LINEAR STIELTJES INTEGRAL EQUATIONS IN BANACH SPACES II; OPERATOR VALUED SOLUTIONS

Štefan Schwabik, Praha

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Abstract. This paper is a continuation of [9]. In [9] results concerning equations of the form

$$x(t) = x(a) + \int_a^t \mathrm{d}[A(s)]x(s) + f(t) - f(a)$$

were presented. The Kurzweil type Stieltjes integration in the setting of [6] for Banach space valued functions was used.

Here we consider operator valued solutions of the homogeneous problem

$$\Phi(t) = I + \int_d^t \mathrm{d}[A(s)]\Phi(s)$$

as well as the variation-of-constants formula for the former equation.

Keywords: linear Stieltjes integral equations, generalized linear differential equation, equation in Banach space

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Assume that X is a Banach space and that $L(X)$ is the Banach space of all bounded linear operators $A\colon X \to X$ with the uniform operator topology. Defining the bilinear form $B\colon L(X) \times X \to X$ by $B(A, x) = Ax \in X$ for $A \in L(X)$ and $x \in X$, we obtain in a natural way the bilinear triple $\mathcal{B} = (L(X), X, X)$ (see [6]) because using the usual operator norm we have

$$\|B(A, x)\|_X \leqslant \|A\|_{L(X)} \|x\|_X.$$

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Similarly, if we define the bilinear form $B^*\colon L(X)\times L(X)\to L(X)$ by the relation $B^*(A,C)=AC\in L(X)$ for $A,C\in L(X)$ where AC is the composition of the linear operators A and C we get the bilinear triple $\mathcal{B}^*=(L(X),L(X),L(X))$ because we have

$$\|B^*(A,C)\|_{L(X)}\leqslant\|AC\|_{L(X)}\leqslant\|A\|_{L(X)}\|C\|_{L(X)}.$$

Assume that $[a,b]\subset\mathbb{R}$ is a bounded interval.

Given $A\colon [a,b]\to L(X)$, the function A is of *bounded variation on* $[a,b]$ if

$$\operatorname*{var}_{[a,b]}(A)=\sup\Big\{\sum_{j=1}^{k}\|A(\alpha_j)-A(\alpha_{j-1})\|_{L(X)}\Big\}<\infty,$$

where the supremum is taken over all finite partitions

$$D\colon\ a=\alpha_0<\alpha_1<\ldots<\alpha_{k-1}<\alpha_k=b$$

of the interval $[a,b]$. The set of all functions $A\colon [a,b]\to L(X)$ with $\operatorname*{var}_{[a,b]}(A)<\infty$ will be denoted by $BV([a,b];L(X))$.

For $A\colon [a,b]\to L(X)$ and a partition D of the interval $[a,b]$ define

$$V_a^b(A,D)=\sup\Big\{\Big\|\sum_{j=1}^{k}[A(\alpha_j)-A(\alpha_{j-1})]y_j\Big\|_X\Big\},$$

where the supremum is taken over all possible choices of $y_j\in X, j=1,\ldots,k$ with $\|y_j\|\leqslant 1$ and similarly

$$\overset{*}{V}{}_a^b(A,D)=\sup\Big\{\Big\|\sum_{j=1}^{k}[A(\alpha_j)-A(\alpha_{j-1})]C_j\Big\|_{L(X)}\Big\},$$

where the supremum is taken over all possible choices of $C_j\in L(X), j=1,\ldots,k$ with $\|C_j\|_{L(X)}\leqslant 1$.

Define

$$(\mathcal{B})\operatorname*{var}_{[a,b]}(A)=\sup V_a^b(A,D)$$

and

$$(\mathcal{B}^*)\operatorname*{var}_{[a,b]}(A)=\sup\overset{*}{V}{}_a^b(A,D)$$

where the supremum is taken over all finite partitions

$$D\colon\ a=\alpha_0<\alpha_1<\ldots<\alpha_{k-1}<\alpha_k=b$$

of the interval $[a,b]$.

The function $A\colon [a,b]\to L(X)$ with $(\mathcal{B})\operatorname*{var}_{[a,b]}(A)<\infty$ is called a *function with bounded $\mathcal{B}$-variation on* $[a,b]$ and similarly if $(\mathcal{B}^*)\operatorname*{var}_{[a,b]}(A)<\infty$ then A is of *bounded $\mathcal{B}^*$-variation on* $[a,b]$ ([3]).

We denote by $(\mathcal{B})BV([a,b];L(X))$ the set of all functions $A\colon [a,b]\to L(X)$ with $(\mathcal{B})\operatorname*{var}_{[a,b]}(A)<\infty$ and by $(\mathcal{B}^*)BV([a,b];L(X))$ the set of all functions $A\colon [a,b]\to L(X)$ with $(\mathcal{B}^*)\operatorname*{var}_{[a,b]}(A)<\infty$.

In [9, Prop. 1.1 and 1.2] it is shown that

$$BV([a,b];L(X))\subset(\mathcal{B})BV([a,b];L(X))=(\mathcal{B}^*)BV([a,b];L(X))$$

holds.

Given $x\colon [a,b]\to X$, the function x is called *regulated on* $[a,b]$ if it has one-sided limits at every point of $[a,b]$, i.e. if for every $s\in[a,b)$ there is a value $x(s+)\in X$ such that

$$\lim_{t\to s+}\|x(t)-x(s+)\|_X=0$$

and if for every $s\in(a,b]$ there is a value $x(s-)\in X$ such that

$$\lim_{t\to s-}\|x(t)-x(s-)\|_X=0.$$

The set of all regulated functions $x\colon [a,b]\to X$ will be denoted by $G([a,b];X)$ and similarly we denote the set of all regulated functions $A\colon [a,b]\to L(X)$ by $G([a,b];L(X))$.

If $\mathcal{B}=(L(X),X,X)$ is the bilinear triple of Banach spaces mentioned above then a function $A\colon [a,b]\to L(X)$ is called *$\mathcal{B}$-regulated on* $[a,b]$ if for every $y\in X$, $\|y\|_X\leqslant 1$, the function $Ay\colon [a,b]\to X$ given by $t\in[a,b]\mapsto A(t)y\in X$ for $t\in[a,b]$ is regulated, i.e. $Ay\in G([a,b];X)$ for every $y\in X$, $\|y\|_X\leqslant 1$.

We denote by $(\mathcal{B})G([a,b];L(X))$ the set of all $\mathcal{B}$-regulated functions $A\colon [a,b]\to L(X)$.

1. Equations with operator valued solutions

For $[a,b]=[0,1]$ we denote shortly

$$BV(L(X))=BV([0,1];L(X)),(\mathcal{B})BV(L(X))=(\mathcal{B})BV([0,1];L(X)),$$

$$G(L(X))=G([0,1];L(X))\text{ and }(\mathcal{B})G(L(X))=(\mathcal{B})G([0,1];L(X)).$$

Assume that $A\colon [0,1]\to L(X)$ satisfies

$$A\in(\mathcal{B})BV(L(X))\cap(\mathcal{B})G(L(X)) \tag{1.1}$$

and the following condition (E) (see [9]):

for every $d\in[0,1]$ there are $0<\varrho=\varrho(d)<1$ and $\Delta=\Delta(d)>0$ such that

$$(\mathcal{B})\operatorname*{var}_{(d,d+\Delta]\cap[0,1]}(A)<\varrho \tag{E+}$$

and

$$(\mathcal{B})\operatorname*{var}_{[d-\Delta,d)\cap[0,1]}(A)<\varrho. \tag{E–}$$

Taking the bilinear triple $\mathcal{B}^*=(L(X),L(X),L(X))$, by Proposition 1.1 in [9] we have

$$(\mathcal{B})BV(L(X))=(\mathcal{B}^*)BV(L(X))$$

and

$$(\mathcal{B})\operatorname*{var}_{[a,b]}(A)=(\mathcal{B}^*)\operatorname*{var}_{[a,b]}(A)$$

for every $[a,b]\subset[0,1]$. Therefore condition (1.1) reads

$$A\in(\mathcal{B}^*)BV(L(X))\cap(\mathcal{B})G(L(X)), \tag{1.1}$$

and in condition (E) the symbol $\mathcal{B}$ can also be replaced by $\mathcal{B}^*$, i.e. condition (E) reads

for every $d\in[0,1]$ there are $0<\varrho=\varrho(d)<1$ and $\Delta=\Delta(d)>0$ such that

$$(\mathcal{B}^*)\operatorname*{var}_{(d,d+\Delta]\cap[0,1]}(A)<\varrho \tag{E+}$$

and

$$(\mathcal{B}^*)\operatorname*{var}_{[d-\Delta,d)\cap[0,1]}(A)<\varrho. \tag{E–}$$

Hence the results presented in Section 2 from [9] can be used for equations of the form

$$Y(t)=\widetilde{Y}+\int_d^t \mathrm{d}[A(s)]Y(s)+F(t)-F(d) \tag{1.2}$$

for every $t\in[0,1]$ where $F\in G(L(X))$, $d\in[0,1]$ and $\widetilde{Y}\in L(X)$.

