ON KURZWEIL-STIELTJES INTEGRAL IN A BANACH SPACE

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Abstract. In the paper we deal with the Kurzweil-Stieltjes integration of functions having values in a Banach space X. We extend results obtained by Štefan Schwabik and complete the theory so that it will be well applicable to prove results on the continuous dependence of solutions to generalized linear differential equations in a Banach space. By Schwabik, the integral $\int_a^b \mathrm{d}[F]g$ exists if $F\colon [a,b] \to L(X)$ has a bounded semi-variation on $[a,b]$ and $g\colon [a,b] \to X$ is regulated on $[a,b]$. We prove that this integral has sense also if F is regulated on $[a,b]$ and g has a bounded semi-variation on $[a,b]$. Furthermore, the integration by parts theorem is presented under the assumptions not covered by Schwabik (2001) and Naralenkov (2004), and the substitution formula is proved.

Keywords: Kurzweil-Stieltjes integral, substitution formula, integration-by-parts

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1. Introduction

It is known that integration processes based on Riemann type sums, such as Kurzweil and McShane integrals, can be extended to Banach space-valued functions. Among other contributions it is worth to highlight the monograph by Schwabik and Ye [13], which studies these types of integrals and their connections e.g. with the classical ones due to Bochner and Pettis.

Concerning integrals of Stieltjes type, Hönig presented a quite complete study in [4] dealing with the interior integral. In [8] and [11] Schwabik investigated some fundamental properties of the Kurzweil-Stieltjes integral.

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The concepts of generalized (nonlinear) Kurzweil or Kurzweil-Stieltjes integrals in a Banach space have been the background of several papers related to generalized differential equations like e.g. [2], [3], [9] and [10].

In this paper we are dealing with the Kurzweil-Stieltjes integral. Our aim is to supplement the existing knowledge by results needed for treating generalized linear differential equations. In particular, we prove that if $F\colon [a,b] \to L(X)$ and $g\colon [a,b] \to X$, then the integral $\int_a^b \mathrm{d}[F]g$ exists provided F is regulated on $[a,b]$ and g has a bounded semi-variation on $[a,b]$, and the integral $\int_a^b F\,\mathrm{d}[g]$ exists provided F has a bounded semi-variation and g is regulated. Furthermore, the integration by parts theorem is presented under the assumptions not covered by those by Schwabik [11] (see Theorems 10, 13, 15 and Corollary 14) or Naralenkov [6] (see Section 3). Finally, the substitution formula is proved.

2. Preliminaries

Throughout these notes X is a Banach space and $L(X)$ is the Banach space of bounded linear operators on X. By $\|\cdot\|_X$ we denote the norm in X. Similarly, $\|\cdot\|_{L(X)}$ denotes the usual operator norm in $L(X)$.

Assume that $-\infty < a < b < +\infty$ and $[a,b]$ denotes the corresponding closed interval. A set $D = \{\alpha_0, \alpha_1, \ldots, \alpha_m\} \subset [a,b]$ is said to be a division of $[a,b]$ if

$$a = \alpha_0 < \alpha_1 < \ldots < \alpha_m = b.$$

The set of all divisions of $[a,b]$ is denoted by $\mathcal{D}[a,b]$.

A function $f\colon [a,b] \to X$ is called a finite step function on $[a,b]$ if there exists a division $D = \{\alpha_0, \alpha_1, \ldots, \alpha_m\}$ of $[a,b]$ such that f is constant on every open interval (α_{j-1}, α_j), $j = 1, 2, \ldots, m$.

For an arbitrary function $f\colon [a,b] \to X$ we set

$$\|f\|_\infty = \sup_{t\in[a,b]} \|f(t)\|_X$$

and

$$\operatorname{var}_a^b f = \sup_{D\in\mathcal{D}[a,b]} \sum_{j=1}^m \|f(\alpha_j) - f(\alpha_{j-1})\|_X$$

is the variation of f over $[a,b]$. If $\operatorname{var}_a^b f < \infty$ we say that f is a function of bounded variation on $[a,b]$. $BV([a,b],X)$ denotes the Banach space of functions $f\colon [a,b] \to X$ of bounded variation on $[a,b]$ equipped with the norm $\|f\|_{BV} = \|f(a)\|_X + \operatorname{var}_a^b f$.

For $F\colon [a,b] \to L(X)$ and a division $D = \{\alpha_0, \alpha_1, \dots, \alpha_m\}$ of the interval $[a,b]$, let

$$V_a^b(F,D) = \sup \left\{ \left\| \sum_{j=1}^{m} [F(\alpha_j) - F(\alpha_{j-1})] y_j \right\|_X \right\},$$

where the supremum is taken over all possible choices of $y_j \in X$, $j = 1,2,\dots,m$, with $\|y_j\|_X \leqslant 1$. Then

$$(\mathcal{B}) \operatorname{var}_a^b(F) = \sup\{V_a^b(x,D);\ D \in \mathcal{D}[a,b]\}$$

is said to be the *semi-variation* of F on $[a,b]$, cf. e.g. [4]. Sometimes it is called also the $\mathcal{B}$-*variation* of F on $[a,b]$ (with respect to the bilinear triple $\mathcal{B} = (L(X), X, X)$, cf. e.g. [8]). Analogously, we can define the $\mathcal{B}$-variation of a function $f\colon [a,b] \to X$ using

$$V_a^b(f,D) = \sup \left\{ \left\| \sum_{j=1}^{m} F_j [f(\alpha_j) - f(\alpha_{j-1})] \right\|_X \right\},$$

where the supremum is taken over all possible choices of operators $F_j \in L(X)$ with $\|F_j\|_{L(X)} \leqslant 1$, $j = 1,2,\dots,m$.

The set of all functions $F\colon [a,b] \to L(X)$ with $(\mathcal{B}) \operatorname{var}_a^b(F) < \infty$ is denoted by $(\mathcal{B})BV([a,b], L(X))$. Analogously to $BV([a,b],X)$, $(\mathcal{B})BV([a,b], L(X))$ is a Banach space with respect to the norm

$$F \in (\mathcal{B})BV([a,b], L(X)) \mapsto \|F\|_{SV} = \|F(a)\|_{L(X)} + (\mathcal{B}) \operatorname{var}_a^b F$$

(cf. [12]).

