ON THE FROBENIUS NUMBER OF A MODULAR DIOPHANTINE INEQUALITY

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Abstract. We present an algorithm for computing the greatest integer that is not a solution of the modular Diophantine inequality $ax \bmod b \leqslant x$, with complexity similar to the complexity of the Euclid algorithm for computing the greatest common divisor of two integers.

Keywords: numerical semigroup, Diophantine inequality, Frobenius number, multiplicity

MSC 2000: 11D75, 20M14

1. Introduction

Given two integers m and n with $n \neq 0$, we denote by $m \bmod n$ the remainder of the division of m by n. Following the terminology used in [6], a proportionally modular Diophantine inequality is an expression of the form $ax \bmod b \leqslant cx$, where a, b and c are positive integers. The set $\mathrm{S}(a, b, c)$ of integer solutions of this inequality is a numerical semigroup, that is, it is a subset of $\mathbb{N}$ (here $\mathbb{N}$ denotes the set of nonnegative integers) that is closed under addition, contains the zero element and its complement in $\mathbb{N}$ is finite. We say that a numerical semigroup is proportionally modular if it is the set of integer solutions of a proportionally modular Diophantine inequality.

The integers a, b and c in the inequality $ax \bmod b \leqslant cx$ are, respectively, the factor, the modulus and the proportion of the inequality. Following the terminology used in [7], proportionally modular Diophantine inequalities with proportion 1, that

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is, such that $c = 1$, are simply called modular Diophantine inequalities. A numerical semigroup is modular if it is the set of integer solutions of a modular Diophantine inequality.

If S is a numerical semigroup, then the greatest integer that does not belong to S is an important invariant of S, called the Frobenius number of S (see [3]) and denoted here by $\mathrm{g}(S)$. Giving a formula for the Frobenius number of $\mathrm{S}(a, b, 1)$, as a function of a and b, is still an open problem. Some progress was made in [7] and [4]. In [2] an algorithm to determine the Frobenius number of $\mathrm{S}(a, b, c)$ is described. The aim of the present paper is to give an algorithm that computes the Frobenius number of $\mathrm{S}(a, b, 1)$, with complexity similar to the complexity of the Euclid algorithm for computing the greatest common divisor of two integers. This algorithm has considerably smaller complexity than the one presented in [2] in most of the cases.

2. Preliminaries

Given a nonempty subset A of $\mathbb{Q}_0^+$ (here $\mathbb{Q}_0^+$ is the set of nonnegative rational numbers), we will denote by $\langle A\rangle$ the submonoid of $(\mathbb{Q}_0^+, +)$ generated by A, that is, $\langle A\rangle = \{\lambda_1 a_1 + \ldots + \lambda_n a_n\,;\ n \in \mathbb{N}\setminus\{0\},\ \lambda_1, \ldots, \lambda_n \in \mathbb{N}$ and $a_1, \ldots, a_n \in A\}$. Clearly, $\langle A\rangle \cap \mathbb{N}$ is a submonoid of $(\mathbb{N}, +)$, represented here by $\mathrm{S}(A)$. We will refer to $\mathrm{S}(A)$ as the submonoid of $\mathbb{N}$ associated to A.

Let p and q be two positive rational numbers with $p < q$. We use the notation

$$[p, q] = \{x \in \mathbb{Q}\,;\ p \leqslant x \leqslant q\} \quad \text{and} \quad]p, q[\, = \{x \in \mathbb{Q}\,;\ p < x < q\}.$$

The following result is a reformulation of [6, Corollary 9].

Proposition 1.

(1) *Let a, b and c be positive integers such that $c < a < b$. Then* $\mathrm{S}([\frac{b}{a}, \frac{b}{a-c}]) = \mathrm{S}(a, b, c)$.

(2) *Conversely, if a_1, b_1, a_2 and b_2 are positive integers such that $\frac{b_1}{a_1} < \frac{b_2}{a_2}$, then* $\mathrm{S}([\frac{b_1}{a_1}, \frac{b_2}{a_2}]) = \mathrm{S}(a_1 b_2, b_1 b_2, a_1 b_2 - a_2 b_1)$.

