A TREE AS A FINITE NONEMPTY SET WITH A BINARY OPERATION

Ladislav Nebeský, Praha

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Abstract. A (finite) acyclic connected graph is called a tree. Let W be a finite nonempty set, and let $\mathbf{H}(W)$ be the set of all trees T with the property that W is the vertex set of T. We will find a one-to-one correspondence between $\mathbf{H}(W)$ and the set of all binary operations on W which satisfy a certain set of three axioms (stated in this note).

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By a graph we mean a finite undirected graph with no loops or multiple edges (i.e. a graph in the sense of [1], for example). If G is a graph, then $V(G)$ and $E(G)$ denote its vertex set and its edge set, respectively.

Let G be a connected graph. We denote by d_G the distance function of G. For every ordered pair of distinct $u, v \in V(G)$ we denote

$$A_G(u,v) = \{w \in V(G);\ d_G(u,w) = 1 \text{ and } d_G(w,v) = d_G(u,v) - 1\}.$$

A graph G is said to be *geodetic* if it is connected and there exists exactly one shortest $u - v$ path in G for every ordered pair of $u, v \in V(G)$. It is not difficult to show that

(1) a connected graph H is geodetic if and only if $|A_H(x,y)| = 1$ for all distinct $x, y \in V(H)$.

A graph is called a *tree* if it is connected and acyclic. It is well-known that a graph G is a tree if and only if there exists exactly one $x - y$ path in G for every ordered pair of $x, y \in V(G)$. Thus, every tree is a geodetic graph.

In [2], the present author proved that a connected graph G is geodetic if and only if there exists a binary operation which "defines" G (in a certain sense) and satisfies a certain set of (four) axioms; the assumption that G is connected cannot be omitted. In the present note we will prove that a graph G is a tree if and only if there exists a binary operation which "defines" G (in the same sense) and satisfies a certain set of (three) axioms. The assumption that G is connected is not needed. Thus our result obtained for trees is stronger than that obtained for geodetic graphs in [2].

Let G be a geodetic graph, and let $+$ be a binary operation on $V(G)$. Following [2] we say that $+$ is the *proper operation* of G if for every ordered pair of $u, v \in V(G)$ we have

$$u + v = u, \quad \text{if} \quad u = v,$$

$$u + v \ \text{ is the second vertex of the shortest } u - v \text{ path } \ \text{provided } u \neq v.$$

This means that if x and y are distinct vertices of G, then $x + y$ is the only element of $A_G(x, y)$.

Lemma 1. *Let T be a tree, and let $+$ be the proper operation of T. Put $W = V(G)$. Then $+$ satisfies the following three Axioms* (A), (B), *and* (C)*:*

(A) $(u + v) + u = u$ *(for all $u, v \in W$);*

(B) *if $(u + v) + v = u$, then $u = v$* *(for all $u, v \in W$);*

(C) *if $u \neq u + v = v \neq u + w$, then $v + w = u$* *(for all $u, v, w \in W$).*

P r o o f. That is very easy. □

Note that the proper operation of any geodetic graph satisfies Axioms (A) and (B).

Let $+$ be a binary operation on a finite nonempty set W, and let $+$ satisfy Axioms (A), (B) and (C). Then we will say that an ordered pair $(W, +)$ is a *tree groupoid.* If $\Gamma = (W, +)$ is a tree groupoid, then we write $V(\Gamma) = W$.

In this note we will show that—roughly speaking—every tree can be considered a tree groupoid, and every tree grupoid can be considered a tree.

Lemma 2. *Let $(W, +)$ be a tree groupoid. Then*

$$u + v = v \ \text{ if and only if } \ v + u = u \quad \text{for all } \ u, v \in W; \tag{2}$$

$$u + v = u \ \text{ if and only if } \ u = v \quad \text{for all } \ u, v \in W. \tag{3}$$

P r o o f. (2) follows from Axiom (A).

Let $u, v \in W$. By Axiom (A), $((u + u) + u) + u = u + u$; and, by Axiom (B), $u + u = u$. Thus, if $u = v$, then $u + v = u$. Conversely, if $u + v = u$, then $(u + v) + v = u + v = u$; and, by Axiom (B), $u = v$. Hence (3) holds. □

Let $\Gamma = (W, +)$ be a tree groupoid, and let G be a graph. We will say that G is *associated* with Γ if $V(G) = W$ and

$$E(G) = \{\{u, v\};\ u, v \in V(G) \text{ such that } u + v = v \neq u\}.$$

As follows from (2), for every tree groupoid Γ there exists exactly one graph associated with Γ.