A function $f\colon [a,b] \to X$, is said to be *regulated* on $[a,b]$ if for each $t \in [a,b)$ there is $f(t+) \in X$ such that

$$\lim_{s \to t+} \|f(s) - f(t+)\|_X = 0$$

and for each $t \in (a,b]$ there is $f(t-) \in X$ such that

$$\lim_{s \to t-} \|f(s) - f(t-)\|_X = 0.$$

By $G([a,b],X)$ we denote the set of all regulated functions $f\colon [a,b] \to X$. For $t \in [a,b)$, $s \in (a,b]$ we put $\Delta^+ f(t) = f(t+) - f(t)$ and $\Delta^- f(s) = f(s) - f(s-)$.

A function $F\colon [a,b] \to L(X)$ is $\mathcal{B}$-*regulated* on $[a,b]$ if for every $x \in X$ with $\|x\|_X \leqslant 1$, the function $t \in [a,b] \longmapsto F(t)x$ is regulated. Similarly we say that $f\colon [a,b] \to X$ is $\mathcal{B}$-regulated on $[a,b]$, if the function $t \in [a,b] \longmapsto Tf(t)$ is regulated for all $T \in L(X)$ with $\|T\|_{L(X)} \leqslant 1$. The set of all $(\mathcal{B})$-regulated functions $F\colon [a,b] \to L(X)$ is denoted by $(\mathcal{B})G([a,b], L(X))$.

Recall that $BV([a,b],X) \subset (\mathcal{B})BV([a,b],X)$ and

$$BV([a,b],X) \subset G([a,b],X) \subset (\mathcal{B})G([a,b],X)$$

while $G([a,b],X) \not\subseteq (\mathcal{B})BV([a,b],X)$ (cf. e.g. [9, 1.5]). If $\dim X < \infty$, then obviously

$$BV([a,b],X) = (\mathcal{B})BV([a,b],X) \quad \text{and} \quad G([a,b],X) = (\mathcal{B})G([a,b],X).$$

Moreover, it is known that regulated functions are uniform limits of finite step functions (see [4, Theorem I.3.1]).

Now, let us recall the definition and some crucial properties of the Kurzweil-Stieltjes integral.

As usual, tagged systems $P = (D,\xi) \in \mathcal{D}[a,b] \times [a,b]^m$ where $D = \{\alpha_0, \alpha_1, \ldots, \alpha_m\}$, $\xi = (\xi_1, \xi_2, \ldots, \xi_m)$, are called *partitions* of $[a,b]$ if

$$\alpha_{j-1} \leqslant \xi_j \leqslant \alpha_j \quad \text{for } j = 1,2,\ldots,m.$$

The set of all partitions of $[a,b]$ is denoted by $\mathcal{P}[a,b]$.

Furthermore, functions $\delta\colon [a,b] \to (0,\infty)$ are said to be *gauges* on $[a,b]$. Given a gauge δ, the partition $P = (D,\xi)$ with $D = \{\alpha_0, \alpha_1, \ldots, \alpha_m\}$, $\xi = (\xi_1, \xi_2, \ldots, \xi_m)$, is *$\delta$-fine* if

$$[\alpha_{j-1}, \alpha_j] \subset (\xi_j - \delta(\xi_j), \xi_j + \delta(\xi_j)) \quad \text{for } j = 1,2,\ldots,m.$$

We remark that for an arbitrary gauge δ on $[a,b]$ there always exists a δ-fine partition of $[a,b]$. This is stated by the Cousin lemma (see [7, Lemma 1.4]).

For given functions $F\colon [a,b] \to L(X)$ and $g\colon [a,b] \to X$ and a partition $P = (D,\xi)$ of $[a,b]$, where $D = \{\alpha_0, \alpha_1, \ldots, \alpha_m\}$, $\xi = (\xi_1, \ldots, \xi_m)$, we define

$$S(F, \mathrm{d}g, P) = \sum_{j=1}^{m} F(\xi_j)[g(\alpha_j) - g(\alpha_{j-1})]$$

and

$$S(\mathrm{d}F, g, P) = \sum_{j=1}^{m} [F(\alpha_j) - F(\alpha_{j-1})]g(\xi_j).$$

We say that $I \in X$ is the Kurzweil-Stieltjes integral (or shortly KS-integral) of F with respect to g on $[a,b]$ and write

$$I = \int_a^b F \,\mathrm{d}[g]$$

if for every $\varepsilon > 0$ there exists a gauge δ on $[a,b]$ such that

$$\|S(F, \mathrm{d}g, P) - I\|_X < \varepsilon \quad \text{for all } \delta\text{-fine partitions } P \text{ of } [a,b].$$

Similarly, $J \in X$ is the KS-integral of g with respect to F on $[a,b]$ if for every $\varepsilon > 0$ there exists a gauge δ on $[a,b]$ such that

$$\|S(\mathrm{d}F, g, P) - J\|_X < \varepsilon \quad \text{for all } \delta\text{-fine partitions } P \text{ of } [a,b].$$

In this case we write $J = \int_a^b \mathrm{d}[F]g$.

Analogously, if $H\colon [a,b] \to L(X)$, we define the integral $\int_a^b H\,\mathrm{d}[F]g$ using sums of the form

$$S(H, \mathrm{d}F, g, P) = \sum_{j=1}^{m} H(\xi_j)[F(\alpha_j) - F(\alpha_{j-1})]g(\xi_j).$$

The KS-integral is linear and additive with respect to intervals. Basic results concerning the KS-integral can be found in [8] and [15]. Obviously, if the Riemann-Stieltjes integral $(\mathrm{RS})\int_a^b F\,\mathrm{d}[g]$ exists, then the KS-integral $\int_a^b F\,\mathrm{d}[g]$ also exists and

$$\int_a^b F\,\mathrm{d}[g] = (\mathrm{RS})\int_a^b F\,\mathrm{d}[g].$$

Some further results needed later are summarized in the following assertions:

Proposition 2.1. *Let $F\colon [a,b] \to L(X)$ and $g\colon [a,b] \to X$.*

(i) [8, Proposition 10] *Let $F \in (\mathcal{B})BV([a,b], L(X))$ and $g\colon [a,b] \to X$ be such that $\int_a^b \mathrm{d}[F]g$ exists. Then*

$$\left\| \int_a^b \mathrm{d}[F]g \right\|_X \leqslant ((\mathcal{B})\operatorname{var}_a^b F)\|g\|_\infty.$$