Since the inequality $ax \bmod b \leqslant cx$ has the same solutions as the inequality $(a \bmod b)x \bmod b \leqslant cx$, we can assume that $a < b$. Moreover, if $c \geqslant a$, then $\mathrm{S}(a, b, c) = \mathbb{N}$. Therefore, we can suppose that a, b and c are positive integers such that $c < a < b$. Consequently, the condition imposed in (1) of the above proposition is not restrictive.

The next proposition is [8, Proposition 5].

Proposition 2. *If I is an interval of positive rational numbers (not necessarily closed), then* $\mathrm{S}(I)$ *is a proportionally modular numerical semigroup.*

As an immediate consequence of Propositions 1 and 2 we have the following result.

Proposition 3. *Let I be an interval of rational numbers greater than one. Then* $\mathrm{S}(I)$ *is a proportionally modular numerical semigroup. Moreover, every proportionally modular numerical semigroup not equal to* $\mathbb{N}$ *is of this form.*

The following lemma can be easily deduced from [8, Lemma 2] and will be used several times in this paper.

Lemma 4. *Let I be an interval of positive rational numbers and let x be a positive integer. Then $x \in \mathrm{S}(I)$ if and only if there exists a positive integer y such that $x/y \in I$.*

If S is a numerical semigroup, then the smallest positive integer that belongs to S is the multiplicity of S (see [1]) and it is denoted by $\mathrm{m}(S)$. If a_1, b_1, a_2 and b_2 are positive integers such that $\frac{b_1}{a_1} < \frac{b_2}{a_2}$, then [9, Algorithm 12] allows us to compute the multiplicity of $\mathrm{S}([\frac{b_1}{a_1}, \frac{b_2}{a_2}])$. In essence, this algorithm follows the steps of the Euclid algorithm for computing the greatest common divisor of two integers.

Note that, by Proposition 1, we have $\mathrm{S}(a,b,1) = \mathrm{S}([\frac{b}{a}, \frac{b}{a-1}])$. In Theorem 18 we will see that

$$\mathrm{g}\Big(\mathrm{S}\Big(\Big[\frac{b}{a}, \frac{b}{a-1}\Big]\Big)\Big) = b - \mathrm{m}\Big(\mathrm{S}\Big(\Big]\frac{b}{a}, \frac{b}{a-1}\Big[\Big)\Big)$$

and in Theorem 9 that

$$\mathrm{S}\Big(\Big]\frac{b}{a}, \frac{b}{a-1}\Big[\Big) = \mathrm{S}\Big(\Big[\frac{2b^2+1}{2ab}, \frac{2b^2-1}{2b(a-1)}\Big]\Big).$$

Therefore

$$\mathrm{g}\Big(\mathrm{S}\Big(\Big[\frac{b}{a}, \frac{b}{a-1}\Big]\Big)\Big) = b - \mathrm{m}\Big(\mathrm{S}\Big(\Big[\frac{2b^2+1}{2ab}, \frac{2b^2-1}{2b(a-1)}\Big]\Big)\Big)$$

and $\mathrm{m}\big(\mathrm{S}\big(\big[\frac{2b^2+1}{2ab}, \frac{2b^2-1}{2b(a-1)}\big]\big)\big)$ can be computed by applying [9, Algorithm 12].

3. A proportionally modular representation for an open modular numerical semigroup

If $x_1 < x_2 < \ldots < x_k$ are integers, then we use $\{x_1, x_2, \ldots, x_k, \rightarrow\}$ to denote the set $\{x_1, x_2, \ldots, x_k\} \cup \{z \in \mathbb{Z};\ z > x_k\}$. Following the terminology used in [8], a numerical semigroup S is a half-line if $S = \{0\} \cup \{\mathrm{m}(S), \rightarrow\}$, and it is open modular if either S is a half-line or there exist integers a and b such that $2 \leqslant a < b$ and $S = \mathrm{S}(]\frac{b}{a}, \frac{b}{a-1}[)$.

