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B-FREDHOLM AND DRAZIN INVERTIBLE OPERATORS THROUGH LOCALIZED SVEP

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Abstract. Let X be a Banach space and T be a bounded linear operator on X. We denote by $S(T)$ the set of all complex $\lambda \in \mathbb{C}$ such that T does not have the single-valued extension property at λ. In this note we prove equality up to $S(T)$ between the left Drazin spectrum, the upper semi-B-Fredholm spectrum and the semi-essential approximate point spectrum. As applications, we investigate generalized Weyl's theorem for operator matrices and multiplier operators.

Keywords: B-Fredholm operator, Drazin invertible operator, single-valued extension property

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1. Introduction

Throughout this paper, X and Y are Banach spaces and $\mathcal{B}(X,Y)$ denotes the space of all bounded linear operators from X to Y. For $Y = X$ we write $\mathcal{B}(X,Y) = \mathcal{B}(X)$. For $T \in \mathcal{B}(X)$, let T^*, $N(T)$, $R(T)$, $\sigma(T)$, $\sigma_{\rm s}(T)$, $\sigma_{\rm p}(T)$ and $\sigma_{\rm a}(T)$ denote the adjoint, the null space, the range, the spectrum, the surjective spectrum, the point spectrum and the approximate point spectrum of T, respectively. Let $\alpha(T)$ and $\beta(T)$ be the nullity and the deficiency of T defined by $\alpha(T) = \dim N(T)$ and $\beta(T) = \operatorname{codim} R(T)$. If the range $R(T)$ is closed and $\alpha(T) < \infty$ (or $\beta(T) < \infty$), then T is called an *upper* (a *lower*) *semi-Fredholm* operator. If $T \in \mathcal{B}(X)$ is either upper or lower semi-Fredholm, then T is called a *semi-Fredholm* operator, and the *index* of T is defined by $\operatorname{ind}(T) = \alpha(T) - \beta(T)$. If both $\alpha(T)$ and $\beta(T)$ are finite, then T is called a *Fredholm* operator. An operator T is called *Weyl* if it is Fredholm of index zero. The Weyl spectrum $\sigma_{\rm W}(T)$ is defined by $\sigma_{\rm W}(T) = \{\lambda \in \mathbb{C}\colon\ T - \lambda I \text{ is not Weyl}\}$.

For $T \in \mathcal{B}(X)$ and a nonnegative integer n define $T_{[n]}$ to be the restriction of T to $R(T^n)$ viewed as a map from $R(T^n)$ into $R(T^n)$ (in particular $T_{[0]} = T$). If for some integer n the range space $R(T^n)$ is closed and $T_{[n]}$ is an upper (or a lower) semi-Fredholm operator, then T is called an *upper* (a *lower*) *semi-B-Fredholm* operator. In this case the *index* of T is defined to be the index of the semi-Fredholm operator $T_{[n]}$. Moreover, if $T_{[n]}$ is a Fredholm operator, then T is called a *B-Fredholm* operator. A *semi-B-Fredholm* operator is an upper or a lower semi-B-Fredholm operator ([6], [8], [13]). The *upper semi-B-Fredholm spectrum* $\sigma_{\mathrm{UBF}}(T)$, the *lower semi-B-Fredholm spectrum* $\sigma_{\mathrm{LBF}}(T)$ and the *B-Fredholm spectrum* $\sigma_{\mathrm{BF}}(T)$ of T are defined by

$$\begin{aligned}
\sigma_{\mathrm{UBF}}(T) &= \{\lambda \in \mathbb{C}\colon\ T-\lambda I \text{ is not an upper semi-B-Fredholm operator}\},\\
\sigma_{\mathrm{LBF}}(T) &= \{\lambda \in \mathbb{C}\colon\ T-\lambda I \text{ is not a lower semi-B-Fredholm operator}\},\\
\sigma_{\mathrm{BF}}(T) &= \{\lambda \in \mathbb{C}\colon\ T-\lambda I \text{ is not a B-Fredholm operator}\}.
\end{aligned}$$

We have

$$\sigma_{\mathrm{BF}}(T) = \sigma_{\mathrm{UBF}}(T) \cup \sigma_{\mathrm{LBF}}(T).$$

An operator $T \in \mathcal{B}(X)$ is said to be a *B-Weyl* operator if it is a B-Fredholm operator of index zero. The *B-Weyl spectrum* $\sigma_{\mathrm{BW}}(T)$ of T is defined by

$$\sigma_{\mathrm{BW}}(T) = \{\lambda \in \mathbb{C}\colon\ T-\lambda I \text{ is not a B-Weyl operator}\}.$$

From [8, Lemma 4.1], T is a B-Weyl operator if and only if $T = F \oplus N$, where F is a Fredholm operator of index zero and N is a nilpotent operator.