(ii) [8, Proposition 11] *Let $F \in (\mathcal{B})BV([a,b], L(X))$ and $g_n\colon [a,b] \to X$ be such that $\int_a^b \mathrm{d}[F]g_n$ exists for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \|g_n - g\|_\infty = 0$. Then*

$$\int_a^b \mathrm{d}[F]g \quad \text{exists} \qquad \text{and} \qquad \int_a^b \mathrm{d}[F]g = \lim_{n\to\infty} \int_a^b \mathrm{d}[F]g_n.$$

(iii) [8, Proposition 15] *Let $F \in (\mathcal{B})BV([a,b], L(X)) \cap (\mathcal{B})G([a,b], L(X))$ and let $g \in G([a,b], X)$. Then $\int_a^b \mathrm{d}[F]g$ exists.*

(iv) [11, Theorem 13] *Let* $F \in G([a,b],L(X)) \cap (\mathcal{B})BV([a,b],L(X))$ *and let* $g \in BV([a,b],X)$. *Then both the integrals* $\int_a^b F\,\mathrm{d}[g]$ *and* $\int_a^b \mathrm{d}[F]g$ *exist, the sum*

$$\sum_{a\leqslant\tau<b} \Delta^+F(\tau)\Delta^+g(\tau) - \sum_{a<\tau\leqslant b} \Delta^-F(\tau)\Delta^-g(\tau)$$

converges in X *and the equality*

$$\begin{aligned}&\int_a^b F\,\mathrm{d}[g] + \int_a^b \mathrm{d}[F]g \\ &\quad = F(b)g(b) - F(a)g(a) - \sum_{a\leqslant t<b} \Delta^+F(t)\Delta^+g(t) + \sum_{a<t\leqslant b} \Delta^-F(t)\Delta^-g(t)\end{aligned}$$

is true.

3. Main results

In this section we will present our main results. First, we will prove some auxiliary properties of the KS-integral which, in the case that $X \neq \mathbb{R}^n$, are not available in literature.

Lemma 3.1. (i) *Let* $F \in (\mathcal{B})BV([a,b],L(X))$, $g \in G([a,b],X)$ *be such that* $\int_a^b F\,\mathrm{d}[g]$ *exists. Then*

$$\|S(F,\mathrm{d}g,P)\|_X \leqslant 2\|F\|_{SV}\|g\|_\infty \quad \text{for each } P \in \mathcal{P}[a,b] \tag{3.1}$$

and

$$\Big\|\int_a^b F\,\mathrm{d}[g]\Big\|_X \leqslant 2\|F\|_{SV}\|g\|_\infty. \tag{3.2}$$

(ii) *Let* $F \in G([a,b],L(X))$, $g \in (\mathcal{B})BV([a,b],X)$ *be such that* $\int_a^b \mathrm{d}[F]g$ *exists. Then*

$$\|S(\mathrm{d}F,g,P)\|_X \leqslant 2\|F\|_\infty\|g\|_{SV} \quad \text{for each } P \in \mathcal{P}[a,b] \tag{3.3}$$

and

$$\Big\|\int_a^b \mathrm{d}[F]g\Big\|_X \leqslant 2\|F\|_\infty\|g\|_{SV}. \tag{3.4}$$

P r o o f. It is easy to check that for an arbitrary partition $P=(D,\xi)$ of $[a,b]$ with $D=\{\alpha_0,\alpha_1,\ldots,\alpha_m\}$ and $\xi=(\xi_1,\xi_2,\ldots,\xi_m)$, we have

$$\begin{aligned}
&S(F,\mathrm{d}g,P)\\
&\quad=F(\xi_1)[g(\alpha_1)-g(a)]+F(\xi_2)[g(\alpha_2)-g(\alpha_1)]+\ldots+F(\xi_m)[g(b)-g(\alpha_{m-1})]\\
&\quad=F(b)g(b)-F(a)g(a)\\
&\qquad-[F(\xi_1)-F(a)]g(a)-[F(\xi_2)-g(\xi_1)]-\ldots-[F(b)-F(\xi_m)]g(b)\\
&\quad=F(b)g(b)-F(a)g(a)-\sum_{j=0}^{m}[F(\xi_{j+1})-F(\xi_j)]g(\alpha_j),
\end{aligned}$$

where $\xi_0=a$ and $\xi_{m+1}=b$. Consequently,

$$\begin{aligned}
&\|S(F,\mathrm{d}g,P)\|_X\\
&\qquad\leqslant(\|F(a)\|_{L(X)}+\|F(b)\|_{L(X)})\|g\|_\infty\\
&\qquad\quad+\left\|\sum_{j=0}^{m}[F(\xi_{j+1})-F(\xi_j)]\frac{g(\alpha_i)}{\|g(\alpha_i)\|_X}\|g(\alpha_i)\|_X\right\|_X\\
&\qquad\leqslant\left(\|F(a)\|_{L(X)}+\|F(b)\|_{L(X)}+\left\|\sum_{j=0}^{m}[F(\xi_{j+1})-F(\xi_j)]\frac{g(\alpha_i)}{\|g(\alpha_i)\|_X}\right\|_X\right)\|g\|_\infty\\
&\qquad\leqslant(\|F(a)\|_{L(X)}+\|F(b)\|_{L(X)}+(\mathcal{B})\operatorname{var}_a^b F)\|g\|_\infty\leqslant 2\|F\|_{SV}\|g\|_\infty,
\end{aligned}$$

i.e. (3.1) is true.

Now, let an arbitrary $\varepsilon>0$ be given. By our assumptions there is a gauge δ on $[a,b]$ such that

$$\left\|S(F,\mathrm{d}g,P)-\int_a^b F\,\mathrm{d}[g]\right\|_X<\varepsilon\quad\text{whenever } P \text{ is } \delta\text{-fine.}$$

Let P_0 be an arbitrary δ-fine partition of $[a,b]$. Then by (3.1) we have

$$\begin{aligned}
\left\|\int_a^b F\,\mathrm{d}[g]\right\|_X\leqslant\left\|S(F,\mathrm{d}g,P_0)-\int_a^b F\,\mathrm{d}[g]\right\|_X&+\|S(F,\mathrm{d}g,P_0)\|_X\\
&<\varepsilon+2\|F\|_{SV}\|g\|_\infty.
\end{aligned}$$

Since $\varepsilon>0$ can be arbitrary, it follows that inequality (3.2) is true.

