CHARACTERIZATIONS OF THE 0-DISTRIBUTIVE SEMILATTICE

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(Received February 22, 2002)

Abstract. The 0-distributive semilattice is characterized in terms of semiideals, ideals and filters. Some sufficient conditions and some necessary conditions for 0-distributivity are obtained. Counterexamples are given to prove that certain conditions are not necessary and certain conditions are not sufficient.

Keywords: semilattice, prime ideal, filter

MSC 2000: 06A12, 06A99, 06B10, 06B99

1. Introduction and preliminaries

The 0-distributive lattice and the 0-distributive semilattice have been studied by Varlet [7], [8], Pawar and Thakare [4], [5], Jayaram [3] and Balasubramani and Venkatanarasimhan [1]. In this paper we obtain some characterizations of the 0-distributive semilattice. For the lattice theoretic concepts which have now become commonplace the reader is referred to Szasz [6] and Grätzer [2].

A semilattice is a partially ordered set in which any two elements have a greatest lower bound. Let S be a semilattice. A semiideal of S is a nonempty subset A of S such that $a \in A$, $b \leqslant a$ $(b \in S) \Rightarrow b \in A$. An ideal of S is a semiideal A of S such that the join of any finite number of elements of A, whenever it exists, belongs to A. If $a \in S$, then $\{x \in S;\ x \leqslant a\}$ is an ideal. It is called the principal ideal generated by a and is denoted by $(a]$. A filter of S is a nonempty subset F of S such that (i) $a \in F$, $b \geqslant a$ $(b \in S) \Rightarrow b \in F$ and (ii) $a, b \in F \Rightarrow a \wedge b \in F$. The dual of a principal ideal is called a principal filter. The principal filter generated by a is denoted by $[a)$. A maximal ideal (filter) of S is a proper ideal (filter) which is not contained in any other proper ideal (filter). A prime semiideal (ideal) is a proper semiideal (ideal)

A such that $a \wedge b \in A \Rightarrow a \in A$ or $b \in A$. A minimal prime semiideal (ideal) is a prime semiideal (ideal) which does not contain any other prime semiideal (ideal). Let $F(S)$ denote the set of filters of S. A prime filter of S is a filter A such that B, $C \in F(S)$, $B \cap C \subseteq A$, $B \cap C \neq \emptyset \Rightarrow B \subseteq A$ or $C \subseteq A$. If A is a prime filter of S and $A_1, \ldots, A_n \in F(S)$, $A_1 \cap \ldots \cap A_n \subseteq A$, $A_1 \cap \ldots \cap A_n \neq \emptyset$, then $A_i \subseteq A$ for some $i \in \{1, \ldots, n\}$.

Let A be a nonempty subset of a semilattice S with 0, $A^* = \{x \in S;\ a \wedge x = 0$ for all $a \in A\}$ and $A^0 = \{x \in S;\ a \wedge x = 0$ for some $a \in A\}$. Then A^* is called the annihilator of A and A^0 is called the pseudoannihilator of A. If $a \in S$, we write $(a)^*$ for $\{a\}^*$ and $(a)^0$ for $\{a\}^0$. We say that a is dense if $(a)^* = \{0\}$. If $\sup(a)^* \in (a)^*$, it is called the pseudocomplement of a and is denoted by a^*. A pseudocomplemented semilattice is a semilattice with 0 in which every element has a pseudocomplement. An ideal (semiideal) A of a semilattice S with 0 is said to be normal if $A^{**} = A$.

The following five lemmas are contained in Venkatanarasimhan [9].

Lemma 1.1. *The set $I(S)$ of all ideals of a semilattice S forms a lattice under set inclusion as the partial ordering relation. The meet in $I(S)$ coincides with the set intersection.*

Lemma 1.2. *Let S be a semilattice and $\{a_i;\ i \in I\}$ any subset of S. Then $\bigwedge a_i$ $\left(\bigvee a_i\right)$ exists if and only if $\bigcap (a_i]$ $\left(\bigcap [a_i)\right)$ is a principal ideal (principal filter). Whenever $\bigwedge a_i$ $\left(\bigvee a_i\right)$ exists then $\bigcap (a_i] = \left(\bigwedge a_i\right]$ $\left(\bigcap [a_i) = \left[\bigvee a_i\right)\right)$.*

Lemma 1.3. *Let S be a semilattice. Then for $a_1, \ldots, a_n \in S$, $a_1 \vee \ldots \vee a_n$ exists if and only if $(a_1] \vee \ldots \vee (a_n]$ is a principal ideal. Whenever $a_1 \vee \ldots \vee a_n$ exists then $(a_1] \vee \ldots \vee (a_n] = (a_1 \vee \ldots \vee a_n]$.*

Lemma 1.4. *If $\{A_i;\ i \in I\}$ is a family of ideals of a semilattice, then $\bigvee A_i = \left\{x;\ (x] \subseteq (a_{i1}] \vee \ldots \vee (a_{in}];\ a_{i1}, \ldots, a_{in} \in \bigcup A_i\right\}$.*

Lemma 1.5. *Every proper filter of a semilattice with 0 is contained in a maximal filter.*

The following lemma is easily proved.

Lemma 1.6. *Let A be a nonempty subset of a semilattice S with 0 and $x \in S$. Then A^* and A^0 are semiideals of S and $(x]^* = [x)^0 = (x)^0 = (x)^*$.*

The following four lemmas are contained in Venkatanarasimhan [10].

Lemma 1.7. *Let A be a nonempty proper subset of a semilattice S with 0. Then A is a filter if and only if $S - A$ is a prime semiideal.*

Lemma 1.8. *Let A be a nonempty subset of a semilattice S with 0. Then A is a maximal filter if and only if $S - A$ is a minimal prime semiideal.*

Lemma 1.9. *Any prime semiideal of a semilattice with 0 contains a minimal prime semiideal.*

Lemma 1.10. *Let A be a nonempty subset of a semilattice with 0. Then A^* is the intersection of all minimal prime semiideals not containing A.*

The following lemma is contained in Pawar and Thakare [4].

Lemma 1.11. *Let A be a proper filter of a semilattice S with 0. Then A is maximal if and only if for each x in $S - A$, there is some a in A such that $a \wedge x = 0$.*

Lemma 1.12. *Let A and B be filters of a semilattice S with 0 such that A and B^0 are disjoint. Then there is a minimal prime semiideal containing B^0 and disjoint from A.*

Proof. It is easily seen that $A \vee B$ is a proper filter of S. Hence by Lemma 1.5, $A \vee B \subseteq M$ for some maximal filter M. Now $B \subseteq M$ and so $M \cap B^0 = \emptyset$. By Lemma 1.8, $S - M$ is a minimal prime semiideal. Clearly $B^0 \subseteq S - M$ and $(S - M) \cap A = \emptyset$. $\square$

Lemma 1.13. *Let A be a filter of a semilattice S with 0. Then A^0 is the intersection of all minimal prime semiideals disjoint from A.*

Proof. Let N be any minimal prime semiideal disjoint from A. If $x \in A^0$, then $x \wedge a = 0$ for some $a \in A$ and so $x \in N$.