If $S = \mathrm{S}(]\frac{b}{a}, \frac{b}{a-1}[)$, then by Proposition 2 we know that S is a proportionally modular numerical semigroup and therefore it admits a proportionally modular representation, that is, there exist positive integers x, y and z such that $S = \mathrm{S}(x, y, z)$. Observe that, in view of Proposition 1, it suffices to find positive integers a_1, b_1, a_2 and b_2 such that $\mathrm{S}(]\frac{b}{a}, \frac{b}{a-1}[) = \mathrm{S}([\frac{b_1}{a_1}, \frac{b_2}{a_2}])$. Finding these positive integers is the fundamental aim of this section. To this end, we need some preliminary results and concepts.

If S is a numerical semigroup, then $\mathbb{N} \setminus S$ is finite. The elements of $\mathbb{N} \setminus S$ are the so called gaps of S. The cardinality of $\mathbb{N} \setminus S$ is known as the singularity degree of S (see [1]).

The Frobenius number and the singularity degree of an open modular numerical semigroup can be easily computed by using the following result.

Lemma 5 [8, Theorem 11]. *Let $2 \leqslant a < b$ be integers, $\alpha = \gcd\{a, b\}$ and $\beta = \gcd\{a-1, b\}$. Then $\mathrm{S}(]\frac{b}{a}, \frac{b}{a-1}[)$ is a proportionally modular numerical semigroup with Frobenius number b and singularity degree $\frac{1}{2}(b - 1 + \alpha + \beta)$.*

The next lemma is straightforward to prove.

Lemma 6. *Let $2 \leqslant a < b$ be integers. Then $b - 1 \in \mathrm{S}(]\frac{b}{a}, \frac{b}{a-1}[)$.*

Proof. A simple check shows that $\frac{b}{a} < \frac{b-1}{a-1} < \frac{b}{a-1}$. By applying Lemma 4 we have that $b - 1 \in \mathrm{S}(]\frac{b}{a}, \frac{b}{a-1}[)$. □

It is well-known (see for instance [5]) that every numerical semigroup S is finitely generated and therefore there exists a finite subset A of $\mathbb{N}$ such that $S = \langle A \rangle$. We say that A is a minimal system of generators of S if no proper subset of A generates S. It is also well-known (see [5]) that $S^* \setminus (S^* + S^*)$ is the unique minimal system of generators of S, with $S^* = S \setminus \{0\}$. The cardinality of the minimal system of generators of S is also an important invariant of S called the embedding dimension of S (see [1]).

From Lemmas 5 and 6 we deduce the following result which gives an upper bound to the minimal generators of $\mathrm{S}(]\frac{b}{a}, \frac{b}{a-1}[)$.

Lemma 7. *Let $2 \leqslant a < b$ be integers. Then every minimal generator of* $\mathrm{S}(]\frac{b}{a}, \frac{b}{a-1}[)$ *is smaller than $2b$.*

Proof. From Lemmas 5 and 6 we know that $\{b-1, b+1, \rightarrow\} \subseteq \mathrm{S}(]\frac{b}{a}, \frac{b}{a-1}[)$. We conclude the proof by pointing out that every positive integer greater than or equal to $2b$ belongs to $\{b-1, b+1, \rightarrow\} + \{b-1, b+1, \rightarrow\}$. $\square$

A simple check proves the next result.

Lemma 8. *Let $2 \leqslant a < b$ be integers. Then* $\frac{b}{a} < \frac{2b^2+1}{2ab} < \frac{2b^2-1}{2b(a-1)} < \frac{b}{a-1}$.

We are now ready to state the principal result of this section.

Theorem 9. *Let $2 \leqslant a < b$ be integers. Then* $\mathrm{S}(]\frac{b}{a}, \frac{b}{a-1}[) = \mathrm{S}([\frac{2b^2+1}{2ab}, \frac{2b^2-1}{2b(a-1)}])$.