We shall denote by $\mathrm{SBF}_+^-(X)$ (or $\mathrm{SBF}_-^+(X)$) the class of all T upper semi-B-Fredholm operators (T lower semi-B-Fredholm operators) such that $\mathrm{ind}(T) \leqslant 0$ ($\mathrm{ind}(T) \geqslant 0$). The spectrum associated with $\mathrm{SBF}_+^-(X)$ is called the *semi-essential approximate point spectrum* and is denoted by $\sigma_{\mathrm{SBF}_+^-}(T) = \{\lambda \in \mathbb{C}\colon\ T-\lambda I \notin \mathrm{SBF}_+^-(X)\}$, while the spectrum associated with $\mathrm{SBF}_-^+(X)$ is denoted by $\sigma_{\mathrm{SBF}_-^+}(T) = \{\lambda \in \mathbb{C}\colon\ T-\lambda I \notin \mathrm{SBF}_-^+(X)\}$.

The *ascent* $a(T)$ and the *descent* $d(T)$ of T are given by $a(T) = \inf\{n\colon\ N(T^n) = N(T^{n+1})\}$ and $d(T) = \inf\{n\colon\ R(T^n) = R(T^{n+1})\}$, with $\inf\emptyset = \infty$. It is well-known that if $a(T)$ and $d(T)$ are both finite then they are equal, see [16, Proposition 38.3].

Recall that an operator T is *Drazin invertible* if it has a finite ascent and descent. It is well known that T is Drazin invertible if and only if $T = R \oplus N$ where R is invertible and N is nilpotent (see [20, Corollary 2.2]). The Drazin spectrum is defined by $\sigma_{\mathrm{D}}(T) = \{\lambda \in \mathbb{C}\colon\ T-\lambda I \text{ is not Drazin invertible}\}$. From [8, Lemma 4.1] and [20, Corollary 2.2] we have

$$\sigma_{\mathrm{BW}}(T) \subseteq \sigma_{\mathrm{D}}(T).$$

Define the set $\mathrm{LD}(X)$ as

$$\mathrm{LD}(X) = \{T \in \mathcal{B}(X) \colon a(T) < \infty \text{ and } R(T^{a(T)+1}) \text{ is closed}\}.$$

From [21], $\mathrm{LD}(X)$ is a regularity and it is the dual version of the regularity $\mathrm{RD}(X) = \{T \in \mathcal{B}(X) \colon d(T) < \infty$ and $R(T^{d(T)})$ is closed$\}$. An operator $T \in \mathcal{B}(X)$ is said to be *left* (or *right*) *Drazin invertible* if $T \in \mathrm{LD}(X)$ ($T \in \mathrm{RD}(X)$). The *left Drazin spectrum* $\sigma_{\mathrm{lD}}(T)$ and the *right Drazin spectrum* $\sigma_{\mathrm{rD}}(T)$ are defined by $\sigma_{\mathrm{lD}}(T) = \{\lambda \in \mathbb{C} \colon T - \lambda I \notin \mathrm{LD}(X)\}$ and $\sigma_{\mathrm{rD}}(T) = \{\lambda \in \mathbb{C} \colon T - \lambda I \notin \mathrm{RD}(X)\}$. It is not difficult to see that

$$\sigma_{\mathrm{D}}(T) = \sigma_{\mathrm{lD}}(T) \cup \sigma_{\mathrm{rD}}(T).$$