Let $y \in S - A^0$. Then $a \wedge y \neq 0$ for all $a \in A$. Hence $A \vee [y) \neq S$. By Lemma 1.5, $A \vee [y) \subseteq M$ for some maximal filter M. By Lemma 1.8, $S - M$ is a minimal prime semiideal. Clearly $(S - M) \cap A = \emptyset$ and $y \notin S - M$. $\square$

Lemma 1.14. *Let S be a semilattice with 0. Then the set complement of a prime filter is a prime ideal. If S is finite, then the set complement of a prime ideal is a prime filter.*

Proof. Let A be a prime filter of S. By Lemma 1.7, $S - A$ is a prime semiideal. Let $x_1, \ldots, x_n \in S - A$ and suppose $x_1 \vee \ldots \vee x_n$ exists. Since A is prime it follows that $x_1 \vee \ldots \vee x_n \in S - A$. Thus $S - A$ is a prime ideal. $\square$

Let S be finite and let A be any prime ideal of S. By Lemma 1.7, $S - A$ is a filter. Since S is finite, every filter of S is principal. Let $a, b \in A$ be such that $[a) \cap [b) \neq \emptyset$. Let $[a) \cap [b) = \{c_1, \ldots, c_n\}$ and $c = c_1 \wedge \ldots \wedge c_n$. Then $c \geqslant a, b$. If $d \geqslant a, b$ then $d = c_j$ for some j and so $d \geqslant c$. Thus $c = a \vee b \in A$. Hence $[a) \cap [b) = [a \vee b) \nsubseteq S - A$ proving $S - A$ is prime.

2. Definition and characterizations

Definition 2.1. A 0-distributive lattice is a lattice with 0 in which $a \wedge b = 0 = a \wedge c$ implies $a \wedge (b \vee c) = 0$.

Varlet [7], has proved that a lattice L bounded below is 0-distributive if and only if the ideal lattice $I(L)$ is pseudocomplemented. He also observed that for an ideal lattice, the two notions of pseudocomplementedness and 0-distributivity are equivalent. These results motivate the following definition.

Definition 2.2. A 0-distributive semilattice is a semilattice S with 0 such that $I(S)$, the lattice of ideals of S, is 0-distributive.

Theorem 2.3. *Let S be a semilattice with 0. Then the following statements are equivalent:*

1. *S is 0-distributive.*
2. *If $A, A_1, \ldots, A_n$ are ideals of S such that $A \cap A_1 = \ldots = A \cap A_n = (0]$, then $A \cap (A_1 \vee \ldots \vee A_n) = (0]$.*
3. *If $a, a_1, \ldots, a_n$ are elements of S such that $(a] \cap (a_1] = \ldots = (a] \cap (a_n] = (0]$, then $(a] \cap ((a_1] \vee \ldots \vee (a_n]) = (0]$.*
4. *If M is a maximal filter of S, then $S - M$ is a minimal prime ideal.*
5. *Every minimal prime semiideal of S is a minimal prime ideal.*
6. *Every prime semiideal of S contains a minimal prime ideal.*
7. *Every proper filter of S is disjoint from a minimal prime ideal.*
8. *For each nonzero element a of S, there is a minimal prime ideal not containing a.*
9. *For each nonzero element a of S, there is a prime ideal not containing a.*

Proof. $1 \Rightarrow 2$: Suppose 1 holds and let $A, A_1, \ldots, A_n \in I(S)$ be such that $A \cap A_1 = \ldots = A \cap A_n = (0]$. By 1, $I(S)$ is 0-distributive. Hence $A \cap (A_1 \vee A_2) = (0]$. Assume $A \cap (A_1 \vee \ldots \vee A_{k-1}) = (0]$ for $2 < k \leqslant n$. Then $A \cap (A_1 \vee \ldots \vee A_{k-1} \vee A_k) = A \cap (B \vee A_k)$ where $B = A_1 \vee \ldots \vee A_{k-1}$. By our induction hypothesis $A \cap B = (0]$. Also $A \cap A_k = (0]$. Consequently $A \cap (A_1 \vee \ldots \vee A_k) = A \cap (B \vee A_k) = (0]$. Thus the result follows by induction.

Obviously $2 \Rightarrow 3$ and $8 \Rightarrow 9$.

$3 \Rightarrow 1$: Suppose 3 holds. Let $A, B, C \in I(S)$ be such that $A \cap B = (0] = A \cap C$. Then $(a] \cap (b] = (0] = (a] \cap (c]$ for all $a \in A$, $b \in B$ and $c \in C$. Let $x \in A \cap (B \vee C)$. Then $x \in B \vee C$. Hence $(x] \subseteq (b_1] \vee \ldots \vee (b_m] \vee (c_1] \vee \ldots \vee (c_n]$ for some $b_1, \ldots, b_m \in B$ and $c_1, \ldots, c_n \in C$. Also $x \in A$. Consequently $(x] \cap (b_i] = (0]$ for $i = 1, \ldots, m$ and $(x] \cap (c_j] = (0]$ for $j = 1, \ldots, n$. By 3, $(x] \cap ((b_1] \vee \ldots \vee (b_m] \vee (c_1] \vee \ldots \vee (c_n]) = (0]$. It follows that $x = 0$. Thus $A \cap (B \vee C) = (0]$.

$3 \Rightarrow 4$: Suppose 3 holds. Let M be any maximal filter of S. By Lemma 1.8, $S - M$ is a minimal prime semiideal. Let $x_1, \ldots, x_n \in S - M$ be such that $x_1 \vee \ldots \vee x_n$ exists. By Lemma 1.11, $a_1 \wedge x_1 = \ldots = a_n \wedge x_n = 0$ for some $a_1, \ldots, a_n \in M$. Let $a = a_1 \wedge \ldots \wedge a_n$. Then $a \in M$ and $a \wedge x_i = 0$ for $i = 1, \ldots, n$. By Lemma 1.2, $(a] \cap (x_i] = (0]$ for $i = 1, \ldots, n$. By Lemma 1.3, $(a] \cap (x_1 \vee \ldots \vee x_n] = (a] \cap ((x_1] \vee \ldots \vee (x_n]) = (0]$ by 3. It follows that $a \wedge (x_1 \vee \ldots \vee x_n) = 0$. Hence $x_1 \vee \ldots \vee x_n \in S - M$. Thus $S - M$ is an ideal.

$4 \Rightarrow 5$: Suppose 4 holds. Let N be any minimal prime semiideal of S. By Lemma 1.8, $S - N$ is a maximal filter. By 4, $N = S - (S - N)$ is a minimal prime ideal.

$5 \Rightarrow 6$: Suppose 5 holds and let A be any prime semiideal of S. By Lemma 1.9, $A \supseteq N$ for some minimal prime semiideal N. By 5, N is a minimal prime ideal.

$6 \Rightarrow 7$: Suppose 6 holds and let A be any proper filter of S. By Lemma 1.7, $S - A$ is a prime semiideal. By 6, $S - A$ contains a minimal prime ideal N. Clearly $A \cap N = \emptyset$.

$7 \Rightarrow 8$: Suppose 7 holds and let a be any nonzero element of S. By 7, $[a)$ is disjoint from a minimal prime ideal N. Clearly $a \notin N$.

