = \bigl\{\, z\in\mathbb{C} \bigm| \overline{d}(A,z) > 0, \overline{d}(\mathbb{C}\setminus A, z) > 0 \,\bigr\}$$

($\partial_{\mathrm{es}}A$ is the so-called essential boundary of A; clearly $\partial_{\mathrm{es}}A \subset \partial A$).

N o t e. Related definitions of double layer potentials $W^A\varphi$ have appeared in [2], [12] (cf. also [10], [15]); for Cauchy's type integrals $\mathcal{K}^A\varphi$ cf. [1].

C o n v e n t i o n. We will suppose in the following that D is a bounded domain in $\mathbb{C}$ with a boundary C and with the reflection mapping defined on an open neighbourhood U of C. Now $A \subset D$ will be a compact set such that $D \setminus U \subset A^\circ$; its boundary $B = \partial A$ is thus contained in $D \cap U$. We will suppose for simplicity that A is massive in the sense that

$$\lambda_2\bigl[B_r(z)\cap A\bigr] > 0$$

for any $z \in A$ and $r > 0$. Define

$$G := (U\setminus A)\cap g(U\setminus A);$$

clearly G is an open set containing $(D \setminus A) \cup C$.

To each $\varphi \in \mathscr{C}^{(1)}(B)$ assign a function $\mathcal{J}^A\varphi(z)$ defined on G by

$$\mathcal{J}^A\varphi(z) = \mathcal{K}^A\varphi(z) - \overline{\mathcal{K}^A\varphi\big(g(z)\big)}, \qquad z \in G.$$

Lemma 1. *For any $\varphi \in \mathscr{C}^{(1)}(B)$ the function*

$$\mathcal{J}^A\varphi\colon z \mapsto \mathcal{J}^A\varphi(z)$$

is holomorphic on G.

P r o o f. The function $\mathcal{K}^A\varphi$ is holomorphic on $\mathbb{C} \setminus B$ and thus on G. Verifying the Cauchy-Riemann equations one can easily prove the following assertion: If f is holomorphic on a neighbourhood of $z_0 \in \mathbb{C}$ and h is holomorphic on a neighbourhood of $\overline{f}(z_0)$, then the function q,

$$q(z) = \overline{h(\overline{f}(z))},$$

is holomorphic on a neighbourhood of z_0. It follows from the definition of G that $g(z) \in G$ for $z \in G$. Indeed, if $z \in G$ then $z \in U \setminus A$ and thus $g(z) \in g(U \setminus A)$; since $z \in g(U \setminus A)$ there is a $z' \in U \setminus A$ with $z = g(z')$ and, consequently, $g(z) = g(g(z')) = z' \in U \setminus A$ and we see that $g(z) \in G$. Applying the above assertion to $h = \mathcal{K}^A\varphi$ and $f = \overline{g}$ we obtain that

$$z \mapsto \overline{\mathcal{K}^A\varphi\big(g(z)\big)}$$

is holomorphic on G so that $\mathcal{J}^A\varphi$ is holomorphic there, too. $\square$

N o t a t i o n. For an open set $V \subset \mathbb{C}$ let $\mathscr{A}(V)$ denote the linear space of all holomorphic functions on V and $\mathscr{H}(V)$ the space of all real-valued harmonic functions on V. By λ_1 we denote the length, that is the Hausdorff one-dimensional measure (cf. chap. II, § 8 in [17]).

Lemma 2. *Let $\mathscr{C}^{(1)}(B) \subset \mathscr{C}(B)$ ($\equiv$ the space of all continuous real-valued functions on B with the supremum norm) be endowed with the topology of uniform convergence induced from $\mathscr{C}(B)$. If*

$$\lambda_1(\partial_{\mathrm{es}}A) < \infty \tag{6}$$

then the mapping $\mathcal{K}^A\colon \mathscr{C}^{(1)}(B) \to \mathscr{A}(\mathbb{C} \setminus B)$,

$$\mathcal{K}^A\colon \varphi \mapsto \mathcal{K}^A\varphi, \tag{7}$$

is continuous under the topology of locally uniform convergence in $\mathscr{A}(\mathbb{C}\setminus B)$, *the mapping* $\mathcal{J}^A\colon \mathscr{C}^{(1)}(B)\to\mathscr{A}(G)$,

$$\mathcal{J}^A\colon \varphi\mapsto \mathcal{J}^A\varphi, \tag{8}$$

is continuous under the topology of locally uniform convergence in $\mathscr{A}(G)$, *and the mapping* $\operatorname{Im}\mathcal{J}^A\colon \mathscr{C}^{(1)}(B)\to\mathscr{H}(G)$,

$$\operatorname{Im}\mathcal{J}^A\colon \varphi\mapsto \operatorname{Im}\mathcal{J}^A\varphi, \tag{9}$$

is continuous under the topology of locally uniform convergence in $\mathscr{H}(G)$.