2. Preliminary results

An operator $T \in \mathcal{B}(X)$ has the single-valued extension property at $\lambda_0 \in \mathbb{C}$ (the SVEP for short) if for every open disc D_{λ_0} centered at λ_0, the only analytic function $f \colon D_{\lambda_0} \longrightarrow X$ which satisfies $(T - \lambda I)f(\lambda) = 0$ for all $\lambda \in D_{\lambda_0}$ is the function $f \equiv 0$. Trivially, every operator T has the SVEP at all points of the resolvent; also T has the SVEP at $\lambda \in \operatorname{iso}\sigma(T)$ ($\operatorname{iso}\sigma(T)$ is the set of all isolated points of $\sigma(T)$). We say that T has SVEP if it has SVEP at every $\lambda \in \mathbb{C}$, [15]. We denote by $\mathcal{S}(T)$ the set of all $\lambda \in \mathbb{C}$ such that T does not have the single-valued extension property at λ. Note that (see [15], [19]) $\mathcal{S}(T) \subseteq \sigma_{\mathrm{p}}(T)$ and $\sigma(T) = \mathcal{S}(T) \cup \sigma_{\mathrm{s}}(T)$. In particular, if T (or T^*) has the SVEP then $\sigma(T) = \sigma_{\mathrm{s}}(T)$ ($\sigma(T) = \sigma_{\mathrm{a}}(T)$).

Recall that if $T - \lambda I$ has a finite ascent then it has the SVEP ([18]). Thus we have

$$\mathcal{S}(T) \subseteq \sigma_{\mathrm{lD}}(T) \text{ and } \mathcal{S}(T^*) \subseteq \sigma_{\mathrm{rD}}(T).$$

In the following theorem, we prove equality up to $\mathcal{S}(T)$ between the left Drazin spectrum, the upper semi-B-Fredholm spectrum and the semi-essential approximate point spectrum.

Theorem 2.1. *Let $T \in \mathcal{B}(X)$. Then*

$$\sigma_{\mathrm{lD}}(T) = \sigma_{\mathrm{UBF}}(T) \cup \mathcal{S}(T) = \sigma_{\mathrm{SBF}_+^-}(T) \cup \mathcal{S}(T).$$

P r o o f. Let $\lambda \notin \sigma_{\mathrm{lD}}(T)$, without loss of generality we assume that $\lambda = 0$. Then $R(T^{a(T)+1})$ is closed. Hence $R(T^{a(T)})$ is closed by [21, Lemma 12]. We shall prove that $T_{[a(T)]}$ is upper semi-Fredholm. Let $x \in N(T_{[a(T)]})$ then $x \in N(T) \cap R(T^{a(T)})$. Hence $x = T^{a(T)}y$ for some $y \in X$. Then $0 = Tx = T^{a(T)+1}y$. Thus

$y \in N(T^{a(T)+1}) = N(T^{a(T)})$. Therefore $x = 0$ and hence $T_{[a(T-\lambda I)]}$ is injective. On the other hand, $R(T_{[a(T)]}) = R(T^{a(T)+1})$ is closed. Thus $T_{[a(T)]}$ is upper semi-Fredholm and hence $0 \notin \sigma_{\mathrm{UBF}}(T)$. Since $\mathcal{S}(T) \subseteq \sigma_{\mathrm{lD}}(T)$ we have

$$\sigma_{\mathrm{UBF}}(T) \cup \mathcal{S}(T) \subseteq \sigma_{\mathrm{lD}}(T).$$

Now let $0 \notin [\sigma_{\mathrm{UBF}}(T) \cup (\mathcal{S}(T)]$, then T is an upper semi-B-Fredholm operator. Hence it follows from [7, Proposition 3.2] that there exist n such that $R(T^n)$ is closed and $T_{[n]}$ is semi-regular. Since T has the SVEP at 0 then $T_{[n]}$ has also the SVEP at 0. Then from [1, Theorem 3.14], we conclude that $T_{[n]}$ is injective with closed range. Let $x \in N(T^{n+1})$, then $TT^n x = 0$. Hence $T^n x \in N(T) \cap R(T^n) = N(T_{[n]}) = \{0\}$. Thus $x \in N(T^n)$, and hence $N(T^n) = N(T^{n+1})$. So T is of finite ascent and $a(T) \leqslant n$. We have $R(T^{n+1}) = R(T_{[n]})$ is closed with $a(T) + 1 \leqslant n + 1$. Hence $R(T^{a(T)+1})$ is closed by [21, Lemma 12]. Thus T is left Drazin invertible. Therefore $\sigma_{\mathrm{lD}}(T) \subseteq \sigma_{\mathrm{UBF}}(T) \cup \mathcal{S}(T)$.