$9 \Rightarrow 3$: Suppose 9 holds. Let $a, a_1, \ldots, a_n \in S$ such that $(a] \cap (a_1] = \ldots = (a] \cap (a_n] = (0]$ and $(a] \cap ((a_1] \vee \ldots \vee (a_n]) \neq (0]$. Then there exists $x \in (a] \cap ((a_1] \vee \ldots \vee (a_n])$ such that $x \neq 0$. By 9 there is a prime ideal A such that $x \notin A$. By Lemma 1.7, $S - A$ is a proper filter and clearly $a \in S - A$. Consequently $a_1, \ldots, a_n \in A$. It follows that $(a_1] \vee \ldots \vee (a_n] \subseteq A$ and so $x \in A$. Thus we get a contradiction. Hence $(a] \cap (a_1] = \ldots = (a] \cap (a_n] = (0] \Rightarrow (a] \cap ((a_1] \vee \ldots \vee (a_n]) = (0]$. □

Theorem 2.4. *Let S be a semilattice with 0. Then the following statements are equivalent:*

1. *S is 0-distributive.*
2. *If A is a nonempty subset of S and B is a proper filter intersecting A, there is a minimal prime ideal containing A^* and disjoint from B.*
3. *If A is a nonempty subset of S and B is a proper filter intersecting A, there is a prime ideal containing A^* and disjoint from B.*
4. *If A is a nonempty subset of S and B is a prime semiideal not containing A, there is a minimal prime ideal containing A^* and contained in B.*

5. *If A is a nonempty subset of S and B is a prime semiideal not containing A, there is a prime ideal containing A^* and contained in B.*
6. *For each nonzero element a of S and each proper filter B containing a, there is a prime ideal containing $(a)^*$ and disjoint from B.*
7. *For each nonzero element a of S and each prime semiideal B not containing a, there is a prime ideal containing $(a)^*$ and contained in B.*
8. *If A and B are filters of S such that A and B^0 are disjoint, there is a minimal prime ideal containing B^0 and disjoint from A.*
9. *If A and B are filters of S such that A and B^0 are disjoint, there is a prime ideal containing B^0 and disjoint from A.*
10. *If A is a filter of S and B is a prime semiideal containing A^0, there is a minimal prime ideal containing A^0 and contained in B.*
11. *If A is a filter of S and B is a prime semiideal containing A^0, there is a prime ideal containing A^0 and contained in B.*
12. *For each nonzero element a in S and each filter A disjoint from $(a)^*$, there is a prime ideal containing $(a)^*$ and disjoint from A.*
13. *For each nonzero element a in S and each prime semiideal B containing $(a)^*$, there is a prime ideal containing $(a)^*$ and contained in B.*

Proof. $1 \Rightarrow 2$: Suppose 1 holds. Let A be a nonempty subset of S and B any proper filter such that $B \cap A \neq \emptyset$. By Lemma 1.7, $S - B$ is a prime semiideal and by Lemma 1.9, $S - B \supseteq N$ for some minimal prime semiideal N. Clearly $N \cap B = \emptyset$. Also $S - B \not\supseteq A$ and so $N \not\supseteq A$. By Lemma 1.10, $N \supseteq A^*$. Since S is 0-distributive, N is a minimal prime ideal [see Theorem 2.3, 5].

By Lemma 1.7, it follows that $2 \Rightarrow 4$, $3 \Rightarrow 5$, $8 \Rightarrow 10$, $9 \Rightarrow 11$ and $12 \Rightarrow 13$.

Obviously $2 \Rightarrow 3$, $2 \Rightarrow 6$, $4 \Rightarrow 5$, $4 \Rightarrow 7$, $8 \Rightarrow 9$, $10 \Rightarrow 11 \Rightarrow 13$ and $5 \Rightarrow 7$.

$1 \Rightarrow 8$: Suppose 1 holds. Let A and B be filters of S such that $A \cap B^0 \neq \emptyset$. By Lemma 1.12, there is a minimal prime semiideal N such that $N \supseteq B^0$ and $N \cap A = \emptyset$. Since S is 0-distributive it follows that N is a minimal prime ideal [see Theorem 2.3, 5].

$8 \Rightarrow 12$: By Lemma 1.6, $(x)^* = [x)^0$ for all $x \in S$. Hence the result.

$6 \Rightarrow 1$: Suppose 6 holds. Let a be any nonzero element of S. Now $[a)$ is a proper filter containing a. By 6, there is a prime ideal N containing $(a)^*$ and disjoint from $[a)$. Clearly $a \notin N$. Thus S is 0-distributive [see Theorem 2.3, 9].

$7 \Rightarrow 1$: Suppose 7 holds. Let a be any nonzero element of S. Now $S - [a)$ is a prime semiideal not containing a. By 7 there is a prime ideal N containing $(a)^*$ and contained in $S - [a)$. Clearly $a \notin N$. Thus S is 0-distributive [See Theorem 2.3, 9].

$13 \Rightarrow 1$: Suppose 13 holds and let a be any nonzero element of S. By Lemma 1.7, $S - [a)$ is a prime semiideal not containing a. Since $(a) \cap (a)^* = (0] \subseteq S - [a)$ it follows that $S - [a)$ contains $(a)^*$. By 13, there is a prime ideal N containing $(a)^*$ and contained in $S - [a)$. Clearly $a \in N$. Thus S is 0-distributive [see Theorem 2.3, 9]. □

Theorem 2.5. *Let S be a semilattice with 0. Then the following statements are equivalent:*

1. *S is 0-distributive.*
2. *For any nonempty subset A of S, A^* is the intersection of all minimal prime ideals not containing A.*
3. *For any filter A of S, A^0 is the intersection of all minimal prime ideals disjoint from A.*
4. *For each a in S, $(a)^*$ is an ideal.*
5. *Every normal semiideal of S is an intersection of minimal prime ideals.*
6. *For any finite number of ideals $A, A_1, \dots, A_n$ of S,*

$$(A \cap (A_1 \vee \dots \vee A_n))^* = (A \cap A_1)^* \cap \dots \cap (A \cap A_n)^*.$$

7. *For any three ideals A, B, C of S,*

$$(A \cap (B \vee C))^* = (A \cap B)^* \cap (A \cap C)^*.$$

8. *For any finite number of ideals $A, A_1, \dots, A_n$ of S,*

$$((A \vee A_1) \cap \dots \cap (A \vee A_n))^* = A^* \cap (A_1 \cap \dots A_n)^*.$$

9. *For any three ideals A, B, C of S,*

$$((A \vee B) \cap (A \vee C))^* = A^* \cap (B \cap C)^*.$$

10. *For any finite number of elements $a, a_1, \dots, a_n$ of S,*

$$((a] \cap ((a_1] \vee \dots \vee (a_n]))^* = ((a] \cap (a_1])^* \cap \dots \cap ((a] \cap (a_n])^*.$$

11. *For any finite number of elements $a_1, \dots, a_n$ of S,*

$$((a_1] \vee \dots \vee (a_n])^* = (a_1]^* \cap \dots \cap (a_n]^*.$$

12. *$I(S)$ is pseudocomplemented.*

P r o o f. $1 \Rightarrow 2$: Follows by Lemma 1.10 and Theorem 2.3, 5.

$1 \Rightarrow 3$: Follows by Lemma 1.13 and Theorem 2.3, 5.

$3 \Rightarrow 4$: By Lemma 1.6, $(a)^* = [a)^0$. Hence the result.