Proof. From Lemma 8 we have that $[\frac{2b^2+1}{2ab}, \frac{2b^2-1}{2b(a-1)}] \subseteq]\frac{b}{a}, \frac{b}{a-1}[$ and so $\mathrm{S}([\frac{2b^2+1}{2ab}, \frac{2b^2-1}{2b(a-1)}]) \subseteq \mathrm{S}(]\frac{b}{a}, \frac{b}{a-1}[)$. To prove the other inclusion we only need to show that every minimal generator of $\mathrm{S}(]\frac{b}{a}, \frac{b}{a-1}[)$ belongs to $\mathrm{S}([\frac{2b^2+1}{2ab}, \frac{2b^2-1}{2b(a-1)}])$.

Let x be a minimal generator of $\mathrm{S}(]\frac{b}{a}, \frac{b}{a-1}[)$. Then by Lemma 4 there exists a positive integer y such that $\frac{b}{a} < \frac{x}{y} < \frac{b}{a-1}$. Moreover, by applying Lemma 7 we have that $x \leqslant 2b-1$ and, since $1 < \frac{b}{a} < \frac{x}{y}$, we deduce that $y < 2b-1$. Let us show that $\frac{2b^2+1}{2ab} \leqslant \frac{x}{y} \leqslant \frac{2b^2-1}{2b(a-1)}$. As $\frac{b}{a} < \frac{x}{y}$, we have $by < ax$ and so $ax - by \geqslant 1$. Hence $2abx - 2b^2y \geqslant 2b$. Since $y < 2b-1$, we infer that $2abx - 2b^2y \geqslant y$, and consequently $\frac{2b^2+1}{2ab} \leqslant \frac{x}{y}$. Arguing in a similar way with $\frac{x}{y} < \frac{b}{a-1}$, we get $2b^2y - 2b(a-1)x \geqslant y$, which is equivalent to $\frac{x}{y} \leqslant \frac{2b^2-1}{2b(a-1)}$. Finally, by applying Lemma 4 we obtain that $x \in \mathrm{S}([\frac{2b^2+1}{2ab}, \frac{2b^2-1}{2b(a-1)}])$. $\square$

As an immediate consequence of the previous theorem we have the following result.

Corollary 10. *Let $2 \leqslant a < b$ be integers and let α and β be rational numbers such that* $\frac{b}{a} < \alpha \leqslant \frac{2b^2+1}{2ab} < \frac{2b^2-1}{2b(a-1)} \leqslant \beta < \frac{b}{a-1}$. *Then* $\mathrm{S}([\alpha, \beta]) = \mathrm{S}(]\frac{b}{a}, \frac{b}{a-1}[)$.

From this we deduce the following.

Corollary 11. *Let $2 \leqslant a < b$ be integers. If k is an integer greater than or equal to $2b^2$, then* $\mathrm{S}(]\frac{b}{a}, \frac{b}{a-1}[) = \{x \in \mathbb{N};\ (ka-1)x \bmod kb \leqslant (k-2)x\}$.

Proof. A simple check shows that

$$\frac{b}{a} < \frac{kb}{ka-1} \leqslant \frac{2b^2+1}{2ab} < \frac{2b^2-1}{2b(a-1)} \leqslant \frac{kb}{k(a-1)+1} < \frac{b}{a-1}.$$

By applying Corollary 10, we have that $\mathrm{S}(]\frac{b}{a}, \frac{b}{a-1}[) = \mathrm{S}([\frac{kb}{ka-1}, \frac{kb}{k(a-1)+1}])$. We conclude the proof by using Proposition 1. $\square$

The next result is an immediate consequence of Lemma 5 and Corollary 11.