From [13, Lemma 2.12] we have $\sigma_{\mathrm{SBF}_+^-}(T) \subseteq \sigma_{\mathrm{lD}}(T)$ and since $\sigma_{\mathrm{UBF}}(T) \subseteq \sigma_{\mathrm{SBF}_+^-}(T)$ we infer $\sigma_{\mathrm{lD}}(T) = \sigma_{\mathrm{UBF}}(T) \cup \mathcal{S}(T) = \sigma_{\mathrm{SBF}_+^-}(T) \cup \mathcal{S}(T)$. □

A useful consequence of the preceding result is that under the assumption of the SVEP for T, the spectra $\sigma_{\mathrm{lD}}(T)$, $\sigma_{\mathrm{UBF}}(T)$ and $\sigma_{\mathrm{SBF}_+^-}(T)$ are equal.

Corollary 2.1. *If $T \in \mathcal{B}(X)$ has the SVEP then*

$$\sigma_{\mathrm{lD}}(T) = \sigma_{\mathrm{UBF}}(T) = \sigma_{\mathrm{SBF}_+^-}(T).$$

By duality we get a similar result for the right Drazin spectrum.

Theorem 2.2. *Let $T \in \mathcal{B}(X)$. Then*

$$\sigma_{\mathrm{rD}}(T) = \sigma_{\mathrm{LBF}}(T) \cup \mathcal{S}(T^*) = \sigma_{\mathrm{SBF}_-^+}(T) \cup \mathcal{S}(T^*).$$

Proof. Since $\sigma_{\mathrm{LBF}}(T) = \sigma_{\mathrm{UBF}}(T^*)$, $\sigma_{\mathrm{SBF}_-^+}(T) = \sigma_{\mathrm{SBF}_+^-}(T^*)$ and $\sigma_{\mathrm{rD}}(T) = \sigma_{\mathrm{lD}}(T^*)$ the assertion follows by Theorem 2.1. □

Corollary 2.2. *If $T^* \in \mathcal{B}(X)$ has the SVEP then*

$$\sigma_{\mathrm{rD}}(T) = \sigma_{\mathrm{LBF}}(T) = \sigma_{\mathrm{SBF}_-^+}(T).$$

From Theorem 2.1 and Theorem 2.2 we get the following corollary.

Corollary 2.3. *Let $T \in \mathcal{B}(X)$. Then*

$$\sigma_{\mathrm{D}}(T) = \sigma_{\mathrm{BF}}(T) \cup [\mathcal{S}(T) \cup \mathcal{S}(T^*)] = \sigma_{\mathrm{BW}}(T) \cup [\mathcal{S}(T) \cup \mathcal{S}(T^*)]. \tag{2.1}$$

In particular if T and T^* have the SVEP then

$$\sigma_{\mathrm{D}}(T) = \sigma_{\mathrm{BF}}(T) = \sigma_{\mathrm{BW}}(T).$$

The equality in (2.1) may be refined for $\sigma_{\mathrm{D}}(T)$ and $\sigma_{\mathrm{BW}}(T)$. More precisely, we have

Theorem 2.3. *Let $T \in \mathcal{B}(X)$ then*

$$\sigma_{\mathrm{D}}(T) = \sigma_{\mathrm{BW}}(T) \cup [\mathcal{S}(T) \cap \mathcal{S}(T^*)].$$

Proof. Since $\sigma_{\mathrm{BW}}(T) \cup (\mathcal{S}(T) \cap \mathcal{S}(T^*)) \subseteq \sigma_{\mathrm{D}}(T)$ always holds, let $\lambda \notin \sigma_{\mathrm{BW}}(T) \cup (\mathcal{S}(T) \cap \mathcal{S}(T^*))$. Without loss of generality we assume that $\lambda = 0$. Then T is a B-Fredholm operator of index zero.