$4 \Rightarrow 1$: Suppose 4 holds. Let $a, a_1, \ldots, a_n \in S$ be such that $(a] \cap (a_1] = \ldots = (a] \cap (a_n] = (0]$. Then $a_1, \ldots, a_n \in (a)^*$. By 4 it follows that $(a_1] \vee \ldots \vee (a_n] \subseteq (a)^*$. Hence $(a] \cap ((a_1] \vee \ldots \vee (a_n]) = (0]$. Thus S is 0-distributive [see Theorem 2.3, 3].

Obviously $6 \Rightarrow 7$, $8 \Rightarrow 9$ and $6 \Rightarrow 10$.

$2 \Rightarrow 5$: Suppose 2 holds. Let A be any normal semiideal of S. Then $A = B^*$ for some semiideal B. By 2, B^* is the intersection of all minimal prime ideals not containing B. Hence the result.

$5 \Rightarrow 4$: By Lemma 1.6, $(a)^* = (a]^*$ for all $a \in S$. Hence the result.

$2 \Rightarrow 6$: Suppose 2 holds. Let $A, A_1, \ldots, A_n \in I(S)$. If Q is any minimal prime ideal of S such that $Q \not\supseteq A \cap (A_1 \vee \ldots \vee A_n)$, then $Q \not\supseteq A \cap A_j$ for some $j \in \{1, \ldots, n\}$. By 2 it follows that $(A \cap (A_1 \vee \ldots \vee A_n))^* \supseteq (A \cap A_1)^* \cap \ldots \cap (A \cap A_n)^*$. The reverse inclusion is obvious.

$7 \Rightarrow 1$: Suppose 7 holds. Then for A, B, $C \in I(S)$ we have $(A \cap (B \vee C))^* = (A \cap B)^* \cap (A \cap C)^*$. By replacing A by $B \vee C$ it follows that $(B \vee C)^* = B^* \cap C^*$. Suppose $A \cap B = (0] = A \cap C$. Then $(a] \cap (b] = (0] = (a] \cap (c]$ for all $a \in A$, $b \in B$ and $c \in C$. Hence $a \in B^* \cap C^*$ for all $a \in A$. Hence $a \in (B \vee C)^*$. Consequently $A \subseteq (B \vee C)^*$. It follows that $A \cap (B \vee C) = (0]$.

$2 \Rightarrow 8$: Suppose 2 holds, let A, $A_1, \ldots, A_n$ be ideals of S and let Q be any minimal prime ideal such that $Q \not\supseteq (A \vee A_1) \cap \ldots \cap (A \vee A_n)$. Then $Q \not\supseteq A \vee A_1, \ldots, A \vee A_n$ and so $Q \not\supseteq A$ or $Q \not\supseteq A_1 \cap \ldots \cap A_n$. By 2 it follows that $((A \vee A_1) \cap \ldots \cap (A \vee A_n))^* \supseteq A^* \cap (A_1 \cap \ldots \cap A_n)^*$. The reverse inclusion is obvious.

$9 \Rightarrow 1$: Suppose 9 holds. Then for any three ideals A, B, C of S, $((A \vee B) \cap (A \vee C))^* = A^* \cap (B \cap C)^*$. By replacing C by B and A by C it follows that $(B \vee C)^* = B^* \cap C^*$. Suppose $A \cap B = (0] = A \cap C$. Then $(a] \cap (b] = (0] = (a] \cap (c]$ for all $a \in A$, $b \in B$ and $c \in C$. Hence $a \in B^* \cap C^*$ for all $a \in A$. Hence $a \in (B \vee C)^*$ for all $a \in A$. Consequently $A \subseteq (B \vee C)^*$. It follows that $A \cap (B \vee C) = (0]$. Thus S is 0-distributive.

$10 \Rightarrow 1$: Suppose 10 holds. Let $a, a_1, \ldots, a_n \in S$ such that $(a] \cap (a_1] = \ldots = (a] \cap (a_n] = (0]$. Then $((a] \cap (a_1])^* = \ldots = ((a] \cap (a_n])^* = S$. Hence $((a] \cap (a_1])^* \cap \ldots \cap ((a] \cap (a_n])^* = S$. By 10, $((a] \cap ((a_1] \vee \ldots \vee (a_n]))^* = S$. Consequently $(a] \cap ((a_1] \vee \ldots \vee (a_n]) = (0]$. It follows that S is 0-distributive [see Theorem 2.3, 3].

$6 \Rightarrow 11$: Suppose 6 holds. Then for any finite number of ideals $A, A_1, \ldots, A_n$ of S, $(A \cap (A_1 \vee \ldots \vee A_n))^* = (A \cap A_1)^* \cap \ldots \cap (A \cap A_n)^*$. By taking $A = A_1 \vee \ldots \vee A_n$ it follows that $(A_1 \vee \ldots \vee A_n)^* = A_1^* \cap \ldots \cap A_n^*$. Hence the result.

$11 \Rightarrow 1$: Suppose 11 holds. Let $a, a_1, \ldots, a_n \in S$ be such that $(a] \cap (a_1] = \ldots = (a] \cap (a_n] = (0]$. Then $a \in (a_1]^* \cap \ldots \cap (a_n]^*$. By 11 it follows that $a \in ((a_1] \vee \ldots \vee (a_n])^*$. Hence $(a] \cap ((a_1] \vee \ldots \vee (a_n]) = (0]$. Thus S is 0-distributive [see Theorem 2.3, 3].

$2 \Rightarrow 12$: Suppose 2 holds. Let $A \in I(S)$. Then by 2 it follows that A^* is an ideal. If $B \in I(S)$ is such that $A \cap B = (0]$ and $x \in B$, then $a \wedge x = 0$ for all $a \in A$ and so $x \in A^*$. Thus $B \subseteq A^*$. It follows that A^* is the pseudocomplement of A.

$12 \Rightarrow 1$: Suppose 12 holds. Then every principal ideal of S has a pseudocomplement in $I(S)$. Let $a, a_1, \ldots, a_n \in S$ be such that $(a] \cap (a_1] = \ldots = (a] \cap (a_n] = (0]$. Then $(a_i] \subseteq (a]^*$ for $i = 1, \ldots, n$ and so $((a_1] \vee \ldots \vee (a_n]) \subseteq (a]^*$. Consequently $(a] \cap ((a_1] \vee \ldots \vee (a_n]) = (0]$. Thus S is 0-distributive [see Theorem 2.3, 3]. $\square$

Remark 2.6. According to Varlet [8], an ideal of a semilattice S is a nonempty subset I of S such that (i) $y \leqslant x$ and $x \in I$ imply $y \in I$; (ii) for any $x, y \in I$ there exists a $z \in I$ such that $z \geqslant x$ and $z \geqslant y$. According to him a semilattice S with 0 is said to be 0-distributive if for any $a \in S$, the subset $(a)^* = \{x \in S;\ x \wedge a = 0\}$ is an ideal.

Let S be a 0-distributive semilattice in Varlet's sense. Then for each $a \in S$, $(a)^*$ is a Varlet ideal and therefore an ideal in our sense. Thus S is 0-distributive in our sense. The converse is not true. Consider the semilattice $S = \{0, a, b, c\}$ in which the ordering is defined by $0 < a, b, c$; $a \| b$; $a \| c$; and $b \| c$. Clearly S is 0-distributive in our sense but not in Varlet's sense.

We give below some additional characterizations when the semilattice is finite.