Corollary 12. *Let $2 \leqslant a < b$ be integers. Set $\alpha = \gcd\{a, b\}$, $\beta = \gcd\{a-1, b\}$, and let k be an integer greater than or equal to $2b^2$. Then the numerical semigroup $\mathrm{S}(ka-1, kb, k-2)$ has Frobenius number b and singularity degree $\frac{1}{2}(b-1+\alpha+\beta)$.*

4. An algorithm for computing the Frobenius number of a modular numerical semigroup

In this section, our first goal will be to prove Theorem 18, which establishes a relationship between the Frobenius number of $\mathrm{S}([\frac{b}{a}, \frac{b}{a-1}])$ and the multiplicity of $\mathrm{S}(]\frac{b}{a}, \frac{b}{a-1}[)$, for a and b integers such that $2 \leqslant a < b$. Before that, we need to recall and establish some results.

The next result is deduced from Proposition 1 and [7, Corollary 6].

Lemma 13. *Let $2 \leqslant a < b$ be integers. If $x \in \mathbb{N} \setminus \mathrm{S}([\frac{b}{a}, \frac{b}{a-1}])$, then $b - x \in \mathrm{S}([\frac{b}{a}, \frac{b}{a-1}])$.*

The following lemma follows from [7, Lemma 11] and describes the integers x for which both x and $b - x$ belong to $\mathrm{S}([\frac{b}{a}, \frac{b}{a-1}])$.

Lemma 14. *Let $2 \leqslant a < b$ be integers. Then $\{x, b-x\} \subseteq \mathrm{S}([\frac{b}{a}, \frac{b}{a-1}])$ if and only if*

$$x \in \Big\{0, \frac{b}{\alpha}, 2\frac{b}{\alpha}, \ldots, (\alpha-1)\frac{b}{\alpha}, \frac{b}{\beta}, 2\frac{b}{\beta}, \ldots, (\beta-1)\frac{b}{\beta}, b\Big\},$$

where $\alpha = \gcd\{a, b\}$ and $\beta = \gcd\{a-1, b\}$.

The next result gives an upper bound for the Frobenius number of $\mathrm{S}([\frac{b}{a}, \frac{b}{a-c}])$.

Lemma 15. *Let $1 \leqslant c < a < b$ be integers. Then the Frobenius number of $\mathrm{S}([\frac{b}{a}, \frac{b}{a-c}])$ is smaller than $b - 1$.*

Proof. By Proposition 1 we know that $\mathrm{S}([\frac{b}{a}, \frac{b}{a-c}]) = \{x \in \mathbb{N};\ ax \bmod b \leqslant cx\}$. Note that, if $x \geqslant b-1$, then $ax \bmod b \leqslant b - 1 \leqslant c(b-1) \leqslant cx$ and therefore $x \in \mathrm{S}([\frac{b}{a}, \frac{b}{a-c}])$. $\square$

Next we discard some values for the multiplicity of $\mathrm{S}(]\frac{b}{a}, \frac{b}{a-1}[)$.

Lemma 16. *Let* $2 \leqslant a < b$ *be integers,* $\alpha = \gcd\{a, b\}$, $\beta = \gcd\{a-1, b\}$ *and* $S' = \mathrm{S}(]\frac{b}{a}, \frac{b}{a-1}[)$. *Then* $\mathrm{m}(S') \notin \{0, \frac{b}{\alpha}, 2\frac{b}{\alpha}, \ldots, (\alpha-1)\frac{b}{\alpha}, \frac{b}{\beta}, 2\frac{b}{\beta}, \ldots, (\beta-1)\frac{b}{\beta}, b\}$.