P r o o f. It is proved in Theorem 1 from [13] that the mapping (7) is continuous (the choice $q\equiv 1$ in [13]). Let I^A denote the mapping from $\mathscr{C}^{(1)}(B)$ into $\mathscr{A}(G)$, I^A: $\varphi\mapsto I^A\varphi$, where

$$I^A\varphi(z)=\overline{\mathcal{K}^A\varphi\bigl(g(z)\bigr)};$$

then $\mathcal{J}^A=\mathcal{K}^A-I^A$. To prove that $\mathcal{J}^A$ is continuous it now suffices to show that I^A is continuous. Let $K\subset G$ be a compact set. Since $Q=g(K)$ is compact and $Q\subset G$ there is a constant $k\in\mathbb{R}$ such that

$$\sup_{z\in Q}\bigl|\mathcal{K}^A\varphi(z)\bigr|\leqslant k\|\varphi\|$$

for any $\varphi\in\mathscr{C}^{(1)}(B)$, where $\|\ldots\|$ stands for the supremum norm in $\mathscr{C}^{(1)}(B)$. Thus we have for $\varphi\in\mathscr{C}^{(1)}(B)$

$$\sup_{z\in K}\bigl|I^A\varphi(z)\bigr|=\sup_{z\in K}\Bigl|\overline{\mathcal{K}^A\varphi\bigl(g(z)\bigr)}\Bigr|=\sup_{z\in Q}\bigl|\mathcal{K}^A\varphi(z)\bigr|\leqslant k\|\varphi\|,$$

which means that I^A is continuous.

The continuity of the mapping (9) follows immediately from the continuity of the mapping (8). □

R e m a r k 1. Condition (6) means that A has a finite perimeter (cf. [6]). Theorem 1 in [13] asserts among other things that (6) is necessary for the continuity of the mapping

$$\operatorname{Im}\mathcal{K}^A\colon \varphi\mapsto\operatorname{Im}\mathcal{K}^A\varphi$$

as a mapping from $\mathscr{C}^{(1)}(B)$ into $\mathscr{H}(\mathbb{R}^2\setminus B)$ [under the topology of locally uniform convergence in $\mathscr{H}(\mathbb{R}^2\setminus B)$]. But the proof relies only on the fact that there are three points $z_1,z_2,z_3\in\mathbb{R}^2\setminus B$ not lying on a line and a constant $k\in\mathbb{R}$ such that

$$\bigl|\operatorname{Im}\mathcal{K}^A\varphi(z_i)\bigr|\leqslant k\|\varphi\|\qquad(1\leqslant i\leqslant 3)$$

for $\varphi\in\mathscr{C}^{(1)}(B)$. Thus one can see that the condition (6) is also necessary for the continuity of that mapping as a mapping from $\mathscr{C}^{(1)}(B)$ into $\mathscr{H}(G)$. By Theorem 1 from [13] the condition (6) is also necessary for the continuity of the mapping (7).

In the following we will always suppose that (6) is valid. Then the mappings (7), (8) [and also (9), of course] have natural extensions to the whole space $\mathscr{C}(B)$. Thus to each $\varphi \in \mathscr{C}(B)$ a function $\mathcal{K}^A\varphi \in \mathscr{A}(\mathbb{C} \setminus B)$ is assigned in a natural way [and the mapping (7) is continuous on $\mathscr{C}(B)$] and also a function $\mathcal{J}^A\varphi \in \mathscr{A}(G)$ is assigned to $\varphi \in \mathscr{C}(B)$ [and the mapping (8) is continuous on $\mathscr{C}(B)$].

By $\widehat{\partial A}$ we denote the reduced boundary of A which consists of all points $\zeta \in \mathbb{R}^2$ for which there exists a vector $\mathbf{n} \equiv \mathbf{n}^A(\zeta) \in \partial B_1(0)$ [where $\partial B_1(0)$ is the boundary of $B_1(0)$] such that the half-plane

$$H_{\mathbf{n}}(\zeta) := \left\{ z \in \mathbb{R}^2 \mid \langle z-\zeta, \mathbf{n}\rangle > 0 \right\}$$

satisfies

$$\overline{d}\big(H_{\mathbf{n}}(\zeta) \setminus A, \zeta\big) = 0, \qquad \overline{d}\big(A \setminus H_{\mathbf{n}}(\zeta), \zeta\big) = 0;$$

the vector $\mathbf{n}^A(\zeta)$ is then uniquely determined and is called the interior Federer normal of A at the point $\zeta \in \widehat{\partial A}$.