Theorem 2.7. *Let S be a finite semilattice. Then the following statements are equivalent:*

1. *S is 0-distributive.*
2. *If a, b, c are elements of S such that $(a] \cap (b] = (0] = (a] \cap (c]$ then $(a] \cap ((b] \vee (c]) = (0]$.*
3. *Every maximal filter of S is prime.*
4. *Each nonzero element of S is contained in a prime filter.*
5. *If A is a nonempty subset of S and B is a proper filter intersecting A, there is a prime filter containing B and disjoint from A^*.*
6. *If A is a nonempty subset of S and B is a prime semiideal not containing A, there is a prime filter containing $S - B$ and disjoint from A^*.*
7. *For each nonzero element a of S and each proper filter B containing a, there is a prime filter containing B and disjoint from $(a)^*$.*
8. *For each nonzero element a of S and each prime semiideal B not containing a, there is a prime filter containing $S - B$ and disjoint from $(a)^*$.*
9. *If A and B are filters of S such that A and B^0 are disjoint, there is a prime filter containing A and disjoint from B^0.*
10. *If A is a filter of S and B is a prime semiideal containing A^0, there is a prime filter containing $S - B$ and disjoint from A^0.*

11. *For each nonzero element a in S and each filter A disjoint from $(a)^*$, there is a prime filter containing A and disjoint from $(a)^*$.*
12. *For each nonzero element a in S and each prime semiideal B containing $(a)^*$, there is a prime filter containing $S - B$ and disjoint from $(a)^*$.*

Proof. Obviously $1 \Rightarrow 2$, $6 \Rightarrow 8$, $10 \Rightarrow 12$, $5 \Rightarrow 7$ and $9 \Rightarrow 11$.

$2 \Rightarrow 1$: Suppose 2 holds and let $a, a_1, \dots, a_n \in S$ be such that $(a] \cap (a_1] = \dots = (a] \cap (a_n] = (0]$. Let $A = (a_1] \cup \dots \cup (a_n]$, let $B = \{b_1, \dots, b_m\}$ be the set of existing suprema of nonempty subsets of A and $b \in B$. Then $(a_1] \vee \dots \vee (a_n] = (b_1] \cup \dots \cup (b_m]$ and $b = c_1 \vee \dots \vee c_k$ for some $c_1, \dots, c_k \in A$. If $p, q \in \{1, \dots, k\}$, clearly b is an upperbound of $\{c_p, c_q\}$. Thus the set C of upperbounds of $\{c_p, c_q\}$ is nonempty and $\inf C = c_p \vee c_q$. Also $(a] \cap (c_p] = (0] = (a] \cap (c_q]$, so that $(a] \cap ((c_p] \vee (c_q]) = (0]$ by 2. It is easily seen that every nonempty subset of $\{c_1, \dots, c_k\}$ has a supremum and by induction it follows that $(a] \cap (b] = (a] \cap ((c_1] \vee \dots \vee (c_k]) = (0]$. Hence $(a] \cap ((a_1] \vee \dots \vee (a_n]) = (a] \cap ((b_1] \cup \dots \cup (b_m]) = ((a] \cap (b_1]) \cup \dots \cup ((a] \cap (b_m]) = (0]$. Consequently S is 0-distributive [see Theorem 2.3, 3].

$1 \Rightarrow 3$: Suppose 1 holds. Let M be any maximal filter of S. Since S is finite, every filter of S is principal. Let $a, b \in S - M$ be such that $[a) \cap [b) \neq \emptyset$. Let $[a) \cap [b) = \{c_1, \dots, c_n\}$ and $c = c_1 \wedge \dots \wedge c_n$. Then $c \geqslant a$, b as $c_i \geqslant a$, b for all i. If $d \in S$ and $d \geqslant a, b$, then $d = c_j$ for some j, so that $d \geqslant c$. Thus $c = a \vee b$. Also $S - M$ is an ideal [see Theorem 2.3, 4]. Hence $a \vee b \in S - M$. It follows that $[a) \cap [b) = [a \vee b) \nsubseteq M$, proving M is prime.

$3 \Rightarrow 4$: Suppose 3 holds. Let a be any nonzero element of S. By Lemma 1.5, $[a)$ is contained in a maximal filter M. By 3, M is prime. Clearly $a \in M$.

$4 \Rightarrow 1$: Suppose 4 holds. Let a be any nonzero element of S. By 4, $a \in B$ for some prime filter B. By Lemma 1.14, $S - B$ is a prime ideal and clearly $a \notin S - B$. It follows that S is 0-distributive [see Theorem 2.3, 9].

$3 \Rightarrow 5$: Suppose 3 holds. Let A be a nonempty subset of S and B a proper filter such that $B \cap A \neq \emptyset$. By Lemma 1.5, $B \subseteq M$ for some maximal filter M. By 3, M is prime. By Lemma 1.8, $S - M$ is a minimal prime semiideal and clearly $S - M \nsupseteq A$. Hence $S - M \supseteq A^*$ and so $M \cap A^* = \emptyset$.

$5 \Rightarrow 6$: Suppose 5 holds. Let A be a nonempty subset of S and B a prime semiideal such that $B \nsupseteq A$. By Lemma 7, $S - B$ is a proper filter and clearly $(S - B) \cap A \neq \emptyset$. By 5 there is a prime filter containing $S - B$ and disjoint from A^*.

$7 \Rightarrow 8$: Similar to $5 \Rightarrow 6$.

$8 \Rightarrow 1$: Suppose 8 holds and let a be any nonzero element of S. Now $S - [a)$ is a prime semiideal not containing a. By 8 there is a prime filter N containing $S - (S - [a)) = [a)$ and disjoint from $(a)^*$. By Lemma 1.14, $S - N$ is a prime ideal and clearly $a \notin S - N$. Thus S is 0-distributive [see Theorem 2.3, 9].

$3 \Rightarrow 9$: Suppose 3 holds. Let A and B be filters of S such that A and B^0 are disjoint. By Lemma 1.12, there is a minimal prime semiideal N such that $N \supseteq B^0$ and $N \cap A = \emptyset$. By Lemma 1.8, $S - N$ is a maximal filter. Clearly $S - N \supseteq A$ and $(S - N) \cap B^0 = \emptyset$. By 3, $S - N$ is prime.

$9 \Rightarrow 10$: Suppose 9 holds. Let A be a filter of S and B a prime semiideal such that $B \supseteq A^0$. By Lemma 1.7, $S - B$ is a proper filter and clearly $(S - B) \cap A^0 = \emptyset$. By 9, there is a prime filter containing $S - B$ and disjoint from A^0.

$11 \Rightarrow 12$: Similar to $5 \Rightarrow 6$.