The operator valued function $Y\colon [\alpha,\beta]\to L(X)$ is called a solution of (1.2) on an interval $[\alpha,\beta]\subset[0,1]$ if Y satisfies (1.2) for every $t\in[\alpha,\beta]$. If $d\in[\alpha,\beta]$ then of course we have $Y(d)=\widetilde{Y}$ for this solution.

With regard to the above mentioned facts we obtain by a simple reformulation of Proposition 2.4 and Theorem 2.10 from [9] the following

1.1. Theorem. *Assume that $A\colon [0,1]\to L(X)$ satisfies (1.1) and condition (E). Then for every $d\in[0,1]$, $\widetilde{Y}\in X$, $F\in G(L(X))$ there is a $\Delta>0$ such that for the interval $J_d=[d-\Delta,d+\Delta]\cap[0,1]$ there is a unique function $Y\in G(J_d;L(X))$ such that*

$$Y(t)=\widetilde{Y}+\int_d^t \mathrm{d}[A(s)]Y(s)+F(t)-F(d),\ t\in J_d,$$

i.e. $Y(t)$ is a local solution of the operator valued equation (1.2) on $J_d=[d-\Delta,d+\Delta]\cap[0,1]$.

If

$$A\in(\mathcal{B})BV(L(X))\cap G(L(X)), \tag{1.3}$$

condition (U):

$$[I+\Delta^+A(t)]^{-1}\in L(X) \text{ exists for every } t\in[0,1) \tag{U+}$$

and

$$[I-\Delta^-A(t)]^{-1}\in L(X) \text{ exists for every } t\in(0,1] \tag{U–}$$

and (E) hold, then for every choice of $d\in[0,1]$, $\widetilde{Y}\in L(X)$, $F\in G([0,1];L(X))$ there exists a unique $Y\in G([0,1];X)$ which is a (global) solution of (1.2) on $[0,1]$.

Let us consider the special case of the equation (1.2) with F a constant, i.e. the so called homogeneous equation

$$Y(t)=\widetilde{Y}+\int_d^t \mathrm{d}[A(s)]Y(s). \tag{1.4}$$

Theorem 1.1 applies to this equation and therefore there is a unique (global) solution to this equation and this operator valued solution is regulated provided $A\colon [0,1]\to L(X)$ satisfies (1.3), (E) and (U).

Together with (1.4) let us consider the equation

$$\Phi(t)=I+\int_d^t \mathrm{d}[A(s)]\Phi(s) \tag{1.5}$$

where $I \in L(X)$ is the identity operator.

Clearly every solution $Y\colon [0,1] \to L(X)$ of (1.4) can be written in the form

$$Y(t) = \Phi(t)\widetilde{Y}, \quad t \in [0,1].$$

Let us now consider the properties of the solution $\Phi\colon [0,1] \to L(X)$ of (1.5).

1.2. Lemma. *Assume that $A\colon [0,1] \to L(X)$ satisfies* (1.3), (E) *and* (U). *Then for the solution $\Phi\colon [0,1] \to L(X)$ of* (1.5) *we have*

$$\Phi \in (\mathcal{B})BV(L(X)) \cap G(L(X))$$

and there is a constant $K > 0$ such that $\|\Phi(t)\| \leqslant K$ for every $t \in [0,1]$.

Proof. By Theorem 1.1 $\Phi \in G([0,1]; L(X))$ and therefore there exists a $K > 0$ such that $\|\Phi(t)\| \leqslant K$ for every $t \in [0,1]$. It remains to show that $\Phi \in (\mathcal{B})BV([0,1]; L(X))$.

Assume that

$$D\colon\ 0 = \alpha_0 < \alpha_1 < \ldots < \alpha_{k-1} < \alpha_k = 1$$

is an arbitrary partition of the interval $[0,1]$.

For any $y_j \in X, j = 1, \ldots, k$ with $\|y_j\| \leqslant 1$ we have

$$\Big\| \sum_{j=1}^{k} [\Phi(\alpha_j) - \Phi(\alpha_{j-1})] y_j \Big\|_X = \Big\| \sum_{j=1}^{k} \int_{\alpha_{j-1}}^{\alpha_j} \mathrm{d}[A(s)]\Phi(s) y_j \Big\|_X.$$

Define

$$\varphi(s) = \Phi(s) y_j \text{ for } s \in (\alpha_{j-1}, \alpha_j) \text{ and } \varphi(s) = 0 \text{ for } s = \alpha_j.$$

Evidently $\|\varphi(s)\| \leqslant K$.

Then by 1.18 from [9] we get

$$\begin{aligned}\int_{\alpha_{j-1}}^{\alpha_j} \mathrm{d}[A(s)]\Phi(s) y_j &= \int_{\alpha_{j-1}}^{\alpha_j} \mathrm{d}[A(s)]\varphi(s) \\ &\quad + [A(\alpha_{j-1}+) - A(\alpha_{j-1})]\Phi(\alpha_{j-1}) y_j + [A(\alpha_j) - A(\alpha_j-)]\Phi(\alpha_j) y_j\end{aligned}$$

and

$$\Big\|\sum_{j=1}^{k}\int_{\alpha_{j-1}}^{\alpha_j}\mathrm{d}[A(s)\Phi(s)y_j\Big\|_X=\Big\|\sum_{j=1}^{k}\int_{\alpha_{j-1}}^{\alpha_j}\mathrm{d}[A(s)]\varphi(s)$$

$$+[A(\alpha_{j-1}+)-A(\alpha_{j-1})]\Phi(\alpha_{j-1})y_j+[A(\alpha_j)-A(\alpha_j-)]\Phi(\alpha_j)y_j\Big\|_X$$

$$=\Big\|\int_0^1\mathrm{d}[A(s)]\varphi(s)+\sum_{j=1}^{k}[A(\alpha_{j-1}+)-A(\alpha_{j-1})]\Phi(\alpha_{j-1})y_j$$

$$+\sum_{j=1}^{k}[A(\alpha_j)-A(\alpha_j-)]\Phi(\alpha_j)y_j\Big\|_X\leqslant\Big\|\int_0^1\mathrm{d}[A(s)]\varphi(s)\Big\|_X$$

$$+\Big\|\sum_{j=1}^{k}[A(\alpha_{j-1}+)-A(\alpha_{j-1})]\Phi(\alpha_{j-1})y_j\Big\|_X+\Big\|\sum_{j=1}^{k}[A(\alpha_j)-A(\alpha_j-)]\Phi(\alpha_j)y_j\Big\|_X.$$

For a given $\eta>0$ let us choose a $\theta>0$ such that

$$\|A(\alpha_{j-1}+\theta)-A(\alpha_{j-1}+)\|_{L(X)}<\frac{\eta}{k+1}$$

and

$$\|A(\alpha_j-\theta)-A(\alpha_j-)\|_{L(X)}<\frac{\eta}{k+1}$$

for all $j=1,\dots,k$. Then

$$\Big\|\sum_{j=1}^{k}[A(\alpha_{j-1}+)-A(\alpha_{j-1})]\Phi(\alpha_{j-1})y_j\Big\|_X$$

$$=\Big\|\sum_{j=1}^{k}[A(\alpha_{j-1}+)-A(\alpha_{j-1}+\theta)+A(\alpha_{j-1}+\theta)-A(\alpha_{j-1})]\Phi(\alpha_{j-1})y_j\Big\|_X$$

$$\leqslant\Big\|\sum_{j=1}^{k}[A(\alpha_{j-1}+)-A(\alpha_{j-1}+\theta)]\Phi(\alpha_{j-1})y_j\Big\|_X$$

$$+\Big\|\sum_{j=1}^{k}[A(\alpha_{j-1}+\theta)-A(\alpha_{j-1})]\Phi(\alpha_{j-1})y_j\Big\|_X$$

$$<\sum_{j=1}^{k}\frac{K\eta}{k+1}+\Big\|\sum_{j=1}^{k}[A(\alpha_{j-1}+\theta)-A(\alpha_{j-1})]\Phi(\alpha_{j-1})y_j\Big\|_X$$

$$<K\eta+K(\mathcal{B})\operatorname*{var}_{[0,1]}(A)$$

and similarly also

$$\Big\|\sum_{j=1}^{k}[A(\alpha_j)-A(\alpha_j-)]\Phi(\alpha_j)y_j\Big\|_X<K\eta+K(\mathcal{B})\operatorname*{var}_{[0,1]}(A).$$