It is known that $\widehat{\partial A} \subset \partial_{\mathrm{es}} A$ is a Borel set, the function $\zeta \mapsto \mathbf{n}^A(\zeta)$ is Borel measurable (see [7]) and, if (6) is fulfilled, then

$$\lambda_1\big(\partial_{\mathrm{es}} A \setminus \widehat{\partial A}\big) = 0.$$

Further, for $\zeta \in \widehat{\partial A}$ let us define the tangent vector of B at the point ζ by

$$\boldsymbol{\tau}^A(\zeta) = -\mathrm{i}\mathbf{n}^A(\zeta).$$

If $\mathrm{Const}(B)$ denotes the subspace of all constant functions in $\mathscr{C}(B)$ then it follows from the definition of $\mathcal{K}^A\varphi$ that (see p. 8 in [13])

$$\varphi \in \mathscr{C}(B) \Longrightarrow \mathcal{K}^A\varphi(z) = \frac{1}{\pi}\int_{\widehat{\partial A}} \frac{\varphi(\zeta)}{\zeta - z} \cdot \boldsymbol{\tau}(\zeta)\,\mathrm{d}\lambda_1(\zeta), \quad z \in \mathbb{C} \setminus A,$$
$$\varphi \in \mathrm{Const}(B) \Longrightarrow \mathcal{K}^A\varphi(z) = 0 \quad \text{for each } z \in \mathbb{C} \setminus A$$

(compare with (26) in [13]). For $\varphi \in \mathscr{C}(B)$ and $z \in G$ we have

$$\begin{aligned}
\mathcal{J}^A\varphi(z) &= \frac{1}{\pi}\int_{\widehat{\partial A}} \frac{\varphi(\zeta)}{\zeta - z}\boldsymbol{\tau}^A(\zeta)\,\mathrm{d}\lambda_1(\zeta) - \overline{\frac{1}{\pi}\int_{\widehat{\partial A}} \frac{\varphi(\zeta)}{\zeta - g(z)}\boldsymbol{\tau}^A(\zeta)\,\mathrm{d}\lambda_1(\zeta)},\\
W^A\varphi(z) \equiv \operatorname{Im}\mathcal{K}^A\varphi(z) &= \frac{1}{\pi}\int_{\widehat{\partial A}} \varphi(\zeta)\cdot \operatorname{Im}\left(\frac{\boldsymbol{\tau}^A}{\zeta - z}\right)\mathrm{d}\lambda_1(\zeta)\\
&= \frac{1}{\pi}\int_{\widehat{\partial A}} \varphi(\zeta)\frac{\operatorname{Re}\big[\mathbf{n}^A(\zeta)\cdot\overline{(z-\zeta)}\,\big]}{|z-\zeta|^2}\,\mathrm{d}\lambda_1(\zeta)\\
&= 2\int_{\widehat{\partial A}} \varphi(\zeta)\big\langle \mathbf{n}^A(\zeta), \operatorname{grad} h_z(\zeta)\big\rangle\,\mathrm{d}\lambda_1(\zeta).
\end{aligned}$$

R e m a r k. We shall look for the solution of the mixed boundary value problem described in the introduction expressed in the form

$$f(z)=\frac{1}{\pi \mathrm{i}}\left\{\int_C \frac{\varphi_C(\zeta)}{\zeta-z}\,\mathrm{d}\zeta+\overline{\int_C \frac{\varphi_C(\zeta)}{\zeta-g(z)}\,\mathrm{d}\zeta}\right\}+\mathcal{J}^A\varphi_B(z), \tag{10}$$

$z\in D\setminus A$, for appropriate $\varphi_C\in\mathscr{C}(C)$, $\varphi_B\in\mathscr{C}(B)$.

In this context we arrive at a question concerning conditions on A under which for any $\varphi\in\mathscr{C}(B)$, $\eta\in B$ the limit

$$\lim_{\substack{z\to\eta\\ z\in G}} \operatorname{Im}\mathcal{J}^A\varphi(z)$$

exists (and is finite), i.e. $\operatorname{Im}\mathcal{J}^A\varphi$ is continuously extendable from $D\setminus A$ to $(D\setminus A)\cup B$. For this purpose let us introduce some notation.