The proof of (3.3) and (3.4) can be obtained in a similar way. $\square$

Lemma 3.2. *Let $g\colon [a,b] \to X$ be a finite step function. Then for any $F\colon [a,b] \to L(X)$ the integral $\int_a^b F\,\mathrm{d}[g]$ exists.*

P r o o f. One can check that $g\colon [a,b] \to L(X)$ is a finite step function if and only if it is a finite linear combination of functions of the form

$$\chi_{[a,\tau]}(t)\tilde{x},\ \chi_{[\sigma,b]}(t)\tilde{y},\ \chi_{[a]}(t)\tilde{z},\ \chi_{[b]}(t)\tilde{w},$$

where τ,σ are some points from (a,b) and $\tilde{x},\tilde{y},\tilde{z},\tilde{w}$ may be arbitrary elements of X. Hence, by the linearity of the integral, it is sufficient to prove that the integral exists for functions g of the form

$$\chi_{[a,\tau]}\tilde{x},\ \chi_{[\tau,b]}\tilde{x},\ \chi_{[a]}\tilde{x},\ \chi_{[b]}\tilde{x},$$

where $\tau \in (a,b)$ and $\tilde{x} \in X$.

Let $\tau \in (a,b)$, $\tilde{x} \in X$ and $g = \tilde{x}\chi_{[a,\tau]}$. Given $\varepsilon > 0$ define

$$\delta(t) = \begin{cases} \varepsilon & \text{if } t = \tau, \\ \frac{1}{2}|\tau - t| & \text{if } t \neq \tau. \end{cases}$$

Then for any δ-fine partition P of $[a,b]$, τ is the tag and $S(F,\mathrm{d}g,P) = -F(\tau)\tilde{x}$. Hence

$$\int_a^b F\,\mathrm{d}[g] = -F(\tau)\tilde{x}.$$

The proofs of the cases $g = \chi_{[\tau,b]}\tilde{x}$, $g = \chi_{[a]}\tilde{x}$ and $g = \chi_{[b]}\tilde{x}$ are analogous. $\square$

The next theorem is the first main result of this paper. It supplements Schwabik's result in Prop. 2.1 (iii) and the results known for $\dim X < \infty$, see [14], [16], [17].

Theorem 3.3. (i) *If $F \in G([a,b],L(X))$, $g \in (\mathcal{B})BV([a,b],X)$, then the integral $\int_a^b \mathrm{d}[F]g$ exists.*

(ii) *If $F \in (\mathcal{B})BV([a,b],L(X))$, $g \in G([a,b],X)$, then the integral $\int_a^b F\,\mathrm{d}[g]$ exists.*

P r o o f. (i) Let $F_n\colon [a,b] \to L(X)$, $n \in \mathbb{N}$, be a sequence of finite step functions such that

$$\lim_{n\to\infty} \|F_n - F\|_\infty = 0.$$

Since $F_n \in BV([a,b],L(X))$ for each $n \in \mathbb{N}$, it follows from Proposition 2.1 (iii) that for each $n \in \mathbb{N}$ the integral $\int_a^b \mathrm{d}[F_n]g$ exists. Moreover, these integrals define a Cauchy sequence in the Banach space X. Indeed, given $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ such that $\|F_n - F\|_\infty < \varepsilon$ for $n \geqslant n_0$. Thus, using Lemma 3.1, we obtain

$$\left\| \int_a^b \mathrm{d}[F_n - F_m]g \right\|_X \leqslant 2\|F_n - F_m\|_\infty \|g\|_{SV} \leqslant 4\varepsilon\|g\|_{SV} \quad \text{for all } m,n \geqslant n_0.$$

Therefore there is $I \in X$ such that $I = \lim_{n\to\infty} \int_a^b \mathrm{d}[F_n]g$. This implies that there exists $N \in \mathbb{N}$ such that $N > n_0$ and

$$\left\| \int_a^b \mathrm{d}[F_N]g - I \right\|_X < \varepsilon.$$

Let δ be a gauge on $[a,b]$ such that

$$\left\| S(\mathrm{d}F_N, g, P) - \int_a^b \mathrm{d}[F_N]g \right\|_X < \varepsilon \quad \text{whenever } P \text{ is } \delta\text{-fine.}$$

Having this in mind and using (3.3), for an arbitrary δ-fine partition P of $[a,b]$ we get

$$\begin{aligned}
&\|S(\mathrm{d}F, g, P) - I\|_X \\
&\qquad \leqslant \|S(\mathrm{d}F, g, P) - S(\mathrm{d}F_N, g, P)\|_X \\
&\qquad\quad + \left\| S(\mathrm{d}F_N, g, P) - \int_a^b \mathrm{d}[F_N]g \right\|_X + \left\| \int_a^b \mathrm{d}[F_N]g - I \right\|_X \\
&\qquad < 2\|F - F_N\|_\infty \|g\|_{SV} + 2\varepsilon < 2\varepsilon(\|g\|_{SV} + 1),
\end{aligned}$$

which concludes the proof of the assertion (i).

The assertion (ii) can be proved by the same arguments using Lemma 3.2 instead of Proposition 2.1. □

The following assertion is a direct consequence of Lemma 3.1 and Theorem 3.3.

Corollary 3.4. (i) *Let $g, g_n \in G([a,b],X)$, $n \in \mathbb{N}$, and $\lim_{n\to\infty} \|g_n - g\|_\infty = 0$. Then for any $F \in (\mathcal{B})BV([a,b],L(X))$, the integrals*

$$\int_a^b F\,\mathrm{d}[g] \quad \textit{and} \quad \int_a^b F\,\mathrm{d}[g_n],\ n \in \mathbb{N},$$

exist and

$$\lim_{n\to\infty} \int_a^b F\,\mathrm{d}g_n = \int_a^b F\,\mathrm{d}g.$$

(ii) *Let $F, F_n \in G([a,b],L(X)), n \in \mathbb{N}$, and $\lim_{n\to\infty} \|F_n - F\|_\infty = 0$. Then for any $g \in (\mathcal{B})BV([a,b],X)$, the integrals*

$$\int_a^b \mathrm{d}[F]g \quad \textit{and} \quad \int_a^b \mathrm{d}[F_n]g,\ n \in \mathbb{N},$$

exist and

$$\lim_{n\to\infty}\int_a^b \mathrm{d}[F_n]g = \int_a^b \mathrm{d}[F]g.$$

Thanks to Theorem 3.3, we are now also able to extend the Integration by Parts Theorem by Schwabik (cf. Proposition 2.1 (iv) or [11, Theorem 10]) and the Substitution Theorems by Federson (cf. [1, Theorems 11 and 12]) to the form suitable for dealing with generalized differential equations in a Banach space as needed in our forthcoming paper [5]. This will be the content of the rest of these notes.