Proof. By Lemma 4 there exists a positive integer y such that $\frac{b}{a} < \frac{\mathrm{m}(S')}{y} < \frac{b}{a-1}$. Let us assume that $\mathrm{m}(S') = k\frac{b}{\alpha}$ with $k \in \{1, \ldots, \alpha\}$. Then $\frac{b}{a} = \frac{kb/\alpha}{ka/\alpha} = \frac{\mathrm{m}(S')}{ka/\alpha} < \frac{\mathrm{m}(S')}{y} < \frac{b}{a-1}$. Hence $\mathrm{S}(]\frac{\mathrm{m}(S')}{ka/\alpha}, \frac{\mathrm{m}(S')}{ka/\alpha-1}[) \subseteq S'$. In view of Lemma 5 we have that $\mathrm{g}(\mathrm{S}(]\frac{\mathrm{m}(S')}{ka/\alpha}, \frac{\mathrm{m}(S')}{ka/\alpha-1}[)) = \mathrm{m}(S')$ and also $\mathrm{g}(S') = b$. So we deduce that $b \leqslant \mathrm{m}(S')$. From Lemma 6 we know that $\mathrm{m}(S') \leqslant b-1$, which is not possible. Similarly we can prove that $\mathrm{m}(S') \neq k\frac{b}{\beta}$ for $k \in \{1, \ldots, \beta\}$. □

Now we study which elements of $\mathrm{S}([\frac{b}{a}, \frac{b}{a-1}])$ belong to $\mathrm{S}(]\frac{b}{a}, \frac{b}{a-1}[)$.

Lemma 17. *Let* $2 \leqslant a < b$ *be integers,* $\alpha = \gcd\{a, b\}$ *and* $\beta = \gcd\{a-1, b\}$. *If* $x \in \mathrm{S}([\frac{b}{a}, \frac{b}{a-1}]) \setminus \{0, \frac{b}{\alpha}, 2\frac{b}{\alpha}, \ldots, (\alpha-1)\frac{b}{\alpha}, \frac{b}{\beta}, 2\frac{b}{\beta}, \ldots, (\beta-1)\frac{b}{\beta}, b\}$, *then* $x \in \mathrm{S}(]\frac{b}{a}, \frac{b}{a-1}[)$.

Proof. Since $x \in \mathrm{S}([\frac{b}{a}, \frac{b}{a-1}])$, by Lemma 4 there exists a positive integer y such that $\frac{b}{a} \leqslant \frac{x}{y} \leqslant \frac{b}{a-1}$. If $\frac{x}{y} = \frac{b}{a}$, then $x = k\frac{b}{\alpha}$ for some positive integer k. Suppose that $k \geqslant \alpha + 1$. Let us prove that $k\frac{b}{\alpha} \in \mathrm{S}(]\frac{b}{a}, \frac{b}{a-1}[)$. To this end, in view of Lemma 4, it suffices to see that $\frac{b}{a} < \frac{kb/\alpha}{ka/\alpha-1} < \frac{b}{a-1}$. But a simple check shows that these inequalities hold. The case $\frac{x}{y} = \frac{b}{a-1}$ is analogous to the previous one. □

We are now ready to state the theorem announced at the beginning of this section.

Theorem 18. *Let* $2 \leqslant a < b$ *be integers. Define* $S = \mathrm{S}([\frac{b}{a}, \frac{b}{a-1}])$ *and* $S' = \mathrm{S}(]\frac{b}{a}, \frac{b}{a-1}[)$. *Then* $\mathrm{g}(S) = b - \mathrm{m}(S')$.

Proof. From Lemmas 14 and 16 we deduce that $b - \mathrm{m}(S') \notin S$. By Lemma 17 we obtain that, if $x \in \{1, \ldots, \mathrm{m}(S') - 1\}$, then either $x \notin S$ or $x \in \{0, \frac{b}{\alpha}, 2\frac{b}{\alpha}, \ldots, (\alpha-1)\frac{b}{\alpha}, \frac{b}{\beta}, 2\frac{b}{\beta}, \ldots, (\beta-1)\frac{b}{\beta}, b\}$. Hence, by Lemmas 13 and 14 we have that $\{b-1, b-2, \ldots, b - (\mathrm{m}(S') - 1)\} \subseteq S$. Moreover, Lemma 15 asserts that $\{b-1, \rightarrow\} \subseteq S$. Therefore $\mathrm{g}(S) = b - \mathrm{m}(S')$. □

Now, we present an algorithm that allows us to compute the Frobenius number of $\mathrm{S}([\frac{b}{a}, \frac{b}{a-1}])$ for a and b integers such that $2 \leqslant a < b$. In view of Proposition 1, we have $\mathrm{S}([\frac{b}{a}, \frac{b}{a-1}]) = \mathrm{S}(a, b, 1)$. Therefore, this algorithm computes the Frobenius number of a modular numerical semigroup.