By 1.11 from [9] we have further

$$\Big\| \int_0^1 \mathrm{d}[A(s)]\varphi(s) \Big\|_X \leqslant K(\mathcal{B}) \operatorname*{var}_{[0,1]}(A)$$

and finally we obtain

$$\Big\| \sum_{j=1}^{k} [\Phi(\alpha_j) - \Phi(\alpha_{j-1}) y_j \Big\|_X = \Big\| \sum_{j=1}^{k} \int_{\alpha_{j-1}}^{\alpha_j} \mathrm{d}[A(s)]\Phi(s) y_j \Big\|_X < 2K\eta + 3K(\mathcal{B}) \operatorname*{var}_{[0,1]}(A).$$

Passing to the corresponding suprema we arrive easily at

$$(\mathcal{B}) \operatorname*{var}_{[0,1]}(\Phi) \leqslant 3K(\mathcal{B}) \operatorname*{var}_{[0,1]}(A) < \infty,$$

i.e. $\Phi \in (\mathcal{B})BV([0,1]; L(X))$. □

1.3. Lemma. *Assume that* $A\colon [0,1] \to L(X)$ *satisfies* (1.3), (E) *and* (U).
Then the solution $\Phi\colon [0,1] \to L(X)$ *of* (1.5) *has an inverse* $[\Phi(t)]^{-1} \in L(X)$ *for every* $t \in [0,1]$.

P r o o f. For $t = d$ we have $\Phi(t) = \Phi(d) = I$ and the inverse $[\Phi(t)]^{-1}$ evidently exists for this value.

Assume that there is a point $t^* \in [0,1]$ such that the inverse $[\Phi(t^*)]^{-1}$ does not exist. Then there exists $y \in X$ such that the equation

$$\Phi(t^*) z = y$$

has no solution in X. Assume that $\Psi\colon [0,1] \to L(X)$ is a solution of the operator valued equation

$$\Psi(t) = I + \int_{t^*}^{t} \mathrm{d}[A(s)]\Psi(s);$$

this solution exists and is uniquely determined by the second part of Theorem 1.1. Let us set $z = \Psi(d)y$. The function $x\colon [0,1] \to X$ given by $x(t) = \Psi(t)y$ is a solution of the equation

$$x(t) = y + \int_{t^*}^{t} \mathrm{d}[A(s)]x(s)$$

with $x(t^*) = y$ and $x(d) = \Psi(d)y$. On the other hand, $\varphi(t) = \Phi(t)z$ is a solution of

$$\varphi(t) = z + \int_{d}^{t} \mathrm{d}[A(s)]\varphi(s)$$

where $\varphi(d) = z = \Psi(d)y = x(d)$ and

$$x(t) = x(d) + \int_d^t \mathrm{d}[A(s)]x(s).$$

Hence by the uniqueness of a solution stated in Theorem 2.10 from [9] we have $x(t) = \varphi(t)$ for all $t \in [0,1]$. Therefore

$$x(t^*) = y = \varphi(t^*) = \Phi(t^*)z = \Phi(t^*)\Psi(d)y,$$

i.e. $z = \Psi(d)y \in X$ is a solution of the equation $\Phi(t^*)z = y$. This contradicts the assumption and proves that the operator $\Phi(t) \in L(X)$ has an inverse for every $t \in [0,1]$. □

1.4. Lemma. *Assume that* $A\colon [0,1] \to L(X)$ *satisfies* (1.3), (E) *and* (U).

Then the inverse $[\Phi(t)]^{-1} = \Phi^{-1}(t)$ *to the solution* $\Phi\colon [0,1] \to L(X)$ *of* (1.5) *belongs to* $G(L(X))$ *and there is a constant* $L > 0$ *such that*

$$\|\Phi^{-1}(t)\|_{L(X)} \leqslant L$$

for every $t \in [0,1]$.

Proof. By Theorem 1.1 we have $\Phi \in G(L(X))$ and therefore the onesided limits of this function exist at every point of $[0,1]$. E. g., the limit $\lim\limits_{r\to t+} \Phi(r)$ exists for every $t \in [0,1)$ and by 1.18 from [9] we have

$$\begin{aligned}\lim_{r\to t+} \Phi(r) = I + \lim_{r\to t+} \int_d^r \mathrm{d}[A(s)]\Phi(s) = I + \int_d^t \mathrm{d}[A(s)]\Phi(s) \\ + \lim_{r\to t+} \int_t^r \mathrm{d}[A(s)]\Phi(s) = \Phi(t) + \lim_{r\to t+} \int_t^r \mathrm{d}[A(s)]\Phi(s) \\ = \Phi(t) + [A(t+) - A(t)]\Phi(t) = [I + \Delta^+ A(t)]\Phi(t).\end{aligned}$$

Hence $\Phi(t+) = [I + \Delta^+ A(t)]\Phi(t)$ and because $\Phi^{-1}(t)$ exists by Lemma 1.3 and the inverse $[I + \Delta^+ A(t)]^{-1}$ exists by (U+) from the assumption (U) the inverse $[\Phi(t+)]^{-1} = \Phi^{-1}(t+)$ also exists and we have the relation

$$[\Phi(t+)]^{-1} = \Phi^{-1}(t+) = \Phi^{-1}(t) \cdot [I + \Delta^+ A(t)]^{-1}, \quad t \in [0,1).$$

Similarly we have also

$$\Phi^{-1}(t-) = \Phi^{-1}(t) \cdot [I - \Delta^- A(t)]^{-1}, \quad t \in (0,1]$$

where $\Phi^{-1}(t-) = [\Phi(t-)]^{-1}$.

Using the continuity of the operation of taking an inverse (see [2], p. 624) we obtain

$$\lim_{r\to t+} \Phi^{-1}(r) = \Phi^{-1}(t+) \text{ for } t \in [0,1)$$

and

$$\lim_{r\to t-} \Phi^{-1}(r) = \Phi^{-1}(t-) \text{ for } t \in (0,1]$$

because $\lim_{r\to t+} \Phi(r) = \Phi(t+)$ for $t \in [0,1)$ and $\lim_{r\to t-} \Phi(r) = \Phi(t-)$ for $t \in (0,1]$.

Hence the operator valued function $\Phi^{-1}\colon [0,1] \to L(X)$ belongs to the space $G(L(X))$ and it is therefore bounded, i.e. there is an $L \geqslant 0$ such that

$$\|\Phi^{-1}(t)\|_{L(X)} \leqslant L$$

for every $t \in [0,1]$. □

1.5. Lemma. *Assume that* $A\colon [0,1] \to L(X)$ *satisfies* (1.3), (E) *and* (U).

Assume that $d \in [0,1]$ *is fixed and that* $\Phi\colon [0,1] \to L(X)$ *is the solution of* (1.5). *Then for every* $t_0 \in [0,1]$ *and* $\widetilde{x} \in X$, *the unique solution* $x\colon [0,1] \to X$ *of the homogeneous equation*

$$x(t) = \widetilde{x} + \int_{t_0}^{t} \mathrm{d}[A(s)]x(s)$$

is given by the relation

$$x(t) = \Phi(t)\Phi^{-1}(t_0)\widetilde{x}, \quad t \in [0,1].$$

Proof. The solution x exists and is unique by Theorem 2.11 in [9]. Using (1.1) we have

$$\begin{aligned} x(t) &= \Phi(t)\Phi^{-1}(t_0)\widetilde{x} = \Big[I + \int_d^t \mathrm{d}[A(s)]\Phi(s)\Big]\Phi^{-1}(t_0)\widetilde{x} \\ &= \Big[I + \int_d^{t_0} \mathrm{d}[A(s)]\Phi(s) + \int_{t_0}^t \mathrm{d}[A(s)]\Phi(s)\Big]\Phi^{-1}(t_0)\widetilde{x} \\ &= \Phi(t_0)\Phi^{-1}(t_0)\widetilde{x} + \int_{t_0}^t \mathrm{d}[A(s)]\Phi(s)\Phi^{-1}(t_0)\widetilde{x} = \widetilde{x} + \int_{t_0}^t \mathrm{d}[A(s)]x(s) \end{aligned}$$

and the lemma is proved. □

2. Variation of constants

2.1. Lemma. *Assume that* $A\colon [0,1] \to L(X)$ *satisfies* (1.3), (E) *and* (U). *Let* $\Phi\colon [0,1] \to L(X)$ *be the solution of* (1.5) *and assume that its inverse* $\Phi^{-1}\colon [0,1] \to L(X)$ *given by Lemma* 1.3 *is such that* $\Phi^{-1} \in (\mathcal{B})BV(L(X))$.