$12 \Rightarrow 4$: Suppose 12 holds. Let a be any nonzero element of S. Now $S - [a)$ is a prime semiideal not containing $(a]$. Since $(a] \cap (a]^* = (0] \subseteq S - [a)$ it follows that $(a)^* \subseteq S - [a)$. By 12 there is a prime filter N containing $S - (S - [a)) = [a)$ and disjoint from $(a)^*$. Clearly $a \in N$. □

Theorem 2.8. *Let S be a finite semilattice. Then the following statements are equivalent:*

1. *S is 0-distributive.*
2. *For any finite number of filters $A, A_1, \ldots, A_n$ of S such that $A \cap A_i \neq \emptyset$ for all $i \in \{1, \ldots, n\}$,*
$$((A \cap A_1) \vee \ldots \vee (A \cap A_n))^0 = A^0 \cap (A_1 \vee \ldots \vee A_n)^0.$$
3. *For any three filters A, B, C of S such that $A \cap B \neq \emptyset$ and $A \cap C \neq \emptyset$,*
$$((A \cap B) \vee (A \cap C))^0 = A^0 \cap (B \vee C)^0.$$
4. *For all a, b, c in S such that $[a) \cap [b) \neq \emptyset$ and $[a) \cap [c) \neq \emptyset$,*
$$(([a) \cap [b)) \vee ([a) \cap [c)))^0 = [a)^0 \cap ([b) \vee [c))^0.$$
5. *For any finite number of filters $A, A_1, \ldots, A_n$ of S such that $A_1 \cap \ldots \cap A_n \neq \emptyset$,*
$$(A \vee (A_1 \cap \ldots \cap A_n))^0 = (A \vee A_1)^0 \cap \ldots \cap (A \vee A_n)^0.$$
6. *For any three filters A, B, C of S such that $B \cap C \neq \emptyset$,*
$$(A \vee (B \cap C))^0 = (A \vee B)^0 \cap (A \vee C)^0.$$
7. *For any finite number of elements $a, a_1, \ldots, a_n$ of S such that $[a_1) \cap \ldots \cap [a_n) \neq \emptyset$,*
$$([a) \vee ([a_1) \cap \ldots \cap [a_n)))^0 = ([a) \vee [a_1))^0 \cap \ldots \cap ([a) \vee [a_n))^0.$$

8. *For all* a, b, c *in* S, *with* $[b) \cap [c) \neq \emptyset$,

$$([a) \vee ([b) \cap [c)))^0 = ([a) \vee [b))^0 \cap ([a) \vee [c))^0.$$

9. *For any finite number of elements* $a_1, \ldots, a_n$ *of* S *such that* $[a_1) \cap \ldots \cap [a_n) \neq \emptyset$,

$$([a_1) \cap \ldots \cap [a_n))^0 = [a_1)^0 \cap \ldots \cap [a_n)^0.$$

10. *For all* a, b *in* S *with* $[a) \cap [b) \neq \emptyset$, $([a) \cap [b))^0 = [a)^0 \cap [b)^0$.
11. *For all* a, b, c *in* S, $((a] \cap ((b] \vee (c]))^* = ((a] \cap (b])^* \cap ((a] \cap (c])^*$.
12. *For all* a, b, c *in* S,

$$(((a] \vee (b]) \cap ((a] \vee (c]))^* = (a]^* \cap ((b] \cap (c])^*.$$

13. *For all* a, b *in* S, $((a] \vee (b])^* = (a]^* \cap (b]^*$.

Proof. $1 \Rightarrow 2$: Suppose 1 holds and let $A, A_1, \ldots, A_n$ be filters of S such that $A \cap A_i \neq \emptyset$ for all $i \in \{1, \ldots, n\}$. If Q is any minimal prime ideal of S such that $Q \cap ((A \cap A_1) \vee \ldots \vee (A \cap A_n)) = \emptyset$, then $Q \cap (A \cap A_1) = \ldots = Q \cap (A \cap A_n) = \emptyset$. By Lemma 1.14, $S - Q$ is a prime filter and $S - Q \supseteq (A \cap A_1), \ldots, (A \cap A_n)$. Hence $S - Q \supseteq A$ or $S - Q \supseteq A_1 \vee \ldots \vee A_n$ and so $Q \cap A = \emptyset$ or $Q \cap (A_1 \vee \ldots \vee A_n) = \emptyset$. It follows that $((A \cap A_1) \vee \ldots \vee (A \cap A_n))^0 \supseteq A^0 \cap (A_1 \vee \ldots \vee A_n)^0$ [see Theorem 2.5, 3]. The reverse inclusion is obvious.

Obviously $2 \Rightarrow 3 \Rightarrow 4$, $5 \Rightarrow 6 \Rightarrow 8$ and $5 \Rightarrow 7 \Rightarrow 8$.

$4 \Rightarrow 10$: Follows by taking $c = b$ in 4.

$1 \Rightarrow 5$: Suppose 1 holds. Let $A, A_1, \ldots, A_n$ be filters of S such that $A_1 \cap \ldots \cap A_n \neq \emptyset$. If Q is any minimal prime ideal of S such that $Q \cap (A \vee (A_1 \cap \ldots \cap A_n)) = \emptyset$, then $Q \cap A = \emptyset = Q \cap (A_1 \cap \ldots \cap A_n)$. By Lemma 1.14, $S - Q$ is a prime filter and clearly $S - Q \supseteq A$, $A_1 \cap \ldots \cap A_n$. Hence $S - Q \supseteq A \vee A_j$ and so $Q \cap (A \vee A_j) = \emptyset$ for some $j \in \{1, \ldots, n\}$. It follows that $(A \vee (A_1 \cap \ldots \cap A_n))^0 \supseteq (A \vee A_1)^0 \cap \ldots \cap (A \vee A_n)^0$ [see Theorem 2.5, 3]. The reverse inclusion is obvious.

$10 \Rightarrow 9$: Suppose 10 holds and let $a_1, \ldots, a_n \in S$ be such that $[a_1) \cap \ldots \cap [a_n) \neq \emptyset$. Then $([a_1) \cap [a_2))^0 = [a_1)^0 \cap [a_2)^0$. Assume $([a_1) \cap \ldots \cap [a_{k-1}))^0 = [a_1)^0 \cap \ldots \cap [a_{k-1})^0$ for $2 < k \leqslant n$. Let $x \in [a_1)^0 \cap \ldots \cap [a_k)^0$. Then $x \in [a_1)^0 \cap \ldots \cap [a_{k-1})^0 = ([a_1) \cap \ldots \cap [a_{k-1}))^0$ by our induction hypothesis. Hence $x \wedge y = 0$ for some $y \in ([a_1) \cap \ldots \cap [a_{k-1}))$. Thus $x \in [y)^0 \cap [a_k)^0 = ([y) \cap [a_k))^0 \subseteq ([a_1) \cap \ldots \cap [a_k))^0$ so that $([a_1)^0 \cap \ldots \cap [a_k)^0) \subseteq ([a_1) \cap \ldots \cap [a_k))^0$. The reverse inclusion is obvious. By induction it follows that $([a_1) \cap \ldots \cap [a_n))^0 = [a_1)^0 \cap \ldots \cap [a_n)^0$.

$9 \Rightarrow 1$: Suppose 9 holds. Let $a \in S$ and let $a_1, \ldots, a_n \in (a)^*$ be such that $a_1 \vee \ldots \vee a_n$ exists. Then $a \wedge a_1 = \ldots = a \wedge a_n = 0$ and so $a \in [a_1)^0 \cap \ldots \cap [a_n)^0 =$

$([a_1) \cap \ldots \cap [a_n))^0$ by 9. That is $a \in [a_1 \vee \ldots \vee a_n)^0$. Hence $a \wedge (a_1 \vee \ldots \vee a_n) = 0$, so that $a_1 \vee \ldots \vee a_n \in (a)^*$. Thus $(a)^*$ is an ideal. It follows that S is 0-distributive [see Theorem 2.5, 4].