Case 1. If $0 \notin \mathcal{S}(T)$: Since T is a B-Fredholm operator of index zero, it follows from [8, Lemma 4.1] that there exists a Fredholm operator F of index zero and a nilpotent operator N such that $T = F \oplus N$. If $0 \notin \sigma(F)$, then F is invertible and hence T is Drazin invertible. Now assume that $0 \in \sigma(F)$. Since T has the SVEP at 0, F has also the SVEP at 0. Hence it follows from [1, Theorem 3.16] that $a(F)$ is finite. F is a Fredholm operator of index zero, hence it follows from [1, Theorem 3.4] that $d(F)$ is also finite. Then $a(F) = d(F) < \infty$ which implies that 0 is a pole of F and hence an isolated point of $\sigma(F)$. Operator N is nilpotent, hence 0 is an isolated point of $\sigma(T)$. From [8, Theorem 4.2] we get $0 \notin \sigma_D(T)$.

Case 2. If $0 \notin \mathcal{S}(T^*)$, the proof goes similarly. □

Corollary 2.4 ([12]). *If T or T^* has the SVEP then*

$$\sigma_{\mathrm{D}}(T) = \sigma_{\mathrm{BW}}(T).$$

Recall that T is a *Browder* operator if T is a Fredholm operator of finite ascent and descent. Let $\sigma_{\mathrm{B}}(T)$ be the *Browder spectrum* defined as the set of all $\lambda \in \mathbb{C}$ such that $T - \lambda I$ is not Browder. Analogously, T is a B-*Browder* operator if for some integer n, $R(T^n)$ is closed and $T_{[n]}$ is Browder. Let $\sigma_{\mathrm{BB}}(T)$ be the B-*Browder spectrum.* In [1, Corollary 3.53] it is proved that if T or T^* has the SVEP, then

$$\sigma_{\mathrm{W}}(T) = \sigma_{\mathrm{B}}(T).$$

From [7, Theorem 3.6] we have $\sigma_{\mathrm{D}}(T) = \sigma_{\mathrm{BB}}(T)$, hence by Corollary 2.4, if T or T^* has the SVEP then

$$\sigma_{\mathrm{BW}}(T) = \sigma_{\mathrm{BB}}(T).$$

Theorem 2.4. *Let $T \in \mathcal{B}(X)$ and let f be an analytic function on some open neighborhood of $\sigma(T)$ which is nonconstant on any connected component of $\sigma(T)$. Then*

$$f(\sigma_{\mathrm{BW}}(T) \cup [\mathcal{S}(T) \cap \mathcal{S}(T^*)]) = \sigma_{\mathrm{BW}}(f(T)) \cup [\mathcal{S}(f(T)) \cap \mathcal{S}(f(T^*))].$$

Proof. According to [21] the Drazin spectrum satisfies the spectral mapping theorem for such a function f, hence the result follows at once from Theorem 2.3. $\square$

It is well known that if T has the SVEP then $f(T)$ has also the SVEP [19]. Now we retrieve the result proved in [2], [23]: $f(\sigma_{\mathrm{BW}}(T)) = \sigma_{\mathrm{BW}}(f(T))$ whenever T or T^* has the SVEP. Note that in [2], [23] the condition "f is nonconstant on any connected component of $\sigma(T)$" is dropped.

3. Applications

3.1. Perturbations.

Lemma 3.1. *Let $T \in \mathcal{B}(X)$. Let $N \in \mathcal{B}(X)$ be a nilpotent operator such that $TN = NT$. Then*

$$\mathcal{S}(T + N) = \mathcal{S}(T).$$

Proof. See for instance [5, Lemma 2.1]. $\square$

Lemma 3.2. *Let $T \in \mathcal{B}(X)$. If $N \in \mathcal{B}(X)$ is a nilpotent operator which commutes with T then*

$$\sigma_{\mathrm{ID}}(T + N) = \sigma_{\mathrm{ID}}(T).$$

Proof. Assume that $\lambda = 0 \notin \sigma_{\mathrm{ID}}(T)$. Then $a(T)$ is finite and $R(T^{a(T)+1})$ is closed. Let m be the nonnegative integer such that $N^m = 0 \neq N^{m-1}$. Let $s = \max(a(T), m)$. Then