Lemma 3. *Let $\Gamma = (W, +)$ be a tree groupoid, let G be the graph associated with Γ, and let H be a component of G. Then*

$$A_H(x, y) = \{x + y\} \text{ for all distinct } x, y \in V(H). \tag{4}$$

P r o o f. If H is trivial, then (4) holds. Let H be nontrivial. Consider arbitrary distinct $x, y \in V(H)$. We will prove that $A_H(x, y) = \{x + y\}$. Put $n = d_H(x, y)$. Then $n \geqslant 1$. We proceed by induction on n. The case when $n = 1$ is obvious. Let $n \geqslant 2$. Assume that

$$A_H(u, v) = \{u + v\} \text{ for all } u, v \in V(H) \text{ such that } d(u, v) = n - 1. \tag{5}$$

Obviously, $A_H(x, y) \neq \emptyset$. Consider an arbitrary $z \in A_H(x, y)$. Then $\{x, z\} \in E(H)$. Since $d_H(z, y) = n-1$, (5) implies that $x \neq z+y$. By virtue of Axiom (C), $z = x+y$. Hence $A_H(x, y) = \{x + y\}$. □

Lemma 4. *Let $\Gamma = (W, +)$ be a tree groupoid, and let G be the graph associated with Γ. Then G is a tree and $+$ is the proper operation of G.*

P r o o f. Consider an arbitrary component H of G. Combining (1) with Lemma 3, we get that H is a geodetic graph. Assume that H contains a cycle of odd length. It is routine to prove that there exist $u, v, w \in V(H)$ such that $d_H(u, v) = d_H(u, w) \geqslant 1$ and $d_H(v, w) = 1$. By Axiom (C), either $v + u = w$ or $w + u = v$, which contradicts (4). Thus H contains no cycle of odd length. Since H is a geodetic graph, we get that H is a tree.

Assume that G has at least two components. Then there exists $y \in W - V(H)$. Consider an arbitrary $x \in V(H)$. We construct an infinite sequence $(x_1, x_2, x_3, \ldots)$ of vertices in G as follows: $x_1 = x$ and

$$x_{n+1} = x_n + y \text{ for all } n = 1, 2, 3 \ldots$$

Since G is associated with Γ, we get

$$\{x_1, x_2\}, \{x_2, x_3\}, \{x_3, x_4\}, \ldots \in E(G).$$

Hence $x_1, x_2, x_3, \ldots \in V(H)$. Note that $y \notin V(H)$. Axiom (B) implies that

$$x_1 \neq x_3, x_2 \neq x_4, x_3 \neq x_5, \ldots$$

Since $V(H)$ is finite, we conclude that H contains a cycle, which is a contradiction. Thus H is the only component of G. We get that G is a tree.

By virtue of (3) and Lemma 3, $+$ is the proper operation of G. $\square$

Let W be a finite nonempty set. We denote by $\mathbf{H}(W)$ the set of all trees T such that $V(T) = W$. Moreover, we denote by $\mathbf{D}(W)$ the set of all tree groupoids Γ such that $V(\Gamma) = W$.

We will now present the main result of this note.

Theorem. *Let W be a finite nonempty set. Then there exists a one-to-one mapping φ of $\mathbf{H}(W)$ onto $\mathbf{D}(W)$ such that*

$$\varphi(T) = (W, +), \quad \text{where } + \text{ is the proper operation of } T,$$

for each $T \in \mathbf{H}(W)$.

Proof. Combining Lemmas 1 and 4, we get the theorem. $\square$

References

[1] *G. Chartrand, L. Lesniak*: Graphs & Digraphs. Third edition. Chapman & Hall, London, 1996.

[2] *L. Nebeský*: An algebraic characterization of geodetic graphs. Czechoslovak Math. J. *48* (1998), 701–710.

Author's address: *Ladislav Nebeský*, Filozofická fakulta Univerzity Karlovy, nám. J. Palacha 2, 116 38 Praha 1, Czech Republic, e-mail: `ladislav.nebesky@ff.cuni.cz`.