$8 \Rightarrow 1$: Suppose 8 holds and let a, b, $c \in S$ such that $(a] \cap (b] = (0] = (a] \cap (c]$. Let $X = \{x_1, \ldots, x_n\}$ be the set of existing suprema of nonempty subsets of $(b] \cup (c]$ and $x \in X$. Then $(b] \vee (c] = (x_1] \cup \ldots \cup (x_n]$ and $x = y_1 \vee \ldots \vee y_m$ for some $y_1, \ldots, y_m \in (b] \cup (c]$. If p, $q \in \{1, \ldots, m\}$, clearly x is an upperbound of $\{y_p, y_q\}$. Thus the set Y of upperbounds of $\{y_p, y_q\}$ is nonempty and $\inf Y = y_p \vee y_q$. Also $a \wedge y_p = 0 = a \wedge y_q$. Hence $([a) \vee [y_p))^0 = S = ([a) \vee [y_q))^0$. Let $z \in (a] \cap ((y_p] \vee (y_q])$. Then $z \leqslant a$ and $z \leqslant y_p \vee y_q$. Now $z \in S = ([a) \vee [y_p))^0 \cap ([a) \vee [y_q))^0 = ([a) \vee ([y_p) \cap [y_q))^0 = ([a) \vee [y_p \vee y_q))^0$ by 8, so that $z \wedge t = 0$ for some $t \in [a) \vee [y_p \vee y_q)$. Thus $z = z \wedge a \wedge (y_p \vee y_q) \leqslant z \wedge t = 0$ and consequently $(a] \cap ((y_p] \vee (y_q]) = (0]$. It is easily seen that every nonempty subset of $\{y_1, \ldots, y_m\}$ has a supremum and by induction it follows that $(a] \cap (x] = (a] \cap ((y_1] \vee \ldots \vee (y_m]) = (0]$. Hence $(a] \cap ((b] \vee (c]) = (a] \cap ((x_1] \cup \ldots \cup (x_n]) = ((a] \cap (x_1]) \cup \ldots \cup ((a] \cap (x_n]) = (0]$. Thus S is 0-distributive [see Theorem 2.7, 2].

$1 \Rightarrow 11$: Suppose 1 holds. Then for all A, B, $C \in I(S)$ we have $(A \cap (B \vee C))^* = (A \cap B)^* \cap (A \cap C)^*$ [see Theorem 2.5, 7]. Hence 11 follows.

$1 \Rightarrow 12$: Suppose 1 holds. Then for all A, B, $C \in I(S)$ we have $((A \vee B) \cap (A \vee C))^* = A^* \cap (B \cap C)^*$ [see Theorem 2.5,9]. Hence 12 follows.

$12 \Rightarrow 13$: Follows by taking $c = b$ in 12.

$13 \Rightarrow 1$: Suppose 13 holds. Let a, b, $c \in S$ be such that $(a] \cap (b] = (0] = (a] \cap (c]$. Then $a \in (b]^* \cap (c]^* = ((b] \vee (c])^*$ by 13. Hence $(a] \cap ((b] \vee (c]) = (0]$. Thus S is 0-distributive [see Theorem 2.7, 2].

$11 \Rightarrow 1$: Suppose 11 holds. Let a, b, $c \in S$ be such that $(a] \cap (b] = (0] = (a] \cap (c]$. Then $((a] \cap (b])^* \cap ((a] \cap (c])^* = S$. Hence By 11, $((a] \cap ((b] \vee (c]))^* = S$. It follows that $(a] \cap ((b] \vee (c]) = (0]$. Thus S is 0-distributive [see Theorem 2.7, 2]. $\square$

Theorem 2.9. *Any one of the conditions* 3 *to* 12 *of Theorem* 2.7 *is sufficient for a semilattice* S *with* 0 *(not necessarily finite) to be* 0*-distributive. These conditions are also necessary in the case of a lattice.*

P r o o f. Suppose 3 of Theorem 2.7 holds and let M be any maximal filter of S. By Lemma 1.8, $S - M$ is a minimal prime semiideal. Let $x_1, \ldots, x_n \in S - M$ and suppose $x_1 \vee \ldots \vee x_n$ exists. By 3, M is prime and clearly $[x_i) \nsubseteq M$ for $i = 1, \ldots, n$. Hence by Lemma 1.2, $[x_1 \vee \ldots \vee x_n) = [x_1) \cap \ldots \cap [x_n) \nsubseteq M$. Consequently $x_1 \vee \ldots \vee x_n \in S - M$ and so $S - M$ is an ideal. It follows that S is 0-distributive [see Theorem 2.3, 4]. $\square$

The sufficiency of the condition 4 of Theorem 2.7 follows by Lemma 1.14 and Theorem 2.3 [see Theorem 2.3, 9]. The sufficiency of the conditions 5 to 12 of Theorem 2.7 follows by Lemma 1.14 and Theorem 2.4 [see Theorem 2.4, 3, 5, 6, 7, 9, 11, 12, 13].

Theorem 2.10. *Any one of the conditions* 2 *to* 10 *of Theorem* 2.8 *is sufficient for a semilattice S with* 0 *(not necessarily finite) to be* 0*-distributive. These conditions are also necessary in the case of a lattice.*

Proof. Obviously $2 \Rightarrow 3 \Rightarrow 4$ and $5 \Rightarrow 6 \Rightarrow 8$.

$4 \Rightarrow 10$: Follows by taking $c = b$ in 4.

$10 \Rightarrow 9$: Same proof as in Theorem 2.8.

Suppose 9 holds. Let $a \in S$ and let $a_1, \ldots, a_n \in (a)^*$ be such that $a_1 \vee \ldots \vee a_n$ exists. Then $a \wedge a_1 = \ldots = a \wedge a_n = 0$ and so $a \in [a_1)^0 \cap \ldots \cap [a_n)^0 = ([a_1) \cap \ldots \cap [a_n))^0$ by 9. That is $a \in [a_1 \vee \ldots \vee a_n)^0$. It follows that $a \wedge (a_1 \vee \ldots \vee a_n) = 0$. Hence $a_1 \vee \ldots \vee a_n \in (a)^*$. Thus $(a)^*$ is an ideal and so S is 0-distributive [see Theorem 2.5, 4].

$8 \Rightarrow 7$: Suppose 8 holds and let $a, a_1, \ldots, a_n \in S$ be such that $[a_1) \cap \ldots \cap [a_n) \neq \emptyset$. Then $([a) \vee ([a_1) \cap [a_2)))^0 = ([a) \vee [a_1))^0 \cap ([a) \vee [a_2))^0$. Assume $([a) \vee ([a_1) \cap \ldots \cap [a_{k-1})))^0 = ([a) \vee [a_1))^0 \cap \ldots \cap ([a) \vee [a_{k-1}))^0$ for $2 < k \leqslant n$. Let $x \in ([a) \vee [a_1))^0 \cap \ldots \cap ([a) \vee [a_k))^0$. Then $x \in ([a) \vee [a_1))^0 \cap \ldots \cap ([a) \vee [a_{k-1}))^0 = ([a) \vee ([a_1) \cap \ldots \cap [a_{k-1})))^0$ by our induction hypothesis and $x \in ([a) \vee [a_k))^0$. Hence $x \wedge y = a$ for some $y \in [a) \vee ([a_1) \cap \ldots \cap [a_{k-1}))$ and $x \wedge z = 0$ for some $z \in [a) \vee [a_k)$. Thus $x \wedge a \wedge t = 0$ for some $t \in [a_1) \cap \ldots \cap [a_{k-1})$ and $x \wedge a \wedge a_k = 0$ so that $x \in [a \wedge t)^0 \cap [a \wedge a_k)^0 = ([a) \vee [t))^0 \cap ([a) \vee [a_k))^0 = ([a) \vee ([t) \cap [a_k)))^0$ by 8. Consequently $x \wedge a \wedge u = 0$ for some $u \in [t) \cap [a_k) \subseteq [a_1) \cap \ldots \cap [a_k)$ and so $x \in ([a) \vee ([a_1) \cap \ldots \cap [a_k)))^0$. Thus $([a) \vee [a_1))^0 \cap \ldots \cap ([a) \vee [a_k))^0 \subseteq ([a) \vee ([a_1) \cap \ldots \cap [a_k)))^0$. The reverse inclusion is obvious. By induction it follows that $([a) \vee [a_1))^0 \cap \ldots \cap ([a) \vee [a_n))^0 = ([a) \vee ([a_1) \cap \ldots \cap [a_n)))^0$.