Then for every $g \in G(X)$, $t \in [0,1]$ *the equality*

$$\int_d^t \mathrm{d}[A(r)]\Phi(r) \int_d^r \mathrm{d}[\Phi^{-1}(s)]g(s) = \Phi(t) \int_d^t \mathrm{d}[\Phi^{-1}(s)]g(s) + \int_d^t \mathrm{d}[A(s)]g(s) \tag{2.1}$$

holds.

Proof. Since $g \in G(X)$ and $\Phi^{-1} \in (\mathcal{B})BV(L(X))$, the integrals on both sides of (2.1) exist by [6, Theorem 11] (see also [9, 1.12]).

To show that the equality (2.1) is valid for every regulated function $g\colon [0,1] \to X$ it is sufficient to prove it for an arbitrary finite step function, because the finite step functions are dense in the space $G(X)$ (see [2]).

For a given $\alpha \in [0,1]$, $c \in X$ and for $s \in [0,1]$ we define

$$\psi_\alpha^+(s) = 0 \text{ if } s \leqslant \alpha, \qquad \psi_\alpha^+(s) = c \text{ if } s > \alpha$$

and

$$\psi_\alpha^-(s) = 0 \text{ if } s < \alpha, \qquad \psi_\alpha^-(s) = c \text{ if } s \geqslant \alpha.$$

It is a matter of routine to verify that every finite step function can be expressed in the form of a finite sum of functions of the the type ψ_α^+ and ψ_α^-. Hence by the linearity of the integral it suffices to show that (2.1) holds for functions of this type.

Let us prove e.g. that (2.1) is satisfied for the function ψ_α^+.

Assume that $\alpha < d$. Then

$$\int_d^r \mathrm{d}_s[\Phi^{-1}(s)]\psi_\alpha^+(s) = [\Phi^{-1}(r) - \Phi^{-1}(d)]c \quad \text{if} \quad r > \alpha$$

and

$$\int_d^r \mathrm{d}_s[\Phi^{-1}(s)]\psi_\alpha^+(s) = [\Phi^{-1}(\alpha+) - \Phi^{-1}(d)]c \quad \text{if} \quad r \leqslant \alpha. \tag{2.2}$$

Hence for $t > \alpha$ we have

$$\begin{aligned} &\int_d^t \mathrm{d}[A(r)]\Phi(r) \int_d^r \mathrm{d}_s[\Phi^{-1}(s)]\psi_\alpha^+(s) \\ &= \int_d^t \mathrm{d}[A(r)]\Phi(r)[\Phi^{-1}(r) - \Phi^{-1}(d)]c = \int_d^t \mathrm{d}[A(r)][I - \Phi(r)\Phi^{-1}(d)]c \\ &= [A(t) - A(d)]c - [\Phi(t) - \Phi(d)]\Phi^{-1}(d)c = [A(t) - A(d)]c + c - \Phi(t)\Phi^{-1}(d)c. \end{aligned} \tag{2.3}$$

If $t \leqslant \alpha$ then

$$\int_d^t \mathrm{d}[A(r)]\Phi(r) \int_d^r \mathrm{d}_s[\Phi^{-1}(s)]\psi_\alpha^+(s) = -\int_t^d \mathrm{d}[A(r)]\Phi(r) \int_d^r \mathrm{d}_s[\Phi^{-1}(s)]\psi_\alpha^+(s)$$
$$= -\Big(\int_t^\alpha \mathrm{d}[A(r)]\Phi(r) \int_d^r \mathrm{d}_s[\Phi^{-1}(s)]\psi_\alpha^+(s) + \int_\alpha^d \mathrm{d}[A(r)]\Phi(r) \int_d^r \mathrm{d}_s[\Phi^{-1}(s)]\psi_\alpha^+(s)\Big)$$

and

$$\begin{aligned}
&\int_\alpha^d \mathrm{d}[A(r)]\Phi(r) \int_d^r \mathrm{d}_s[\Phi^{-1}(s)]\psi_\alpha^+(s) \\
&= [A(\alpha+) - A(\alpha)]\Phi(\alpha)[\Phi^{-1}(\alpha+) - \Phi^{-1}(d)]c \\
&\quad + \lim_{\delta\to 0+} \int_{\alpha+\delta}^d \mathrm{d}[A(r)]\Phi(r)[\Phi^{-1}(r) - \Phi^{-1}(d)]c \\
&= [A(\alpha+) - A(\alpha)]\Phi(\alpha)[\Phi^{-1}(\alpha+) - \Phi^{-1}(d)]c \\
&\quad + \lim_{\delta\to 0+} \int_{\alpha+\delta}^d \mathrm{d}[A(r)]c - \lim_{\delta\to 0+} \int_{\alpha+\delta}^d \mathrm{d}[A(r)]\Phi(r)\Phi^{-1}(d)c \\
&= [A(\alpha+) - A(\alpha)]\Phi(\alpha)[\Phi^{-1}(\alpha+) - \Phi^{-1}(d)]c + [A(d) - A(\alpha+)]c \\
&\quad - [\Phi(d) - \Phi(\alpha+)]\Phi^{-1}(d)c.
\end{aligned}$$

Further we have

$$\int_t^\alpha \mathrm{d}[A(r)]\Phi(r) \int_d^r \mathrm{d}_s[\Phi^{-1}(s)]\psi_\alpha^+(s) = [\Phi(\alpha) - \Phi(t)][\Phi^{-1}(\alpha+) - \Phi^{-1}(d)]c$$

and

$$\begin{aligned}
&\int_d^t \mathrm{d}[A(r)]\Phi(r) \int_d^r \mathrm{d}_s[\Phi^{-1}(s)]\psi_\alpha^+(s) \\
&= -\{[A(\alpha+) - A(\alpha)]\Phi(\alpha)[\Phi^{-1}(\alpha+) - \Phi^{-1}(d)]c + [A(d) - A(\alpha+)]c \\
&\quad - [\Phi(d) - \Phi(\alpha+)]\Phi^{-1}(d)c + [\Phi(\alpha) - \Phi(t)][\Phi^{-1}(\alpha+) - \Phi^{-1}(d)]c\}.
\end{aligned}$$

Since $[A(\alpha+) - A(\alpha)]\Phi(\alpha) = \Delta^+ A(\alpha)\Phi(\alpha) = \Phi(\alpha+) - \Phi(\alpha)$ we have

$$\begin{aligned}
&\int_d^t \mathrm{d}[A(r)]\Phi(r) \int_d^r \mathrm{d}_s[\Phi^{-1}(s)]\psi_\alpha^+(s) \\
&= -\{[\Phi(\alpha+) - \Phi(\alpha)][\Phi^{-1}(\alpha+) - \Phi^{-1}(d)] + [A(d) - A(\alpha+)] \\
&\quad - I + \Phi(\alpha+)\Phi^{-1}(d) + \Phi(\alpha)\Phi^{-1}(\alpha+) - \Phi(\alpha)\Phi^{-1}(d) \\
&\quad - \Phi(t)\Phi^{-1}(\alpha+) + \Phi(t)\Phi^{-1}(d)\}c \\
&= -\{[A(d) - A(\alpha+)] - \Phi(t)[\Phi^{-1}(\alpha+) - \Phi^{-1}(d)]\}c \\
&= [A(\alpha+) - A(d)]c + \Phi(t)[\Phi^{-1}(\alpha+) - \Phi^{-1}(d)]c
\end{aligned} \tag{2.4}$$

for $t \leqslant \alpha$.