Lemma 3.5. (i) *If $F \in G([a,b],L(X))$ and $g \in (\mathcal{B})BV([a,b],X)\cap G([a,b],X)$, then*

$$\Big\|\sum_{t\in[a,b)}\Delta^+F(t)\Delta^+g(t)-\sum_{t\in(a,b]}\Delta^-F(t)\Delta^-g(t)\Big\|_X \leqslant 4\|F\|_\infty((\mathcal{B})\operatorname{var}_a^b g). \tag{3.5}$$

(ii) *If $F \in (\mathcal{B})BV([a,b],L(X))\cap G([a,b],L(X))$ and $g\in G([a,b],X)$, then*

$$\Big\|\sum_{t\in[a,b)}\Delta^+F(t)\Delta^+g(t)-\sum_{t\in(a,b]}\Delta^-F(t)\Delta^-g(t)\Big\|_X \leqslant 4((\mathcal{B})\operatorname{var}_a^b F)\|g\|_\infty.$$

P r o o f. (i) Let $F \in G([a,b],L(X))$ and $g \in (\mathcal{B})BV([a,b],X)\cap G([a,b],X)$. By [11, Lemma 11] the series in (3.5) converge. It is known that the points of discontinuities of a regulated function are at most countable (see [4, Corollary I.3.2.b]). Let $\{s_k\}$ be the set of common points of discontinuity of the functions F and g in (a,b), so we can write

$$\begin{aligned}\sum_{t\in[a,b)}&\Delta^+F(t)\Delta^+g(t)-\sum_{t\in(a,b]}\Delta^-F(t)\Delta^-g(t)\\ &=\Delta^+F(a)\Delta^+g(a)-\Delta^-F(b)\Delta^-g(b)\\ &\quad+\sum_{k=1}^{\infty}[\Delta^+F(s_k)\Delta^+g(s_k)-\Delta^-F(s_k)\Delta^-g(s_k)].\end{aligned}$$

For $n\in\mathbb{N}$, define

$$\begin{aligned}S_n&=\Delta^+F(a)\Delta^+g(a)-\Delta^-F(b)\Delta^-g(b)\\ &\quad+\sum_{k=1}^{n}[\Delta^+F(s_k)\Delta^+g(s_k)-\Delta^-F(s_k)\Delta^-g(s_k)].\end{aligned}$$

Let $\varepsilon>0$ and $n\in\mathbb{N}$ be given and let $\{t_1,t_2,\dots,t_n\}\subset(a,b)$ be such that

$$\{t_1,t_2,\dots,t_n\}=\{s_1,s_2,\dots,s_n\}\quad\text{and}\quad a<t_1<t_2<\ldots<t_n<b.$$

Then

$$S_n = \Delta^+ F(a)\Delta^+ g(a) - \Delta^- F(b)\Delta^- g(b) + \sum_{k=1}^{n} \left[\Delta^+ F(t_k)\Delta^+ g(t_k) - \Delta^- F(t_k)\Delta^- g(t_k)\right].$$

Furthermore, for each $k = 1, 2, \ldots, n$, choose $\delta_k > 0$ in such a way that

$$\|g(t_k + \delta_k) - g(t_k+)\|_X < \frac{\varepsilon}{8(n+1)\|F\|_\infty}, \ \|g(t_k - \delta_k) - g(t_k-)\|_X < \frac{\varepsilon}{8(n+1)\|F\|_\infty}$$

and

$$[t_k - \delta_k, t_k + \delta_k] \cap \{t_1, t_2, \ldots, t_n\} = \{t_k\}.$$

Analogously, let $\delta_0 > 0$ be such that

$$a + \delta_0 < t_1 \quad \text{and} \quad \|g(a + \delta_0) - g(a+)\|_X < \frac{\varepsilon}{8\|F\|_\infty}$$

and

$$b - \delta_0 > t_n \quad \text{and} \quad \|g(b-) - g(b - \delta_0)\|_X < \frac{\varepsilon}{8\|F\|_\infty}.$$

Now, noting that

$$\|\Delta^+ F(t)\|_{L(X)} \leqslant 2\|F\|_\infty \quad \text{for } t \in [a, b)$$

and

$$\|\Delta^- F(t)\|_{L(X)} \leqslant 2\|F\|_\infty \quad \text{for } t \in (a, b],$$

we can see that

$$\begin{aligned}
&\|S_n\|_X \\
&< 2\|F\|_\infty\Big(\|g(a+) - g(a + \delta_0)\|_X + \sum_{k=1}^{n} \|g(t_k+) - g(t_k + \delta_k)\|_X\Big) \\
&\quad + \Big\|\Delta^+ F(a)[g(a + \delta_0) - g(a)] + \sum_{k=1}^{n} \Delta^+ F(t_k)[g(t_k + \delta_k) - g(t_k)]\Big\|_X \\
&\quad + 2\|F\|_\infty\Big(\|g(b - \delta_0) - g(b-)\|_X + \sum_{k=1}^{n} \|g(t_k - \delta_k) - g(t_k-)\|_X\Big) \\
&\quad + \Big\|\Delta^- F(b)[g(b) - g(b - \delta_0)] + \sum_{k=1}^{n} \Delta^- F(t_k)[g(t_k) - g(t_k - \delta_k)]\Big\|_X \\
&< \frac{\varepsilon}{4} + \frac{n\varepsilon}{4(n+1)} + \frac{\varepsilon}{4} + \frac{n\varepsilon}{4(n+1)} \\
&\quad + 2\|F\|_\infty\Big\|\frac{\Delta^+ F(a)}{2\|F\|_\infty}[g(a + \delta_0) - g(a)] + \sum_{k=1}^{n} \frac{\Delta^+ F(t_k)}{2\|F\|_\infty}[g(t_k + \delta_k) - g(t_k)]\Big\|_X \\
&\quad + 2\|F\|_\infty\Big\|\frac{\Delta^- F(b)}{2\|F\|_\infty}[g(b) - g(b - \delta_0)] + \sum_{k=1}^{n} \frac{\Delta^+ F(t_k)}{2\|F\|_\infty}[g(t_k) - g(t_k - \delta_k)]\Big\|_X.
\end{aligned}$$