Suppose 7 holds. Let $a \in S$ and let $a_1, \ldots, a_n \in (a)^*$ be such that $a_1 \vee \ldots \vee a_n$ exsits. Then $a \wedge a_1 = \ldots = a \wedge a_n = 0$ and so $a \in [a_1)^0 \cap \ldots \cap [a_n)^0$. Replacing a by $a_1 \vee \ldots \vee a_n$ in 7, we have $([a_1) \cap \ldots \cap [a_n))^0 = [a_1)^0 \cap \ldots \cap [a_n)^0$. Thus $a \in ([a_1) \cap \ldots \cap [a_n))^0 = [a_1 \vee \ldots \vee a_n)^0$. Hence $a \wedge (a_1 \vee \ldots \vee a_n) = 0$ and consequently $a_1 \vee \ldots \vee a_n \in (a)^*$. Thus $(a)^*$ is an ideal. It follows that S is 0-distributive [see Theorem 2.5, 4]. □

Remark 2.11. The conditions 3 to 12 of Theorem 2.7 are not necessary for an infinite semilattice to be 0-distributive. These conditions are both necessary and sufficient in the case of a lattice.

Clearly each of the conditions 3 to 12 implies the condition 4. Hence it is enough to prove that 4 is not necessary.

Let C be an infinite chain without the least element and $S = C \cup \{0, a, b, d\}$. Define an ordering on S as follows: $0 < a, b, d$; $a \| b$; $a \| d$; $b \| d$ and $a, b, d < c$ for all $c \in C$. Clearly S is a 0-distributive semilattice with respect to this ordering. But no prime filter of S contains the nonzero element a. Thus 4 is not necessary.

Remark 2.12. The conditions 2 to 10 of Theorem 2.8 are not necessary for an infinite semilattice to be 0-distributive. These conditions are both necessary and sufficient in the case of a lattice.

Clearly $2 \Rightarrow 3 \Rightarrow 4 \Rightarrow 10$, $5 \Rightarrow 6 \Rightarrow 8$, $7 \Rightarrow 8$, and $9 \Rightarrow 10$. Hence it is enough to prove that 8 and 10 are not necessary.

Let C be an infinite chain without the least element and $S = C \cup \{0, a, b, d, e\}$. Define an ordering on S as follows: $0 < a, b, d, e$; $a < e$; $a \| b$; $a \| d$; $b \| d$; $b \| e$; $d \| e$; $a, b, d, < c$ for all $c \in C$; $e \| c$ for all $c \in C$. It is easily seen that S is a 0-distributive semilattice with respect to this ordering. Now $[e) \vee [b) = S = [e) \vee [d)$, so that $([e) \vee [b))^0 \cap ([e) \vee [d))^0 = S$. Also $[e) \vee ([b) \cap [d)) = [a)$ and hence $([e) \vee ([b) \cap [d)))^0 = \{0, b, d\}$. Thus $([e) \vee ([b) \cap [d)))^0 \neq ([e) \vee [b))^0 \cap ([e) \vee [d))^0$, proving 8 is not necessary.

Consider the 0-distributive semilattice S from Remark 2.11. Now $([a) \cap [b))^0 = \{0\}$ and $[a)^0 \cap [b)^0 = \{0, d\}$. Thus $([a) \cap [b))^0 \neq [a)^0 \cap [b)^0$, proving 10 is not necessary.

Remark 2.13. The condition 2 of Theorem 2.7 and the conditions 11, 12, 13 of Theorem 2.8 are necessary for a semilattice (not necessarily finite) to be 0-distributive.

Proof. The necessity of the condition 2 of Theorem 2.7 is obvious. The necessity of the conditions 11, 12, 13 of Theorem 2.8 follows by Theorem 2.5 [see Theorem 2.5, 10, 8, 11]. $\square$

Remark 2.14. The condition 2 of Theorem 2.7 and the conditions 11, 12, 13 of Theorem 2.8 are not sufficient for an infinite semilattice with 0 to be 0-distributive.

Clearly the condition 12 of Theorem 2.8 implies the condition 13 of Theorem 2.8 and the condition 13 of Theorem 2.8 implies the condition 2 of Theorem 2.7. Hence it is enough to show that the conditions 11 and 12 of Theorem 2.8 are not sufficient.

Let C_1, C_2, C_3 be infinite chains without greatest and least elements and let $S = C_1 \cup C_2 \cup C_3 \cup \{0, a, b, c, d, e, f, g, 1\}$. Define an ordering on S as follows. $0 < a, b, c, d$; $a < e$; $b < f$; $c < g$; $d < e$; $d < f$; $d < g$; $e < c_1 < 1$ for all $c_1 \in C_1$; $e < c_2 < 1$ for all $c_2 \in C_2$; $f < c_1$ for all $c_1 \in C_1$; $f < c_3 < 1$ for all $c_3 \in C_3$; $g < c_2$ for all $c_2 \in C_2$; $g < c_3$ for all $c_3 \in C_3$; $a \| b$; $a \| c$; $a \| d$; $a \| f$; $a \| g$; $a \| c_3$ for all $c_3 \in C_3$; $b \| c$; $b \| d$; $b \| e$; $b \| g$; $b \| c_2$ for all $c_2 \in C_2$; $c \| d$; $c \| e$; $c \| f$; $c \| c_1$ for all $c_1 \in C_1$; $c_1 \| c_2$ for all $c_1 \in C_1$ and $c_2 \in C_2$; $c_1 \| c_3$ for all $c_1 \in C_1$ and $c_3 \in C_3$; $c_2 \| c_3$ for all $c_2 \in C_2$ and $c_3 \in C_3$. Clearly S is a semilattice with respect to this ordering.

Also for all $x, y, z \in S$, we have $((x] \cap ((y] \vee (z]))^* = ((x] \cap (y])^* \cap ((x] \cap (z])^*$ and $((x] \vee (y]) \cap ((x] \vee (z])^* = (x]^* \cap ((y] \cap (z])^*$. Now $(d] \cap (a] = (0] = (d] \cap B$ where $B = (b] \vee (c]$. But $(d] \cap ((a] \vee B) \neq (0]$. Thus S is not 0-distributive.

I would like to thank Prof. P. V. Venkatanarasimhan for his valuable suggestions in the preparation of this paper. I also thank the referee whose valuable comments helped in shaping the paper into its present form.

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