For the right hand side of (2.1) we use (2.2) for obtaining

$$\Phi(t)\int_d^t \mathrm{d}[\Phi^{-1}(s)]\psi_\alpha^+(s) = \Phi(t)[\Phi^{-1}(t) - \Phi^{-1}(d)]c \quad \text{if} \quad t > \alpha$$

and

$$\Phi(t)\int_d^t \mathrm{d}[\Phi^{-1}(s)]\psi_\alpha^+(s) = [\Phi^{-1}(\alpha+) - \Phi^{-1}(d)]c \quad \text{if} \quad t \leqslant \alpha. \tag{2.5}$$

Now it is a matter of routine to show that

$$\int_d^t \mathrm{d}[A(s)]\psi_\alpha^+(s) = [A(t) - A(d)]c \text{ if } t > \alpha$$

and

$$\int_d^t \mathrm{d}[A(s)]\psi_\alpha^+(s) = [A(\alpha+) - A(d)]c \text{ if } t \leqslant \alpha. \tag{2.6}$$

Using (2.5) and (2.6) we obtain

$$\begin{aligned}\Phi(t)\int_d^t \mathrm{d}[\Phi^{-1}(s)]\psi_\alpha^+(s) + \int_d^t \mathrm{d}[A(s)]\psi_\alpha^+(s)\\ = -\Phi(t)[\Phi^{-1}(t) - \Phi^{-1}(d)]c + [A(t) - A(d)]c \quad \text{if} \quad t > \alpha\end{aligned}$$

and

$$\begin{aligned}\Phi(t)\int_d^t \mathrm{d}[\Phi^{-1}(s)]\psi_\alpha^+(s) + \int_d^t \mathrm{d}[A(s)]\psi_\alpha^+(s)\\ = [\Phi^{-1}(\alpha+) - \Phi^{-1}(d)]c + [A(\alpha+) - A(d)]c \quad \text{if} \quad t \leqslant \alpha.\end{aligned}$$

Looking at (2.3) and (2.4) we can see immediately that the equality (2.1) holds for the function ψ_α^+ if $\alpha < d$.

For $\alpha \geqslant d$ as well as for the case of the function ψ_α^- the result can be proved similarly. The computations are straightforward but slightly tedious. □

Let us assume that $A\colon [0,1] \to L(X)$ satisfies (1.3), (E) and (U).

Let us consider the equation

$$x(t) = \widetilde{x} + \int_{t_0}^t \mathrm{d}[A(s)]x(s) + f(t) - f(t_0). \tag{2.7}$$

By [9, Theorem 2.10] we obtain that

for every choice of $t_0 \in [0,1]$, $\widetilde{x} \in X$, $f \in G(X)$ *there exists* $x \in G(X)$ *such that*

$$x(t) = \widetilde{x} + \int_{t_0}^{t} \mathrm{d}[A(s)]x(s) + f(t) - f(t_0)$$

for every $t \in [0,1]$.

This solution of (2.7) *is determined uniquely.*

2.2. Theorem. *Assume that* $A\colon [0,1] \to L(X)$ *satisfies* (1.3), (E) *and* (U). *Let* $\Phi\colon [0,1] \to L(X)$ *be the solution of* (1.5) *and assume that its inverse* $\Phi^{-1}\colon [0,1] \to L(X)$ *given by Lemma* 1.3 *is such that* $\Phi^{-1} \in (\mathcal{B})BV(L(X))$.

Then for every $t_0 \in [0,1]$, $\widetilde{x} \in X$ *and* $f \in G(X)$ *the formula*

$$x(t) = \Phi(t)\Phi^{-1}(t_0)\widetilde{x} + f(t) - f(t_0) - \Phi(t) \int_{t_0}^{t} \mathrm{d}[\Phi^{-1}(s)](f(s) - f(t_0)), \tag{2.8}$$

$t \in [0,1]$, *represents a solution of* (2.7).

P r o o f. Using (2.8) we have for $t \in [0,1]$

$$\begin{aligned}
&\int_{t_0}^{t} \mathrm{d}[A(r)]x(r) \\
&\quad = \int_{t_0}^{t} \mathrm{d}[A(r)]\Big\{\Phi(r)\Phi^{-1}(t_0)\widetilde{x} + f(r) - f(t_0) - \Phi(r) \int_{t_0}^{r} \mathrm{d}[\Phi^{-1}(s)](f(s) - f(t_0))\Big\} \\
&\quad = \int_{t_0}^{t} \mathrm{d}[A(r)]\Phi(r)\Phi^{-1}(t_0)\widetilde{x} + \int_{t_0}^{t} \mathrm{d}[A(r)](f(r) - f(t_0)) \\
&\qquad - \int_{t_0}^{t} \mathrm{d}[A(r)]\Phi(r) \int_{t_0}^{r} \mathrm{d}[\Phi^{-1}(s)](f(s) - f(t_0)).
\end{aligned}$$

For a solution Φ of (1.5) we have

$$\int_{t_0}^{t} \mathrm{d}[A(r)]\Phi(r) = \Phi(t) - \Phi(t_0)$$

and by Lemma 2.1 we have

$$\begin{aligned}
&\int_{t_0}^{t} \mathrm{d}[A(r)]\Phi(r) \int_{t_0}^{r} \mathrm{d}[\Phi^{-1}(s)](f(s) - f(t_0)) \\
&\quad = \Phi(t) \int_{t_0}^{t} \mathrm{d}[\Phi^{-1}(s)](f(s) - f(t_0)) + \int_{t_0}^{t} \mathrm{d}[A(s)](f(s) - f(t_0)).
\end{aligned}$$

Therefore

$$\begin{aligned}\int_{t_0}^{t} & \mathrm{d}[A(r)]x(r) \\ &= [\Phi(t)-\Phi(t_0)]\Phi^{-1}(t_0)\widetilde{x} + \int_{t_0}^{t} \mathrm{d}[A(r)](f(r)-f(t_0)) \\ &\quad - \Phi(t)\int_{t_0}^{t} \mathrm{d}[\Phi^{-1}(s)](f(s)-f(t_0)) - \int_{t_0}^{t} \mathrm{d}[A(s)](f(s)-f(t_0)) = \Phi(t)\Phi^{-1}(t_0)\widetilde{x} - \widetilde{x} \\ &\quad - \Phi(t)\int_{t_0}^{t} \mathrm{d}[\Phi^{-1}(s)](f(s)-f(t_0)).\end{aligned}$$

Hence

$$\int_{t_0}^{t} \mathrm{d}[A(r)]x(r) = x(t) - \widetilde{x} - (f(s)-f(t_0))$$

for every $t\in[0,1]$ and this means that the function $x\colon [0,1]\to X$ given by (2.8) is a solution of the equation (2.7). □

R e m a r k. From the point of view of the variation-of-constants formula (2.8) presented in Theorem 2.2 the assumption that the inverse $\Phi^{-1}\colon [0,1]\to L(X)$ to $\Phi\colon [0,1]\to L(X)$ given by Lemma 1.3 is such that $\Phi^{-1}\in(\mathcal{B})BV(L(X))$ is very unnatural. It would be nice if the property $\Phi^{-1}\in(\mathcal{B})BV(L(X))$ could be derived from the general assumptions, i.e. from the fact that $A\colon [0,1]\to L(X)$ satisfies (1.3), (E) and (U).

In the next section we will show that in the special situation of $A\in BV(L(X))$ the variation-of-constants formula (2.8) holds without any further assumption.

3. The variation-of-constants formula for the case $A\in BV(L(X))$

Assume throughout this section that $A\in BV(L(X))$.

First of all it should be mentioned that by [9, 1.5] we have $A\in G(L(X))$ and therefore $A\colon [0,1]\to L(X)$ evidently satisfies (1.3) because, as was already mentioned in the introductory part of this note, we have $BV(L(X))\subset(\mathcal{B})BV(L(X))$ by [9, Prop. 1.1 and 1.2].

As was mentioned in the last Remark in [9], if $A\in BV(L(X))$ then A satisfies also condition (E).

Let us now prove the following proposition.

3.1. Proposition. *Assume that* $A\colon [0,1]\to L(X)$.

Then $A \in BV(L(X))$ if and only if

$$\sup_P \Big\{ \sup_{C_j, D_j} \Big\| \sum_{j=1}^{k} D_j [A(\alpha_j - A(\alpha_{j-1})] C_j \Big\|_{L(X)} \Big\} < \infty \tag{3.1}$$

where $P\colon\ 0 = \alpha_0 < \alpha_1 < \ldots < \alpha_{k-1} < \alpha_k = 1$ is a partition of $[0,1]$, $C_j, D_j \in L(X)$ with $\|C_j\|_{L(X)} \leqslant 1, \|D_j\|_{L(X)} \leqslant 1$, $j = 1, \ldots, k$, and

$$\operatorname*{var}_{[0,1]}(A) = \sup_P \Big\{ \sup_{C_j, D_j} \Big\| \sum_{j=1}^{k} D_j [A(\alpha_j - A(\alpha_{j-1})] C_j \Big\|_{L(X)} \Big\}.$$

P r o o f. Assume that

$$P\colon\ 0 = \alpha_0 < \alpha_1 < \ldots < \alpha_{k-1} < \alpha_k = 1$$

is an arbitrary partition of $[0,1]$.