N o t a t i o n. For $\eta\in\mathbb{R}^2$ and $\theta\in\partial B_1(0)$ let

$$H(\eta,\theta)=\{\,\eta+t\theta\mid t>0\,\}$$

be the half-line with the origin η and the direction θ. For $r\in\,]0,+\infty]$ by

$$v_r(\theta,\eta)$$

we mean the number of points of $\partial_{\mathrm{es}}A\cap H(\eta,\theta)\cap B_r(\eta)$. Then the function

$$\theta\mapsto v_r(\theta,\eta])$$

is λ_1-measurable on $\partial B_1(0)$ and (compare pp. 27, 28 in [13])

$$\frac{1}{2\pi}\int_{\partial B_1(0)} v_r(\theta,\eta)\,\mathrm{d}\lambda_1(\theta)=\int_{\widehat{\partial A}}\bigl|\bigl\langle \mathbf{n}^A(\zeta),\operatorname{grad}h_\eta(\zeta)\bigr\rangle\bigr|\,\mathrm{d}\lambda_1(\zeta).$$

Put

$$v_r(\eta)=\frac{1}{\pi}\int_{\partial B_1(0)} v_r(\theta,\eta)\,\mathrm{d}\lambda_1(\theta).$$

Proposition 2. *The limit*

$$\lim_{\substack{z\to\eta\\ z\in D\setminus A}} \operatorname{Im}\mathcal{J}^A\varphi(z)$$

exists and is finite for each $\varphi \in \mathscr{C}(B)$ *and each* $\eta \in B$ *[that is for each* $\varphi \in \mathscr{C}(B)$ *the function* $\operatorname{Im} \mathcal{J}^A \varphi$ *is continuously extendable from* $D \setminus A$ *to* $(D \setminus A) \cup B$*] if and only if*

$$\sup_{\eta \in B} v_\infty(\eta) < \infty. \tag{11}$$

P r o o f. For each $\varphi \in \mathscr{C}(B)$ the function $\mathcal{K}^A \varphi$ is continuous on $\mathbb{C} \setminus A$ and thus the function

$$z \mapsto \mathcal{K}^A \varphi\big(g(z)\big)$$

is continuous on $U \cap D$ [as $g(U \cap D) = U \setminus \operatorname{cl} D \subset \mathbb{C} \setminus A$] and hence also on $(D \setminus A) \cup B$ (as $B \subset U \cap D$). This means that $\operatorname{Im} \mathcal{J}^A \varphi$ can be continuously extended from $D \setminus A$ to $(D \setminus A) \cup B$ if and only if $\operatorname{Im} \mathcal{K}^A \varphi$ has this property. One can see easily that the condition (11) is fulfilled if and only if for each $\eta \in B$

$$\limsup_{\substack{z \to \eta \\ z \in B}} v_\infty(\eta) < \infty \tag{12}$$

(using the fact that v_∞ is lower semicontinuous on $\mathbb{R}^2$—see Corollary 1 from [14], for example). But by Theorem 4 from [14] (choice $q \equiv 1$) (12) is a sufficient and necessary condition for the existence of a finite limit

$$\lim_{\substack{z \to \eta \\ z \in D \setminus A}} \operatorname{Im} \mathcal{K}^A \varphi(z)$$

for any $\varphi \in \mathscr{C}(B)$. □

C o n v e n t i o n. From now on we will always suppose that (11) is fulfilled. Then the following is valid.

Lemma 3. *For each* $\eta \in B$ *the density of* A *at* η

$$d(A, \eta) := \lim_{r \downarrow 0} \frac{\lambda_2\big[B_r(\eta) \cap A\big]}{\lambda_2\big[B_r(\eta)\big]}$$

exists, the function

$$\zeta \mapsto \big\langle \mathbf{n}^A(\zeta), \operatorname{grad} h_\eta(\zeta)\big\rangle$$

is integrable on $\widehat{\partial} A$ *with respect to* λ_1 *and for any* $\varphi \in \mathscr{C}(B)$

$$\lim_{\substack{z \to \eta \\ z \in D \setminus A}} \operatorname{Im} \mathcal{K}^A \varphi(z) = -\varphi(\eta) + T\varphi(\eta),$$

where

$$(13)\qquad T\varphi(\eta) = \big[1 - 2d(A,\eta)\big]\varphi(\eta) + 2\int_{\widehat{\partial A}} \varphi(\zeta)\big\langle \mathbf{n}^A(\zeta), \operatorname{grad} h_\eta(\zeta)\big\rangle \,\mathrm{d}\lambda_1(\zeta).$$

P r o o f. The assertion follows from Theorem 2.19 [see also formula (51) on p. 72 and formulas (1)–(3) on p. 73] in [10]. □

R e m a r k. The operator $T\colon \varphi \mapsto T\varphi$ is continuous on $\mathscr{C}(B)$.