$$\begin{aligned}
(T+N)^{2s} &= \sum_{k=0}^{2s} \binom{k}{2s} T^k N^{2s-k} \\
&= \binom{0}{2s} N^{2s} + \ldots + \binom{s}{2s} T^s N^s + \binom{s+1}{2s} T^{s+1} N^{s-1} + \ldots + \binom{2s}{2s} T^{2s} \\
&= \binom{s+1}{2s} T^{s+1} N^{s-1} + \ldots + \binom{2s}{2s} T^{2s} \\
&= T^s \left[\binom{s+1}{2s} T^1 N^{s-1} + \ldots + \binom{2s}{2s} T^s \right].
\end{aligned}$$

Now let $x \in N(T)^{2s} = N(T)^s$ that is $(T)^{2s}x = 0$. Then it follows from the above equality that $(T+N)^{2s}x = 0$. Hence $N(T)^{2s} \subseteq N(T+N)^{2s}$. With the same argument for $T + N$ and $-N$ we have $N(T + N)^{2s} \subseteq N(T)^{2s}$. Thus $N(T)^{2s} = N(T + N)^{2s}$. Since $N(T^s) = N(T^{2s}) = N(T^{2s+1})$, we get $N(T+N)^{2s} = N(T+N)^{2s+1}$. Therefore $T+N$ is of finite ascent. On the other hand, $R(T+N)^{2s} \subseteq R(T^s)$ is closed. Hence by [21, Lemma 12] $R(T+N)^{2s+1}$ is closed. Thus $0 \notin \sigma_{\mathrm{lD}}(T+N)$. Hence $\sigma_{\mathrm{lD}}(T+N) \subseteq \sigma_{\mathrm{lD}}(T)$. With the same argument for $T + N$ and $-N$ we get $\sigma_{\mathrm{lD}}(T) \subseteq \sigma_{\mathrm{lD}}(T + N)$. □

The next result follows from Theorem 2.1, Lemma 3.1 and Lemma 3.2.

Theorem 3.1. *Let $T \in \mathcal{B}(X)$. Let $N \in \mathcal{B}(X)$ be a nilpotent operator which commutes with T. Then*

$$\sigma_{\mathrm{SBF}_+^-}(T + N) \cup \mathcal{S}(T) = \sigma_{\mathrm{SBF}_+^-}(T) \cup \mathcal{S}(T).$$

The following corollary which is proved in [3] gives an affirmative answer to the question posed by Berkani-Amouch [9] in the case when T has the SVEP.

Corollary 3.1. *Let $T \in \mathcal{B}(X)$ have the SVEP. Let $N \in \mathcal{B}(X)$ be a nilpotent operator which commutes with T. Then*

$$\sigma_{\mathrm{SBF}_+^-}(T + N) = \sigma_{\mathrm{SBF}_+^-}(T).$$

3.2. Generalized Weyl's theorem for operator matrices. Berkani [8, Theorem 4.5] has shown that every normal operator T acting on a Hilbert space H satisfies

$$\sigma(T) \setminus E(T) = \sigma_{\mathrm{BW}}(T), \tag{3.1}$$

where $E(T)$ is the set of all isolated eigenvalues of T. We say that the *generalized Weyl's theorem* holds for T if equality (3.1) holds. This gives a generalization of the classical Weyl's theorem. Recall that $T \in \mathcal{B}(X)$ obeys *Weyl's theorem* if

$$\sigma(T) \setminus E_0(T) = \sigma_{\mathrm{W}}(T) \tag{3.2}$$

where $E_0(T)$ denotes the set of the isolated points of $\sigma(T)$ which are eigenvalues of finite multiplicity. By [13, Theorem 3.9] the generalized Weyl's theorem implies Weyl's theorem and generally the reverse is not true.