If $C_j, D_j \in L(X)$ with $\|C_j\|_{L(X)} \leqslant 1, \|D_j\|_{L(X)} \leqslant 1$, $j = 1, \ldots, k$ then

$$\begin{aligned} &\Big\| \sum_{j=1}^{k} D_j [A(\alpha_j) - A(\alpha_{j-1})] C_j \Big\|_{L(X)} \\ &\qquad \leqslant \sum_{j=1}^{k} \|D_j\|_{L(X)} \|A(\alpha_j) - A(\alpha_{j-1})\|_{L(X)} \|C_j\|_{L(X)} \\ &\qquad \leqslant \sum_{j=1}^{k} \|A(\alpha_j) - A(\alpha_{j-1})\|_{L(X)}. \end{aligned}$$

Hence

$$\sup_{C_j, D_j} \Big\| \sum_{j=1}^{k} D_j [A(\alpha_j) - A(\alpha_{j-1})] C_j \Big\|_{L(X)} \leqslant \sum_{j=1}^{k} \|A(\alpha_j) - A(\alpha_{j-1})\|_{L(X)}$$

where the supremum on the left hand side is taken over all $C_j, D_j \in L(X)$ with $\|C_j\|_{L(X)} \leqslant 1, \|D_j\|_{L(X)} \leqslant 1$. Consequently,

$$\begin{aligned} &\sup_P \Big\{ \sup_{C_j, D_j} \Big\| \sum_{j=1}^{k} D_j [A(\alpha_j) - A(\alpha_{j-1})] C_j \Big\|_{L(X)} \Big\} \\ &\qquad \leqslant \sup_P \sum_{j=1}^{k} \|A(\alpha_j) - A(\alpha_{j-1})\|_{L(X)} = \operatorname*{var}_{[0,1]}(A). \end{aligned} \tag{3.2}$$

Assume that $\widehat{D}_j \in L(X)$ with $\|\widehat{D}_j\|_{L(X)} \leqslant 1$ and $x_j \in X$ with $\|x_j\|_X \leqslant 1$, $j = 1, \dots, k$. Let us take $w \in X$ such that $\|w\|_X = 1$. Then for all $j = 1, \dots, k$ there exist $\widehat{C}_j \in L(X)$ with $\|\widehat{C}_j\|_{L(X)} \leqslant 1$ such that $\widehat{C}_j w = x_j$. Hence

$$\begin{aligned} \Big\| \sum_{j=1}^{k} \widehat{D}_j [A(\alpha_j) - A(\alpha_{j-1})] x_j \Big\|_X &= \Big\| \sum_{j=1}^{k} \widehat{D}_j [A(\alpha_j) - A(\alpha_{j-1})] \widehat{C}_j w \Big\|_X \\ &\leqslant \sup_{\|y\|_X \leqslant 1} \Big\| \sum_{j=1}^{k} \widehat{D}_j [A(\alpha_j) - A(\alpha_{j-1})] \widehat{C}_j y \Big\|_X \\ &= \Big\| \sum_{j=1}^{k} \widehat{D}_j [A(\alpha_j) - A(\alpha_{j-1})] \widehat{C}_j \Big\|_{L(X)} \\ &\leqslant \sup_{C_j, D_j} \Big\| \sum_{j=1}^{k} D_j [A(\alpha_j) - A(\alpha_{j-1})] C_j \Big\|_{L(X)} \end{aligned}$$

where the supremum on the right hand side is taken over all $C_j, D_j \in L(X)$ with $\|C_j\|_{L(X)} \leqslant 1, \|D_j\|_{L(X)} \leqslant 1$. Passing to the supremum over all $\widehat{D}_j \in L(X)$ with $\|\widehat{D}_j\|_{L(X)} \leqslant 1$ and $x_j \in X$ with $\|x_j\|_X \leqslant 1$, $j = 1, \dots, k$ we get

$$\begin{aligned} &\sup_{x_j, D_j} \Big\| \sum_{j=1}^{k} D_j [A(\alpha_j) - A(\alpha_{j-1})] x_j \Big\|_X \\ &\qquad \leqslant \sup_{C_j, D_j} \Big\| \sum_{j=1}^{k} D_j [A(\alpha_j) - A(\alpha_{j-1})] C_j \Big\|_{L(X)}. \end{aligned} \tag{3.3}$$

Assume that $\varepsilon > 0$ is given. Choose vectors $x_j \in X$ with $\|x_j\|_X \leqslant 1$, $j = 1, \dots, k$ such that

$$\|[A(\alpha_j) - A(\alpha_{j-1})] x_j\|_X > \|[A(\alpha_j) - A(\alpha_{j-1})]\|_{L(X)} - \tfrac{\varepsilon}{k}. \tag{3.4}$$

Let us set

$$v_j = \frac{[A(\alpha_j) - A(\alpha_{j-1})] x_j}{\|[A(\alpha_j) - A(\alpha_{j-1})] x_j\|_X} \text{ if } [A(\alpha_j) - A(\alpha_{j-1})] x_j \neq 0$$

and

$$v_j = 0 \text{ if } [A(\alpha_j) - A(\alpha_{j-1})] x_j = 0.$$

For $v_j \neq 0$ let Y_j be the onedimensional subspace of X given by

$$Y_j = \{\lambda v_j; \lambda \in \mathbb{R}\}$$

and assume that $\widetilde{f}_j$ is a bounded linear functional on Y_j such that $\widetilde{f}_j(v_j) = 1$ and denote by $f_j \in X^*$ its extension onto X with $\|f_j\| = 1$.

Assume that $w \in X$ is fixed such that $\|w\|_X = 1$ and define the linear operator $D_j \in L(X)$ by the relation

$$D_j x = f_j(x)w, \ x \in X, \ j = 1, \dots, k.$$

Then certainly

$$\|D_j\|_{L(X)} = \|f_j\| \|w\| = 1$$

and

$$\begin{aligned} D_j[A(\alpha_j) - A(\alpha_{j-1})]x_j &= \|A(\alpha_j) - A(\alpha_{j-1})]x_j\|_X D_j v_j \\ &= \|A(\alpha_j) - A(\alpha_{j-1})]x_j\|_X f_j(v_j) w = \|A(\alpha_j) - A(\alpha_{j-1})]x_j\|_X w. \end{aligned}$$

Hence by (3.4) we get

$$\begin{aligned} \Big\| \sum_{j=1}^k D_j[A(\alpha_j) - A(\alpha_{j-1})]x_j \Big\|_X &= \Big\| \sum_{j=1}^k \Big\| A(\alpha_j) - A(\alpha_{j-1})]x_j\|_X w\|_X \\ &= \sum_{j=1}^k \|A(\alpha_j) - A(\alpha_{j-1})]x_j\|_X > \sum_{j=1}^k \big(\|A(\alpha_j) - A(\alpha_{j-1})]\|_{L(X)} - \tfrac{\varepsilon}{k} \big) \\ &= \sum_{j=1}^k \|A(\alpha_j) - A(\alpha_{j-1})]\|_{L(X)} - \varepsilon. \end{aligned}$$

Taking the supremum over all $D_j \in L(X)$ with $\|D_j\|_{L(X)} \leqslant 1$ and $x_j \in X$ with $\|x_j\|_X \leqslant 1$, $j = 1, \dots, k$ we get

$$\sup_{x_j, D_j} \Big\| \sum_{j=1}^k D_j[A(\alpha_j) - A(\alpha_{j-1})]x_j \Big\|_X > \sum_{j=1}^k \|A(\alpha_j) - A(\alpha_{j-1})]\|_{L(X)} - \varepsilon$$

and using (3.3) we finally obtain

$$\sup_{C_j, D_j} \Big\| \sum_{j=1}^k D_j[A(\alpha_j) - A(\alpha_{j-1})]C_j \Big\|_{L(X)} \geqslant \sum_{j=1}^k \|A(\alpha_j) - A(\alpha_{j-1})]\|_{L(X)} - \varepsilon.$$

Taking the supremum over all partitions P of $[0,1]$ we obtain together with (3.2) for every $\varepsilon > 0$ the inequality

$$\operatorname*{var}_{[0,1]}(A) - \varepsilon < \sup_P \Big\{ \sup_{C_j, D_j} \Big\| \sum_{j=1}^k D_j[A(\alpha_j) - A(\alpha_{j-1})]C_j \Big\|_{L(X)} \Big\} \leqslant \operatorname*{var}_{[0,1]}(A)$$

and therefore

$$\operatorname*{var}_{[0,1]}(A) = \sup_{P}\Big\{ \sup_{C_j, D_j} \Big\| \sum_{j=1}^{k} D_j[A(\alpha_j) - A(\alpha_{j-1})]C_j \Big\|_{L(X)} \Big\}.$$

□

R e m a r k. It has to be mentioned that the characterization of the space $BV(L(X))$ given by Proposition 3.1 is interesting independently of the context of the equations studied in this paper.