Lemma 4. *If f is defined on $D \setminus A$ by (10) where $\varphi_C \in \mathscr{C}(C)$ and $\varphi_B \in \mathscr{C}(B)$ then*

$$(14)\qquad \lim_{\substack{z\to\eta\\ z\in D\setminus A}} \operatorname{Re} f(z) = 2\varphi_C(\eta) \qquad \text{for each } \eta \in C.$$

P r o o f. If $z \in D \setminus A$ approaches $\eta \in C$ then $g(z) \in \mathbb{C} \setminus \operatorname{cl} D$ approaches η too but through the complementary set $E = \mathbb{C} \setminus \operatorname{cl} D$. It follows from the theorem on the jump of the double layer potential (compare 2.19 in [10]) that

$$\begin{aligned}
&\lim_{\substack{z\to\eta\\ z\in D\setminus A}} \operatorname{Re} \frac{1}{\pi \mathrm{i}}\left[\int_C \frac{\varphi_C(\zeta)}{\zeta - z}\,\mathrm{d}\zeta + \overline{\int_C \frac{\varphi_C(\zeta)}{\zeta - g(z)}\,\mathrm{d}\zeta}\right]\\
&\quad = \lim_{\substack{z\to\eta\\ z\in D}} \frac{1}{\pi}\operatorname{Im}\int_C \frac{\varphi_C(\zeta)}{\zeta - z}\,\mathrm{d}\zeta - \lim_{\substack{w\to\eta\\ w\in E}} \frac{1}{\pi}\operatorname{Im}\int_C \frac{\varphi_C(\zeta)}{\zeta - w}\,\mathrm{d}\zeta = 2\varphi_C(\eta).
\end{aligned}$$

On the other hand, it follows from the continuity of $\mathcal{K}^A\varphi_B$ at $\eta = g(\eta) \in C \subset \mathbb{C} \setminus B$ that

$$\lim_{\substack{z\to\eta\\ z\in D}} \operatorname{Re}\mathcal{J}^A\varphi_B(z) = \operatorname{Re}\mathcal{K}^A\varphi_B(\eta) - \operatorname{Re}\overline{\mathcal{K}^A\varphi_B\big(g(\eta)\big)} = 0,$$

which yields (14). □

Lemma 5. *If f is defined on $D \setminus A$ by (10) and if (1), (2) are fulfilled for given $f_C \in \mathscr{C}(C)$, $f_B \in \mathscr{C}(B)$ then for $\eta \in B$*

$$(15)\qquad -\varphi_B(\eta) + T\varphi_B(\eta) + \operatorname{Im}\mathcal{K}^A\varphi_B\big(g(\eta)\big) = f_B(\eta) - h(\eta) + k,$$

where T is defined by (13) and

$$(16)\qquad h(\eta) = \operatorname{Im}\frac{1}{2\pi\mathrm{i}}\left[\int_C \frac{f_C(\zeta)}{\zeta - \eta}\,\mathrm{d}\zeta + \overline{\int_C \frac{f_C(\zeta)}{\zeta - g(\eta)}\,\mathrm{d}\zeta}\right]$$

for $\eta \in B$.

P r o o f. By (14) necessarily $\varphi_C = \frac{1}{2} f_C$; with this choice we have for $\eta \in B$

$$\lim_{\substack{z\to\eta\\ z\in D\setminus A}} \frac{1}{\pi \mathrm{i}}\left[\int_C \frac{\varphi_C(\zeta)}{\zeta - z}\,\mathrm{d}\zeta + \overline{\int_C \frac{\varphi_C(\zeta)}{\zeta - g(z)}\,\mathrm{d}\zeta}\right] = \frac{1}{2\pi \mathrm{i}}\left[\int_C \frac{f_C(\zeta)}{\zeta - \eta}\,\mathrm{d}\zeta + \overline{\int_C \frac{f_C(\zeta)}{\zeta - g(\eta)}\,\mathrm{d}\zeta}\right]$$

and by Lemma 3

$$\begin{aligned}\lim_{\substack{z\to\eta\\ z\in D\setminus A}} \operatorname{Im} \mathcal{J}^A\varphi_B(z) &= \lim_{\substack{z\to\eta\\ z\in D\setminus A}} \left[\operatorname{Im} \mathcal{K}^A\varphi_B(z) - \operatorname{Im} \overline{\mathcal{K}^A\varphi_B\big(g(z)\big)}\right]\\ &= -\varphi_B(\eta) + T\varphi_B(\eta) + \operatorname{Im}\mathcal{K}^A\varphi_B\big(g(\eta)\big).\end{aligned}$$

Now (15) follows from (2). $\square$

R e m a r k 2. Define an operator $H\colon \varphi \mapsto H\varphi$ acting on $\mathscr{C}(B)$ by setting for $\varphi \in \mathscr{C}(B)$

$$H\varphi(\eta) = \operatorname{Im}\mathcal{K}^A\varphi\big(g(\eta)\big), \qquad \eta \in B.$$

The operator H is bounded on $\mathscr{C}(B)$. As we have seen above a function f of the form (10) fulfils the conditions (1), (2) if and only if $\varphi_C = \frac{1}{2} f_C$ and the function $\varphi_B = \varphi$ is a solution of the equation

$$(I - T - H)\varphi = h - f_B - k\cdot \mathbf{1}_B, \tag{17}$$

where $h \in \mathscr{C}(B)$ is given by (16) and $\mathbf{1}_B$ denotes the constant function equal to one on B; here I is the identity operator.