Summarizing, we have

$$\begin{aligned}
&\|S_n\|_X \\
&\quad < \varepsilon + 2\|F\|_\infty \left\| \frac{\Delta^+ F(a)}{2\|F\|_\infty}[g(a+\delta_0) - g(a)] + \sum_{k=1}^{n} \frac{\Delta^+ F(t_k)}{2\|F\|_\infty}[g(t_k+\delta_k) - g(t_k)] \right\|_X \\
&\qquad + 2\|F\|_\infty \left\| \frac{\Delta^- F(b)}{2\|F\|_\infty}[g(b) - g(b-\delta_0)] + \sum_{k=1}^{n} \frac{\Delta^- F(t_k)}{2\|F\|_\infty}[g(t_k) - g(t_k-\delta_k)] \right\|_X \\
&\quad \leqslant \varepsilon + 4\|F\|_\infty((\mathcal{B})\operatorname{var}_a^b g).
\end{aligned}$$

As $\varepsilon > 0$ can be arbitrarily small, we finally deduce that the estimate

$$\|S_n\|_X \leqslant 4\|F\|_\infty\big((\mathcal{B})\operatorname{var}_a^b g\big)$$

is true for each $n \in \mathbb{N}$, wherefrom the desired estimate (3.5) follows.

(ii) Similarly we can proceed in the case $F \in (\mathcal{B})BV([a,b], L(X)) \cap G([a,b], L(X))$ and $g \in G([a,b], X)$. □

Corollary 3.6 (Integration by parts). *Let $F \in (\mathcal{B})BV([a,b], L(X)) \cap G([a,b], L(X))$ and $g \in G([a,b], X)$ (or $F \in G([a,b], L(X))$ and $g \in (\mathcal{B})BV([a,b], X) \cap G([a,b], X)$). Then both the integrals*

$$\int_a^b F\,\mathrm{d}[g] \quad \textit{and} \quad \int_a^b \mathrm{d}[F]g$$

exist and

$$\begin{aligned}
(3.6)\qquad &\int_a^b F\,\mathrm{d}[g] + \int_a^b \mathrm{d}[F]g \\
&\quad = F(b)g(b) - F(a)g(a) - \sum_{a \leqslant t < b} \Delta^+ F(t)\Delta^+ g(t) + \sum_{a < t \leqslant b} \Delta^- F(t)\Delta^- g(t)
\end{aligned}$$

holds.

Proof. a) Let $F \in (\mathcal{B})BV([a,b], L(X)) \cap G([a,b], L(X))$ $g \in G([a,b], X)$ and let $\{g_n\}$ be a sequence of finite step functions on $[a,b]$ which tends uniformly to g on $[a,b]$. Then by Proposition 2.1 (iv) we have

$$\begin{aligned}
(3.7)\qquad &\int_a^b F\,\mathrm{d}[g_n] + \int_a^b \mathrm{d}[F]g_n - F(b)g_n(b) + F(a)g_n(a) \\
&\quad = - \sum_{a \leqslant t < b} \Delta^+ F(t)\Delta^+ g_n(t) + \sum_{a < t \leqslant b} \Delta^- F(t)\Delta^- g_n(t)
\end{aligned}$$

for any $n \in \mathbb{N}$. By Proposition 2.1(ii) and Corollary 3.4(i), the relation

$$\lim_{n\to\infty}\Bigl(\int_a^b F\,\mathrm{d}[g_n] + \int_a^b \mathrm{d}[F]g_n - F(b)g_n(b) + F(a)g_n(a)\Bigr)$$
$$= \int_a^b F\,\mathrm{d}[g] + \int_a^b \mathrm{d}[F]g - F(b)g(b) + F(a)g(a)$$

holds. Further, by Lemma 3.5 (ii) the estimate

$$\Bigl\|\sum_{a\leqslant t<b}\Delta^+F(t)\Delta^+(g(t)-g_n(t)) - \sum_{a<t\leqslant b}\Delta^-F(t)\Delta^-(g(t)-g_n(t))\Bigr\|_X$$
$$\leqslant 4((\mathcal{B})\operatorname{var}_a^b F)\|g-g_n\|_\infty$$

is true. Consequently,

$$\lim_{n\to\infty}\Bigl(\sum_{a\leqslant t<b}\Delta^+F(t)\Delta^+g_n(t) - \sum_{a<t\leqslant b}\Delta^-F(t)\Delta^-g_n(t)\Bigr)$$
$$= \sum_{a\leqslant t<b}\Delta^+F(t)\Delta^+g(t) - \sum_{a<t\leqslant b}\Delta^-F(t)\Delta^-g(t).$$

Summarizing, letting $n \to \infty$ in (3.7) we obtain (3.6).

b) Similarly we can proceed if $F \in G([a,b],L(X))$ and $g \in BV([a,b],X)$. □

Theorem 3.7 (Substitution Theorem)**.** *Asume that $H \in (\mathcal{B})BV([a,b],L(X))$. Let $F\colon [a,b] \to L(X)$ and let $g\colon [a,b] \to X$ be such that the integral $\int_a^b \mathrm{d}[F]g$ and at least one of the integrals*

$$\int_a^b H(t)\,\mathrm{d}_t\Bigl[\int_a^t \mathrm{d}[F]g\Bigr], \quad \int_a^b H\,\mathrm{d}[F]g$$

exist. Then also the other one has sense and the equality

$$\int_a^b H(t)\,\mathrm{d}_t\Bigl[\int_a^t \mathrm{d}[F]g\Bigr] = \int_a^b H\,\mathrm{d}[F]g \tag{3.8}$$

holds.