3.2. Lemma. *Assume that* $A\colon [0,1] \to L(X)$ *satisfies* $A \in BV(L(X))$ *and* (U). *Then for the solution* $\Phi\colon [0,1] \to L(X)$ *of* (1.5) *we have* $\Phi \in BV(L(X))$.

P r o o f. Since $BV(L(X)) \subset (\mathcal{B}^*)BV(L(X))$ the conclusion of Lemma 1.2 holds and there exists a $K > 0$ such that $\|\Phi(t)\| \leqslant K$ for every $t \in [0,1]$. It remains to show that the relation $\Phi \in BV(L(X))$ holds.

Assume that

$$P\colon\ 0 = \alpha_0 < \alpha_1 < \ldots < \alpha_{k-1} < \alpha_k = 1$$

is an arbitrary partition of the interval $[0,1]$ and that $C_j, D_j \in L(X), j = 1, \ldots, k$ with $\|C_j\|_{L(X)} \leqslant 1$, $\|D_j\|_{L(X)} \leqslant 1$ are given.

The fact that $\Phi \in G(L(X))$ yields by [6, Prop. 15] the existence of the integral $\int_0^1 \mathrm{d}[A(r)]\Phi(r)$ and therefore by definition for every $\varepsilon > 0$ there is a gauge $\delta\colon [0,1] \to (0,\infty)$ such that

$$\Big\| \sum_{i=1}^{l} [A(\beta_i) - A(\beta_{i-1})]\Phi(\sigma_i) - \int_0^1 \mathrm{d}[A(r)]\Phi(r) \Big\|_{L(X)} < \frac{\varepsilon}{k+1}$$

for every δ-fine P-partition

$$\{\beta_0, \sigma_1, \beta_1, \ldots, \beta_{l-1}, \sigma_l, \beta_l\}$$

of the interval $[0,1]$.

By the Saks-Henstock Lemma (see [6, Lemma 16]) we have

$$\Big\| \sum_{i=1}^{l_j} [A(\beta_i^j) - A(\beta_{i-1}^j)]\Phi(\sigma_i^j) - \int_{\alpha_{j-1}}^{\alpha_j} \mathrm{d}[A(r)]\Phi(r) \Big\|_{L(X)} \leqslant \frac{\varepsilon}{k+1} \tag{3.5}$$

for every δ-fine P-partition

$$\{\beta_0^j, \sigma_1^j, \beta_1^j, \ldots, \beta_{l_j-1}^j, \sigma_{l_j}^j, \beta_{l_j}^j\}$$

of the interval $[\alpha_{j-1}, \alpha_j]$, $j = 1, \dots, k$.

Further, we have

$$\Phi(\alpha_j) - \Phi(\alpha_{j-1}) = \int_{\alpha_{j-1}}^{\alpha_j} \mathrm{d}[A(r)]\Phi(r)$$

for every $j = 1, \dots, k$ by the definition of a solution of (1.5) and therefore

$$\begin{aligned}
&\Big\| \sum_{j=1}^{k} D_j[\Phi(\alpha_j) - \Phi(\alpha_{j-1})]C_j \Big\|_{L(X)} = \Big\| \sum_{j=1}^{k} D_j \Big[\int_{\alpha_{j-1}}^{\alpha_j} \mathrm{d}[A(r)]\Phi(r) \Big] C_j \Big\|_{L(X)} \\
&\quad = \Big\| \sum_{j=1}^{k} \Big\{ D_j \Big[\int_{\alpha_{j-1}}^{\alpha_j} \mathrm{d}[A(r)]\Phi(r) - \sum_{i=1}^{l_j} [A(\beta_i^j) - A(\beta_{i-1}^j)]\Phi(\sigma_i^j) \Big] C_j \Big\} \\
&\qquad + \sum_{j=1}^{k} \sum_{i=1}^{l_j} D_j [A(\beta_i^j) - A(\beta_{i-1}^j)]\Phi(\sigma_i^j) C_j \Big\|_{L(X)} \\
&\quad \leqslant \Big\| \sum_{j=1}^{k} \Big\{ D_j \Big[\int_{\alpha_{j-1}}^{\alpha_j} \mathrm{d}[A(r)]\Phi(r) - \sum_{i=1}^{l_j} [A(\beta_i^j) - A(\beta_{i-1}^j)]\Phi(\sigma_i^j) \Big] C_j \Big\} \Big\|_{L(X)} \\
&\qquad + \Big\| \sum_{j=1}^{k} \sum_{i=1}^{l_j} D_j [A(\beta_i^j) - A(\beta_{i-1}^j)]\Phi(\sigma_i^j) C_j \Big\|_{L(X)} \\
&\quad \leqslant \sum_{j=1}^{k} \Big\| \Big[\int_{\alpha_{j-1}}^{\alpha_j} \mathrm{d}[A(r)]\Phi(r) - \sum_{i=1}^{l_j} [A(\beta_i^j) - A(\beta_{i-1}^j)]\Phi(\sigma_i^j) \Big] \Big\|_{L(X)} \\
&\qquad + \Big\| \sum_{j=1}^{k} \sum_{i=1}^{l_j} D_j [A(\beta_i^j) - A(\beta_{i-1}^j)]\Phi(\sigma_i^j) C_j \Big\|_{L(X)}
\end{aligned}$$

provided

$$\{\beta_0^j, \sigma_1^j, \beta_1^j, \dots, \beta_{l_j-1}^j, \sigma_{l_j}^j, \beta_{l_j}^j\}$$

is a δ-fine P-partition of the interval $[\alpha_{j-1}, \alpha_j]$, $j = 1, \dots, k$. Hence using (3.5) we obtain by the last inequalities

$$\begin{aligned}
&\Big\| \sum_{j=1}^{k} D_j[\Phi(\alpha_j) - \Phi(\alpha_{j-1})]C_j \Big\|_{L(X)} \\
&\qquad \leqslant \sum_{j=1}^{k} \frac{\varepsilon}{k+1} + \Big\| \sum_{j=1}^{k} \sum_{i=1}^{l_j} D_j [A(\beta_i^j) - A(\beta_{i-1}^j)]\Phi(\sigma_i^j) C_j \Big\|_{L(X)} \\
&\qquad < \varepsilon + \Big\| \sum_{j=1}^{k} \sum_{i=1}^{l_j} D_j [A(\beta_i^j) - A(\beta_{i-1}^j)]\Phi(\sigma_i^j) C_j \Big\|_{L(X)}.
\end{aligned}$$

For the second term on the right hand side we have

$$\Big\| \sum_{j=1}^{k} \sum_{i=1}^{l_j} D_j [A(\beta_i^j) - A(\beta_{i-1}^j)] \Phi(\sigma_i^j) C_j \Big\|_{L(X)}$$
$$\leqslant \sum_{j=1}^{k} \sum_{i=1}^{l_j} \|D_j\|_{L(X)} \|A(\beta_i^j) - A(\beta_{i-1}^j)\|_{L(X)} \|\Phi(\sigma_i^j)\|_{L(X)} \|C_j\|_{L(X)}$$
$$\leqslant K \cdot \sum_{j=1}^{k} \sum_{i=1}^{l_j} \|A(\beta_i^j) - A(\beta_{i-1}^j)\|_{L(X)} \leqslant K \cdot \operatorname*{var}_{[0,1]}(A).$$

Hence

$$\Big\| \sum_{j=1}^{k} D_j [\Phi(\alpha_j) - \Phi(\alpha_{j-1})] C_j \Big\|_{L(X)} < \varepsilon + K \cdot \operatorname*{var}_{[0,1]}(A)$$

and since $\varepsilon > 0$ can be taken arbitrarily small, we get

$$\Big\| \sum_{j=1}^{k} D_j [\Phi(\alpha_j) - \Phi(\alpha_{j-1})] C_j \Big\|_{L(X)} \leqslant K \cdot \operatorname*{var}_{[0,1]}(A)$$

for any partition

$$P\colon\ 0 = \alpha_0 < \alpha_1 < \ldots < \alpha_{k-1} < \alpha_k = 1$$

of the interval $[0,1]$ and any choice of $C_j, D_j \in L(X), j = 1, \ldots, k$ with $\|C_j\|_{L(X)} \leqslant 1$, $\|D_j\|_{L(X)} \leqslant 1$.