In connection with equation (17) it is useful to evaluate the essential norm of the operator $\mathcal{U} = T + H$ which is defined by

$$\omega(\mathcal{U}) := \inf\big\{\, \|\mathcal{U} - \mathcal{Q}\| \;\big|\; \mathcal{Q} \in \mathfrak{M}\,\big\},$$

where $\mathfrak{M}$ is the space of all compact operators on $\mathscr{C}(B)$.

Lemma 6. $\omega(T + H) = \lim\limits_{r\downarrow 0}\big[\sup\limits_{\eta\in B} v_r(\eta)\big]$.

P r o o f. First note that H is a compact operator on $\mathscr{C}(B)$. In order to prove this it suffices to show that if $\mathscr{B}$ is the unit ball in $\mathscr{C}(B)$ then $H(\mathscr{B})$ is a set of uniformly bounded and equicontinuous functions (on B). For this purpose use the following expression of $H\varphi(\eta)$ for $\varphi \in \mathscr{C}(B)$, $\eta \in B$,

$$H\varphi(\eta) = \operatorname{Im}\mathcal{K}^A\varphi\big(g(\eta)\big) = \frac{1}{\pi}\int_{\widehat{\partial A}} \varphi(\zeta)\cdot \operatorname{Im}\left(\frac{\boldsymbol{\tau}^A(\zeta)}{\zeta - g(\eta)}\right)\,\mathrm{d}\lambda_1(\zeta)$$

mentioned above.

Put $\delta = \operatorname{dist}(B, g(B))$; clearly $\delta > 0$. Further let $q = \lambda_1(\widehat{\partial A})$; we have $q < \infty$ by assumption. For $\varphi \in \mathscr{B}$, $\eta \in B$ we obtain

$$\bigl|H\varphi(\eta)\bigr| = \frac{1}{\pi}\biggl|\int_{\widehat{\partial A}} \varphi(\zeta) \cdot \operatorname{Im}\left(\frac{\boldsymbol{\tau}^A(\zeta)}{\zeta - g(\eta)}\right) \,\mathrm{d}\lambda_1(\zeta)\biggr| \leqslant \frac{q}{\pi\delta},$$

which shows that $H(\mathscr{B})$ is a set of uniformly bounded functions.

For $\zeta \in B$, $\eta_1, \eta_2 \in B$ we have

$$\left|\frac{1}{\zeta - g(\eta_1)} - \frac{1}{\zeta - g(\eta_2)}\right| = \frac{\bigl|g(\eta_1) - g(\eta_2)\bigr|}{\bigl|(\zeta - g(\eta_1))(\zeta - g(\eta_2))\bigr|} \leqslant \frac{\bigl|g(\eta_1) - g(\eta_2)\bigr|}{\delta^2}.$$

Thus we get for $\eta_1, \eta_2 \in B$, $\varphi \in \mathscr{B}$

$$\begin{aligned}\bigl|H\varphi(\eta_1) - H\varphi(\eta_2)\bigr| &= \frac{1}{\pi}\biggl|\int_{\widehat{\partial A}} \varphi(\zeta) \cdot \operatorname{Im}\left[\boldsymbol{\tau}^A(\zeta)\left(\frac{1}{\zeta - g(\eta_1)} - \frac{1}{\zeta - g(\eta_2)}\right)\right] \,\mathrm{d}\lambda_1(\zeta)\biggr| \\ &\leqslant \frac{q}{\pi\delta^2}\bigl|g(\eta_1) - g(\eta_2)\bigr|.\end{aligned}$$

Since g is continuous we see that $H(\mathscr{B})$ is a set of equicontinuous functions, which completes the proof of compactness of H.