For $A \in \mathcal{B}(X)$, $B \in \mathcal{B}(Y)$ and $C \in \mathcal{B}(Y, X)$ we denote by M_C the operator defined on $X \oplus Y$ by

$$M_C = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix}.$$

In general the fact that the generalized Weyl's theorem holds for A and B does not imply that the generalized Weyl's theorem holds for $M_0 = \left[\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right]$. Indeed, let I_1 and I_2 be the identities on $\mathbb{C}$ and l_2, respectively. Let S_1 and S_2 be defined on l_2 by

$$S_1(x_1, x_2, \ldots) = (0, \tfrac{1}{3}x_1, \tfrac{1}{3}x_2, \ldots), \quad S_2(x_1, x_2, \ldots) = (0, \tfrac{1}{2}x_1, \tfrac{1}{3}x_2, \ldots).$$

Let $T_1 = I_1 \oplus S_1$, $T_2 = S_2 - I_2$, $A = T_1^2$ and $B = T_2^2$, then from [23, Example 1] we have A and B obey the generalized Weyl's theorem but M_0 does not obey it. It also may happen that M_C obeys the generalized Weyl's theorem while M_0 does not obey it. Let A be the unilateral unweighted shift operator. For $B = A^*$ and $C = I - AA^*$, we have that M_C is unitary without eigenvalues. Hence M_C satisfies the generalized Weyl's theorem (see [10, Remark 3.5]). But $\sigma_{\mathrm{W}}(M_0) = \{\lambda\colon\ |\lambda| = 1\}$ and $\sigma(M_0) \setminus E_0(M_0) = \{\lambda\colon\ |\lambda| \leqslant 1\}$. Hence M_0 does not satisfy the Weyl's theorem and so by [13, Theorem 3.9] it does not satisfy the generalized Weyls theorem either.

A bounded linear operator T is said to be *isoloid* if every isolated point of $\sigma(T)$ is an eigenvalue of T.

Proposition 3.1. *Let A and B be isoloids. Assume that $\sigma_{\mathrm{BW}}(M_0) = \sigma_{\mathrm{BW}}(A) \cup \sigma_{\mathrm{BW}}(B)$. If A and B obey the generalized Weyl's theorem, then M_0 obeys the generalized Weyl's theorem.*

P r o o f. Since A and B are isoloids, we have

$$E(M_0) = [E(A) \cap \varrho(B)] \cup [\varrho(A) \cap E(B)] \cup [E(A) \cap E(B)].$$

Now if A and B obey the generalized Weyl's theorem, then

$$\begin{aligned} E(M_0) &= [\sigma(A) \cup \sigma(B)] \setminus [\sigma_{\mathrm{BW}}(A) \cup \sigma_{\mathrm{BW}}(B)] \\ &= \sigma(M_0) \setminus \sigma_{\mathrm{BW}}(M_0). \end{aligned}$$

Then M_0 obeys the generalized Weyl's theorem. □

Lemma 3.3. *Let $A \in \mathcal{B}(X)$ and $B \in \mathcal{B}(Y)$ have the SVEP. Then*

$$\sigma_{\mathrm{BW}}(M_C) = \sigma_{\mathrm{BW}}(A) \cup \sigma_{\mathrm{BW}}(B)$$

for all $C \in \mathcal{B}(Y, X)$.

P r o o f. Since A and B have the SVEP, then it follows from [17, Proposition 3.1] that M_C also has the SVEP. Hence $\sigma_{\mathrm{BW}}(M_C) = \sigma_{\mathrm{D}}(M_C)$ by Corollary 2.4. Also since A and B have the SVEP, it follows from [24, Corollary 2.1] that $\sigma_{\mathrm{D}}(M_C) = \sigma_{\mathrm{D}}(A) \cup \sigma_{\mathrm{D}}(B)$. Therefore $\sigma_{\mathrm{BW}}(M_C) = \sigma_{\mathrm{BW}}(A) \cup \sigma_{\mathrm{BW}}(B)$ by Corollary 2.4. □

Theorem 3.2. *Let A and B be isoloids with the SVEP. If A and B obey the generalized Weyl's theorem, then M_C obeys the generalized Weyl's theorem for every $C \in \mathcal{B}(Y, X)$.*

Proof. It follows from Proposition 3.1 and Lemma 3.3 that

$$E(M_0) = \sigma(M_0) \setminus \sigma_{\mathrm{BW}}(M_0) = \sigma(M_C) \setminus \sigma_{\mathrm{BW}}(M_C).$$