P r o o f. Put $K(t) = \int_a^t \mathrm{d}[F]g$ for $t \in [a,b]$. Let $P = (D,\xi)$, where $D = \{\alpha_0, \alpha_1, \ldots, \alpha_m\}$, $\xi = (\xi_1, \ldots, \xi_m)$, be an arbitrary partition of $[a,b]$. Then

$$\begin{aligned}
(3.9)\qquad & \|S(H, \mathrm{d}F, g, P) - S(H, \mathrm{d}K, P)\|_X \\
&= \Big\| \sum_{j=1}^m H(\xi_j)\Big([F(\alpha_j) - F(\alpha_{j-1})]g(\xi_j) - \int_{\alpha_{j-1}}^{\alpha_j} \mathrm{d}[F]g\Big)\Big\|_X \\
&\leqslant \Big\| \sum_{j=1}^m [H(\xi_j) - H(\xi_{j-1})]\Big(\sum_{k=j}^m [F(\alpha_k) - F(\alpha_{k-1})]g(\xi_k) - \int_{\alpha_{k-1}}^{\alpha_k} \mathrm{d}[F]g\Big)\Big\|_X \\
&\quad + \Big\| H(a)\Big(\sum_{k=1}^m [F(\alpha_k) - F(\alpha_{k-1})]g(\xi_k) - \int_{\alpha_{k-1}}^{\alpha_k} \mathrm{d}[F]g\Big)\Big\|_X.
\end{aligned}$$

(Here, analogously to the proof of Theorem 11 in [1], we made use of the relation

$$\sum_{j=1}^m A_j B_j = \sum_{j=1}^m [A_j - A_{j-1}]\Big(\sum_{k=j}^m B_k\Big) + A_0\Big(\sum_{k=1}^m B_k\Big)$$

valid for arbitrary collections of operators A_j, $j = 0, 1, \ldots, m$, B_j, $j = 1, 2, \ldots, m$.)

Now, let an $\varepsilon > 0$ be given, let δ_ε be a gauge on $[a,b]$ such that

$$\Big\| S(\mathrm{d}F, g, P) - \int_a^b \mathrm{d}[F]g \Big\|_X < \varepsilon \quad \text{for all } \delta_\varepsilon\text{-fine partitions } P \text{ of } [a,b]$$

and let $P = (D,\xi)$, where $D = \{\alpha_0, \alpha_1, \ldots, \alpha_m\}$, $\xi = (\xi_1, \ldots, \xi_m)$, be an arbitrary δ_ε-fine partition of $[a,b]$. Then, by the Saks-Henstock lemma (cf. e.g. [8, Lemma 16]), we have

$$\Big\| \sum_{k=j}^m [F(\alpha_k) - F(\alpha_{k-1})]g(\xi_k) - \int_{\alpha_{k-1}}^{\alpha_k} \mathrm{d}[F]g \Big\|_X \leqslant \varepsilon \quad \text{for } j = 1, 2, \ldots, m.$$

Inserting this into (3.9), we get

$$\begin{aligned}
&\|S(H \mathrm{d}F, g, P) - S(H, \mathrm{d}K, P)\|_X \\
&\quad\leqslant \Big\| \varepsilon \sum_{j=1}^m [H(\xi_j) - H(\xi_{j-1})]\Big(\frac{1}{\varepsilon}\sum_{k=j}^m ([F(\alpha_k) - F(\alpha_{k-1})]g(\xi_k) - \int_{\alpha_{k-1}}^{\alpha_k} \mathrm{d}[F]g)\Big)\Big\|_X \\
&\qquad + \Big\| H(a)\Big(\sum_{k=1}^m [F(\alpha_k) - F(\alpha_{k-1})]g(\xi_k) - \int_{\alpha_{k-1}}^{\alpha_k} \mathrm{d}[F]g\Big)\Big\|_X \\
&\quad\leqslant \varepsilon((\mathcal{B}) \operatorname{var}_a^b H) + \|H(a)\|_X),
\end{aligned}$$

wherefrom the proof immediately follows. $\square$

The last result of this paper provides a different version of the Substitution Theorem not covered by Theorem 3.7 and which is also applicable to generalized differential equations. On the one hand, it assumes $H \in G([a,b], L(X))$ instead of $H \in (\mathcal{B})BV([a,b], L(X))$. On the other hand, F should be of bounded variation on $[a,b]$ and g bounded on $[a,b]$.

Theorem 3.8 (Second Substitution Theorem). *Assume $F \in BV([a,b], L(X))$, the function $g\colon [a,b] \to X$ is bounded and let the integral $\int_a^b \mathrm{d}[F]g$ exist. Then, for each $H \in G([a,b], L(X))$, both the integrals in* (3.8) *exist and the equality* (3.8) *is true.*

Proof. *Step 1.* First, we show that (3.8) holds for every finite step function $H\colon [a,b] \to L(X)$.

By the linearity of the integral and since a finite step function $H\colon [a,b] \to L(X)$ is a finite linear combination of functions of the form

$$\chi_{[a,\tau]}(t)\widetilde{H}_1,\ \chi_{[\sigma,b]}(t)\widetilde{H}_2,\ \chi_{[a]}(t)\widetilde{H}_3,\ \chi_{[b]}(t)\widetilde{H}_4,$$

where $\tau, \sigma \in (a,b)$ and $\widetilde{H}_i \in L(X)$, $i = 1,2,3,4$, it is enough to justify (3.8) for functions H of such a form.

Let $\tau \in (a,b)$, $\widetilde{H} \in L(X)$, $H = \chi_{[a,\tau]}(t)\widetilde{H}$ and $K(t) = \int_a^t \mathrm{d}[F]g$ for $t \in [a,b]$.