As H is compact we have

$$\omega(T + H) = \omega(T).$$

It follows from Theorem 4.1 in [10] that

$$\omega(T) = \lim_{r\downarrow 0}\, \sup_{\eta \in B} v_r(\eta).$$

The assertion is proved. □

In the following we will always suppose that

$$V_0^B := \lim_{r\downarrow 0}\, \sup_{\eta \in B} v_r(\eta) < 1. \tag{18}$$

It is shown in [11] that then each point $\eta \in B$ has a neighbourhood in B of the form $J_1 \cup \{\eta\} \cup J_2$ where J_1, J_2 are disjoint non-parametric Lipschitz open arcs with a common end-point η; further

$$V_0^B = \lim_{r\downarrow 0}\, \sup_{\eta \in B} \mathbb{V}_r(\eta),$$

where

$$\mathbb{V}_r(\eta) = \frac{1}{\pi}\int_{\partial B_1(0)} V_r(\theta, \eta) \,\mathrm{d}\lambda_1(\theta)$$

with $V_r(\theta,\eta)$ denoting the number of points of the set

$$B \cap H(\eta,\theta) \cap B_r(\eta)$$

[$\theta \mapsto V_r(\theta,\eta)$ is a Baire function on $\partial B_1(0)$].

If A is a continuum not separating D then $B = \partial A$ is a rectifiable Jordan curve as a consequence of (18) and A is the closure of its bounded complementary domain.

Proposition 3. *Let A be a simply connected continuum satisfying conditions specified above. Then for $\varphi \in \mathscr{C}(B)$*

$$\begin{gathered}(I-T-H)\varphi \in \operatorname{Const}(B) \iff \varphi \in \operatorname{Const}(B),\\ \varphi_0 \in \operatorname{Const}(B) \Longrightarrow (I-T-H)\varphi_0 = 0.\end{gathered}$$

Proof. If we define for $\varphi \in \mathscr{C}(B)$

$$f(z) = \frac{1}{\pi}\left[\int_B \frac{\varphi(\zeta)}{\zeta - z}\,\mathrm{d}\zeta - \overline{\int_B \frac{\varphi(\zeta)}{\zeta - g(z)}\,\mathrm{d}\zeta}\right], \qquad z \in D \setminus A,$$

then, as we have seen,

$$\eta \in B \Longrightarrow (I-T-H)\varphi(\eta) = -\lim_{\substack{z\to\eta\\ z\in D\setminus A}} \operatorname{Im} f(z).$$

Thus if $(I-T-H)\varphi = c\mathbf{1}_B$ $(c \in \mathbb{R})$ then

$$\begin{aligned}&\lim_{\substack{z\to\eta\\ z\in D\setminus A}} \operatorname{Re} f(z) = 0 \quad \text{for each } \eta \in C,\\ &\lim_{\substack{z\to\eta\\ z\in D\setminus A}} \operatorname{Im} f(z) = -c \quad \text{for each } \eta \in B.\end{aligned}$$

By Proposition 1 then

$$f(z) = -\mathrm{i}c \quad \text{for each } z \in D \setminus A,$$

whence

$$\operatorname{Re}\int_B \frac{\varphi(\zeta)}{\zeta - z}\,\mathrm{d}\zeta = \operatorname{Re}\int_B \frac{\varphi(\zeta)}{\zeta - g(z)}\,\mathrm{d}\zeta \quad \text{for each } z \in D \setminus A.$$

If $z \in D \setminus A$, $z \to \eta \in B$ then $g(z) \to g(\eta) \in \mathbb{R}^2 \setminus B$ and, consequently,

$$\lim_{\substack{z\to\eta\\ z\in D\setminus A}} \operatorname{Re}\int_B \frac{\varphi(\zeta)}{\zeta - z}\,\mathrm{d}\zeta = \operatorname{Re}\int_B \frac{\varphi(\zeta)}{\zeta - g(\eta)}\,\mathrm{d}\zeta.$$

The function

$$d\colon z \mapsto \operatorname{Re} \int_B \frac{\varphi(\zeta)}{\zeta - z}\, \mathrm{d}\zeta, \qquad z \in \mathbb{R}^2 \setminus A,$$

is harmonic on $\mathbb{R}^2 \setminus A$ and continuously extendable from $\mathbb{R}^2 \setminus A$ to $(\mathbb{R}^2 \setminus A) \cup B$. Let $\eta_0 \in B$ be such that

$$|d|(\eta_0) = \max_{\eta \in B} \bigl|d(\eta)\bigr|.$$

Since d has the zero limit at ∞ we have also

$$|d|(\eta_0) = \sup_{\eta \in (\mathbb{R}^2 \setminus A) \cup B} \bigl|d(\eta)\bigr|.$$

But we have $d(g(\eta_0)) = d(\eta_0)$ and since $g(\eta_0)$ is an interior point of $\mathbb{R}^2 \setminus A$ the function d is constant on $\mathbb{R}^2 \setminus A$ (compare [3]). Thus by [9] the integral $\int_B \frac{\varphi(\zeta)}{\zeta - z}\, \mathrm{d}\zeta$ is constant on $\mathbb{R}^2 \setminus A$ and also on the Jordan domain A°, and $\varphi \in \operatorname{Const}(B)$. But if $\varphi \in \operatorname{Const}(B)$ then the corresponding function f vanishes and

$$0 = - \lim_{\substack{z \to \eta \\ z \in D \setminus A}} \operatorname{Im} f(z) = (I - T - H)\varphi(\eta) \quad \text{for each } \eta \in B.$$

The assertion is proved. $\square$

Summarizing our investigation we arrive at the main result of this paper.