Hence it is enough to show that $E(M_0) = E(M_C)$. Let $\lambda \in E(M_C)$. Then $\lambda \in \sigma_{\mathrm{p}}(M_C) \subseteq \sigma_{\mathrm{p}}(A) \cup \sigma_{\mathrm{p}}(B)$. Hence $\lambda \in \sigma_{\mathrm{p}}(M_0)$. Since $\lambda \in \operatorname{iso}\sigma(M_C) = \operatorname{iso}\sigma(M_0)$ we have $\lambda \in E(M_0)$. Now let $\lambda \in E(M_0)$. If $\lambda \in \sigma(A)$ then $\lambda \in \operatorname{iso}\sigma(A)$. Since A is an isoloid, we have $\lambda \in \sigma_{\mathrm{p}}(A) \subseteq \sigma_{\mathrm{p}}(M_C)$. Hence $\lambda \in E(M_C)$. If $\lambda \in \sigma(B) \setminus \sigma(A)$, then $\lambda \in \sigma_{\mathrm{p}}(B)$. Since A is invertible, we conclude that $\lambda \in \sigma_{\mathrm{p}}(M_C)$. Thus $\lambda \in E(M_C)$. Therefore $E(M_0) = E(M_C)$. $\square$

Let $\pi(T)$ be the set of all poles of the resolvent of T. Recall from [14] that T is a *polaroid* if $\operatorname{iso}\sigma(T) \subseteq \pi(T)$. Since $\pi(T) \subseteq E(T)$ holds without restriction on T, then if T is a polaroid then $E(T) = \pi(T)$.

Corollary 3.2. *Let A and B be polaroids with the SVEP. Then M_C obeys the generalized Weyl's theorem for every $C \in \mathcal{B}(Y, X)$.*

Proof. A and B are polaroids hence $E(A) = \pi(A)$ and $E(B) = \pi(B)$. Since A and B have the SVEP, we have by [4] that A and B satisfy the generalized Weyl's theorem. Hence we complete the proof by Theorem 3.2. $\square$

3.3. Multipliers on a commutative Banach algebra. Let $\mathcal{A}$ be a semi-simple commutative Banach algebra. A mapping $T\colon \mathcal{A} \longrightarrow \mathcal{A}$ is called a *multiplier* if

$$T(x)y = xT(y) \quad \text{for all } x,\ y \in \mathcal{A}.$$

By semi-simplicity of $\mathcal{A}$, every multiplier is a bounded linear operator on $\mathcal{A}$. Also the semi-simplicity of $\mathcal{A}$ implies that every multiplier has the SVEP (see [1], [19]).

By [1, Theorem 4.36], for every multiplier T on a semi-simple commutative Banach algebra $\mathcal{A}$, $E(T) = \pi(T)$ and since T has the SVEP we get from [4]

Proposition 3.2. *Every multiplier on a semi-simple commutative Banach algebra $\mathcal{A}$ obeys the generalized Weyl's theorem.*

From Corollary 2.4 we have

Proposition 3.3 ([11]). *Let T be a multiplier on a semi-simple commutative Banach algebra $\mathcal{A}$. Then the following assertions are equivalent:*

i) *T is B-Fredholm of index zero.*

ii) *T is Drazin invertible.*

Now if we assume in addition that $\mathcal{A}$ is regular and Tauberian (see [19] for definition) then every multiplier T has the weak decomposition property ($\delta_{\rm w}$) and then T^* has also the SVEP (see [22] for definition and details). Hence we get from Corollary 2.3

Proposition 3.4. *Let T be a multiplier on a semi-simple regular Tauberian commutative Banach algebra $\mathcal{A}$. Then the following assertions are equivalent:*

i) *T is B-Fredholm.*

ii) *T is Drazin invertible.*

For G a locally compact abelian group, let $L^1(G)$ be the space of $\mathbb{C}$-valued functions on G integrable with respect to Haar measure and $M(G)$ the Banach algebra of regular complex Borel measures on G. We recall that $L^1(G)$ is a regular semi-simple Tauberian commutative Banach algebra. Then we have

Corollary 3.3. *Let G be a locally compact abelian group, $\mu \in M(G)$ and $X = L^1(G)$. Then every convolution operator $T_\mu\colon X \longrightarrow X$, $T_\mu(k) = \mu \star k$ is B-Fredholm if and only if it is Drazin invertible.*

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