Passing to the suprema over all $C_j, D_j \in L(X), j = 1, \ldots, k$ with $\|C_j\|_{L(X)} \leqslant 1$, $\|D_j\|_{L(X)} \leqslant 1$ and all partitions P of $[0,1]$ we obtain

$$\sup_{P} \sup_{C_j, D_j} \Big\| \sum_{j=1}^{k} D_j [\Phi(\alpha_j) - \Phi(\alpha_{j-1})] C_j \Big\|_{L(X)} \leqslant K \cdot \operatorname*{var}_{[0,1]}(A)$$

and this together with Proposition 3.1 yields the result. □

3.3. Lemma. *Assume that $A\colon [0,1] \to L(X)$ satisfies $A \in BV(L(X))$ and* (U).
Then the inverse $[\Phi(t)]^{-1} = \Phi^{-1}(t)$ to the solution $\Phi\colon [0,1] \to L(X)$ of (1.5) *exists for every $t \in [0,1]$ and we have $\Phi^{-1} \in BV(L(X))$.*

P r o o f. By the results given in Lemma 1.3 and 1.4 the inverse Φ^{-1} exists and $\Phi^{-1} \in G(L(X))$. Hence there is a constant $L > 0$ such that

$$\|\Phi^{-1}(t)\|_{L(X)} \leqslant L$$

for every $t \in [0,1]$.

It remains to show that $\Phi^{-1} \in BV(L(X))$.

Assume that

$$P\colon\ 0 = \alpha_0 < \alpha_1 < \ldots < \alpha_{k-1} < \alpha_k = 1$$

is an arbitrary partition of the interval $[0,1]$ and that $C_j, D_j \in L(X), j = 1, \ldots, k$ with $\|C_j\|_{L(X)} \leqslant 1$, $\|D_j\|_{L(X)} \leqslant 1$ are given.

We have

$$\begin{aligned}
\Big\| \sum_{j=1}^{k} D_j[\Phi^{-1}(\alpha_j) - \Phi^{-1}(\alpha_{j-1})]C_j \Big\| &= \Big\| \sum_{j=1}^{k} D_j \Phi^{-1}(\alpha_j)[I - \Phi(\alpha_j)\Phi^{-1}(\alpha_{j-1})]C_j \Big\| \\
&= \Big\| \sum_{j=1}^{k} D_j \Phi^{-1}(\alpha_j)[\Phi(\alpha_{j-1}) - \Phi(\alpha_j)]\Phi^{-1}(\alpha_{j-1})C_j \Big\| \\
&= \Big\| \sum_{j=1}^{k} D_j \Phi^{-1}(\alpha_j)[\Phi(\alpha_j) - \Phi(\alpha_{j-1})]\Phi^{-1}(\alpha_{j-1})C_j \Big\| \\
&\leqslant L^2 \cdot \operatorname*{var}_{[0,1]}(\Phi) \leqslant L^2 \cdot K \cdot \operatorname*{var}_{[0,1]}(A).
\end{aligned}$$

Passing to the suprema over all $C_j, D_j \in L(X), j = 1, \ldots, k$ with $\|C_j\|_{L(X)} \leqslant 1$, $\|D_j\|_{L(X)} \leqslant 1$ and all partitions P of $[0,1]$ we obtain

$$\sup_{P} \sup_{C_j, D_j} \Big\| \sum_{j=1}^{k} D_j[\Phi^{-1}(\alpha_j) - \Phi^{-1}(\alpha_{j-1})]C_j \Big\|_{L(X)} \leqslant L^2 \cdot K \cdot \operatorname*{var}_{[0,1]}(A).$$

and this together with Proposition 3.1 yields $\Phi^{-1} \in BV(L(X))$. □

3.4. Theorem. *Assume that* $A\colon [0,1] \to L(X)$ *satisfies* $A \in BV(L(X))$ *and* (U). *Let* $\Phi\colon [0,1] \to L(X)$ *be the solution of* (1.5).

Then for every $t_0 \in [0,1]$, $\widetilde{x} \in X$ *and* $f \in G(X)$ *the formula*

$$x(t) = \Phi(t)\Phi^{-1}(t_0)\widetilde{x} + f(t) - f(t_0) - \Phi(t) \int_{t_0}^{t} \mathrm{d}[\Phi^{-1}(s)](f(s) - f(t_0)), \tag{2.8}$$

$t \in [0,1]$, *represents a solution of* (2.7).

P r o o f. By Lemma 3.3 the inverse $\Phi^{-1}\colon [0,1] \to L(X)$ given by Lemma 1.3 belongs to $BV(L(X))$ and therefore we have also $\Phi^{-1} \in (\mathcal{B})BV(L(X))$. All the assumptions of Theorem 2.2 being satisfied we obtain the result by this theorem. □

3.5 Example. Let us consider the abstract linear differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = a(t)x + \varphi(t) \tag{3.6}$$

on $[0,1]$ where $a\colon [0,1] \to L(X)$, $\varphi\colon [0,1] \to X$ and both a and φ are Bochner integrable. For equations of this kind see e.g. [1].

A solution of (3.6) is understood to be a solution of the integral equation

$$x(t) = x_0 + \int_d^t a(s)x(s)\,\mathrm{d}s + \int_a^t \varphi(s)\,\mathrm{d}s \tag{3.7}$$

where $d \in [0,1]$ and $x_0 = x(d)$.

More generally we can consider the integral equation of the form

$$x(t) = \int_d^t a(s)x(s)\,\mathrm{d}s + g(t) \tag{3.8}$$

with $g \in G(X)$.

Let us set

$$A(t) = \int_d^t a(s)\,\mathrm{d}s \text{ and } f(t) = \int_d^t \varphi(s)\,\mathrm{d}s, \quad t \in [0,1].$$

Assume that $D\colon 0 = \alpha_0 < \alpha_1 < \ldots < \alpha_{k-1} < \alpha_k = 1$ is an arbitrary partition of $[0,1]$. Then using the properties of the Bochner integral we get

$$\begin{aligned} \sum_{j=1}^k \|A(\alpha_j) - A(\alpha_{j-1})\| &= \sum_{j=1}^k \Big\| \int_{\alpha_{j-1}}^{\alpha_j} a(s)\,\mathrm{d}s \Big\| \\ &\leqslant \sum_{j=1}^k \int_{\alpha_{j-1}}^{\alpha_j} \|a(s)\|\,\mathrm{d}s = \int_0^1 \|a(s)\|\,\mathrm{d}s < \infty \end{aligned}$$

and therefore $A \in BV(L(X))$. Since the function $\|a\|$ is Lebesgue integrable over $[0,1]$ we have

$$\|A(t) - A(r)\| \leqslant \Big| \int_r^t \|a(s)\|\,\mathrm{d}s \Big|$$

for $t, r \in [0,1]$ and this yields the continuity of A on $[0,1]$. Hence $\lim\limits_{t\to r+} A(t) = A(r)$ for $r \in [0,1)$ and $\lim\limits_{t\to r-} A(t) = A(r)$ for $r \in (0,1]$ and consequently we have $\Delta^+ A(r) = 0$ for $r \in [0,1)$ and $\Delta^- A(r) = 0$ for $r \in (0,1]$ and the function $A\colon [0,1] \to L(X)$ satisfies the condition (U) given in Theorem 1.1. Similarly the function $f\colon [0,1] \to X$ is also continuous and belongs trivially to $G(X)$.

It is a matter of routine to show that
if $x \in G(X)$ *then the integrals* $\int_0^1 \mathrm{d}[A(s)]x(s)$ *and* $\int_0^1 a(s)x(s)\,\mathrm{d}s$ *both exist and*

$$\int_0^1 \mathrm{d}[A(s)]x(s) = \int_0^1 a(s)x(s)\,\mathrm{d}s.$$

Since g is assumed to belong to $G(X)$, every solution of (3.8) also belongs to $G(X)$ and therefore the equation (3.8) is equivalent to

$$x(t) = \int_d^t \mathrm{d}[A(s)]x(s) + g(t) = g(d) + \int_d^t \mathrm{d}[A(s)]x(s) + g(t) - g(d).$$

Hence by Theorem 2.10 in [9] there exists a unique solution $x\colon [0,1] \to X$, $x \in G(X)$ of (3.8) and by Theorem 3.4 we get after a straightforward calculation

$$\begin{aligned} x(t) &= \Phi(t)\Phi^{-1}(t_0)g(d) + g(t) - g(d) - \Phi(t)\int_d^t \mathrm{d}[\Phi^{-1}(s)](g(s) - g(d)) \\ &= g(t) - \Phi(t)\int_d^t \mathrm{d}[\Phi^{-1}(s)]g(s) \end{aligned}$$

where the function $\Phi\colon [0,1] \to L(X)$ is a solution of (1.5) with A given by $A(t) = \int_d^t a(s)\,\mathrm{d}s$ for $t \in [0,1]$.

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Author's address: *Štefan Schwabik*, Mathematical Institute, Academy of Sciences of the Czech Republic, Žitná 25, 115 67 Praha 1, Czech Republic, e-mail: `schwabik@math.cas.cz`.