Theorem. *Let $D \subset \mathbb{C}$ be a bounded domain with the reflection mapping g defined on $U \supset \partial D = C$. Let $A \subset D$ be a simply connected massive continuum with $B = \partial A$ satisfying $V_0^B < 1$ [cf. (18)] such that $D \setminus U \subset A^\circ$. Then for any $f_C \in \mathscr{C}(C)$, $f_B \in \mathscr{C}(B)$ there are $k \in \mathbb{R}$ and $\varphi_C \in \mathscr{C}(C)$, $\varphi_B \in \mathscr{C}(B)$ such that the function*

$$f(z) = \frac{1}{\pi \mathrm{i}} \left[\int_C \frac{\varphi_C(\zeta)}{\zeta - z}\, \mathrm{d}\zeta + \overline{\int_C \frac{\varphi_C(\zeta)}{\zeta - g(z)}\, \mathrm{d}\zeta} \right] + \frac{1}{\pi} \left[\int_B \frac{\varphi_B(\zeta)}{\zeta - z}\, \mathrm{d}\zeta - \overline{\int_B \frac{\varphi_B(\zeta)}{\zeta - g(z)}\, \mathrm{d}\zeta} \right]$$

satisfies (1), (2); *$k \in \mathbb{R}$ and φ_C $(= \frac{1}{2} f_C) \in \mathscr{C}(C)$ are determined uniquely and $\varphi_B \in \mathscr{C}(B)$ is determined uniquely up to an additive constant from* $\operatorname{Const}(B)$.

P r o o f. Let us first remark that under our assumptions on A the condition $V_0^B < 1$ implies that B is a rectifiable Jordan curve satisfying (11) and $\lambda_1(B) < \infty$ guarantees (6) (cf. [11]). We see that all the requirements on A specified above are fulfilled.

We have seen that f of the given form is a solution of the problem if and only if $\varphi_C = \frac{1}{2} f_C$ and φ_B is a solution of the equation (17) where h is given by (16). Choose a linear mapping P which continuously projects $\mathscr{C}(B)$ onto $\operatorname{Const}(B)$ [for example a mapping of the form $P\varphi = \varphi(w_0)\mathbf{1}_B$, where $w_0 \in B$ is fixed]. Let us show that the equation

$$(I - T - H - P)\varphi = e \tag{19}$$

has for each $e \in \mathscr{C}(B)$ a unique solution $\varphi \in \mathscr{C}(B)$. According to the Fredholm alternative it suffices to verify that the only solution of the equation

$$(I - T - H - P)\varphi_0 = 0 \tag{20}$$

is the trivial solution $\varphi_0 = 0$. Indeed, if (20) is valid then $(I-T-H)\varphi_0 \in \mathrm{Const}(B)$, which by Proposition 3 implies that $\varphi_0 \in \mathrm{Const}(B)$ and $(I - T - H)\varphi_0 = 0$, whence $P\varphi_0 = \varphi_0 = 0$.

For the right hand side $e = h - f_B$ [where h is given by (16)] the equation (19) has a unique solution $\varphi = \varphi_B$; the function f corresponding to φ_B fulfils

$$\lim_{\substack{z\to\eta \\ z\in D\setminus A}} \operatorname{Im} f(z) = f_B(\eta) - P\varphi_B(\eta) \tag{21}$$

for each $\eta \in B$. Due to Proposition 3 the constant $P\varphi_B \in \mathrm{Const}(B)$ is determined uniquely and $\varphi_B \in \mathscr{C}(B)$ is determined uniquely up to a constant. Indeed, let P_1, P_2 be two such projections and let

$$(I - T - H - P_1)\varphi_1 = e, \qquad (I - T - H - P_2)\varphi_2 = e;$$

then

$$(I - T - H)(\varphi_1 - \varphi_2) = P_2\varphi_2 - P_1\varphi_1 \in \mathrm{Const}(B)$$

and by Proposition 3 we conclude that $\varphi_1 - \varphi_2 \in \mathrm{Const}(B)$ and $P_1\varphi_1 = P_2\varphi_2$. □

R e m a r k 3. The integral representation obtained in our theorem for the solution f of the mixed boundary value problem described in the introduction appears useful also from the point of view of numerical analysis. Under the same assumption (18) it is possible to propose a numerical method solving the mixed boundary value problem and to establish its convergence using methods from [23], [3]. These topics will be discussed in a forthcoming paper.

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