Obviously,

$$\int_a^\tau H \,\mathrm{d}[F]g = \int_a^\tau H \,\mathrm{d}[K] = \widetilde{H} \int_a^\tau \mathrm{d}[F]g. \tag{3.10}$$

Let $\varepsilon > 0$ be given and let

$$\delta(t) = \begin{cases} \varepsilon & \text{if } t = \tau, \\ \frac{1}{2}|\tau - t| & \text{if } \tau < t \leqslant b. \end{cases}$$

Then for any δ-fine partition P of $[\tau, b]$ with

$$D = \{\alpha_0, \alpha_1, \ldots, \alpha_m\} \quad \text{and} \quad \xi = (\xi_1, \xi_2 \ldots, \xi_m)$$

we have $\xi_1 = \alpha_0 = \tau$, $\alpha_1 < \tau + \varepsilon$ and

$$S(H, \mathrm{d}F, g, P) = \widetilde{H}[F(\alpha_1) - F(\tau)]g(\tau) \quad \text{and} \quad S(H, \mathrm{d}K, P) = \widetilde{H}[K(\alpha_1) - K(\tau)].$$

As a result and as a consequence of the Hake theorem for KS-integrals (cf. e.g. [8, Corollary 24]) we get

$$\int_\tau^b H \,\mathrm{d}[F]g = \widetilde{H}\Delta^+ F(\tau)g(\tau) \quad \text{and} \quad \int_\tau^b H \,\mathrm{d}[K] = \widetilde{H}\Delta^+ K(\tau) = \widetilde{H}\Delta^+ F(\tau)g(\tau),$$

i.e.

$$\int_\tau^b H\,\mathrm{d}[F]g = \int_\tau^b H\,\mathrm{d}[K] = \widetilde{H}\Delta^+F(\tau)g(\tau).$$

This together with (3.10) yields (3.8).

The proofs of the remaining cases $H = \chi_{[\tau,b]}\widetilde{H}$, $H = \chi_{[a]}\widetilde{H}$ and $H = \chi_{[b]}\widetilde{H}$ can be done in a similar way.

Step 2. Next, notice that using arguments analogous to those from the proof of the assertion (i) in Proposition 2.1, we can show that if $F \in BV([a,b], L(X))$, $H\colon [a,b] \to L(X)$ and $g\colon [a,b] \to X$ are such that $\int_a^b H\,\mathrm{d}[F]g$ exists, then the estimate

$$\left\| \int_a^b H\,\mathrm{d}[F(s)]g(s) \right\|_X \leqslant \|H\|_\infty (\operatorname{var}_a^b F)\|g\|_\infty$$

holds. As a consequence, the following assertion is true:

Let $F \in BV([a,b], L(X))$, $H_n\colon [a,b] \to L(X)$ for $n \in \mathbb{N}$ and let $g\colon [a,b] \to X$ be bounded. Moreover, let $\int_a^b H_n\,\mathrm{d}[F]g$ exist for $n \in \mathbb{N}$ and $\lim\limits_{n\to\infty} \|H_n - H\|_\infty = 0$. Then the integral

$$\int_a^b H\,\mathrm{d}[F]g$$

exists and

$$\lim_{n\to\infty} \left\| \int_a^b H_n\,\mathrm{d}[F]g - \int_a^b H\,\mathrm{d}[F]g \right\|_X = 0.$$

Step 3. Let $H \in G([a,b], L(X))$. Denote again $K(t) = \int_a^t \mathrm{d}[F]g$ for $t \in [a,b]$ and consider the sequence $H_n\colon [a,b] \to L(X)$, $n \in \mathbb{N}$, of finite step functions such that $\lim\limits_{n\to\infty} \|H_n - H\|_\infty = 0$. As $K \in BV([a,b], L(X))$, by Proposition 2.1 (ii) and steps 1 and 2 we have

$$\begin{aligned} \int_a^b H(t)\,\mathrm{d}_t\Big[\int_a^t \mathrm{d}[F]g\Big] &= \int_a^b H\,\mathrm{d}[K] = \lim_{n\to\infty}\int_a^b H_n\,\mathrm{d}[K] \\ &= \lim_{n\to\infty}\int_a^b H_n\,\mathrm{d}[F]g = \int_a^b H\,\mathrm{d}[F]g, \end{aligned}$$

i.e., (3.8) is true. □

References

[1] *M. Federson*: Substitution formulas for the Kurzweil and Henstock vector integrals. Math. Bohem. *127* (2002), 15–26. zbl

[2] *M. Federson, R. Bianconi, L. Barbanti*: Linear Volterra integral equations as the limit of discrete systems. Acta Math. Appl. Sin. Engl. Ser. *20* (2004), 623–640. zbl

[3] *M. Federson, P. Táboas*: Topological dynamics of retarded functional differential equations. J. Differ. Equations *195* (2003), 313–331. zbl

[4] *Ch. S. Hönig*: Volterra Stieltjes-Integral Equations. North Holland and American Elsevier, Mathematics Studies 16. Amsterdam and New York, 1975. zbl

[5] *G. A. Monteiro, M. Tvrdý*: Generalized linear differential equations in a Banach space: Continuous dependence on a parameter. Discrete Contin. Dyn. Syst. *33* (2013), 283–303.

[6] *K. M. Naralenkov*: On integration by parts for Stieltjes-type integrals of Banach space-valued functions. Real Anal. Exch. *30* (2004/2005), 235–260.

[7] *Š. Schwabik*: Generalized Ordinary Differential Equations. World Scientific. Singapore, 1992. zbl

[8] *Š. Schwabik*: Abstract Perron-Stieltjes integral. Math. Bohem. *121* (1996), 425–447. zbl

[9] *Š. Schwabik*: Linear Stieltjes integral equations in Banach spaces. Math. Bohem. *124* (1999), 433–457. zbl

[10] *Š. Schwabik*: Linear Stieltjes integral equations in Banach spaces II; Operator valued solutions. Math. Bohem. *125* (2000), 431–454. zbl

[11] *Š. Schwabik*: A note on integration by parts for abstract Perron-Stieltjes integrals. Math. Bohem. *126* (2001), 613–629. zbl

[12] *Š. Schwabik*: Operator-valued functions of bounded semivariation and convolutions. Math. Bohem. *126* (2001), 745–777. zbl

[13] *Š. Schwabik, G. Ye*: Topics in Banach Space Integration. World Scientific. Singapore, 2005. zbl

[14] *Š. Schwabik, M. Tvrdý, O. Vejvoda*: Differential and Integral Equations: Boundary Value Problems and Adjoints. Academia and Reidel, Praha and Dordrecht, 1979. zbl

[15] *M. Tvrdý*: Regulated functions and the Perron-Stieltjes integral. Čas. Pěst. Mat. *114* (1989), 187–209. zbl

[16] *M. Tvrdý*: Differential and integral equations in the space of regulated functions. Mem. Differ. Equ. Math. Phys. *25* (2002), 1–104. zbl

[17] *R.M. McLeod*: The Generalized Riemann Integral. Carus Monographs, Mathematical Association of America, Washington, 1980. zbl

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