In [9] we gave an algorithm for computing the multiplicity of a proportionally modular numerical semigroup defined by a closed interval. Thus the idea is to combine this algorithm with Theorems 9 and 18.

Algorithm 19. Input: a and b integers such that $2 \leqslant a < b$.
Output: The Frobenius number of $\mathrm{S}([\frac{b}{a}, \frac{b}{a-1}])$.

(1) Compute the multiplicity m of $\mathrm{S}([\frac{2b^2+1}{2ab}, \frac{2b^2-1}{2b(a-1)}])$ by using [9, Algorithm 12].
(2) Return $b - \mathrm{m}$.

Next we briefly recall [9, Algorithm 12]. In order to do this, we need to introduce some concepts.

Let a_1, b_1, a_2 and b_2 be positive integers. Define

$$R\Big(\Big[\frac{b_1}{a_1}, \frac{b_2}{a_2}\Big]\Big) = \Big[\frac{a_2}{b_2 \bmod a_2}, \frac{a_1}{b_1 \bmod a_1}\Big].$$

Given a closed interval I of positive rational numbers we can construct recursively the following sequence of closed intervals:

$I_1 = I$,

$I_{n+1} = R(I_n)$, if I_n contains no integers, and $I_{n+1} = I_n$, otherwise.

We will refer to $\{I_n\}_{n\in\mathbb{N}\setminus\{0\}}$ as the sequence of intervals associated with I.

Given a rational number q we denote by $\lfloor q \rfloor$ the integer $\max\{z \in \mathbb{Z};\ z \leqslant q\}$ and by $\lceil q \rceil$ the integer $\min\{z \in \mathbb{Z};\ q \leqslant z\}$. Let I be a closed interval. If I does not contain an integer, then $\lfloor x \rfloor = \lfloor y \rfloor$ for every $x, y \in I$. This integer is denoted by $\lfloor I \rfloor$. We are now ready to recall [9, Algorithm 12].

Algorithm 20. Input: I a closed interval of positive rational numbers such that $\mathrm{S}(I) \neq \mathbb{N}$.

Output: The multiplicity of the semigroup $\mathrm{S}(I)$.

(1) Compute the sequence of intervals associated to I until we find the first interval of the sequence that contains an integer. Let us denote such intervals by $I_1, I_2, \ldots, I_l$.
(2) If $I_l = [\alpha, \beta]$, then $P(I_l) = \lceil \alpha \rceil / 1$.
(3) Calculate $P(I_1)$ by applying successively $P(I_{n-1}) = P(I_n)^{-1} + \lfloor I_{n-1} \rfloor$.
(4) The multiplicity of $\mathrm{S}(I)$ is the numerator of $P(I_1)$.

We end this section with an example that illustrates Algorithm 19.

Example 21. Let us compute the Frobenius number of the modular numerical semigroup $\mathrm{S}(17, 108, 1)$. By Proposition 1, we have $\mathrm{S}(17, 108, 1) = \mathrm{S}([\frac{108}{17}, \frac{108}{16}])$.

(1) (a)
$$I_1 = \Big[\frac{23329}{3672}, \frac{23327}{3456}\Big], \quad I_2 = \Big[\frac{3456}{2591}, \frac{3672}{1297}\Big].$$
Note that $2 \in I_2$.
  (b) $P(I_2) = \frac{2}{1}$.
  (c) $P(I_1) = \frac{1}{2} + 6 = \frac{13}{2}$.
  (d) The multiplicity of $\mathrm{S}([\frac{23329}{3672}, \frac{23327}{3456}])$ is 13.
(2) The Frobenius number of $\mathrm{S}([\frac{108}{17}, \frac{108}{16}])$ is $108 - 13 = 95$.

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